The purpose of this course is basically to introduce you the developments in the study of Hamilton-Jacobi equations in these 5 years or so. It will also cover the recent developments in the uniqueness questions of solutions of fully nonlinear elliptic PDEs. About 8 years ago Crandall and Lions introduced the notion of viscosity solution and showed this solution is very nice to have existence and uniqueness of solution of H-J equations.

I will explain the definition of visc. solution, some of uniqueness results of solutions of H-J equation how to prove them, some of existence results, the extensions of those uniqueness and existence results to 2nd order, fully nonlinear elliptic or parabolic

PDEs, the role of visc. solutions in optimal control or differential games and applications of viscosity solution to asymptotic problems.

## §1. Hamilton-Jacobi equations

I am going to tell you why we need "viscosity solution". In calculus of variations, geometrical optics, optimal control and differential games we encounter H-J equations

$$u_t + H(x, t, Du) = 0 \qquad x \in \mathbb{R}^N, \quad t > 0,$$

and also

$$\lambda u + H(x, Du) = 0 \qquad x \in \mathbb{R}^N.$$

Here $u = u(x,t)$ is supposed to be unknown, $u_t = \partial u/\partial t$, $Du = (\partial u/\partial x_1, \cdots, \partial u/\partial x_N)$, $H$ and $u$ are real-valued functions and $\lambda \geq 0$ is a constant. I will call $H$

a hamiltonian.

We find Hamiltonians of the form

(1) $$H(x,p) = \sum_{i,j=1}^{N} a_{ij}(x) p_i p_j - V(x)$$

in the calculus of variations and in the geometrical optics, (where $(a_{ij}(x))_{1 \le i,j \le N} \ge 0$.) Hamiltonians which appears in optimal control take the form

(2) $$H(x,p) = \max_{a \in A} \{-g(x,a) \cdot p - f(x,a)\},$$

where the set $A$ corresponds to the control set, $f$ corresponds to running cost and $g$ corresponds to the generator of the dynamics. Notice that you can express Hamiltonians (1) in the form of (2).

Hamiltonians in differential games have the form

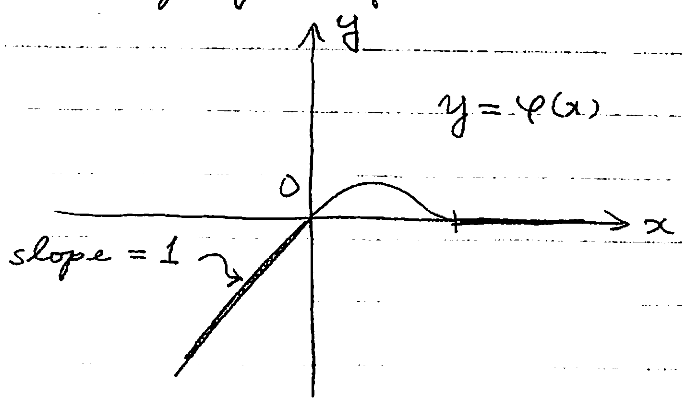
(3) $$H(x,p) = \min_{b \in B} \max_{a \in A} \{-g(x,a,b) \cdot p - f(x,a,b)\}.$$

Let us consider some of simple examples of H-J equations.

<u>Example 1</u> Let $N=1$ and consider the initial value problem

$$(4) \quad \begin{cases} u_t + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R} \times [0,1], \\ u(x,0) = \varphi(x) & x \in \mathbb{R}. \end{cases}$$

Assume the graph of $\varphi$ has the following shape:

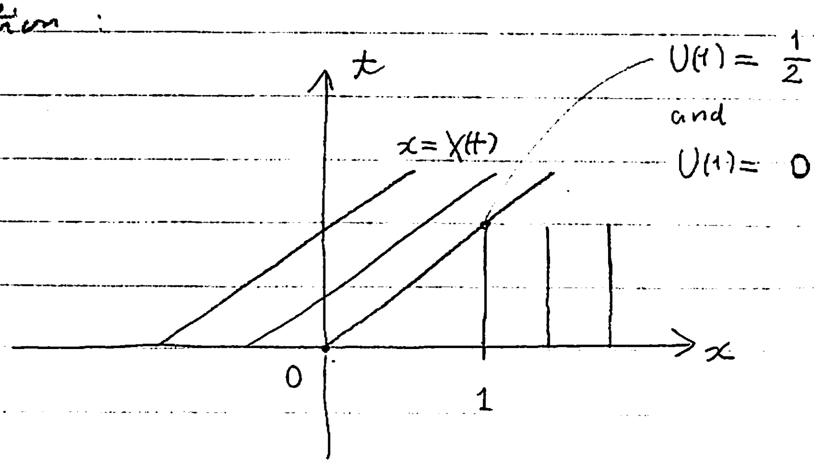



Then the problem does not admit a $C^1$ solution.

In the category of $C^2$ solutions, it is easy to see this kind of break down of the existence of solutions. Indeed, the characteristic equations for (4) are:

$$\dot{X}(t) = P(t), \quad \dot{P}(t) = 0, \quad X(0) = x, \quad P(0) = D\varphi(x).$$
$$\dot{U}(t) = p^2 - \tfrac{1}{2}p^2 = \tfrac{1}{2}p^2, \quad U(0) = \varphi(x)$$

The method of characteristics tells you that along the characteristic line $x = X(t)$, you have $u(x,t) = U(t) \equiv \varphi(x) + \frac{t}{2}|D\varphi|^2$ and $Du(x,t) = P(t) \equiv D\varphi(x)$. That is, we are in the following situation:




This analysis works only for $C^2$ solutions, and the problem (4) requires for solutions to be $C^1$ in order that (4) has a classical meaning. I need a lemma to show Example 1.

Lemma 1 If $v \in C^1([a,b]\times[0,1])$ satisfies

$$\begin{cases} v_t + \frac{1}{2}|Dv|^2 = 0 & \text{in } [a,b]\times[0,1], \\ v(x,0) = 0 & \text{in } [a,b], \end{cases}$$

definition of $M$, we have

$$|X(s)-x| \leq \int_s^t \left|\tfrac{1}{2} Dv(X(\tau),\tau)\right| d\tau \leqq \frac{M}{2}(t-s),$$

and so

$$\left|X(s)-\frac{a+b}{2}\right| \leq |X(s)-x| + \left|x-\frac{a+b}{2}\right| \leq \frac{b-a}{2} - \frac{M}{2}s.$$

That is, $(X(s),s) \in \Delta(\frac{M}{2})$ and in particular, $X(s)$ is continued up to $s=0$. Now, we have

$$\frac{d}{ds} v(X(s),s) = Dv\cdot\dot{X} + v_t = v_t + \frac{1}{2}|Dv|^2 = 0,$$

and so

$$v(x,t) = v(X(0),0) = 0.$$

Thus, $v \equiv 0$ in $\Delta(\frac{M}{2})$. We now set

$$m_0 = \inf\left\{0 \leq m \leq \frac{M}{2} : v \equiv 0 \text{ in } \Delta(m)\right\}.$$

Obviously, $0 \leq m_0 \leq \frac{M}{2}$. If $m_0 = 0$, then we are done. We assume $m_0 > 0$. By the continuity of $Dv$, we can find $0 < m < m_0$ such that

$$|Dv| \leq m \quad \text{on} \quad \Delta(m).$$

Arguing as before, we find

$$v \equiv 0 \quad \text{on} \quad \Delta(m).$$

This is a contradiction. That is, we always have $m_1 = 0$. $\blacksquare$

$\Longrightarrow$ Insert here 8+1

When we can not expect any classical solution, a standard way in PDE theory to define "weak solution" is the use of distributions. The basic idea there is to remove the differential operation from solutions and put it to test functions using the formula of integration by parts. If the equation under consideration has the divergence form, then it is easy to adapt this idea. However, H-J equations do not have the divergence form, and therefore we have to give up this idea.

The second natural idea to define "weak solution" would be the requirement that a solution be

Proof of Example 1 Suppose there is a $C^1$ solution $u$ of (4). By Lemma 1 we have

$$u \equiv 0 \quad \text{on} \quad [1,\infty) \times [0,1].$$

Put $v(x,t) = u(x+t,t) - x - \frac{t}{2}$ for $x \le 0$ and $0 \le t \le 1$.

Then we have

$$\begin{cases} v_t + \frac{1}{2}|Dv|^2 = u_t + Du - \frac{1}{2} + \frac{1}{2}|Du-1|^2 = u_t + \frac{1}{2}|Du|^2 = 0 \\ v(x,0) = 0 \quad \text{for } x \le 0 \end{cases}$$

Therefore, by Lemma 1 we have

$$v(x,t) \equiv 0 \quad \text{for } x \le 0, \quad 0 \le t \le 1.$$

Thus

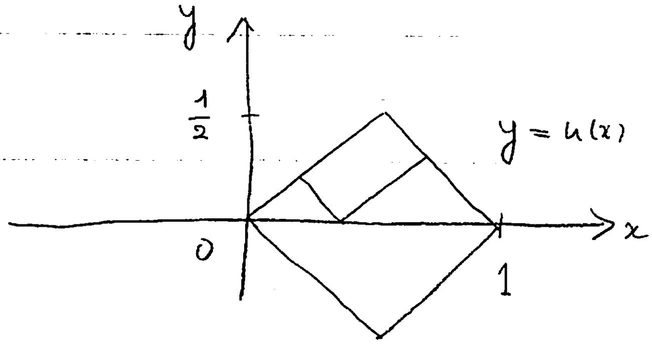
$$0 = v(0,1) = u(1,1) - \frac{1}{2} = -\frac{1}{2}; \quad \text{a contradiction.} \ \blacksquare$$

absolutely continuous and it satisfies the given equation in the a.e. sense. However, simple examples show that this is not good enough.

Example 2 Let $N = 1$, and consider the problem

$$|Du| = 1 \text{ a.e. in } (0,1) \quad \text{and} \quad u(0) = u(1) = 0.$$

Absolutely continuous functions which satisfy these requirements are depicted in the following drawing.

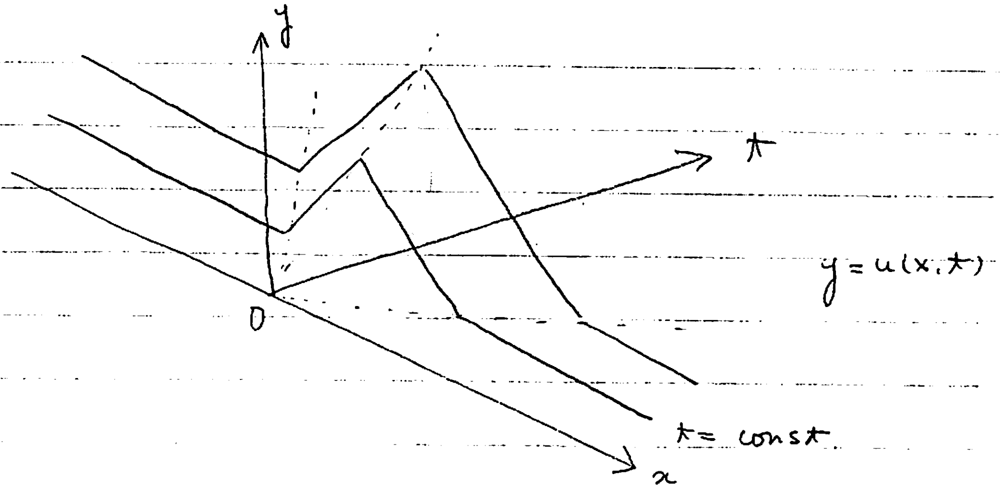



Example 3 Again, let $N = 1$, and consider the problem

$$u_t - |Du| = 0 \text{ a.e. in } \mathbb{R} \times [0,\infty) \quad \text{and} \quad u(x,0) = 0 \quad \forall x \in \mathbb{R}.$$

The functions $u \equiv 0$ and 
$$u = \max\{0, t - |x|\} = \max\{0, \min\{t-x, t+x\}\}$$

satisfy our requirements.




A good answer was given independently by Kruzkov and Douglis in '60s to this uniqueness problem, i.e. how to resolve the lack of uniqueness. In addition to the requirement that a solution should satisfy the given equation a.e., they required that it should be semiconcave. The semiconcavity of $u(x)$ (on $\mathbb{R}^N$) means that the inequality

$$u(x+h) - 2u(x) + u(x-h) \le C|h|^2$$

holds for $\forall x, h \in \mathbb{R}^n$ and some $C > 0$.

This notion of weak solution was also successful in existence of solutions. However, for the existence of such weak solutions, stringent regularity assumptions on Hamiltonians and initial data are usually required. Especially, the convexity of Hamiltonians is needed. This is not the case for Hamiltonians from differential games.

At the beginning of 1980's, the notion of viscosity solutions is introduced by Crandall and Lions. This notion made it possible not only to get a nice existence and uniqueness result for solutions of H-J equations including those from differential games, but also provided an easy way to access such a hard subject.

## §2. Definition of viscosity solution and its basic properties

Let $\Omega$ be an open subset of $\mathbb{R}^N$. I'm going to define viscosity solutions of

(2.1) $\quad F(x, u, Du) = 0 \quad$ in $\Omega$.

~~Throughout this section we assume F is continuous.~~

Definition 2.1 Let $u : \Omega \to \mathbb{R}$ and $y \in \Omega$.

Define

$$D^+u(y) = \{p \in \mathbb{R}^N : u(x) \leq u(y) + p\cdot(x-y) + o(|x-y|) \quad \text{as } x \to y\},$$

$$D^-u(y) = \{p \in \mathbb{R}^N : u(x) \geq u(y) + p\cdot(x-y) + o(|x-y|) \quad \text{as } x \to y\},$$

where $o(r)/r \to 0$ as $r \to 0$.

The sets $D^+u(y)$ and $D^-u(y)$ are called, resp., super- and subdifferentials of $u$ at $y$.

It is easy to see that

$u$ differentiable at $y \Rightarrow D^+u(y) = D^-u(y) = \{Du(y)\}$,

$D^+u(y) \cap D^-u(y) \neq \emptyset \Rightarrow u$ differentiable at $y$.

Proposition 2.1 (i) $p \in D^+u(y) \Leftrightarrow$ There is a $\varphi \in C^1(\Omega)$

such that $D\varphi(y) = p$ and $u-\varphi$ has a local maximum at $y$. (ii) $p \in D^-u(y) \Leftrightarrow$ There is a $\varphi \in C^1(\Omega)$ such that $D\varphi(y) = p$ and $u-\varphi$ has a local minimum at $y$.

<u>Proof</u> The proof of $(\Leftarrow)$ part is easy. We prove $(\Rightarrow)$ for (i). Let $p \in D^+u(y)$. By definition,

$$u(x) \leq u(y) + p\cdot(x-y) + \omega(|x-y|)\,|x-y| \quad \forall x \in B(y,\delta)$$

for some $\delta > 0$ and $\omega \in C([0,\infty))$ satisfying $\omega(0) = 0$. We may assume $\omega$ is increasing. Define

$$\rho(r) = \int_0^r \omega(s)\,ds \qquad \text{for} \quad r \geq 0.$$

Clearly, $\rho \in C^1([0,\infty))$, $\rho(0) = \rho'(0) = 0$ and

$$\rho(2r) \geq \int_r^{2r} \omega(s)\,ds \geq \omega(r)\,r.$$

Setting $\varphi(x) = u(y) + p\cdot(x-y) + \rho(2|x-y|)$, we see that $\varphi \in C^1(\mathbb{R}^N)$, $D\varphi(y) = p$ and

$$\max_{x \in B(y,\delta)} (u-\varphi)(x) = 0 = (u-\varphi)(0).$$

To see (ii, $\Rightarrow$), one has just to note that

$$D^- u(y) = -\left(D^+(-u)(y)\right)$$

and apply (i, $\Rightarrow$) to $-u$. ∎

If $u \in C(\Omega)$, then $\{x \in \Omega : D^+u(x) \neq \phi\}$ is dense in $\Omega$. Indeed, let $y \in \Omega$ and consider the function

$$x \to u(x) - \frac{1}{\varepsilon}|x-y|^2 \qquad \varepsilon > 0.$$

This function has a local maximum at a point as close (to $y$) as you desire by choosing $\varepsilon$ small enough. It is interesting to compare (this fact) with the fact that there are continuous functions which are nowhere differentiable.

<u>Definition 2.2.</u> ~~Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous.~~

Let $u \in C(\Omega)$. (i) We call $u$ a <u>viscosity subsolution</u> of (2.1) (or a viscosity solution of $F(x, u, Du) \leq 0$ in $\Omega$) if

$$F(x, u(x), p) \leq 0 \qquad \forall x \in \Omega, \quad \forall p \in D^+u(x).$$

(ii) $u$ is called a <u>visc.</u> <u>supersolution</u> of (2.1) (or visc. solution of $F(x,u,Du) \geq 0$ in $\Omega$) if

$$F(x,u(x),p) \geq 0 \quad \forall x \in \Omega, \quad \forall p \in D^- u(x).$$

(iii) $u$ is called a <u>viscosity</u> <u>solution</u> of (2.1) if it is a visc. sub- and supersolution of (2.1).

$\Rightarrow$ Insert here 15+1

<u>Proposition 2.2</u> Let $u \in C$. (i) $u$ is a visc. subsolution of (2.1) $\Leftrightarrow$ For $\forall \varphi \in C^1(\Omega)$ and $\forall$ local maximum point $y$ of $u - \varphi$, $F(y, u(y), D\varphi(y)) \leq 0$.

(ii) $u$ is a visc. supersolution of (2.1) $\Leftrightarrow$ For $\forall \varphi \in C^1(\Omega)$ and $\forall$ local minimum point $y$ of $u-\varphi$, $F(y, u(y), D\varphi(y)) \geq 0$.

This is a direct consequence of Prop. 2.1.

In the above proposition "local maximum (minimum)" can be replaced by "local strict maximum (minimum)" by adding $\pm |x-y|^2$ to $\varphi$. Also it can be replaced by "global (strict) maximum"

Observe here that if $u$ is a visc. subsolution of (2.1) and differentiable at $y \in \Omega$, then $F(y, u(y), Du(y)) \le 0$, and ~~also that~~ 15+1 if $u$ is a $C^1$ subsolution of (2.1), then it is a visc. subsolution.

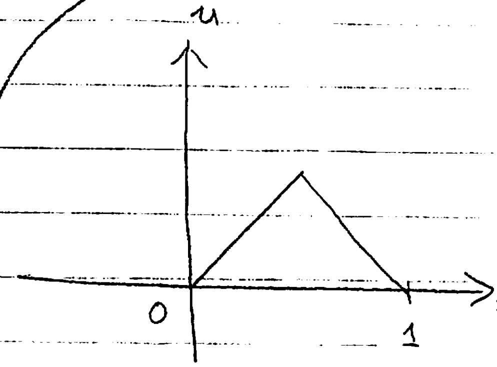
Let us go back to a simple example.

Example 2.1 Let $N = 1$, and consider the problem

$$|Du| = 1 \quad \text{in} \quad (0, 1) \quad \text{and} \quad u(0) = u(1) = 0.$$

Let

$$u(x) = \begin{cases} x & (0 \le x \le 1/2) \\ 1 - x & (1/2 \le x \le 1). \end{cases}$$




Then

$$\begin{aligned} D^+u(x) &= \{1\} & (0 < x < 1/2) \\ &= [-1, 1] & (x = 1/2) \\ &= \{-1\} & (1/2 < x < 1), \end{aligned}$$

Similar assertions are valid also for supersolutions and solutions.

and

$$D^-u(x) = \begin{cases} \{1\} & (0 < x < 1/2) \\ \phi & (x = 1/2) \\ \{-1\} & (1/2 < x < 1). \end{cases}$$

Therefore $u$ is a visc. solution of the above problem. (Notice that $u$ is concave.) On the other hand, $v = -u$ is not a visc. supersolution of the problem since $D^-v(1/2) = [-1, 1] \ni 0$ and $|0| < 1$. We will see that $u$ is the unique visc. solution of the above problem.

by adding a suitable $\psi \in C^1(\Omega)$ with $D\psi(y) = 0$ to $\varphi$. Moreover by adding the constant $-(u-\varphi)(y)$ to $\varphi$, the maximum (minimum) value can be assumed to be zero. Then the inequalities can be replaced by

$$F(y, \varphi(y), D\varphi(y)) \le 0 \quad (F(y, \varphi(y), D\varphi(y)) \ge 0).$$

Note also that $u \in C(\Omega)$ is a viscosity subsolution of (3.1) iff $v = -u$ is a visc. supersolution of $-F(x, -v, -Dv) = 0$ in $\Omega$. Notice that a visc. solution of (2.1) is not a visc. solution of $-F(x, u, Du) = 0$ in $\Omega$ in general. E.g., consider the case $N = 1$ and $F = |Du| - 1$. The function

$$u(x) = \begin{cases} x & (0 < x \le 1/2) \\ 1-x & (1/2 \le x < 1) \end{cases}$$

is a visc. solution of $F = 0$ in $(0, 1)$ but it is not a visc. solution of $-F = 0$. Indeed, we have

$D^+u(1/2) = [-1,1] \ni 0$ and $-|0|+1 \not\leq 0$. A natural question is when we have

$$F(x,u,Du)=0 \quad \Rightarrow \quad -F(x,u,Du)=0$$

in the viscosity sense. An obvious sufficient condition is that $u \in C^1(\Omega)$.

The next proposition explains why our weak solutions are called viscosity solutions.

<u>Proposition 2.3</u> For $\varepsilon > 0$ let $F_\varepsilon \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$, and let $u_\varepsilon \in C^2(\Omega)$ be a solution of

$$-\varepsilon \Delta u_\varepsilon + F_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \leq 0 \quad \text{in } \Omega .$$

Assume that for some $u$, $F \in C$,

$$u_\varepsilon \to u \quad \text{in } C(\Omega),$$

$$F_\varepsilon \to F \quad \text{in } C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$$

as $\varepsilon \downarrow 0$. Then $u$ is a visc. subsolution of (2.1).

<u>Proof</u> Let $\varphi \in C^2(\Omega)$, $y \in \Omega$ be a local strict

maximum of $u-\varphi$. For sufficiently small $\varepsilon>0$ $u_\varepsilon-\varphi$ attains a local maximum at a point $y_\varepsilon$ close to $y$. We may assume $y_\varepsilon \to y$ as $\varepsilon \searrow 0$. We have

$$-\Delta(u_\varepsilon-\varphi)(y_\varepsilon) \geqslant 0, \qquad D(u_\varepsilon-\varphi)(y_\varepsilon)=0.$$

Hence

$$-\varepsilon\Delta\varphi(y_\varepsilon)+F_\varepsilon(y_\varepsilon, u_\varepsilon(y_\varepsilon), D\varphi(y_\varepsilon)) \leq 0.$$

Sending $\varepsilon \searrow 0$, we get $F(y, u(y), D\varphi(y)) \leq 0$.

Next, we show that this inequality is true for all $\varphi \in C^1(\Omega)$. We choose $\{\varphi_j\}_{j=1}^\infty \subset C^2(\Omega)$ so that

$$\varphi_j \to \varphi \quad \text{in } C^1(\Omega) \quad \text{as } j \to \infty.$$

We are now assuming that $y$ is a local strict maximum point of $u-\varphi$. The following argument is very similar to the above. For large $j \in \mathbb{N}$, $u-\varphi_j$ attains a local maximum at a point $y_j$.

We may assume that $y_j \to y$. Then we have $F(y_j, u(y_j), D\varphi_j(y_j)) \le 0$ as we saw ~~have seen~~ above. Sending $j \to \infty$, we conclude the proof. ▩

The last half of the above proof shows that we may replace the condition "$\varphi \in C^1(\Omega)$" by "$\varphi \in C^\infty(\Omega)$" in Proposition 2.2.

A similar argument to the above shows that the next proposition holds.

<u>Proposition 2.4.</u> Let $\{u_k\} \subset C(\Omega)$, $\{F_k\} \subset C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ and $u_k$ be a viscosity solution of $F_k(x, u_k, Du_k) \le 0$ in $\Omega$. Assume

$$u_k \to u \quad \text{in } C(\Omega),$$

$$F_k \to F \quad \text{in } C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$$

for some $u, F \in C$. Then $u$ is a viscosity solution of (2.1).

The following properties are also easy to check.

<u>Proposition 2.5.</u> Let $u \in C(\Omega)$ ~~and $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N$~~ (i) If $u$ is a visc. subsolution of (2.1) and $\Omega' \subset \Omega$ is open, then $u$ is a visc. subsolution of (2.1) in $\Omega'$. (ii) If, for each $y \in \Omega$, there is a neighborhood $\Omega_y$ such that $u$ is a viscosity solution of $F(x, u, Du) \le 0$ in $\Omega_y$, then $u$ is a visc. subsolution of (2.1).

<u>Proposition 2.6.</u> ~~Let $F \in C$.~~ Let $u_1, u_2 \in C(\Omega)$ be viscosity subsolutions of (2.1). Then $u_1 \vee u_2 \equiv \max\{u_1, u_2\}$ is a viscosity subsolution of (2.1).

<u>Proof</u> Let $\varphi \in C^1(\Omega)$ and $y \in \Omega$ be a maximum point of $u_1 \vee u_2 - \varphi$. We have $u_1 \vee u_2(y) = u_i(y)$ for $i=1$ or $2$. Let us assume $u_1 \vee u_2 = u_1$ at $y$.

Then

$$(u_1-\varphi)(x) \le (u_1 \vee u_2 - \varphi)(x) \le (u_1-\varphi)(y) \qquad \forall x \in \Omega.$$

That is, $u_1-\varphi$ attains a maximum at $y$. Hence

$F(y, u_1 \vee u_2(y), D\varphi(y)) = F(y, u_1(y), D\varphi(y)) \le 0.$ ■

<u>Proposition 2.7</u> Let $u \in C$ ~~and v~~ be a viscosity subsolution of (2.1). (i) Let $\widetilde{\Omega} \subseteq \mathbb{R}^N$, and $\Phi : \Omega \to \widetilde{\Omega}$ be a $C^1$ diffeomorphism. Set

$$u(x) = v(\Phi(x)), \qquad D\Phi(x) = \begin{pmatrix} \Phi_{1x_1} & \cdots & \Phi_{1x_N} \\ \vdots & & \\ \Phi_{Nx_1} & \cdots & \Phi_{Nx_N} \end{pmatrix},$$

$$G(y,r,p) = F(\Phi^{-1}(y), r, {}^{t}D\Phi(\Phi^{-1}(y))\,p).$$

Then $v$ is a viscosity subsolution of $G(y,v,Dv)=0$ in $\widetilde{\Omega}$. (ii) Let $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ be $C^1$ function. Assume $\Phi_r(x,r) > 0 \quad \forall (x,r) \in \Omega \times \mathbb{R}$. Set

$$G(x,r,p) = F(x, \Phi(x,r), D_x\Phi(x,r) + \Phi_r(x,r)\,p).$$

Let $v \in C(\Omega)$ satisfy $u(x) = \Phi(x, v(x)) \quad \forall x \in \Omega.$

Then $v$ is a viscosity solution of $G(x, v, Dv) \le 0$ in $\Omega$.

Proof (i) Let $y_0 \in \tilde{\Omega}$, $p \in D^+v(y_0)$. We have

$$v(y) \le v(y_0) + p\cdot(y-y_0) + o(|y-y_0|) \quad \text{as } y \to y_0.$$

Let $x_0 = \Phi^{-1}(y_0)$. Then

$$u(x) \le u(x_0) + p\cdot(\Phi(x) - \Phi(x_0)) + o(|x-x_0|) =$$

$$= u(x_0) + p\cdot D\Phi(x_0)(x-x_0) + o(|x-x_0|) \quad \text{as } x \to x_0.$$

Therefore, ${}^TD\Phi(x_0)p \in D^+u(x_0)$ and hence

$F(x_0, u(x_0), {}^TD\Phi(x_0)p) \le 0$. That is, $F(\Phi^{-1}(y_0), v(y_0), {}^TD\Phi(\Phi^{-1}(y_0))p) \le 0$.

(ii) Let $x_0 \in \Omega$ and $p \in D^+v(x_0)$. Then

$$v(x) \le v(x_0) + p\cdot(x-x_0) + o(|x-x_0|).$$

Since $\Phi_r(x,r) \ge 0$, we have

$$u(x) = \Phi(x, v(x)) \le \Phi(x, v(x_0) + p\cdot(x-x_0) + o(|x-x_0|))$$

$$\le \Phi(x_0, v(x_0)) + D_x\Phi(x_0, v(x_0))\cdot(x-x_0) + \Phi_r(x_0, v(x_0))\, p\cdot(x-x_0) + o(|x-x_0|) =$$

$$= u(x_0) + (D_x\Phi(x_0, v(x_0)) + \Phi_r(x_0, v(x_0))p)\cdot(x-x_0) + o(|x-x_0|).$$

Hence $D_x\Phi(x_0, v(x_0)) + \Phi_r(x_0, v(x_0))p \in D^+u(x_0)$, and so

$$F(x_0, u(x_0), D_x\Phi(x_0, v(x_0)) + \Phi_r(x_0, v(x_0))p) \le 0.$$

Thus $F(x_0, \Phi(x_0, v(x_0)), D_x\Phi(x_0, v(x_0)) + \Phi_r(x_0, v(x_0))p) \le 0$. ∎

<u>Examples</u> Let $u$ be a visc. solution of

(2.2) $u_t + H(x, t, u, Du) \le 0$.

Set $v(x,t) = e^{\lambda t}u(x,t)$, with $\lambda \in \mathbb{R}$. Then by Prop. 2.7, (ii) we have

$$\lambda e^{\lambda t}v + e^{\lambda t}v_t + H(x, t, e^{\lambda t}v, e^{\lambda t}Dv) \le 0.$$

That is, $v$ is a visc. solution of

$$\lambda v + v_t + e^{-\lambda t}H(x, t, e^{\lambda t}v, e^{\lambda t}Dv) \le 0.$$

Indeed, setting

$$F(x, t, r, p, q) = q + H(x, t, r, p),$$

$$\Phi(x, t, r) = e^{\lambda t}r,$$

we have

$$\Phi_r = e^{\lambda t}, \quad D_x \Phi = 0, \quad \Phi_t = \lambda e^{\lambda t} r,$$

$$v = \Phi(x,t,u(x,t)),$$

and so the corresponding $G$ is:

$$G(x,t,r,p,q) = \lambda e^{\lambda t} r + e^{\lambda t} q + H(x,t,e^{\lambda t} r, e^{\lambda t} p).$$

Next, we consider the change of variables

$v(x,t) = u(x,-t)$, where $u$ is a visc. solution of (2.2).

Define $\Phi$ by $\Phi(x,t) = (x,-t)$. We have

$$u(x,t) = v(\Phi(x,t)), \qquad D\Phi = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $F$ be as above, and set

$$G(y,s,r,p,q) = F(y,-s,r,p,-q).$$

Then, by Prop. 2.7, we see that $v$ is a visc. solution of $G(x,t,v,Dv) \le 0$, i.e.

$$-v_t + H(x,-t,v,Dv) \le 0.$$

## §3. Viscosity solutions in optimal control

We consider the following optimal control problem: Let $A \subset \mathbb{R}^m$ be a compact set, and $g : \mathbb{R}^N \times A \to \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $f : \mathbb{R}^N \times A \to \mathbb{R}$ be continuous functions. Let

$$\mathcal{A} = \{ \alpha ; \ \alpha : [0, \infty) \to A \ \text{measurable} \}.$$

Consider the initial value problem

$$(3.1) \quad \begin{cases} \dot{X}(t) = g(X(t), \alpha(t)) & (t > 0), \\ X(0) = x \end{cases}$$

for $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A}$. Let $X(t; x, \alpha)$ denote the solution of (3.1). Moreover, we define

$$(3.2) \quad J(x, \alpha) = \int_0^\infty e^{-\lambda t} f(X(t; x, \alpha), \alpha(t))\, dt,$$

$$(3.3) \quad V(x) = \inf \{ J(x, \alpha) ; \ \alpha \in \mathcal{A} \},$$

where $\lambda \geq 0$ is a positive constant.

We pose the following questions: (i) Find an $\alpha \in \mathcal{A}$ such that $V(x) = J(x, \alpha)$. (ii)

Find $V(x)$. Here we give an answer to (ii), a characterization of the function $V$. In the terminologies of optimal control, ODE (3.1) is called the <u>state equation</u> (<u>dynamics</u>), $J$ is called the <u>cost functional</u>, $\lambda$ is <u>discount factor</u>, $f$ is called the <u>running cost</u> and $V$ is called the <u>value function</u>. $\alpha \in \mathcal{A}$ is called a <u>control</u> and if it satisfies $V(x) = J(x, \alpha)$, then it is called an <u>optimal control</u>.

In order for $J$ and $V$ to be well-defined, we assume:

(A.1) There is a constant $M > 0$ such that

$$|f(x,a)| \le M, \quad |g(x,a)| \le M, \quad |f(x,a) - f(y,a)| \le M|x-y|,$$

$$|g(x,a) - g(y,a)| \le M|x-y| \qquad (\forall x, y \in \mathbb{R}^N, \ \forall a \in A).$$

(A.2) $\lambda > 0$.

<u>Proposition 3.1</u> Assume (A.1), (A.2). Then $V \in BUC(\mathbb{R}^N)$, and $V$ is a visc solution of

(3.4) $\lambda u + \max\{-g(x,a)\cdot Du - f(x,a)\} = 0$ in $\mathbb{R}^N$.

This equation (3.4) is called <u>Bellman equation</u> of optimal control problem (3.1) – (3.3). Later we prove a uniqueness theorem for solutions of H-J equations which ensures the uniqueness of viscosity solution of (3.4). That is, the value function $V$ is characterized as a unique solution of (3.4).
$\Rightarrow$ Insert here 27+1
This is our answer to the above question (ii).

<u>Lemma 3.1</u> (Dynamic programming principle) For $x \in \mathbb{R}^N$, $\tau > 0$ we have

(3.5) $$V(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))\, dt + e^{-\lambda\tau} V(X(\tau)) \right\},$$

where $X(t) = X(t, x, \alpha)$.

The following continuity of the function

$$H(x,p) = \max_{a \in A} \{ -g(x,a) \cdot p - f(x,a) \}$$

is important when we apply the uniqueness resul[t]

We have

$$|H(x,p) - H(y,p)| \le \max_{a \in A} \{ |g(x,a) - g(y,a)|\,|p| + |f(x,a) - f(y,a)| \}$$

$$\le M|x-y|\,(|p|+1),$$

$$|H(x,p) - H(x,q)| \le \max_{a \in A} \{ |g(x,a)|\,|p-q| \} \le M|p-q|.$$

<u>Sketch of proof</u> Let $\beta(t) = \alpha(t+\tau)$, $Y(t) = X(t+\tau)$.

Then, $Y(t) = X(t, X(\tau), \beta)$ by the uniqueness of solutions of IVP (3.1). Also, we have

$$J(x,\alpha) = \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))\,dt + \int_\tau^\infty e^{-\lambda t} f(X(t), \alpha(t))\,dt$$

$$= \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))\,dt + e^{-\lambda\tau}\int_0^\infty e^{-\lambda t} f(Y(t), \beta(t))\,dt$$

$$= \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))\,dt + e^{-\lambda\tau} J(X(\tau), \beta).$$

From this we easily conclude (3.5). ▩

<u>Proof of Proposition 3.1</u> To see the boundedness of $V$, we compute

$$|V(x)| \le \sup_{\alpha\in\mathcal{A}} \int_0^\infty e^{-\lambda t} |f(X(t), \alpha(t))|\,dt \le M\int_0^\infty e^{-\lambda t}\,dt = \frac{M}{\lambda}$$

for $x \in \mathbb{R}^N$, where $X(t) = X(t; x, \alpha)$. To see the continuity of $V$, we set

$$X(t) = X(t; x, \alpha) \quad \text{and} \quad Y(t) = X(t; y, \alpha)$$

for $x, y \in \mathbb{R}^N$, $\alpha \in \mathcal{A}$. Using Gronwall's inequality we see

$$|X(t) - Y(t)| \le |x-y|\, e^{Mt}.$$

Fix any $T > 0$, and compute

$$(3.6) \quad |J(x,\alpha) - J(y,\alpha)| \le \int_0^T e^{-\lambda t} |f(X(t),\alpha(t)) - f(Y(t),\alpha(t))|\, dt + \int_T^\infty \cdots\, dt$$

$$\le M \int_0^T e^{-\lambda t} |x-y| e^{Mt}\, dt + 2M \int_T^\infty e^{-\lambda t}\, dt =$$

$$= \begin{cases} M|x-y| \dfrac{1}{M-\lambda}\left(e^{(M-\lambda)T} - 1\right) + 2M \dfrac{e^{-\lambda T}}{\lambda} & \text{if } \lambda \ne M, \\ MT|x-y| + 2M \dfrac{e^{-\lambda T}}{\lambda} & \text{if } \lambda = M. \end{cases}$$

We can make the right hand side as close to 0 as possible by choosing $T$ large enough and then $|x-y|$ small enough. This proves that $V$ is unif. continuous on $\mathbb{R}^N$ since $|V(x) - V(y)| \le \sup_{\alpha \in A} |J(x,\alpha) - J(y,\alpha)|$.

Now, we prove that $V$ is a visc. subsolution of (3.4). To this end, we suppose the contrary. That is, we assume

$$\max_{\mathbb{R}^N} (V - \varphi) = (V-\varphi)(y) = 0$$

$$\lambda V(y) + H(y, D\varphi(y)) > 0$$

for some $\varphi \in C^1(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$. We choose $a \in A$

and $\delta > 0$ so that

$$\lambda \varphi(x) - g(x,a)\cdot D\varphi(x) - f(x,a) > 0 \qquad \text{for } {}^\forall x \in B(y,\delta).$$

Let $\alpha(t) \equiv a$ and $X(t) = X(t; y, \alpha)$. We have

$X(t) \in B(y,\delta)$ $(0 \leq {}^\forall t \leq \tau)$ for some $\tau > 0$. Then

$$\lambda \varphi(X(t)) - g(X(t),a)\cdot D\varphi(X(t)) - f(X(t),a) > 0 \qquad (0 \leq t \leq \tau).$$

Multiply this by $e^{-\lambda t}$ and integrate, to get

$$\int_0^\tau \left\{ -\frac{d}{dt} e^{-\lambda t}\varphi(X(t)) - e^{-\lambda t} f(X(t),a) \right\} dt > 0 .$$

Hence

$$V(y) = \varphi(y) > e^{-\lambda\tau}\varphi(X(h)) + \int_0^\tau e^{-\lambda t} f(X(t),a)\,dt \geq e^{-\lambda\tau} V(X(t))$$
$$+ \int_0^\tau e^{-\lambda\tau} f(X(t),a)\,dt.$$

This contradicts Lemma 3.1. Thus $V$ is a visc. sub-solution of (3.4).

Next, we prove that $V$ is a visc. supersolution of (3.4). Suppose there are $\varphi \in C^1(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$ such that

$$\min_{\mathbb{R}^N} (V-\varphi) = (V-\varphi)(y) = 0,$$

$$\lambda V(y) + H(y, D\varphi(y)) < 0.$$

By the continuity of the function $H$, there is a $\delta > 0$ such that

$$\lambda \varphi(x) + H(x, D\varphi(x)) \le -\delta \qquad \forall x \in B(y,\delta).$$

Fix any $\alpha \in \mathcal{A}$ and set $X(t) = X(t; y, \alpha)$. Since $\sup |g| \le M$, there is a $\tau > 0$, independent of $\alpha$, such that $X(t) \in B(y,\delta)$ for $0 \le t \le \tau$. Thus we have

$$\lambda \varphi(X(t)) - g(X(t), \alpha(t)) \cdot D\varphi(X(t), \alpha(t)) - f(X(t), \alpha(t)) < -\delta$$
$$(0 \le t \le \tau).$$

Calculating as before, we have

$$V(y) = \varphi(y) < -\delta \int_0^\tau e^{-\lambda t} dt + e^{-\lambda\tau} \varphi(X(h)) + \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t)) dt \le$$

$$\le -\delta \int_0^\tau e^{-\lambda t} dt + e^{-\lambda\tau} V(X(h)) + \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t)) dt.$$

Hence,

$$V(y) < \inf_{\alpha} \left\{ \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))\, dt + e^{-\lambda \tau} V(X(\tau)) \right\},$$

a contradiction to Lemma 3.1. ∎

⟹ ~~Insert here~~ ~~32+1~~

Next, we give a formal answer to the question (i). We assume (3.4) has a $C^1$ solution, which we denote by $u$. For each $x \in \mathbb{R}^N$ we choose $a(x) \in A$ so that

$$-g(x, a(x)) \cdot Du(x) - f(x, a(x)) = H(x, Du(x)).$$

We now assume $x \to g(x, a(x))$ is Lipschitz continuous on $\mathbb{R}^N$. We solve the IVP

$$\begin{cases} \dot{X}(t) = g(X(t), a(X(t))), & t > 0, \\ X(0) = x \in \mathbb{R}^N. \end{cases}$$

We set $\alpha(t) = a(X(t))$ for $t \geq 0$. Again, we assume $\alpha \in \mathcal{A}$. (Also we assume $u$ is bounded on $\mathbb{R}^N$.) Then $V(x) = J(x, \alpha) = u(x)$. Indeed, we have $X(t) = X(t; x, \alpha)$ for $t \geq 0$ and

$$\lambda u(X(t)) - g(X(t), \alpha(t)) \cdot Du(X(t)) - f(X(t), \alpha(t)) = 0 \qquad \forall\, t \geq 0.$$

<u>Remarks</u> 1. From (3.6) we have

$$(3.7) \qquad |V(x) - V(y)| \le \frac{M}{\lambda - M}|x-y| \quad \text{if } \lambda > M,$$
$$\le C(M,\lambda)\,|x-y|^{\lambda/M} \quad \text{if } \lambda < M,$$
$$\le C(M,\lambda)\,|x-y|\,|\log|x-y|| \quad \text{if } \lambda = M,$$

for $|x-y|$ small, where $C(M,\lambda)$ is a positive constant.

2. In the above proof, once the continuity of $V$ and Lemma 3.1 are established, the properties concerning $f$ and $g$ which we really need are: (i) $H$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N$ and (ii) for any $\varepsilon > 0$ (and $x \in \mathbb{R}^N$) there is a $\delta > 0$ such that $X(t; x, \alpha)$ exists and $X(t: x, \alpha) \in B(x, \varepsilon)$ for $0 \le t \le \delta$ and $\alpha \in \mathcal{A}$.

Calculating as in the proof of Prop. 3.1, we get

$$u(x) = e^{-\lambda\tau} u(X(\tau)) + \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))\, dt \qquad (\forall \tau > 0),$$

and so

$$u(x) = \int_0^\infty e^{-\lambda t} f(X(t), \alpha(t))\, dt = J(x, \alpha).$$

If we perform the same computation as above for arbitrary $\beta \in \mathcal{A}$, then we have

$$u(x) \geq \int_0^\infty e^{-\lambda t} f(X(t, x; \beta), \beta(t))\, dt = J(x, \beta).$$

Thus we see that $V(x) = u(x) = J(x, \alpha)$. ∎

An optimal control problem with cost functional $J$ defined by an integral over an infinite interval, like (3.2), is called an <u>infinite horizon problem</u>. It is not realistic, and more realistic one is a <u>finite horizon problem</u>, where the cost functional is defined by an integral over a finite interval.

We next consider a finite horizon problem.

Now, let $T > 0$ and $g : \mathbb{R}^N \times [0,T] \times A \to \mathbb{R}^N$, $f : \mathbb{R}^N \times [0,T] \times A \to \mathbb{R}$, $h : \mathbb{R}^N \to \mathbb{R}$ be continuous functions.

For $(x,t) \in \mathbb{R}^N \times [0,T)$ and $\alpha \in \mathcal{A}$, we consider IVP

$$(3.8) \quad \begin{cases} \dot{X}(s) = g(X(s), s, \alpha(s)) & (t < s < T), \\ X(t) = x. \end{cases}$$

Let $X(s; x, t, \alpha)$ denote the solution of (3.8).

We define

$$(3.9) \quad J(x,t,\alpha) = \int_t^T f(X(s;x,t,\alpha), \alpha(s))\, ds + \varphi(X(T)),$$

$$(3.10) \quad V(x,t) = \inf\{J(x,t,\alpha) : \alpha \in \mathcal{A}\}$$

for $(x,t) \in \mathbb{R}^N \times [0,T]$. Here, $J$ and $V$ are called resp., the cost functional and the value function. The function $\varphi$ is called the terminal cost (?).

$\Longrightarrow$ Insert here 34+1

For this problem, the dynamic programming principle is formulate as follows:

Our assumptions here are:

(A.3) For some constant $M > 0$,

$$|f(x,t,a)| \le M, \quad |g(x,t,a)| \le M, \quad |h(x)| \le M,$$

$$|f(x,t,a) - f(y,t,a)| \le M|x-y|$$

$$|g(x,t,a) - g(y,t,a)| \le M|x-y|$$

$$|h(x) - h(y)| \le M|x-y|$$

Under these assumptions $V$ is well defined.

<u>Lemma 3.2.</u> Let $(x,t) \in \mathbb{R}^N \times [0,T)$ and $t \leqslant \tau \leqslant T$. Then

$$(3.11) \quad V(x,t) = \inf_{\alpha \in \mathcal{A}} \Big\{ \int_t^\tau f(X(s), s, \alpha(s))\,ds + V(X(\tau), \tau) \Big\},$$

where $X(s) = X(s; x, t, \alpha)$.

The proof of this lemma is similar to that of Lemma 3.1.

Our main assertion is stated as follows:

<u>Proposition 3.2.</u> Assume (A.3). Then $V \in BUC(\mathbb{R}^N \times [0,T])$ and $V$ is a viscosity solution of

$$(3.12) \quad -u_t + \max_{a \in A} \{ -g(x,t,a) \cdot Du - f(x,t,a) \} = 0 \quad \text{in } \mathbb{R}^N \times (0,T).$$

We denote

$$H(x,t,p) = \max_{a \in A} \{ -g(x,t,a) \cdot p - f(x,t,a) \}.$$

Assumption (A.3) yields

$$|H(x,t,p) - H(y,t,p)| \leq M|x-y|(|p|+1),$$

$$|H(x,t,p) - H(x,t,q)| \leq M|p-q|,$$

and that $H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R}^N)$.

We will prove a uniqueness result which implies the uniqueness of viscosity solutions of the Cauchy problem for (3.12). Note that $V$ satisfies the terminal condition

$$(3.13) \qquad V(x,T) = h(x) \qquad \forall x \in \mathbb{R}^N.$$

Observe also that if we set $v(x,t) = V(x, T-t)$, then it is equivalent for $v$ to satisfy

$$(3.14) \qquad \begin{cases} v_t + H(x,t,Dv) = 0 & \text{in } \mathbb{R}^N \times (0,T), \\ v(x,0) = h(x) & x \in \mathbb{R}^N, \end{cases}$$

in the visc. sense to that $V$ is a visc. solution of (3.12) and satisfies (3.13).

<u>Proof of Prop. 3.2</u> The boundedness and uniform continuity in $x$ of $V$ is proved as in the same way as in the proof of Prop. 3.1. We show that

$V$ is Lipschitz continuous in $t$ uniformly in $\mathbb{R}^N \times [0,T]$. Fix $x \in \mathbb{R}^N$ and $0 \le t < \tau \le T$.

Then we have

$$|V(x,\tau) - V(x,t)| \le \sup_{\alpha \in \mathcal{A}} \int_t^\tau |f(X(s), s, \alpha(s))| ds \le M(\tau - t),$$

where $X(s) = X(s : x, t, \alpha)$. Thus we see that $V \in BUC(\mathbb{R}^N \times [0,T])$.

Next we prove that $V$ is a visc. supersolution of (3.12) and will skip to show that $V$ is a visc. supersolution of (3.12). As in the proof of Prop. 3.1 we suppose that

$$\max(V - \varphi) = (V - \varphi)(y, \sigma) = 0,$$

$$-\varphi_t(y,\sigma) + H(y, \sigma, D\varphi(y,\sigma)) > 0$$

hold for some $\varphi \in C^1(\mathbb{R}^N \times (0,T))$. For some $a \in A$ and $r > 0$ we have

$$-\varphi_t(x,t) - g(x,t,a)\cdot D\varphi(x,t) - f(x,t,a) > 0$$

for $(x,t) \in B(y,r) \times [\sigma, (\sigma+r)\wedge T]$. For some $\sigma < \tau \leq (\sigma+r)\wedge T$, setting $\alpha(s) \equiv a$ and $X(s) = X(s; y, \sigma, \alpha)$, we have $X(s) \in B(y,r)$ for $\sigma \leq s \leq \tau$. Then we have

$$(3.15)\quad -\varphi_t(X(s),s) - g(X(s),s,a)\cdot D\varphi(X(s),s) - f(X(s),s,a) > 0.$$

Noting that

$$\frac{d}{ds}\varphi(X(s),s) = \varphi_t(X(s),s) + D\varphi(X(s),s)\cdot \dot{X}(s)$$

$$= \varphi_t(X(s),s) + g(X(s),s,a)\cdot D\varphi(X(s),s)$$

and integrating (3.15), we obtain

$$V(y,\sigma) = \varphi(y,\sigma) > \varphi(X(\tau),\tau) + \int_\sigma^\tau f(X(s),s,a)\,ds \geq$$

$$\geq \inf_{\alpha\in\mathcal{A}} \left\{ \int_\sigma^\tau f(X(s;y,\sigma,\alpha),s,\alpha(s))\,ds + V(X(\tau;y,\sigma,\alpha),\tau) \right\},$$

which contradicts Lemma 3.2. This proves our claim. ∎

Remarks analoguous to Remarks after the proof of

Prop. 3.1 applies to the above proof.

## §4. Generalization of the definition of visc. solutions

We will generalize the definition of visc. solutions in order that it applies even to discontinuous functions. In Prop. 2.6, the maximum is taken over at most finitely many visc. subsolutions simply because otherwise the resulting function may not be continuous any more.

Let $E$ be a subset of $\mathbb{R}^N$ and $H : E \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be locally bounded. For locally bounded $u : E \to \mathbb{R}$ we define

$$u^*(x) = \lim_{r \searrow 0} \sup \{u(y) : y \in E,\ |y - x| < r\} \equiv \overline{\lim_{y \to x}}\ u(y),$$

$$u_*(x) = \lim_{r \searrow 0} \inf \{u(y) : y \in E,\ |y - x| < r\} \equiv \underline{\lim_{y \to x}}\ u(y).$$

The functions $u^*$ and $u_*$ will be called, resp., upper semicontinuous and lower semicontinuous envelopes. We remark here that $u_* = -(-u)^*$.

Observe that $u_* \leq u \leq u^*$ in $E$ and that $u \in C(E)$ iff $u^* \leq u_*$ in $E$. Note also that if $u$ is u.s.c., then $u^* = u$.

These $u^*$ and $u_*$ are in fact u.s.c. and l.s.c. in $E$, respectively. Notice that $f: E \to \mathbb{R}$ is u.s.c. iff $f(x) \geq \overline{\lim}_{E \ni y \to x} f(y)$ for $x \in E$ and $f$ is l.s.c. iff $f(x) \leq \underline{\lim}_{E \ni y \to x} f(y)$ for $x \in E$ by definition. Let us just check that $u^*$ is u.s.c on $E$. Fix $x \in E$. By definition, for any $\varepsilon > 0$, there is $r > 0$ such that

$$|u^*(x) - \sup\{u(y) : y \in E,\ |y-x| < r\}| < \varepsilon$$

and so

$$u^*(x) + \varepsilon > \sup\{u(y) : y \in E,\ |y-x| < r\} \geq \sup\{u(y) : y \in E,\ |y-z| < r - |x-z|\} \geq$$

$$\geq u^*(z) \qquad \text{for } z \in E, \text{ with } |z-x| < r.$$

Therefore

$$\overline{\lim}_{z \to x} u^*(z) \leq u^*(x) \qquad \forall x \in E.$$

$\Longleftarrow$ Insert here (41+1)

We denote the spaces of real-valued u.s.c. functions and l.s.c. functions on $E$, resp., by

$$USC(E) \quad \text{and} \quad LSC(E).$$

Other definitions of u.s.c. and l.s.c. envelopes are given through these identities:

$$u^*(x) = \inf\{v(x) : v \in USC(E), \quad v \geq u \quad \text{in } E\},$$

$$u_*(x) = \sup\{v(x) : v \in LSC(E), \quad v \leq u \quad \text{in } E\}.$$

Let us prove just the first identity. We write

$$\hat{u}(x) = \inf\{v(x) : v \in USC(E), \quad v \geq u \quad \text{in } E\}.$$

As $u^* \in USC(E)$ and $u^* \geq u$ in $E$, we see that $u^* \geq \hat{u}$ in $E$. To prove that $u^* \leq \hat{u}$ in $E$, we let $v \in USC(E)$ satisfy $v \geq u$ in $E$. Then we have

$$v(x) \geq \overline{\lim_{E \ni y \to x}} v(y) \geq \overline{\lim_{E \ni y \to x}} u(y) = u^*(x) \qquad \forall x \in E.$$

This implies that $\hat{u} \geq u^*$ in $E$.

We remark also that if $E$ is paracompact, then

$$u^*(x) = \inf\{v(x) : v \in C(E), \quad v \geq u \quad \text{in } E\},$$

$$u_*(x) = \sup\{v(x) : v \in C(E), \quad v \leq u \quad \text{in } E\}.$$

The definition of sub- and superdifferentials is easily modified to make sense for functions on $E$. Indeed, if $u : E \to \mathbb{R}$ and $y \in E$, then $D^+u(y)$ and

$D^- u(y)$ are defined by

(4.1) $D^+ u(y) = \{ p \in \mathbb{R}^N : u(x) \le u(y) + p\cdot(x-y) + o(|x-y|)$

as $x \in E \to y \}$,

(4.2) $D^- u(y) = \{ p \in \mathbb{R}^N : u(x) \ge u(y) + p\cdot(x-y) + o(|x-y|)$

as $x \in E \to y \}$.

*Definition 4.1* Let $u : E \to \mathbb{R}$. (i) We call $u$ a visc. subsolution (resp. supersolution) of

(4.3) $$F(x, u, Du) = 0 \quad \text{in } E$$

if $u$ is locally bounded and satisfies

$$F_*(x, u^*(x), p) \le 0 \quad \forall x \in E, \quad \forall p \in D^+ u^*(x)$$

(resp. $F^*(x, u_*(x), p) \ge 0 \quad \forall x \in E, \quad \forall p \in D^- u_*(x)$ ).

(ii) We call $u$ a visc. solution of (4.3) if it is both a visc. subsolution and a visc. supersolution of (4.3).

For $k \in \mathbb{N}$ we define

$C^k(E) = \{ \varphi \in C(E) : \exists\, U$ open neighborhood of $E$ in $\mathbb{R}^N$

such that $\varphi \in C^k(U)$ }.

<u>Remark</u> Henceforth we call a visc. subsolution (resp., visc supersolution and visc. solution) just a subsolution (resp., supersolution and solution).

<u>Proposition 4.1</u> Let $u: E \to \mathbb{R}$ be locally bounded. Then, $u$ is a subsolution (resp., supersolution) of (43) iff whenever $y \in E$, $\varphi \in C^1(\{y\})$ and $u^* - \varphi$ (resp. $u_* - \varphi$) attains a local maximum (resp. minimum) at $y$, then

$$F_*(y, u^*(y), D\varphi(y)) \leq 0 \quad (\text{resp. } F^*(y, u_*(y), D\varphi(y)) \geq 0).$$

In this proposition, if $E$ is paracompact, then we may replace "$\varphi \in C^1(\{y\})$" by "$\varphi \in C^1(E)$" as in Proposition 2.2 and also "local" by "global". ⌋

Assume that $E$ is locally compact.

<u>Proposition 4.2</u> Let $\mathcal{S}$ be a family of subsolutions of (4.3). Put

(4.4) $\quad u(x) = \sup\{v(x) : v \in \mathcal{S}\} \quad \forall x \in E.$

Assume $u$ is locally bounded. Then $u$ is a subsolution of (4.3).

<u>Proof</u> Let $u$ be defined by (4.4) and locally bounded in $E$. Let $y \in E$, $\varphi \in C^1(B(y,r))$ for some $r>0$, and assume $\max_{E \cap B(y,r)} (u^* - \varphi) = (u^* - \varphi)(y) = 0$.

Furthermore we assume

$$(u^* - \varphi)(x) \le -|x-y|^2 \qquad \text{for } x \in E \cap B(y,r)$$

by adding $|x-y|^2$ to $\varphi$ if necessary. By the definition of $u^*$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that

$$(u - \varphi)(x_n) > -\frac{1}{n}, \qquad x_n \to y \quad \text{as } n \to \infty.$$

By the definition of $u$, there is a sequence $\{v_n\}_{n=1}^{\infty}$ $\subset S$ such that

$$u(x_n) - \frac{1}{n} < v_n(x_n) \qquad \text{for } n \in \mathbb{N}.$$

Then we have

$$(v_n - \varphi)(x_n) > -\frac{2}{n}, \qquad (v_n - \varphi)(x) \leq -|x-y|^2 \quad \forall x \in E \cap B(y,r)$$

for $n \in \mathbb{N}$. From these

$$(v_n^* - \varphi)(x_n) > -\frac{2}{n}, \qquad (v_n^* - \varphi)(x) \leq -|x-y|^2 \quad \forall x \in E \cap B(y,r)$$

for $n \in \mathbb{N}$. (Since $E$ is locally compact, we may assume that $E \cap B(y,r)$ is compact) From these we see that there is a point $y_n \in E \cap B(y,r)$ such that

$$\max(v_n^* - \varphi) = (v_n^* - \varphi)(y_n)$$

if $n$ is large enough. Then we have

$$-\frac{2}{n} < (v_n^* - \varphi)(y_n) \leq -|y_n - y|^2,$$

and so

$$y_n \to y, \qquad v_n^*(y_n) \to \varphi(y) \qquad \text{as } n \to \infty.$$

Since $v_n$ is a visc. subsolution of (4.3), we get

$$F_*(y_n, v_n^*(y_n), D\varphi(y_n)) \le 0.$$

Sending $n \to \infty$, we conclude

$$F_*(y, \varphi(y), D\varphi(y)) \le 0. \qquad \blacksquare$$

We notice that if $u$ is a subsolution of (4.3), then $v = -u$ is a supersolution of

$$-F(x, -v, -Dv) = 0 \quad \text{in } E,$$

and vice versa. This allows you to conclude that any proposition on subsolutions has its "dual" proposition on supersolutions. For instance, the same assertion as Prop. 4.2 but with "supersolution" and "inf" in place of "subsolution" and "sup" is valid.

<u>Proposition 4.3</u> Assume $E$ is locally compact. Let $u_n \in USC(E)$, $n \in N$, be subsolutions of (4.3). Let $u : E \to R$ be locally bounded. Assume that $u_n(x) \searrow u(x)$ for $x \in E$

as $n \to \infty$. Then $u$ is a subsolution of (4.3).

From the above two propositions we may conclude that if we assume in addition to the hypotheses of Prop. 4.3 that $u_n$ is supersolution of (4.3), then $u$ is a solution of (4.3).

Dini's lemma Let $\{u_n\}$ be a sequence of u.s.c. functions on a compact subset $K$ of $\mathbb{R}^N$. Assume that $u_n(x) \searrow u(x)$ for $x \in K$ and some $u \in C(K)$ as $n \to \infty$. Then the convergence is uniform on $K$.

Proof of Prop. 4.2 Let $\{u_n\}$ and $u$ be as in Prop. 4.3. Since $u(x) = \inf u_n(x)$ for $x \in E$, we find that $u \in USC(E)$. Let $r > 0$, $y \in E$, $\varphi \in C^1(B(y,r))$, and assume $\max_{E \cap B(y,r)} (u - \varphi) = (u - \varphi)(y) = 0$.

We may assume that

$$(u-\varphi)(x) \leq -|x-y|^2 \qquad \text{for } x \in E \cap B(y,r),$$

and that $E \cap B(y,r)$ is compact. Consider the functions

$$x \to (u_n(x) - \varphi(x) + |x-y|^2)^+.$$

These are u.s.c. on $E$ and converge to $0$ monotonically from above. By Dini's lemma we see that the above convergence is uniform on $E \cap B(y,r)$.

Relabeling if necessary, we may assume that

$$u_n(x) - \varphi(x) \leq \frac{1}{n} - |x-y|^2 \quad \text{for } x \in E \cap B(y,r)$$
$$\text{and } n \in \mathbb{N}.$$

Let $y_n$ be a maximum point of $u_n - \varphi$ over $E \cap B(y,r)$. We have

$$0 = (u-\varphi)(y) \leq (u_n - \varphi)(y) \leq (u_n - \varphi)(y_n) \leq \frac{1}{n} - |y_n - y|^2 \qquad \forall n \in \mathbb{N}.$$

Therefore

$$y_n \to y \quad \text{and} \quad u_n(y_n) \to u(y) \quad \text{as } n \to \infty.$$

Since $u_n$ is a solution of (4.3), we have

$$F_*(y_n, u_n(y_n), D\varphi(y_n)) \le 0 \quad \text{for } n \text{ large enough,}$$

and so

$$F_*(y, u(y), D\varphi(y)) \le 0. \qquad \blacksquare$$

<u>Remark</u> The notion of subsolution (resp. supersolution) is directly extended to functions locally bounded from above (below). It is not clear how to extend these notions beyond that scope.

<u>Example</u> (discontinuous Hamiltonians). Let $\Omega$ be an open subset of $\mathbb{R}^N$. Let $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$ and $B \in C(\partial\Omega \times \mathbb{R} \times \mathbb{R}^N)$. Set

$$F(x,r,p) = \begin{cases} H(x,r,p) & \text{if } x \in \Omega, \\ B(x,r,p) & \text{otherwise.} \end{cases}$$

Then $F$ is locally bounded on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ and

$$F_*(x,r,p) = \begin{cases} H(x,r,p) & \text{if } x \in \Omega, \\ H(x,r,p) \wedge B(x,r,p) & \text{otherwise,} \end{cases}$$

and

$$F^*(x,r,p) = \begin{cases} H(x,r,p) & \text{if } x \in \Omega, \\ H(x,r,p) \vee B(x,r,p) & \text{otherwise.} \end{cases}$$

## §5. Existence of solutions

Here we will see that Perron's method, originally introduced to 2nd order elliptic equations, yields solutions of Hamilton-Jacobi equations. Those solutions may not be continuous. We will find two cases where the continuity of those solutions can be shown. The two cases in fact cover most of the existing existence results for H-J equations.

There are roughly two other methods of construction of solutions to H-J equations. One is a well-known method, the method of vanishing viscosity. The other way is to construct a solution of a given H-J equation as the value function of the associated optimal control (or differential game) problem to the H-J equation. We will discuss these methods in a later section.

Theorem 5.1 Let $E$ be a locally compact subset of $\mathbb{R}^N$ and $F: E \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ locally bounded.
Let $f, g: E \to \mathbb{R}$ satisfy $f \le g$ in $E$ and be, respectively, sub- and supersolutions of

(5.1) $\quad F(x, u, Du) = 0 \quad$ in $E$.

Define $u: E \to \mathbb{R}$ by

(5.2) $u(x) = \sup\{v(x) : v \le g$ in $E$, $v$ subsolution of (5.1)$\}$.

Then $u$ is a solution of (5.1) and satisfies $f \le u \le g$ in $E$.

We stress once again that the solution given by (5.2) is not necessarily continuous.

Proof It is immediate from (5.2) that $f \le u \le g$ in $E$. In particular, $u$ is locally bounded in $E$. By Proposition 4.2 $u$ is a subsolution of (5.1).

It remains to see that $u$ is a super-solution of (5.1). To this end, we assume that $u$ is not a viscosity supersolution of (5.1), i.e. for some $\varphi \in C^1(E)$ and $y \in E$,

$$\min (u_* - \varphi) = (u_* - \varphi)(y) = 0, \qquad F^*(y, \varphi(y), D\varphi(y)) < 0. \tag{5.3}$$

We may and will assume that $(u_* - \varphi)(x) \leq -|x-y|^2$ in $E$. (5.4)
We will get a contradiction. Since $u \leq g$ in $E$, we have $u_* \leq g_*$ in $E$. To show that $u_*(y) < g_*(y)$, we suppose the contrary, $u_*(y) = g_*(y)$. Then

$$\min (g_* - \varphi) = (g_* - \varphi)(y) = 0$$

and so $F^*(y, \varphi(y), D\varphi(y)) \geq 0$ because $g$ is a supersolution of (5.1). This contradiction shows $u_*(y) < g_*(y)$. Let $\delta = \frac{1}{2}(g_* - \varphi)(y)$ $(> 0)$ and consider the set $\{x \in E : (g_* - \varphi)(x) > 0\}$ which is a neighborhood of $y$. We then find by the semicontinuity of $F^*$ that there is an $r > 0$ such that

$$F^*(x, \varphi(x)+s, D\varphi(x)) \le 0 \quad \forall x \in E \cap B(y,r), \quad 0 \le \forall s \le r,$$

$$\varphi(x) + \delta \le g(x) \qquad \forall x \in E \cap B(y,r).$$

Meanwhile, we have

$$\varphi(x) + \left(\frac{r}{2}\right)^2 \le u_*(x) \le u(x) \qquad \forall x \in E \setminus B(y, \tfrac{r}{2}).$$

Thus, putting $\varepsilon = r \wedge \delta \wedge \left(\frac{r}{2}\right)^2$, we have

$$F_*(x, \varphi(x)+\varepsilon, D\varphi(x)) \le 0 \qquad \forall x \in E \cap B(y,r),$$

$$\varphi(x) + \varepsilon \le \begin{cases} g(x) & \forall x \in E \cap B(y,r), \\ u(x) & \forall x \in E \setminus B(y, \frac{r}{2}). \end{cases}$$

Define $w(x) = \max\{u(x), \varphi(x)+\varepsilon\}$ for $x \in E$. By Prop. 4.2 $w$ is a subsolution of (5.1) in $B(y,r)^\circ$. As $w = u$ in $E \setminus B(y, \frac{r}{2})$, $w$ is a subsolution of (5.1) in $E \setminus B(y, \frac{r}{2})$. By Proposition 2.5 we see that $w$ is a subsolution of (5.1) in $E$. Also, $w$ satisfies: $w(y) = u_*(y) + \varepsilon$, which implies $\exists\, y_n \to y$ such that $w(y_n) > u(y_n)$, and $w \le g$ in $E$. This

Since $u \leq g$ in $E$ and $g$ is u.s.c. in $E$, we find that $u^* \leq g$ in $E$ and hence $u = u^*$ by the definition of $u$.

is a contradiction. ∎

We give an example where the above method yields a continuous solution.

<u>Example</u> Assume the hypotheses of Theorem 5.1. Suppose $g \in USC(E)$ and

$$(5.3) \qquad \lim_{R\to\infty} \inf\{F(x,r,p) : (x,r) \in E\times\mathbb{R},\ |p| \leq R\} > 0.$$

Let $u$ be the function given by (5.2). Then $u \in C(E)$, and moreover it is locally Lipschitz continuous in $E$. Also, if $f$ and $g$ are, respectively, l.s.c. and u.s.c. on $\overline{E}$ and $f = g$ on $\partial E$, then $u \in C(\overline{E})$ (more precisely, $u$ is extended uniquely to $\overline{E}$ by continuity and $u = f = g$ on $\partial E$).

Insert here 55+1

<u>Proof</u> We choose $L > 0$ so that

$$(5.4) \qquad F_*(x,r,p) > 0 \qquad \forall x \in E,\ \forall r \in \mathbb{R},\ \forall p \in B(0,L),$$

and show that for each $z \in E$ there is $\delta > 0$ such that

$$|u(x) - u(y)| \le L|x-y| \qquad \forall x, y \in E \cap B(z, \delta)$$

Fix $z \in E$, and choose $\delta > 0$ so that $E \cap B(z, 3\delta)$ is compact.

Choose $\theta \in C^1([0, 2\delta))$ so that

$$\theta(r) = Lr \quad \text{for } 0 \le r \le \delta, \qquad \theta'(r) \ge L \quad \text{for } 0 \le r < 2\delta,$$

$$\lim_{r \nearrow 2\delta} \theta(r) = \infty.$$

Fix $y \in E \cap B(z, 2\delta)$, and put

$$v(x) = u(y) + \theta(|x-y|) \qquad \text{for } x \in B(y, 2\delta)^\circ.$$

Then, $v \in C^1(B(y, 2\delta)^\circ \setminus \{y\}) \cap C(B(y, 2\delta)^\circ)$ and

$$F_*(x, v(x), Dv(x)) > 0 \qquad \text{for } \forall x \in B(y, 2\delta)^\circ \setminus \{y\}, \ \forall r \in \mathbb{R}.$$

Let us suppose $\sup_{E \cap B(y, 2\delta)^\circ} (u - v) > 0$. As $u - v$ is u.s.c. in $E \cap B(y, 2\delta)^\circ$ and

$$(u-v)(y) = 0, \qquad \lim (u-v)(x) = -\infty,$$

there is a maximum point $\xi \in E \cap B(y, 2\delta)^\circ \setminus \{y\}$ of $u - v$ over $E \cap B(y, 2\delta)^\circ$. And we have

$$F(\xi, v(\xi), Dv(\xi)) \le 0,$$

which contradicts (5.4). Thus we see that $u-v \leq 0$ in $B(y,2\delta)^\circ$ and in particular

$$u(x) \leq u(y) + L|x-y| \qquad \forall x \in E \cap B(y,\delta)^\circ$$

That is, we have

$$u(x) - u(y) \leq L|x-y| \qquad \forall x,y \in E \cap B(z,\delta)^\circ.$$

If $f$ and $g$ are, resp., l.s.c. and u.s.c. on $\overline{E}$ and $f=g$ on $\partial E$, then, setting $u=g$ on $\partial E$, we have $u \in C(\overline{E})$ and $u=f=g$ on $\partial E$. ∎

A typical example of functions $F$ satisfying (5.3) is

$$F(x,p) \equiv F(x,r,p) = \sum_{i,j=1}^{N} a_{ij}(x)\, p_i p_j - V(x) \qquad \text{on } E \times \mathbb{R} \times \mathbb{R}^N,$$

where $a_{ij}$ and $V$ are bounded in $E$ and $(a_{ij}(x))_{1\leq i,j\leq N} \geq \theta I$ is assumed to hold for $x \in E$ and some constant $\theta > 0$. Obviously, (5.3) is satisfied.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$. Let $h \in C(\partial\Omega)$ and consider the Dirichlet problem

$$(5.5)\qquad \begin{cases} F(x, Du) = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

Choose $L > 0$ so that $F(x,p) > 0$ for $x \in \Omega$ and $p \notin B(0, L)$. For $\varepsilon > 0$ and $z \in \partial\Omega$ we choose $L(\varepsilon, z) \geq L$ so that

$$(5.6)\qquad h(x) \leq h(z) + \varepsilon + L(\varepsilon, z)\,|x - z| \qquad \forall x \in \partial\Omega.$$

For $\varepsilon > 0$ and $z \in \partial\Omega$ we set

$$g_{z,\varepsilon}(x) = h(z) + \varepsilon + L(\varepsilon, z)\,|x - z| \qquad \text{for } x \in \overline{\Omega}.$$

This function is a supersolution of (5.1) (with $E = \Omega$) and continuous on $\overline{\Omega}$. Hence, $g$ defined by

$$g(x) = \inf\{ g_{z,\varepsilon}(x) : \ \varepsilon > 0,\ z \in \partial\Omega \} \qquad \forall x \in \overline{\Omega}$$

is a supersolution of (5.1) and u.s.c. on $\overline{\Omega}$. Moreover, $g \geq h$ on $\partial\Omega$ by (5.6) and $g(z) \le g_{z,\varepsilon}(z) = h(z) + \varepsilon$ for $z \in \partial\Omega$ and $\varepsilon > 0$, and therefore $g = h$ on $\partial\Omega$. Thus we find a supersolution $g$ of (5.1) which is u.s.c. on $\overline{\Omega}$ and satisfies $g = h$ on $\partial\Omega$.

On the other hand, we need a fortune to find a subsolution $f$ of (5.1) with properties similar to those for $g$ above. Indeed, if $V(x) < 0$ for some $x \in \Omega$, then we have no chance to find any subsolution of (5.1). If $V(x) \geq 0$ in $\Omega$ and $h = 0$ on $\partial\Omega$ then $f = 0$ satisfies the required properties, $f \leq g$ and $f = g = 0$ on $\partial\Omega$. Even if $V \geq 0$ in $\Omega$, the solvability of the problem (5.5) requires some compatibility condition on $h$ which we will discuss later.

## §6. Comparison of viscosity solutions

We consider the following H-J equations

(6.1) $\quad u + H(x, u, Du) = 0 \quad$ in $\Omega$,

(6.2) $\quad u_t + H(x, t, u, Du) = 0 \quad$ in $\Omega \times (0, T)$,

where $T > 0$ $(\neq \infty)$ and $\Omega$ is an open subset of $\mathbb{R}^N$.

Our assumptions are :

(H0) $\quad H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R})$.

(H1) $\quad r \to H(x, t, r, p)$ is nondecreasing for $(x, t, p) \in \mathbb{R}^N \times [0, T] \times \mathbb{R}^N$.

(H2) $\quad$ There is $m \in C([0,\infty), [0,\infty))$ such that $m(0) = 0$ and

$$|H(x, t, r, p) - H(y, t, r, p)| \leq m(|x-y|(|p|+1))$$

for $x, y \in \Omega$ and $(t, r, p) \in [0, T] \times \mathbb{R} \times \mathbb{R}^N$.

(H3) For each $R>0$ there is $\sigma_R \in C([0,\infty),[0,\infty))$ such that $\sigma_R(0)=0$ and

$$|H(x,t,r,p)-H(x,t,r,q)| \le \sigma_R(|p-q|)$$

for $(x,t,r)\in \mathbb{R}^N\times[0,T]\times\mathbb{R}$ and $p,q\in B(0,R)$.

If we regard the Hamiltonian $H$ in (6.1) as a function on $\mathbb{R}^N\times[0,T]\times\mathbb{R}\times\mathbb{R}^N$ which is independent of $t\in[0,T]$, then conditions (H0) – (H3) make sense for this function.

$\Longrightarrow$ Insert here 61+1, ....

Theorem 6.1 Assume (H0) – (H3). (i) Let $u$ and $v$ be, respectively, a subsolution and a supersolution of (6.1). Assume that $u\in USC(\overline{\Omega})$, $v\in LSC(\overline{\Omega})$, $u$ and $v$ are bounded on $\overline{\Omega}$, and

(6.3) $$\lim_{r\searrow 0}\sup\{u(x)-v(y) : (x,y)\in\partial(\Omega\times\Omega),\ |x-y|\le r\}\le 0.$$

Then $u\le v$ in $\Omega$.

(ii) Let $u$ and $v$ be, resp., a subsolution and a supersolution of (6.2). Assume that $u,v$ are

We begin with the following result.

Theorem 6.~~0~~1. Assume (H0) – (H2) and that $\Omega$ is bounded. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., a subsolution of

(6.0.~~1~~3) $$H(x, u, Du) = 0 \quad \text{in } \Omega$$

and a supersolution of

(6.0.~~2~~4) $$H(x, u, Du) = a \quad \text{in } \Omega$$

for some $a > 0$. Assume $u \le v$ on $\partial\Omega$. Then $u \le v$ in $\Omega$.

We will give three kind of proofs of the theorem.

Proof, I We set

$$w(x,y) = u(x) - v(y) \qquad \text{for } (x,y) \in \overline{\Omega} \times \overline{\Omega}.$$

Then $w$ is a subsolution of

(6.0.3) $$\tilde{H}(x, y, w, Dw) = -a \quad \text{in } \Omega \times \Omega,$$

where $Dw = (D_x w, D_y w)$ and

$$\tilde{H}(x,y,r,p,q) = H(x, r+v(y), p) - H(y, u(x)-r, -q).$$

Indeed, if $\varphi \in C^1(\Omega\times\Omega)$ and $(\bar{x},\bar{y})$ is a maximum point of $w-\varphi$, then we have

$$H(\bar{x}, u(\bar{x}), D_x\varphi(\bar{x},\bar{y})) \le 0,$$

$$H(\bar{y}, v(\bar{y}), -D_y\varphi(\bar{x},\bar{y})) \ge a,$$

and so

$$\tilde{H}(\bar{x},\bar{y}, w(\bar{x},\bar{y}), D\varphi(\bar{x},\bar{y})) = H(\bar{x}, u(\bar{x}), D_x\varphi(\bar{x},\bar{y}))$$
$$- H(\bar{y}, v(\bar{y}), -D_y\varphi(\bar{x},\bar{y})) \le -a.$$

Let $\varepsilon > 0$, and set

$$\Phi(x,y) = u(x) - v(y) - \tfrac{1}{\varepsilon}|x-y|^2 \quad \text{for } (x,y)\in\bar{\Omega}\times\bar{\Omega}.$$

Let $(\bar{x},\bar{y})\in\bar{\Omega}\times\bar{\Omega}$ be a maximum point of $\Phi$. We have

$$\Phi(x,y) \le \Phi(\bar{x},\bar{y}) \qquad \text{for } (x,y)\in\bar{\Omega}\times\bar{\Omega}.$$

In particular, taking $x=y$, we have

$$\max_{\bar{\Omega}}(u-v) \le \Phi(\bar{x},\bar{y}) \le \max_{\bar{\Omega}\times\bar{\Omega}} w - \tfrac{1}{\varepsilon}|\bar{x}-\bar{y}|^2.$$

This implies that $\bar{x} - \bar{y} \to 0$ as $\varepsilon \searrow 0$. Using the fact that $\max_{\bar{\Omega}}(u-v) \le \Phi(\bar{x}, \bar{y}) \le u(\bar{x}) - v(\bar{y})$ and the semicontinuity, we see that $u(\bar{x}) - v(\bar{y}) \to \max_{\bar{\Omega}}(u-v)$ as $\varepsilon \searrow 0$ and hence

(6.0.4) $$\frac{1}{\varepsilon}|\bar{x}-\bar{y}|^2 \to 0 \quad \text{as } \varepsilon \searrow 0.$$

Now, we suppose

$$\max_{\bar{\Omega}}(u-v) > 0.$$

As $u \le v$ on $\partial\Omega$, using (6.0.4), we find that $\bar{x}, \bar{y} \in$ if $\varepsilon > 0$ is small enough. By using (6.0.3) we see that if $\varepsilon > 0$ is small enough, then

$$\begin{aligned} -a &\ge \tilde{H}(\bar{x}, \bar{y}, w(\bar{x}, \bar{y}), \tfrac{2}{\varepsilon}(\bar{x}-\bar{y}), \tfrac{2}{\varepsilon}(\bar{y}-\bar{x})) \\ &\ge H(\bar{x}, u(\bar{x}), \tfrac{2}{\varepsilon}(\bar{x}-\bar{y})) - H(\bar{y}, v(\bar{y}), \tfrac{2}{\varepsilon}(\bar{x}-\bar{y})) \\ &\ge H(\bar{x}, v(\bar{y}), \tfrac{2}{\varepsilon}(\bar{x}-\bar{y})) - H(\bar{y}, v(\bar{y}), \tfrac{2}{\varepsilon}(\bar{x}-\bar{y})) \quad (\text{by (H1) since } u(\bar{x}) > v(\bar{y})) \\ &\ge -m\left(|\bar{x}-\bar{y}|\left(\tfrac{2}{\varepsilon}|\bar{x}-\bar{y}|+1\right)\right) \quad (\text{by (H2)}). \end{aligned}$$

This is a contradiction in view of (6.0.4). ∎

Proof, II We will use Lemma 6.1. Define $w$ and $\tilde{H}$ as in Proof, I. For $d > 0$, set

$$\Delta_d = \{(x,y) \in \Omega \times \Omega : |x-y| < d\}.$$

We will construct a function $z \in C^1(\bar{\Delta}_d)$ which satisfies: (i) $z$ is a supersolution of (6.0.3), (ii) $z(x,x)$ is very small for $x \in \Omega$, (iii) $w \le z$ on $\partial\Delta_d$.

To do this, we fix $0<\varepsilon<a$, and choose $C_\varepsilon > 0$ so that

$$\begin{cases} m(r) \le \varepsilon + C_\varepsilon r & \text{for } r \ge 0, \\ w(x,y) \le \varepsilon + C_\varepsilon |x-y| & \text{for } (x,y) \in \partial(\Omega\times\Omega). \end{cases}$$

For $A, \delta > 0$ and $0 < \gamma < 1$, we define

$$z(x,y) = A\langle x-y\rangle_\delta^\gamma, \qquad \text{where } \langle x\rangle_\delta = (|x|^2 + \delta^2)^{1/2}.$$

Assuming $|x-y| \le 1$ and that $w(x,y) > z(x,y)$, we compute

$$\begin{aligned} a + \tilde{H}(x,y,w,Dz) = {} & a + H(x, u(x), \gamma A\langle x-y\rangle_\delta^{\gamma-2}(x-y)) \\ & - H(y, v(y), \gamma A\langle x-y\rangle_\delta^{\gamma-2}(x-y)) \end{aligned}$$

$$\geqslant a - m(|x-y|(\gamma A\langle x-y\rangle_\delta^{\gamma-1}+1)) \qquad (\text{by } (H2))$$

$$\geqslant a-\varepsilon - C_\varepsilon(\gamma A+1)\langle x-y\rangle_\delta^\gamma \qquad (\text{since } |x-y|\leq 1).$$

We now fix $d^\gamma$, $A$, $\gamma$ and $\delta$ in this order so that

$2C_\varepsilon d^\gamma < a-\varepsilon$ and $d^\gamma \leq 1$;

$A d^\gamma \geqslant \max_{\overline{\Omega}\times\overline{\Omega}} w$ ; $\gamma A < 1$ ; and

$2C_\varepsilon (d^2+\delta^2)^{\gamma/2} < a-\varepsilon$.

Then we have

$$\widetilde{H}(x,y,w,Dz) > -a \quad \text{if } (x,y)\in\Delta_d \text{ and } w>z \text{ at } (x,y).$$

Also, we have

$$z(x,y)\geqslant w(x,y) \quad \text{if } (x,y)\in\overline{\Delta}_d \text{ and } |x-y|=d,$$

$$z(x,y)\geqslant w(x,y) \quad \text{if } (x,y)\in\overline{\Delta}_d \text{ and } (x,y)\in\partial(\Omega\times\Omega),$$

and hence $z\geqslant w$ on $\partial\Delta_d$. Applying Lemma 6.1, we find that $z\geqslant w$ in $\Delta_d$ and, in particular,

$u(x)-v(x)\leqslant A\delta^\gamma$ for $x\in\Omega$. This implies $u\leqslant v$ in $\Omega$.

__Proof, III__ Let $\varepsilon > 0$, and define

$$u^\varepsilon(x) = \sup\{u(y) - \tfrac{1}{\varepsilon}|x-y|^2 : y \in \overline{\Omega}\} \quad \text{for } x \in \mathbb{R}^N,$$

$$v_\varepsilon(x) = \inf\{v(y) + \tfrac{1}{\varepsilon}|x-y|^2 : y \in \overline{\Omega}\} \quad \text{for } x \in \mathbb{R}^N.$$

Notice that $v_\varepsilon = -(-v)^\varepsilon$. Observe that

$$x \to \tfrac{1}{\varepsilon}|x|^2 + u(y) - \tfrac{1}{\varepsilon}|x-y|^2 \quad \text{is affine on } \mathbb{R}^N,$$

and hence

$$x \to u^\varepsilon(x) + \tfrac{1}{\varepsilon}|x|^2 \quad \text{is convex on } \mathbb{R}^N.$$

Also, we see that $u(x) \leq u^\varepsilon(x)$ for $x \in \overline{\Omega}$, ($\max u^\varepsilon \leq \max u$,) $u^\varepsilon(x) \in \mathbb{R}$ for $x \in \mathbb{R}^N$, and $u^\varepsilon(x) \searrow u(x)$ for $x \in \overline{\Omega}$ as $\varepsilon \searrow 0$.

For simplicity we assume that $u, v \in C(\overline{\Omega})$. Set $C_0 = 2 \max_{\overline{\Omega}} |u| \vee \max_{\overline{\Omega}} |v|$. If $x, y \in \overline{\Omega}$ and

$$u^\varepsilon(x) = u(y) - \tfrac{1}{\varepsilon}|x-y|^2,$$

then

$$-\frac{C_0}{2} \leq u^\varepsilon(x) \leq \frac{C_0}{2} - \frac{1}{\varepsilon}|x-y|^2,$$

i.e. $|x-y| \leq (C_0\varepsilon)^{1/2}$. Therefore, setting

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < (C_0\varepsilon)^{1/2}\},$$

we have

$$u^\varepsilon(x) = \sup\{u(y) - \tfrac{1}{\varepsilon}|x-y|^2 : y \in \Omega\} \quad \text{for } x \in \Omega_\varepsilon.$$

We now claim that $u^\varepsilon$ is a subsolution of

$$H(x + \tfrac{\varepsilon}{2}Du, u, Du) = 0 \quad \text{in } \Omega_\varepsilon.$$

To see this, let $\varphi \in C^1(\Omega_\varepsilon)$ and $\bar{x} \in \Omega_\varepsilon$ be a maximum point of $u^\varepsilon - \varphi$. For some $\bar{y} \in \Omega$, we have

$$u(y) - \tfrac{1}{\varepsilon}|x-y|^2 \le (u^\varepsilon - \varphi)(\bar{x}) = u(\bar{y}) - \tfrac{1}{\varepsilon}|\bar{x}-\bar{y}|^2 - \varphi(\bar{x}) \quad \forall y \in \Omega, \forall x \in$$

That is, $(x, y) \to u(y) - \frac{1}{\varepsilon}|x-y|^2 - \varphi(x)$ attains a maximum at $(\bar{x}, \bar{y})$. Therefore,

$$H(\bar{y}, u(\bar{y}), \tfrac{2}{\varepsilon}(\bar{y} - \bar{x})) \le 0,$$

$$\tfrac{2}{\varepsilon}(\bar{x} - \bar{y}) + D\varphi(\bar{x}) = 0,$$

and so

$$H(\bar{x} + \tfrac{\varepsilon}{2}D\varphi(\bar{x}), u(\bar{y}), D\varphi(\bar{x})) \le 0.$$

Since $u^\varepsilon(\bar{x}) = u(\bar{y}) - \frac{1}{\varepsilon}|\bar{x}-\bar{y}|^2 \leq u(\bar{y})$, we obtain

$$H(\bar{x} + \tfrac{\varepsilon}{2} D\varphi(\bar{x}),\ u^\varepsilon(\bar{x}),\ D\varphi(\bar{x})) \leq 0.$$

Similarly we find that $v_\varepsilon$ is a supersolution of

$$H(x - \tfrac{\varepsilon}{2} Du,\ u,\ Du) = a \quad \text{in } \Omega_\varepsilon.$$

Since $u^\varepsilon(x) \to u(x)$ and $v_\varepsilon(x) \to v(x)$ uniformly on $\bar{\Omega}$ as $\varepsilon \downarrow 0$ (and $u \leq v$ on $\partial\Omega$,) if we assume

$$\max_{\bar{\Omega}} (u - v) > 0,$$

then, for sufficiently small $\varepsilon > 0$, $u^\varepsilon - v_\varepsilon$ attains a positive maximum at some point in $\Omega_\varepsilon$. Let $y \in \Omega_\varepsilon$ be a positive maximum point of $u^\varepsilon - v_\varepsilon$. Observe that $x \to u^\varepsilon(x) + \frac{1}{\varepsilon}|x|^2$ is convex and so $u^\varepsilon$ is subdifferentiable everywhere. Similarly $v_\varepsilon$ is superdifferentiable everywhere. Let $p \in D^- u^\varepsilon(y)$ and $q \in D^+ v_\varepsilon(y)$. By definition we have

$$u^\varepsilon(x) \geq u^\varepsilon(y) + p\cdot(x-y) + o(|x-y|), \quad v_\varepsilon(x) \leq v_\varepsilon(y) + q\cdot(x-y) + o(|x-y|)$$

as $x \to y$. Thus

$$(u^\varepsilon - v_\varepsilon)(y) \geq (u^\varepsilon - v_\varepsilon)(x) \geq (u^\varepsilon - v_\varepsilon)(y) + (p-q)\cdot(x-y) + o(|x-y|)$$

as $x \to y$. Therefore, $p = q$ and

$$u^\varepsilon(y) + p\cdot(x-y) + o(|x-y|) \leq u^\varepsilon(x) \leq v_\varepsilon(x) + (u^\varepsilon - v_\varepsilon)(y) \leq$$
$$\leq u^\varepsilon(y) + p\cdot(x-y) + o(|x-y|)$$

as $x \to y$. That is, $u^\varepsilon$ is differentiable at $y$. Also, $v_\varepsilon$ is differentiable at $y$ and $Du(y) = Dv(y)$.

Hence, we have

$$(6.0.5) \quad \begin{cases} H(y + \frac{\varepsilon}{2} p_\varepsilon,\ u^\varepsilon(y),\ p_\varepsilon) \leq 0, \\ H(y - \frac{\varepsilon}{2} p_\varepsilon,\ u^\varepsilon(y),\ p_\varepsilon) \geq a, \end{cases}$$

where $p_\varepsilon = Du^\varepsilon(y) = Dv_\varepsilon(y)$. Let us write $x_\varepsilon = y$, and show that $Du^\varepsilon(x_\varepsilon) = o\left(\frac{1}{\sqrt{\varepsilon}}\right)$ as $\varepsilon \searrow 0$. Let $y_\varepsilon,\ z_\varepsilon \in \overline{\Omega}$ be points such that

$$u^\varepsilon(x_\varepsilon) = u(y_\varepsilon) - \frac{1}{\varepsilon}|y_\varepsilon - x_\varepsilon|^2, \qquad v_\varepsilon(x_\varepsilon) = v(z_\varepsilon) + \frac{1}{\varepsilon}|z_\varepsilon - x_\varepsilon|^2.$$

Then we have

$$\max_{\overline{\Omega}} (u - v) \leq u(y_\varepsilon) - \frac{1}{\varepsilon}|y_\varepsilon - x_\varepsilon|^2 - v(z_\varepsilon) - \frac{1}{\varepsilon}|z_\varepsilon - x_\varepsilon|^2.$$

We may assume that $x_\varepsilon \to \bar{x}$ as $\varepsilon \searrow 0$. From the above inequality we see that $x_\varepsilon - y_\varepsilon$, $x_\varepsilon - z_\varepsilon \to 0$ as $\varepsilon \searrow 0$ and so $y_\varepsilon, z_\varepsilon \to \bar{x}$ as $\varepsilon \searrow 0$. Moreover, we find that $u(y_\varepsilon) - v(z_\varepsilon) \to \max(u-v)$ as $\varepsilon \searrow 0$, and hence $\frac{1}{\varepsilon}|y_\varepsilon - x_\varepsilon|^2, \frac{1}{\varepsilon}|z_\varepsilon - x_\varepsilon|^2 \to 0$ as $\varepsilon \searrow 0$. We know that $Du^\varepsilon(x_\varepsilon) = \frac{2}{\varepsilon}(y_\varepsilon - x_\varepsilon)$. Therefore

$$|Du^\varepsilon(x_\varepsilon)| = \frac{2}{\sqrt{\varepsilon}}\left(\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon}\right)^{1/2} = o\left(\frac{1}{\sqrt{\varepsilon}}\right) \quad \text{as } \varepsilon \searrow 0.$$

From (6.0.5), using (H2), we have

$$-a \geq H(x_\varepsilon + \tfrac{\varepsilon}{2}p_\varepsilon, u^\varepsilon(x_\varepsilon), p_\varepsilon) - H(x_\varepsilon - \tfrac{\varepsilon}{2}p_\varepsilon, u^\varepsilon(x_\varepsilon), p_\varepsilon)$$

$$\geq -m(\varepsilon|p_\varepsilon|(|p_\varepsilon| + 1)) = -m(o(1)) \quad \text{as } \varepsilon \searrow 0.$$

This is a contradiction. ∎

Example This example shows that if the inequality in (H2) is replaced by

$$|H(x,r,p)-H(y,r,p)| \le m(|x-y|^{\theta}(|p|+1)), \quad \text{with } 0<\theta<1,$$

then Theorem 6.0.1 is not valid any more.

Let $0<\alpha<1$. Define

$$u(x) = e^{|x|^{\alpha}} \qquad \text{for } x \in \mathbb{R}^N,$$

$$v(x) = \begin{cases} e^{|x|^{\alpha}} & \text{for } x \in \mathbb{R}^N \setminus \{0\}, \\ 0 & \text{for } x = 0, \end{cases}$$

$$g(x) = \frac{1}{\alpha}|x|^{-\alpha}x \qquad \text{for } x \in \mathbb{R}^N.$$

Then $u = v \in C^{\infty}(\mathbb{R}^N \setminus \{0\})$ and for $x \neq 0$,

$$u(x) - g(x)\cdot Du(x) = u(x) - (g(x)\cdot x)\,\alpha|x|^{\alpha-2}e^{|x|^{\alpha}}$$

$$= e^{|x|^{\alpha}}\left(1 - \frac{1}{\alpha}|x|^{-\alpha}|x|^2\alpha|x|^{\alpha-2}\right) = 0.$$

Also, $D^+u(0) = \emptyset$, and

$$v(0) - g(0)\cdot p = 0 \qquad \forall p \in D^-v(0) = \mathbb{R}^N.$$

Therefore, $u$ and $v$ are solutions of

$$u - g(x)\cdot Du = 0 \qquad \text{in } \mathbb{R}^N.$$

Choose any open bounded subset $\Omega$ of $\mathbb{R}^N$ such that $0 \in \Omega$. Then

$u \in USC(\overline{\Omega})$, $v \in LSC(\overline{\Omega})$, $u$ and $v$ are solutions of $u - g(x)\cdot Du = 0$ in $\Omega$, and $u = v$ on $\partial\Omega$.

Notice that $g \in C^{0,1-\alpha}$ but $g \notin Lip$.

Theorem 6.0.a  Assume (H0) – (H2) and that $\Omega$ is bounded. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., sub- and supersolutions of (6.1). Assume $u \le v$ on $\partial\Omega$. Then $u \le v$ in $\Omega$.

Proof. For $a > 0$, define

$$v_a(x) = v(x) + a \quad \text{for } x \in \overline{\Omega}.$$

Then $v_a \in LSC(\overline{\Omega})$, $u \le v_a$ on $\partial\Omega$ and $v_a$ is a supersolution of

$$v_a + H(x, v_a, Dv_a) = a \quad \text{in } \Omega.$$

By Theorem 6.0, we see that $u \le v_a$ in $\Omega$, and so, $u \le v$ in $\Omega$. $\blacksquare$

This theorem is used to show the continuity of solutions of (6.1).

Theorem 6.0.b  Assume (H0) – (H2) and that $\Omega$ is bounded. Assume there is a subsolution $f$ and a supersolution $g$ of (6.1) such that $f \le g$ in $\Omega$, $f = g$ on $\partial\Omega$ and $f \in LSC(\overline{\Omega})$, $g \in USC(\overline{\Omega})$. Let $u$ be a solution of (6.1) satisfying $f \le u \le g$ on $\overline{\Omega}$. Then $u \in C(\overline{\Omega})$.

Proof By Theorem 5.1, there is a solution $u$ of (6.1) satisfying $f \le u \le g$ in $\overline{\Omega}$. We have $f \le u_* \le u^* \le g$ in $\overline{\Omega}$. As $f = g$ on $\partial\Omega$, $u^* = u_*$ on $\partial\Omega$. By Theorem 6.0.a we have $u^* \le u_*$ in $\Omega$, which implies $u \in C(\overline{\Omega})$. ▩

Theorem 6.0.c. Assume (H0)–(H2) and that $\Omega$ is bounded. Assume that $p \to H(x, r, p)$ is convex on $\mathbb{R}^N$ for $(x, r) \in \overline{\Omega} \times \mathbb{R}$ and that there is a $C^1$ function $\psi$ on $\overline{\Omega}$ for which

$$H(x, r, D\psi(x)) < 0 \qquad \text{for } x \in \overline{\Omega} \text{ and } r \in \mathbb{R}.$$

Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., suband supersolutions of (6.0.1). Then, if $u \le v$ on $\partial\Omega$, then $u \le v$ in $\Omega$.

<u>Proof</u> We may assume that $\psi(x) \le \inf_{\overline{\Omega}} u$ for $x \in \overline{\Omega}$. For $0 < \theta < 1$ define

$$u_\theta = \theta u + (1-\theta)\psi \qquad \text{on } \overline{\Omega}.$$

Using Prop. 2.7, we see that $u_\theta$ is a subsolution of

$$u_\theta + H(x, u_\theta, Du_\theta) = f(x) \qquad \text{in } \Omega,$$

where $f(x) = (1-\theta)H(x, u_\theta(x), D\psi(x))$. Choose $a > 0$ so that $f(x) \le -a$. Then $u_\theta$ is a subsolution of

$$u_\theta + H(x, u_\theta, Du_\theta) - a = 0 \qquad \text{in } \Omega.$$

Applying Theorem 6.0, we get $u_\theta \le v$ in $\Omega$, from which we conclude $u \le v$ in $\Omega$. ∎

<u>Example</u> Consider the equation

$$(6.0.6) \qquad |Du| = n(x) \qquad \text{in } \Omega,$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $n \in C(\overline{\Omega})$

and $n(x) > 0$ in $\overline{\Omega}$. Let $\psi(x) \equiv 0$. Then

$$|D\psi(x)| - n(x) = -n(x) < 0 \qquad \text{for } x \in \overline{\Omega}.$$

Therefore Theorem 6.0.c applies to (6.0.6), to conclude that the solution of (6.0.6) with prescribed Dirichlet data is unique.

bounded on $\overline{\Omega} \times [0,T)$, $u$ and $v$ are, resp., u.s.c and l.s.c. on $\overline{\Omega} \times [0,T)$, and

(6.4) $$\lim_{r \downarrow 0} \sup \{ u(x,t) - v(y,t) : (x,y,t) \in \partial(\Omega \times \Omega) \times (0,T) \cup \overline{\Omega} \times \overline{\Omega} \times \{0\}, \; |x-y| \leq r \} \leq 0.$$

Then $u \leq v$ in $\Omega \times \Omega \times (0,T)$.

<u>Lemma 6.1</u> Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ (be locally bounded). Let $u$ be an (u.s.c.) solution of $F(x,u,Du) \leq 0$ in $\Omega$ and $v \in C^1(\Omega)$ satisfy

$F_*(x, u(x), Dv(x)) > 0$ if $x \in \Omega$ and $u(x) > v(x)$. Assume

$$\limsup_{\Omega \ni x \to y} (u-v)(x) \leq 0 \qquad \forall y \in \partial\Omega.$$

Then $u \leq v$ in $\Omega$.

<u>Proof</u> Suppose

$$\sup_{\Omega} (u-v) > 0.$$

Then, for some $y \in \Omega$;

$$(u-v)(y) = \sup (u-v).$$

By the definition of subsolution, we have

$$F_*(x, u(y), Dv(y)) \leq 0,$$

which is a contradiction. Thus $u \leq v$ in $\Omega$. ∎

<u>Proof of Thm 6.1, (i)</u> We set

$$\tilde{H}(x, y, r, p, q) = H(x, r + v(y), p) - H(y, u(x) - r, -q)$$

for $x, y \in \overline{\Omega}$, $r \in \mathbb{R}$, and $p, q \in \mathbb{R}^N$, and

$$w(x, y) = u(x) - v(y) \quad \text{for } x, y \in \overline{\Omega}.$$

Note that $\tilde{H}$ is l.s.c. and locally bounded in $\Omega \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N$ and $w \in USC(\overline{\Omega} \times \overline{\Omega})$.

It is easy to see that $w$ is a subsolution of

(6.5) $\quad w + \tilde{H}(x, y, w, Dw) = 0$ in $\Omega \times \Omega$, where $Dw = (D_x w, D_y$

We suppose

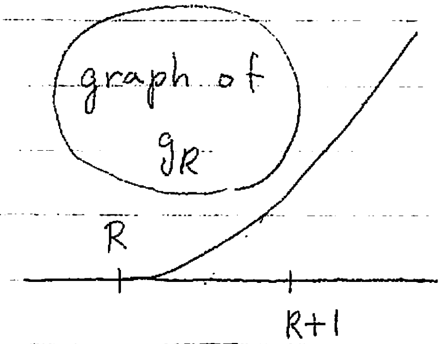
$$\sup_{\Omega} (u - v) > 0.$$

There is $R_0 > 0$ such that

$$\sup_{\Omega \cap B(0, R_0)} (u - v) > 0.$$

Choose $g_R \in C^1(\mathbb{R})$ for each $R > 0$ so that

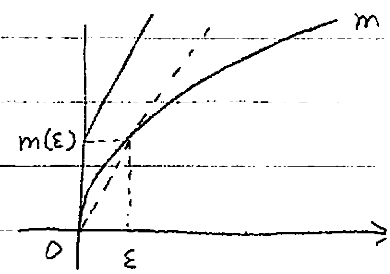
$$\begin{cases} g_R(r) = 0 & \text{for } r \leq R, \\ g_R(r) = r - R - \frac{1}{2} & \text{for } r \geq r+1, \\ 0 \leq g_R'(r) \leq 1 & \text{for } r \in \mathbb{R}. \end{cases}$$




For constants $A, \alpha, \delta > 0$ and $0 < \gamma < 1$ we set

$$z(x,y) = A\langle x-y\rangle_\delta^\gamma + \alpha, \quad \text{where } \langle x\rangle_\delta = (|x|^2 + \delta^2)^{1/2}.$$

We may assume that $m$ from (H2) is increasing and concave on $[0,\infty)$. We have




$$(6.6) \qquad m(r) \leq m(\varepsilon) + \frac{m(\varepsilon)}{\varepsilon} r \qquad \forall r \geq 0, \forall \varepsilon > 0.$$

Fix $\varepsilon > 0$, and compute that if $|x-y| \leq 1$ and $w(x,y) > z(x,y)$, then

$$z + \tilde{H}(x,y,w,Dz) = z + H(x, u(x), \gamma A\langle x-y\rangle_\delta^{\gamma-2}(x-y))$$

$$-H(y, v(y), -\gamma A\langle x-y\rangle_\delta^{\gamma-2}(y-x)) \geq z + H(x, v(y), \gamma A\langle x-y\rangle_\delta^{\gamma-2}(x-y))$$

$$-H(y, v(y), \gamma A\langle x-y\rangle_\delta^{\gamma-2}(x-y)) \qquad (\text{by (H1) since } u(x) > v(y))$$

$$\geq z - m(|x-y|(\gamma A\langle x-y\rangle_\delta^{\gamma-2}|x-y| + 1)) \qquad (\text{by (H2)})$$

$$\geq z - m(\varepsilon) - \frac{m(\varepsilon)}{\varepsilon}\left(\gamma A \langle x-y\rangle_\delta^\gamma + \langle x-y\rangle_\delta^\gamma\right) = \qquad (\text{by } (6.6))$$

$$= \left\{\left(1 - \frac{m(\varepsilon)}{\varepsilon}\gamma A\right) - \frac{m(\varepsilon)}{\varepsilon}\right\}\langle x-y\rangle_\delta^\gamma + \alpha - m(\varepsilon).$$

Set $C_1 = \sup_\Omega |u| + \sup_\Omega |v|$, choose $C_\varepsilon > 0$ so that $w(x,y) \leq \varepsilon + C_\varepsilon |x-y|$ for $(x,y) \in \partial(\Omega\times\Omega)$, and fix

$$\alpha = m(\varepsilon) + \varepsilon, \qquad \gamma = \frac{\varepsilon}{2m(\varepsilon)} \wedge 1, \qquad A = C_1 \vee \frac{2m(\varepsilon)}{\varepsilon} \vee C_\varepsilon.$$

Thus we have

$$\begin{cases} z + \tilde{H}(x,y,w,Dz) > 0 & \text{if } |x-y| \leq 1 \text{ and } w > z. \\ w \leq z & \text{if } (x,y) \in \partial(\Omega\times\Omega) \text{ and } |x-y| \leq 1. \end{cases}$$

For $\beta > 0$ and $R > R_0$ we set

$$z_1(x,y) = z(x,y) + \beta g_R(|x|).$$

Then, if $|x-y| \leq 1$ and $w(x,y) > z_1(x,y)$, then

$$z_1 + \tilde{H}(x,y,w,Dz_1) \geq z + \tilde{H}(x,y,w,Dz) - \sigma_{L+\beta}(\beta) > -\sigma_{L+\beta}(\beta),$$

where $L = \gamma A \delta^{\gamma-2}$ (note that $|D_x z| \leq \gamma A \delta^{\gamma-1}$). Observe that

$$\lim_{|x|+|y|\to\infty} z_1(x,y) = -\infty,$$

and that if $|x-y| = 1$, then $w(x,y) \leq C_1 \leq z(x,y) \leq z_1(x,y)$

Therefore, for some $R_1 > R$, setting

$$\Delta = \{(x,y) \in \Omega\times\Omega : |x-y| < 1, \ |x| < R_1, \ |y| < R_1\},$$

we have

$$w \le z_1 \quad \text{on } \partial\Delta = \overline{\Delta} \cap \left[\{|x-y|=1\} \cup \{|x|=R_1\} \cup \{|y|=R_1\} \cup \partial(\Omega\times\Omega\right.$$

Using Lemma 6.1, we compare $w$ with $z_1 + \sigma_{L+\beta}(\beta)$ on the set $\Delta$, to get

$$w(x,y) \le A\langle x-y\rangle_\delta^\gamma + \alpha + \sigma_{L+\beta}(\beta) \quad \text{for } (x,y) \in \Delta.$$

Send $R_1 \to \infty$, $R \to \infty$, $\beta \searrow 0$ and $\delta \searrow 0$ in this order, to get

(6.7) $u(x) - v(y) \le m(\varepsilon) + \varepsilon + A|x-y|^\gamma$ for $x, y \in \Omega$ with $|x-y| <$

Taking $x = y$ and letting $\varepsilon \searrow 0$, we obtain $u \le v$ in $\Omega$ ∎

Lemma 6.2 Under the assumptions of Theorem 6.1, (ii), setting $w(x,y,t) = u(x,t) - v(y,t)$ and

(6.8) $\tilde{H}(x,y,t,p,q) = H(x,t,r+v(y),p) - H(y,t,u(x)-r,-q)$,

we have

(6.9) $w_t + \tilde{H}(x,y,w,Dw) \le 0$ in $\Omega\times\Omega\times(0,T)$

in the viscosity sense.

*Proof* Let $\varphi \in C^1(\Omega\times\Omega\times(0,T))$ and $(\bar{x},\bar{y},\bar{t}) \in \Omega\times\Omega\times(0,T)$ be a strict maximum point of $w-\varphi$. Choose $r>0$ so that $B((\bar{x},\bar{y},\bar{t},\bar{t}),r) \subset \Omega\times\Omega\times(0,T)\times(0,T)$. For $n\in N$ we set

$$\Phi_n(x,y,t,s) = u(x,t) - v(y,s) - \varphi(x,y,t) - n(t-s)^2$$

for $(x,y,t,s) \in B((\bar{x},\bar{y},\bar{t},\bar{t}),r)$. Let $(x_n,y_n,t_n,s_n)$ be a maximum point of $\Phi_n$. Then

$$u(x_n,t_n) - v(y_n,s_n) - \varphi(x_n,y_n,t_n) \geq \Phi_n(x_n,y_n,t_n,s_n)$$
$$\geq u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - \varphi(\bar{x},\bar{y},\bar{t}) \quad \text{for } n\in N.$$

From this we find that $t_n - s_n \to 0$ as $n\to\infty$.

There is a subsequence of $\{(x_n,y_n,t_n,s_n)\}$ which converges to some point $(x_0,y_0,t_0,s_0)$. We have $t_0=s_0$ and

$$u(x_0,t_0) - v(y_0,t_0) - \varphi(x_0,y_0,t_0) \geq u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - \varphi(\bar{x},\bar{y},\bar{t}).$$

This implies that $x_n\to\bar{x}$, $y_n\to\bar{y}$, $t_n, s_n\to\bar{t}$ (and $u(x_n,t_n) - v(y_n,s_n) \to u(\bar{x},\bar{t}) - v(\bar{y},\bar{s})$) as $n\to\infty$. (Moreover, we see that $u(x_n,t_n)\to u(\bar{x},\bar{t})$ and $v(y_n,s_n)\to v(\bar{y},\bar{s})$ as $n\to\infty$.) On the other hand, as $u$ and $v$ are, resp., sub- and supersolutions of (6.2), we have

$$\varphi_t(x_n, y_n, t_n) + 2n(t_n - s_n) + H(x_n, t_n, u(x_n, t_n), D_x\varphi(x_n, y_n, t_n)) \le 0,$$

$$-2n(s_n - t_n) + H(y_n, s_n, v(y_n, s_n), -D_y\varphi(x_n, y_n, s_n)) \ge 0$$

for large $n$. Subtract one from the other and send $n \to \infty$, to get

$$\varphi_t(\bar{x}, \bar{y}, \bar{t}) + H(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D_x\varphi(\bar{x}, \bar{y}, \bar{t}))$$

$$- H(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), -D_y\varphi(\bar{x}, \bar{y}, \bar{t})) \le 0,$$

i.e.,

$$\varphi_t(\bar{x}, \bar{y}, \bar{t}) + \tilde{H}(\bar{x}, \bar{y}, \bar{t}, w(\bar{x}, \bar{y}, \bar{t}), D\varphi(\bar{x}, \bar{y}, \bar{t})) \le 0. \quad \blacksquare$$

Proof of Theorem 6.1, (ii) Set

$$w(x, y, t) = u(x, t) - v(y, t) \quad \text{for } (x, y, t) \in \bar{\Omega} \times \bar{\Omega} \times [0, T)$$

By Lemma 6.2, $w$ is a solution of (6.9). Fix $\varepsilon > 0$, and choose $C_\varepsilon > 0$ so that

(6.10) $$w(x, y, t) \le \varepsilon + C_\varepsilon |x - y| \quad \text{for } (x, y, t) \in \partial(\Omega \times \Omega) \times (0, T) \cup \bar{\Omega} \times \bar{\Omega} \times \{0\}.$$

Let $A, R, \alpha, \beta, \delta, \eta > 0$ and $0 < \gamma < 1$.

Define

$$z(x,y,t) = e^t \left( A\langle x-y \rangle_\delta^\gamma + \alpha \right) + \beta g_R(|x|) + \frac{\eta}{T-t}$$

where $g_R$ is as in the proof of Theorem 6.1, (i). If $|x-y| \le 1$ and $w(x,y) \ge 0$, then

$$z_t + \widetilde{H}(x,y,t,w,Dz) \geqslant \frac{\eta}{(T-t)^2} + A\langle x-y \rangle_\delta^\gamma + \alpha + H(x,t,u(x,t), e^t \gamma A \langle x-y \rangle_\delta^{\gamma-2}(x-$$

$$- H(y,t,v(y,t), \gamma A\langle x-y \rangle_\delta^{\gamma-2}(x-y)) - \sigma_{L+\beta}(\beta)$$

$$\geqslant -m\left(|x-y|\left(e^T \gamma A \langle x-y \rangle_\delta^{\gamma-1} + 1\right)\right) - \sigma_{L+\beta}(\beta) + A\langle x-y \rangle_\delta^\gamma + \alpha$$

$$\geqslant -m(\varepsilon) - \frac{m(\varepsilon)}{\varepsilon}\left(e^T \gamma A + 1\right)\langle x-y \rangle_\delta^\gamma - \sigma_{L+\beta}(\beta) + A\langle x-y \rangle_\delta^\gamma + \alpha =$$

$$= \left\{ \left(\frac{A}{2} - \frac{m(\varepsilon)}{\varepsilon}\right) + A\left(\frac{1}{2} - \frac{m(\varepsilon)}{\varepsilon}\gamma e^T\right) \right\} \langle x-y \rangle_\delta^\gamma + \alpha - m(\varepsilon) - \sigma_{L+\beta}(\beta),$$

where $L = e^T \gamma A \delta^{\gamma-2}$. We choose

$$\alpha = m(\varepsilon) + \varepsilon, \quad \gamma = \frac{\varepsilon}{2m(\varepsilon)} e^{-T} \wedge 1 \quad \text{and} \quad A = \frac{2m(\varepsilon)}{\varepsilon} \vee C_\varepsilon .$$

Then we choose $\beta > 0$ so small that $\sigma_{L+\beta}(\beta) < \varepsilon$.

We have

$$(w - z)(x,y,t) \to -\infty \text{ uniformly (on compact subsets of } \overline{\Omega}\times\overline{\Omega}) \text{ as } t \nearrow T,$$

$$w(x,y,t) \le z(x,y,t)$$

if $(x,y,t) \in \partial(\Omega\times\Omega)\times(0,T) \cup \overline{\Omega}\times\overline{\Omega}\times\{0\}$ and $|x-y|:$

$$z_t + \widetilde{H}(x,y,t,w,Dz) > 0$$

if $(x,y,t) \in \Omega\times\Omega\times(0,T)$, $|x-y| \le 1$ and $w > z$ at $(x,y,t)$.

We suppose

$$\sup_{\Omega\times(0,T)} (u - v) > 0.$$

Choosing $R > 0$ and then $R_1 > R$ large enough, $\eta > 0$ small enough

and setting

$$\Delta = \{(x,y,t) \in \Omega\times\Omega\times(0,T) : |x-y| < 1,\ |x| < R_1,\ |y| < R_1\},$$

we have

$$\lim_{(x,y,t)\to(\bar{x},\bar{y},\bar{t})} (w - z)^+(x,y,t) = 0 \quad \text{for } (\bar{x},\bar{y},\bar{t}) \in \partial\Delta,$$

$$\sup_{\Delta} (w - z) > 0.$$

Using Lemma 6.1, we have

$w \leq z$ in $\Delta$.

Sending $R_1 \to \infty$, $R \to \infty$, $\beta \searrow 0$, $\delta \searrow 0$ and $\eta \searrow 0$, we obtain

(6.11) $u(x,t) - v(y,t) \leq \alpha + A|x-y|^\gamma$

for $(x,y,t) \in \Omega \times \Omega \times (0,T)$ with $|x-y| < 1$.

From this we conclude that $u \leq v$ in $\Omega \times \Omega \times (0,T)$.

In order to carry out the above proof, it is enough to assume

(6.12) $\displaystyle\sup_{\substack{x,y \in \Omega \\ |x-y| \leq \delta}} (u(x) - v(y)) < \infty$

(resp.,

(6.13) $\displaystyle\sup_{\substack{x,y \in \Omega \\ |x-y| \leq \delta \\ 0 \leq t < T}} (u(x,t) - v(y,t)) < \infty$ )

for some $\delta > 0$ instead of requiring the boundedness of $u, v$. The condition (6.12) (resp., (6.13)) is satisfies if $u, v$ grow at most linearly as $|x| \to$

*Theorem 6.2.* Assume the hypotheses of (i) (resp., (ii)) of Theorem 6.1 except that $u$, $v$ are bounded. Assume that

$$(6.14)\qquad u(x), -v(x) \leq C(|x|+1) \qquad \text{for } x, y \in \Omega$$

(resp.,

$$(6.15)\qquad u(x,t), -v(x,t) \leq C(|x|+1) \qquad \text{for } x, y \in \Omega \text{ and } 0 \leq t$$

for some constant $C > 0$. Then $u \leq v$ in $\Omega$ (resp., $\Omega \times (0,T)$).

*Proof.* We only prove (6.13) for $\delta = 1$ under the assumption (6.15). Let $w$, $\widetilde{H}$ and $g_R$ be as in the proof of Theorem 6.1, (ii). Let $A, B, \eta >$ and define

$$z(x,y,t) = e^t\,(A\langle x-y\rangle + B) + (C+1)\,g_R(|x|) + \frac{\eta}{T-t}, \text{ where } \langle x\rangle = (|x|^2 +$$

Assuming $w > z$ at $(x,y,t)$, we compute

$$z_t + \widetilde{H}(x,y,t,w,Dz) \geq A\langle x-y\rangle + B + H(x,t,u(x,t),0)$$

$$-H(y,t,u(x),0) - 2\sigma_{Ae^T+C+1}(C+1)$$

$$\geq A\langle x-y\rangle - m(|x-y|) + B - 2\sigma_{Ae^T+C+1}(C+1).$$

We assume that $A$, $B$ are so large that

$$A\langle x-y\rangle \geq m(|x-y|) \quad \text{for } x, y \in \mathbb{R}^N, \text{ and}$$

$$B > 2\sigma_{Ae^T+C+1}(C+1).$$

These assures the right side of the above inequality is positive.

In view of (6.4) we may assume by choosing $A$ large enough

(6.16) $\quad w(x,y,t) \leq A\langle x-y\rangle$

for $(x,y,t) \in \partial(\Omega\times\Omega)\times(0,T) \cup \bar{\Omega}\times\bar{\Omega}\times\{0\}$. We also

— $A > 3C$ 程度とする。

assume $A > C$, which guarantees

$$(w-z)(x,y,t) \to -\infty \quad \text{uniformly in } t \text{ as } |x|+|y| \to \infty.$$

Moreover,

$$(w-z)(x,y,t) \to -\infty \quad \text{uniformly on compact subsets of } \bar{\Omega}\times\bar{\Omega} \text{ as } t \nearrow T.$$

These observations show that $w-z$ attains a maximum at a point in $\Omega\times\Omega\times(0,T)$. Applying Lemma 6. we find that $w \le z$ on $\Omega\times\Omega\times(0,T)$. Sending $\gamma \downarrow 0$, we have

$$w \le A\langle x-y\rangle + B \quad \text{in } \Omega\times\Omega\times(0,T). \quad \blacksquare$$

Remark In the proofs of Theorems 6.1 and 6.2, we considered a positive maximum point of functions taking form

$$\Phi(x,y) = u(x) - v(y) - \varphi(x-y) - \beta g_R(|x|)$$

or

$$\Phi(x,y,t) = u(x,t) - v(y,t) - \varphi(x-y) - \frac{\gamma}{T-t} - \beta g_R(|x|),$$

where $u$ and $v$ are sub- and supersolutions of (6.1) or (6.2), respectively, and $\varphi \in C^1$. If, e.g., $u$ is Lipschitz continuous with Lipschitz constant $L$, then

$$D_x \Phi = D\varphi(x-y) + \beta g_R'(|x|)\frac{x}{|x|}$$

is bounded by $L$ in its norm at the maximum point of $\Phi$. As

$$|D_y(-\Phi)| = |D\varphi(x-y)| \leq L + \beta$$

at the maximum point, if we notice that we had an upper bound on $\beta$ in the above proofs, we can replace (H2) by a weaker assumption in this case. That is, (H2) can be replaced by the following (H2)′ in Theorems 6.1 and 6.2 if $u$ or $v$ is assumed to be Lipschitz continuous.

(H2)′ For each $R > 0$ there is $m_R \in C([0,\infty))$ such that $m_R(0) = 0$ and

$$|H(x,t,r,p) - H(y,t,r,p)| \leq m_R(|x-y|)$$

for $x, y \in \Omega$, $(t,r) \in [0,T] \times \mathbb{R}$ and $p \in B(0,R)$.

<u>Remark</u> In the proof of Theorem 6.1, we have used this observation: For any $\varepsilon > 0$ there is

a constant $C_\varepsilon > 0$ such that

$$m(r) \leq \varepsilon + C_\varepsilon r \qquad \text{for } r \geq 0,$$

$$w(x,y) \leq \varepsilon + C_\varepsilon |x-y| \qquad \text{for } (x,y) \in \partial(\Omega \times \Omega).$$

If, e.g., $\Omega$ is convex, then this is true. However this is not true in general. This difficulty is solved by introducing

$$\{(x,y) \in \Omega \times \Omega : \ l(x,y) \subseteq \Omega, \ |x-y| < 1, \ |x| < R_1, \ |y| < R_1\}$$

and using it in place of $\Delta$, where $l(x,y)$ denotes the line segment between $x$ and $y$, i.e. $\{z : z = \lambda x + (1-\lambda) y, \ 0 \leq \lambda \leq 1\}$.

The role of (H3) is just in the case of unbounded domains $\Omega$. Condition (H3) seems appropriate to the uniqueness result for solution which grows at most linearly. Next, we examine uniqueness for some other conditions.

The first one is:

(H4) There is a constant $A > 0$ such that

$$|H(x,t,r,p) - H(x,t,r,q)| \leq A|p-q|$$

for $(x,t,r) \in \Omega \times [0,T] \times \mathbb{R}$ and $p, q \in \mathbb{R}^N$.

$\Longrightarrow$ Insert here 77+1

Theorem 6.3. Assume (H0), (H1), (H4) and (H5). Let $u$ and $v$ be, resp., sub- and supersolutions of (6.1). Assume $u \in USC(\overline{\Omega})$, $v \in LSC(\overline{\Omega})$, $u \leq v$ on $\partial\Omega$ and that

$$(6.17) \quad \varlimsup_{\substack{x \in \overline{\Omega} \\ |x| \to \infty}} \left[u(x) \vee (-v(x)) \vee 0\right] e^{-\frac{1}{A}|x|} = 0.$$

Then $u \leq v$ in $\Omega$.

Proof Set $\lambda = \frac{1}{A}$. For $\varepsilon > 0$ define

$$u_\varepsilon(x) = u(x) - \varepsilon e^{\lambda\langle x\rangle} \quad \text{and} \quad v_\varepsilon(x) = v(x) + \varepsilon e^{\lambda\langle x\rangle}.$$

Notice that $|De^{\lambda\langle x\rangle}| \leq \lambda e^{\lambda\langle x\rangle}$. Using this, we see that $u_\varepsilon$ and $v_\varepsilon$ are, resp., sub- and super

In the proofs below condition (H2) is required only on each bounded subsets of $\Omega$. That is, we need a weaker assumption

(H5) For each $R > 0$ there is a function $m_R \in C([0,\infty))$ such that $m_R(0) = 0$ and

$$|H(x,t,r,p) - H(y,t,r,p)| \leq m_R(|x-y|(|p|+1))$$

holds for $x, y \in \Omega \cap B(0,R)$, $0 \leq t \leq T$, $r \in \mathbb{R}$, $p \in \mathbb{R}$

solutions of (6.1). By (6.17) we have

$$\overline{\lim_{\substack{x \in \overline{\Omega} \\ |x| \to \infty}}} \; u_\varepsilon(x) \vee (-v_\varepsilon(x)) = -\infty.$$

This implies that for some $R > 0$,

$$u_\varepsilon(x), \; -v_\varepsilon(x) \leq 0 \qquad \forall x \in \overline{\Omega} \setminus B(0,R)^\circ.$$

Let $\Omega_R = \Omega \cap B(0,R)^\circ$. Then we have

$$u_\varepsilon \leq v_\varepsilon \qquad \text{on } \partial\Omega_R.$$

As $\Omega_R$ is bounded, we may apply Theorem 6.0 to conclude

$$u_\varepsilon(x) \leq v_\varepsilon(x) \qquad \text{in } \Omega.$$

Thus, $u \leq v$ in $\Omega$. ∎

Remark Let $N = 1$. $u(x) \equiv 0$ and $u(x) = e^{x/A}$ solve $u - ADu = 0$ in

Remark In case where $H$ has the form

$$H(x,p) = \max_{a \in A} \{-g(x,a)\cdot p - f(x,a)\},$$

if $g$ is bounded in $x$ and $a$, then (H4) is satisfied. The next assumption concerns the case where we have the Lipschitz continuity of $g$ in $x$ uniformly for $a \in A$.

(H6) There are constants $A, B > 0$ such that

$$|H(x,t,r,p) - H(x,t,r,q)| \leq (A|x| + B)|p-q|$$

for $(x,t,r) \in \Omega \times [0,T] \times R$ and $p, q \in \mathbb{R}^N$.

<u>Theorem 6.4</u> Assume (H0), (H1), (H5) and (H6). Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., sub- and supersolutions of (6.1). Suppose that $u \leq v$ on $\partial\Omega$ and

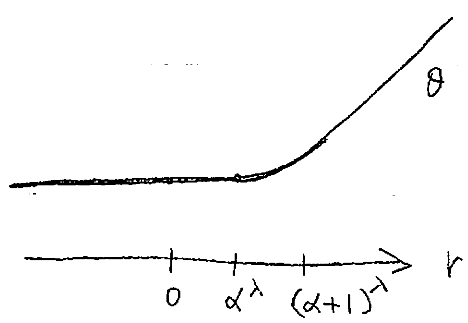
$$(6.18) \qquad \overline{\lim_{x \in \overline{\Omega},\ |x| \to \infty}} \, [u(x) \vee (-v(x)) \vee 0]\, |x|^{-\frac{1}{A}} = 0. \qquad |Dg| \leq g(x)$$

Then $u \leq v$ in $\Omega$.

$g(x) = (A\langle x \rangle + B)^{\frac{1}{A}}$ の方が良い.

<u>Proof</u> Set $\lambda = \frac{1}{A}$ and $\alpha = \frac{B}{A}$. Choose $\theta \in C^1(\mathbb{R})$ so that $\theta(r) = r$ for $r \geq (\alpha+1)^\lambda$, $0 \leq \theta'(r) \leq 1$ for $r \in \mathbb{R}$ and $\theta(r) = 0$ for $r \leq \alpha^\lambda$.

Of course, $\theta(r) \geq 0$ for $r \in \mathbb{R}$.



Then $g(x) = \theta((|x|+\alpha)^{\lambda})$ is a $C^1$ function on $\mathbb{R}^N$, and

$$|Dg(x)| = \theta'((|x|+\alpha)^{\lambda})\,\lambda(|x|+\alpha)^{\lambda-1}.$$

If $|x| \geqslant 1$, then $|Dg(x)|(A|x|+B) = (|x|+\alpha)^{\lambda} = g(x)$. And there is a constant $C > 0$ such that

$$|Dg(x)|(A|x|+B) \leq C \qquad \text{for } x \in B(0,1).$$

For $\varepsilon > 0$ define

$$u_\varepsilon(x) = u(x) - \varepsilon(g(x) + C) \quad \text{and} \quad v_\varepsilon(x) = v(x) + \varepsilon(g(x) + C).$$

Notice that for $(x, r, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, we have

$$r \pm \varepsilon(g(x)+C) + H(x, r \pm \varepsilon(g(x)+C), p \pm \varepsilon Dg(x))$$

$$\gtreqless r + H(x,r,p) \pm \varepsilon(g(x)+C) \mp |Dg(x)|(A|x|+B) \gtreqless r + H(x,r,p).$$

Therefore, $u_\varepsilon$ and $v_\varepsilon$ are, resp., sub- and super-solutions of (6.1). Moreover,

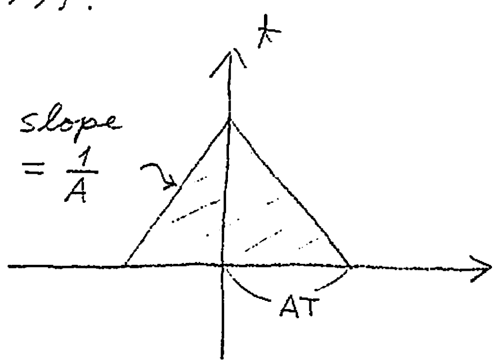
$$\varlimsup_{x\in\overline{\Omega},\,|x|\to\infty} (u_\varepsilon - v_\varepsilon)(x) \leqslant -2\varepsilon C.$$

Applying Lemma 6.1, we obtain $u_\varepsilon \leq v_\varepsilon$ in $\Omega$, from which follows $u \leqslant v$ in $\Omega$. ∎

<u>Remark</u> Functions $u(x) \equiv 0$ and $u(x) = |x|^{\lambda}$, with $\lambda > c$ satisfy $u - \frac{1}{\lambda} x \cdot Du = 0$ in $\mathbb{R}^N$.

We now turn our attension to (6.2) again and prove the existence of cone of dependence for solutions of (6.2) under assumption (H4). Let $A$ be a positive constant from (H4), and define

$$C_{A,T} = \{(x,t) \in \mathbb{R}^N \times (0,T) : |x| < A(T-t)\}.$$



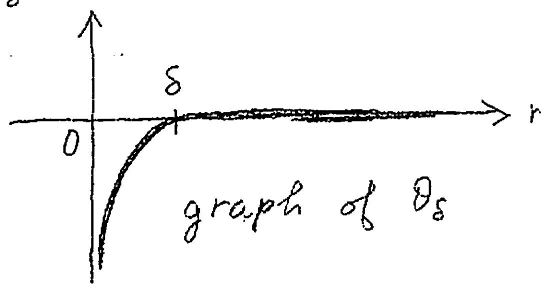

<u>Theorem 6.5</u> Assume (H0)-(H2), and (H4). Let $u \in USC(\overline{C}_{A,T})$ and $v \in LSC(\overline{C}_{A,T})$ be, resp., sub- and super-solutions of (6.2) in $C_{A,T}$. Assume $u \le v$ on $B(0, AT) \times \{0\}$. Then $u \le v$ in $C_{A,T}$.

<u>Proof</u> For each small $\delta > 0$ we choose a $C^1$

function $\theta_\delta$ in $(0, \infty)$ so that

$\theta_\delta' \geq 0$ in $(0, \infty)$, $\quad \lim_{r \searrow 0} \theta_\delta(r) = -\infty$ and

$\theta_\delta(r) = 0$ for $r \geq \delta$.




We define

$$u_\delta(x,t) = u(x,t) + \theta_\delta\left(A^2(T-t)^2 - |x|^2\right),$$

$$v_\delta(x,t) = v(x,t) - \theta_\delta\left(A^2(T-t)^2 - |x|^2\right)$$

in $C_{A,T}$. Notice that $A^2(T-t)^2 - |x|^2 > 0$ in $C_{A,T}$.

$$\overline{\lim_{(x,t) \in C_{A,T},\ (x,t) \to (y,s)}} (u_\delta - v_\delta)(x,t) \leq 0 \quad \text{for } (y,s) \in \partial C_{A,}$$

If we set $g(x,t) = \theta_\delta\left(A^2(T-t)^2 - |x|^2\right)$, then

$$q \pm g_t(x,t) + H(x,t,r \pm g(x,t), p \pm D_x g(x,t))$$

$$\lesseqgtr q \pm g_t(x,t) + H(x,t,r,p) \pm A|D_x g(x,t)|$$

$$\lesseqgtr q + H(x,t,r,p) \mp 2A^2(T-t)\theta_\delta'(\cdots) \pm 2A|x|\,\theta_\delta'(\cdots) =$$

$$= q + H(x,t,r,p) \mp 2A\,\theta_\delta'(\cdots)\left[A(T-t) - |x|\right]$$

$$\lesseqgtr q + H(x,t,r,p)$$

for $(x,t) \in C_{A,T}$, $r \in \mathbb{R}$ and $(q,p) \in \mathbb{R}^{N+1}$. From th we see that $u_\delta$ and $v_\delta$ are, resp., sub- and super-solutions of (6.2) in $C_{A,T}$. By using Lemma 6.1 we obtain $u_\delta \le v_\delta$ in $C_{A,T}$, from which $u \le v$ in $C_{A,T}$ is concluded. ∎

The above proof is easily modified in order to get the following result.

<u>Theorem 6.6</u> Assume (H0)–(H2) and (H4). Let $u \in USC(\bar{\Omega} \times [0,T])$ and $v \in LSC(\bar{\Omega} \times [0,T])$ be, resp., sub- and supersolutions of (6.2). Assume $u \le v$ on $\partial\Omega \times (0,T) \cup \bar{\Omega} \cap B(0,AT) \times \{0\}$. Then $u \le v$ in $C_{A,T} \cap \Omega \times (0,T)$.

We apply this result to get a comparison result under assumption (H6).

<u>Theorem 6.7</u> Assume (H0), (H1), (H5) and (H6). Let $u \in USC(\bar{\Omega} \times [0,T])$ and $v \in LSC(\bar{\Omega} \times [0,T])$ be, resp.

sub- and supersolutions of (6.2). Assume $u \leq v$ on $\partial\Omega \times (0,T) \cup \bar{\Omega} \times \{0\}$. Then $u \leq v$ in $\Omega \times (0,T)$.

<u>Proof</u> We may assume that $T < \frac{1}{A}$. If this is not the case, we have only to repeat the argument below in order to get the general result. Fix $x_0 \in \Omega$, and set

$$S = \frac{(A|x_0| + B)T}{1 - AT}.$$

Define

$$C_{S,T} = \{(x,t) \in \mathbb{R}^N \times (0,T) : |x - x_0| < S(T-t)\}.$$

Note that if $(x,t) \in C_{S,T} \cap \Omega \times (0,T)$, then

$$|x| < |x_0| + ST,$$

and so

$$|H(x,t,r,p) - H(x,t,r,q)| \leq (A|x| + B)|p-q|$$

$$\leq (A|x_0| + ATS + B)|p-q| = S|p-q| \quad \text{for } r \in \mathbb{R} \text{ and } p,q \in \mathbb{R}^N$$

Using Theorem 6.7, we have

$$u \leq v \quad \text{in} \quad C_{s,T} \cap \Omega \times (0,T).$$

In particular,

$$u(x_0, t) \leq v(x_0, t) \qquad \text{for} \quad 0 < t < T.$$

From this we conclude $u \leq v$ in $\Omega \times (0,T)$. ∎

If we review carefully the above proofs, then we find that condition (H2) in Theorems 6.0, 6.0.a 6.0.c, 6.1, 6.2, 6.3, 6.5 and 6.6 can be replaced by

(H7) There is a function $m \in C([0,\infty))$ such that $m(0) = 0$ and

$$H(y,t,r,\lambda(x-y)) - H(x,t,r,\lambda(x-y)) \leq m(\lambda|x-y|^2 + |x-y|)$$

for $x, y \in \Omega$, $0 \leq t \leq T$, $r \in \mathbb{R}$ and $0 < \lambda < 1$.

A remark concerning (H5) similar to this is, of course, valid.

We also notice that the following condition replaces (H3) in Theorems 6.1 and 6.2.

(H8) For each $R > 0$ there is a function $\sigma_R \in C([0,\infty))$ such that $\sigma_R(0) = 0$ and

$$H(x,t,r,p) - H(x,t,r,p+\lambda x) \leq \sigma_R(\lambda)$$

for $(x,t,r) \in \Omega \times [0,T] \times \mathbb{R}$, $p \in B(0,R)$ and $0 < \lambda \leq R$.

These observations are important when we treat this example: Let $N = 1$, and consider the equation $u + x^3 Du = 0$ in $\mathbb{R}$. Our Hamiltonian is given by

$$H(x,p) = x^3 p.$$

This Hamiltonian does not satisfy neither (H2) nor (H3). On the other hand, if $\lambda > 0$, then

$$H(y, \lambda(x-y)) - H(x, \lambda(x-y)) = \lambda(x-y)(y^3 - x^3) = -\lambda(x-y)^2(x^2+xy+$$
$$\leq 0$$

and

$$H(x,p) - H(x, p+\lambda x) = -\lambda x^4 \leq 0.$$

Therefore, (H7) and (H8) are satisfied, and so the conclusion of Theorem 6.1 is valid to

$$u + x^3 Du = 0 \quad \text{in } \mathbb{R}.$$

More generally, assumption (H3) in Theorem 6.1 can be replaced by:

(H9) There is a Lipschitz continuous differentiable function $\mu : \overline{\Omega} \longrightarrow [0, \infty)$ and for each $R > 0$ a function $\sigma_R : [0, \infty) \to [0, \infty)$ such that $\sigma_R(0) =$

$$H(x, t, r, p) - H(x, t, r, p + \lambda D\mu(x)) \leq \sigma_R(\lambda)$$

for $x \in \Omega$, $r \in \mathbb{R}$, $p \in B(0, R)$ and $0 \leq \lambda \leq 1$ and

$$\lim_{|x| \to \infty} \mu(x) = \infty.$$

Also, the following (H10) in the stationary case (6.1) and (H11) in the Cauchy problem case (6.2) take the place of (H2) in Theorem

6.1 with (H9) in place of (H3).

(H10) There is a $d_0 > 0$ and for each $\varepsilon > 0$ a Lipschitz continuous differentiable function $w_\varepsilon : \overline{\Delta} \to [0, \infty)$, where $\Delta = \{(x,y) \in \Omega \times \Omega : |x-y| < d_0\}$, which satisfies

$$w_\varepsilon(x,y) + H(x, r, D_x w_\varepsilon(x,y)) - H(y, r, -D_y w_\varepsilon(x,y)) \geq 0$$

for $(x, y, r) \in \Delta \times \mathbb{R}$,

$$w_\varepsilon(x,x) \leq \varepsilon \quad \text{for } x \in \Omega, \quad \text{and}$$

$$w_\varepsilon(x,y) \geq \frac{1}{\varepsilon} \quad \text{for } (x,y) \in \overline{\Delta} \quad \text{with } |x-y| \geq \varepsilon.$$

(H11) There is a $d_0 > 0$ and for each $\varepsilon > 0$ a Lipschitz continuous differentiable function $w_\varepsilon : \overline{\Delta} \times [0,T] \to [0, \infty)$, where $\Delta$ is as in (H10), which satisfies

$$w_{\varepsilon,t} + H(x, t, r, D_x w_\varepsilon) - H(y, t, r, -D_y w_\varepsilon) \geq 0$$

for $(x,y) \in \Delta$, $0 < t < T$ and $r \in \mathbb{R}$,

$$w_\varepsilon(x,x,t) \le \varepsilon \quad \text{for } (x,t) \in \Omega \times (0,T), \quad \text{and}$$

$$w_\varepsilon(x,y,t) \ge \frac{1}{\varepsilon} \quad \text{for } (x,y,t) \in \bar{\Delta} \times [0,T] \text{ with } |x-y| \ge \varepsilon.$$

Conditions (H2) and (H3) in Theorem 6.2 can be replaced by (H10), (H9) and the following (H12) and (H13) in the stationary case and by (H11), (H9), (H12) and the following (H14) in the Cauchy problem case.

(H12) There is a Lipschitz continuous differentiable function $v : \bar{\Omega} \to [0,\infty)$ and for each $R > 0$ a constant $C_R > 0$ such that

$$H(x,t,r,p) - H(x,t,r,p + \lambda Dv(x)) \le C_R$$

for $x \in \Omega$, $0 < t < T$, $p \in B(0,R)$ and $0 \le \lambda \le R$, and $v(x) \ge |x|$ for $x \in \bar{\Omega}$ with $|x|$ large enough

(H13) For each $\varepsilon > 0$ there is a Lipschitz continuous differentiable function $w_\varepsilon : \overline{\Omega} \times \overline{\Omega} \to [0,\infty)$ which satisfies

$$w_\varepsilon(x,y) + H(x, r, D_x w_\varepsilon) - H(y, r, -D_y w_\varepsilon) \geq 0$$

for $x, y \in \Omega$ and $r \in \mathbb{R}$, and

$$w_\varepsilon(x,y) \geq \frac{1}{\varepsilon}|x-y| \qquad \text{for } x, y \in \overline{\Omega}.$$

(H14) For each $\varepsilon > 0$ there is a differentiable function $w_\varepsilon : \overline{\Omega} \times \overline{\Omega} \times [0,T] \to [0,\infty)$ such that $D_x w_\varepsilon$ and $D_y w_\varepsilon$ are bounded on $\overline{\Omega} \times \overline{\Omega} \times [0,T]$,

$$w_{\varepsilon,t} + H(x,t,r, D_x w_\varepsilon) - H(y,t,r,-D_y w_\varepsilon) \geq 0$$

for $x, y \in \Omega$, $0 < t < T$ and $r \in \mathbb{R}$, and

$$w_\varepsilon(x,y,t) \geq \frac{1}{\varepsilon}|x-y| \qquad \text{for } x, y \in \overline{\Omega} \text{ and } 0 \leq t \leq T.$$

## §7 Existence in the whole space

We discuss the existence of continuous solution to the stationary and Cauchy problems in $\mathbb{R}^N$. That is, we will consider equations (6.1) and (6.2 with $\Omega = \mathbb{R}^N$.

*Theorem 7.1* Let $\Omega = \mathbb{R}^N$. Assume (H0) – (H3).

(i) There is a solution of (6.1) which is uniformly continuous on $\mathbb{R}^N$. If $x \longrightarrow H(x,0,0)$ is bounded then the solution is also bounded on $\mathbb{R}^N$.

(ii) For each uniformly continuous function $u_0$ on $\mathbb{R}^N$ there is a solution $u$ of (6.1) which satisfies $u(x,0) = u_0(x)$ for $x \in \mathbb{R}^N$, $u \in C(\mathbb{R}^N \times [0,T])$, and

$$\lim_{r \searrow 0} \sup\{|u(x,t) - u(y,t)| : x, y \in \mathbb{R}^N,\ 0 \le t \le T,\ |x-y| \le r\} =$$

Moreover, if $u_0$ and $(x,t) \longrightarrow H(x,t,0,0)$ is bounded, then $u$ is bounded on $\mathbb{R}^N \times [0,T]$.

*Proof* We begin with part (i). In order to apply Perron's method, we have to show the existence of appropriate sub- and supersolutions. Let $A$, $B > 0$ be sufficiently large numbers. Setting

$$g(x) = A\langle x\rangle + B \qquad \text{for } x \in \mathbb{R}^N,$$

we compute that

$$g(x) + H(x, g(x), Dg(x)) \geq g(x) + H(x, 0, A\tfrac{x}{\langle x\rangle}) \geq$$

$$\geq g(x) + H(x,0,0) - \sigma_A(A) \geq g(x) + H(0,0,0) - m(|x|) - \sigma_A(A).$$

We may assume

$$m(r) \leq C(r+1) \qquad \forall r \geq 0,$$

for some $C > 0$ since $\Omega = \mathbb{R}^N$. Now we choose

$$A = C \qquad \text{and} \qquad B = |H(0,0,0)| + C + \sigma_C(C)$$

so that $g(x) - |H(0,0,0)| - C(|x|+1) - \sigma_A(A) \geq 0$

Of course, we have

$$g(x) + H(x, g(x), Dg(x)) \geq 0 \qquad \text{in } \mathbb{R}^N.$$

Similarly, setting $f = -g$, we find that

$$f(x) + H(x, f(x), Df(x)) \leq 0 \qquad \text{in } \mathbb{R}^N.$$

Obviously, we have $f \leq g$ in $\mathbb{R}^N$. Therefore, from Theorem 5.1 we see that there is a solution $u$ of (6.1) satisfying $f \leq u \leq g$ in $\mathbb{R}^N$. Because of the linear growth of $f$ and $g$, we can apply Theorem 6.2 to $u$, to conclude that $u^* \leq u_*$ in $\mathbb{R}^N$, whence $u \in C(\mathbb{R}^N)$. Moreover, (6.7) guarantees the uniform continuity of $u$. Finally, if $H(x, 0, 0)$ is bounded, then $g(x) = B$ and $f(x) = -B$ with $B$ large enough are, resp., super- and subsolutions of (6.1) and so $B \leq u(x) \leq -B$ by comparison.

We now turn to part (ii). For large $A, B > 0$ we set

$$f_1(x,t) = -(A\langle x\rangle + B)e^t \quad \text{and} \quad g(x,t) = (A\langle x\rangle + B)e^t.$$

We compute

$$g_{1,t} + H(x,t,g_1,Dg_1) \geq g_1 + H(x,t,0,e^t A\tfrac{x}{\langle x\rangle})$$

$$\geq g_1 + H(x,t,0,0) - \sigma_{Ae^T}(Ae^T)$$

$$\geq g_1 + H(0,t,0,0) - m(|x|) - \sigma_{Ae^T}(Ae^T).$$

Then we choose $A, B > 0$ so that

$$A\langle x\rangle \geq m(|x|) \vee |u_0(x)| \qquad \text{for } x \in \mathbb{R}^N, \text{ and}$$

$$B \geq \max_{0\leq t\leq T} |H(0,t,0,0)| + \sigma_{Ae^T}(Ae^T).$$

Obviously, we have

$$g_{1,t} + H(x,t,g_1,Dg_1) \geq 0 \qquad \text{in } \mathbb{R}^N\times(0,T), \text{ and}$$

$$f_1(x,0) \leq u_0(x) \leq g_1(x,0) \qquad \text{for } x\in\mathbb{R}^N.$$

Also, we have

$$f_{1,t} + H(x,t,f_1,Df_1) \leq 0 \qquad \text{in } \mathbb{R}^N\times(0,T)$$

By the uniform continuity of $u_0$, for each $0<\varepsilon<1$ there is a constant $C_\varepsilon > 0$ such that

$$|u_0(x) - u_0(y)| \leq \varepsilon + C_\varepsilon |x-y| \quad \text{for } x, y \in \mathbb{R}^N.$$

We set

$$A_\varepsilon = \max\{g_1(x,T) : |x| \leq \tfrac{1}{\varepsilon}+2,\ 0 \leq t \leq T\},$$

$$B_\varepsilon = C_\varepsilon \vee Ae^T \vee 2A_\varepsilon,$$

$$D_\varepsilon = \max\{|H(x,t,r,p)| : |x| \leq \tfrac{1}{\varepsilon}+2,\ |r| \leq A_\varepsilon,\ |p| \leq B_\varepsilon\}.$$

For $y \in \mathbb{R}^N$ we set

$$g_{\varepsilon,y}(x,t) = u_0(y) + \varepsilon + B_\varepsilon |x-y| + D_\varepsilon t,$$

$$f_{\varepsilon,y}(x,t) = u_0(y) - \varepsilon - B_\varepsilon |x-y| - D_\varepsilon t$$

Then we see that (if $|y| \leq \frac{1}{\varepsilon}$, then) $f_{\varepsilon,y}$ and $g_{\varepsilon,y}$ are, resp., sub- and supersolutions of

$$u_t + H(x,t,u,Du) = 0 \quad \text{in } B(0, \tfrac{1}{\varepsilon}+2)^\circ \times (0,T)$$

and satisfy

$$f_1(x,t) \leq f_{\varepsilon,y}(x,t) \leq g_{\varepsilon,y}(x,t) \leq g_1(x,t)$$

for $(\mathbb{R}^N \setminus B(y,1)^\circ) \times [0,T]$.

$\therefore$ for $\mathbb{R}^N \setminus B(0, \frac{1}{\varepsilon}+1)^\circ \times [0,T]$

Moreover we have

$$f_{\varepsilon,y}(y,0) = u_0(y) - \varepsilon, \quad g_{\varepsilon,y}(y,0) = u_0(y) + \varepsilon \quad \text{for } y \in \mathbb{R}^N$$

and

$$f_{\varepsilon,y}(x,t) \le u_0(x) \le g_{\varepsilon,y}(x,t).$$

We now set

$$f(x,t) = \max\{f_1(x,t), \sup\{f_{\varepsilon,y}(x,t) : 0 < \varepsilon < 1, |y| \le \tfrac{1}{\varepsilon}\}\},$$

$$g(x,t) = \min\{g_1(x,t), \inf\{g_{\varepsilon,y}(x,t) : 0 < \varepsilon < 1, |y| \le \tfrac{1}{\varepsilon}\}\}.$$

From the above observations we see that $f$ and $g$ are resp. sub- and supersolutions of

$$u_t + H(x,t,u,Du) = 0 \quad \text{in } \mathbb{R}^N \times (0,T)$$

and satisfy

$$f(x,t) \le u_0(x) \le g(x,t) \quad \text{for } (x,t) \in \mathbb{R}^N \times [0,T],$$

$$f(x,0) = g(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}^N.$$

Also, we see that $f \in LSC(\mathbb{R}^N \times [0,T])$ and $g \in USC(\mathbb{R}^N \times [0,T])$. Now we apply Theorem 5.1, to get a solution $u$ of (6.2) satisfying