§7 Existence in the whole space

We discuss the existence of continuous solutions to the stationary and Cauchy problems in $\mathbb{R}^N$. That is, we will consider equations (6.1) and (6.2) with $\Omega = \mathbb{R}^N$.

Theorem 7.1 Let $\Omega = \mathbb{R}^N$. Assume (H0) – (H3).

(i) There is a solution of (6.1) which is uniformly continuous on $\mathbb{R}^N$. If $x \longrightarrow H(x,0,0)$ is bounded, then the solution is also bounded on $\mathbb{R}^N$.

(ii) For each uniformly continuous function u_0 on $\mathbb{R}^N$, there is a solution u of (6.2) which satisfies $u(x,0) = u_0(x)$ for $x \in \mathbb{R}^N$, $u \in C(\mathbb{R}^N \times [0,T])$, and

$$\lim_{r \downarrow 0} \sup\{|u(x,t) - u(y,t)| : x, y \in \mathbb{R}^N,\ 0 \leq t \leq T,\ |x-y| \leq r\} = 0.$$

Moreover, if u_0 and $(x,t) \longrightarrow H(x,t,0,0)$ is bounded, then u is bounded on $\mathbb{R}^N \times [0,T]$.

Proof We begin with part (i). In order to apply Perron's method, we have to show the existence of appropriate sub- and supersolutions. Let $A, B > 0$ be sufficiently large numbers. Setting

$$g(x) = A\langle x\rangle + B \qquad \text{for } x \in \mathbb{R}^N,$$

we compute that

$$g(x) + H(x, g(x), Dg(x)) \geq g(x) + H(x, 0, A\tfrac{x}{\langle x\rangle}) \geq$$

$$\geq g(x) + H(x,0,0) - \sigma_A(A) \geq g(x) + H(0,0,0) - m(|x|) - \sigma_A(A).$$

We may assume

$$m(r) \leq C(r+1) \qquad \forall r \geq 0,$$

for some $C > 0$ since $\Omega = \mathbb{R}^N$. Now we choose

$$A = C \quad \text{and} \quad B = |H(0,0,0)| + C + \sigma_C(C)$$

so that $g(x) - |H(0,0,0)| - C(|x|+1) - \sigma_A(A) \geq 0$

Of course, we have

$$g(x) + H(x, g(x), Dg(x)) \geq 0 \qquad \text{in } \mathbb{R}^N.$$

Similarly, setting $f = -g$, we find that

$$f(x) + H(x, f(x), Df(x)) \le 0 \quad \text{in } \mathbb{R}^N.$$

Obviously, we have $f \le g$ in $\mathbb{R}^N$. Therefore, from Theorem 5.1 we see that there is a solution u of (6.1) satisfying $f \le u \le g$ in $\mathbb{R}^N$. Because of the linear growth of f and g, we can apply Theorem 6.2 to u, to conclude that $u^* \le u_*$ in $\mathbb{R}^N$, whence $u \in C(\mathbb{R}^N)$. Moreover, (6.7) guarantees the uniform continuity of u. Finally, if $H(x,0,0)$ is bounded, then $g(x) = B$ and $f(x) = -B$ with B large enough are, resp., super- and subsolutions of (6.1) and so $B \le u(x) \le -B$ by comparison.

We now turn to part (ii). For large $A, B > 0$ we set

$$f_1(x,t) = -(A\langle x\rangle + B)e^t \quad \text{and} \quad g(x,t) = (A\langle x\rangle + B)e^t.$$

We compute

$$g_{1,t} + H(x,t,g_1,Dg_1) \geq g_1 + H(x,t,0,e^t A\tfrac{x}{\langle x\rangle})$$

$$\geq g_1 + H(x,t,0,0) - \sigma_{Ae^T}(Ae^T)$$

$$\geq g_1 + H(0,t,0,0) - m(|x|) - \sigma_{Ae^T}(Ae^T).$$

Then we choose $A, B > 0$ so that

$$A\langle x\rangle \geq m(|x|) \vee |u_0(x)| \quad \text{for } x \in \mathbb{R}^N, \text{ and}$$

$$B \geq \max_{0\leq t\leq T} |H(0,t,0,0)| + \sigma_{Ae^T}(Ae^T).$$

Obviously, we have

$$g_{1,t} + H(x,t,g_1,Dg_1) \geq 0 \quad \text{in } \mathbb{R}^N \times (0,T), \text{ and}$$

$$f_1(x,0) \leq u_0(x) \leq g_1(x,0) \quad \text{for } x \in \mathbb{R}^N.$$

Also, we have

$$f_{1,t} + H(x,t,f_1,Df_1) \leq 0 \quad \text{in } \mathbb{R}^N \times (0,T)$$

By the uniform continuity of u_0, for each $0 < \varepsilon < 1$ there is a constant $C_\varepsilon > 0$ such that

$$|u_0(x) - u_0(y)| \leqslant \varepsilon + C_\varepsilon |x-y| \qquad \text{for} \quad x, y \in \mathbb{R}^N.$$

We set

$$A_\varepsilon = \max\{ g_1(x,t) : |x| \leqslant \tfrac{1}{\varepsilon} + 2, \quad 0 \leqslant t \leqslant T \},$$

$$B_\varepsilon = C_\varepsilon \vee A e^T \vee 2A_\varepsilon,$$

$$D_\varepsilon = \max\{ |H(x,t,r,p)| : |x| \leqslant \tfrac{1}{\varepsilon} + 2, \ |r| \leqslant A_\varepsilon, \ |p| \leqslant B_\varepsilon \}.$$

For $y \in \mathbb{R}^N$ we set

$$g_{\varepsilon,y}(x,t) = u_0(y) + \varepsilon + B_\varepsilon |x-y| + D_\varepsilon t,$$

$$f_{\varepsilon,y}(x,t) = u_0(y) - \varepsilon - B_\varepsilon |x-y| - D_\varepsilon t.$$

Then we see that if $|y| \leqslant \frac{1}{\varepsilon}$, then $f_{\varepsilon,y}$ and $g_{\varepsilon,y}$ are, resp., sub- and supersolutions of

$$u_t + H(x,t,u,Du) = 0 \quad \text{in} \quad B(0, \tfrac{1}{\varepsilon} + 2)^\circ \times (0,T)$$

and satisfy

$$f_1(x,t) \leqslant f_{\varepsilon,y}(x,t) \leqslant g_{\varepsilon,y}(x,t) \leqslant g_1(x,t)$$

$$\text{for} \quad (\mathbb{R}^N \setminus B(y,1)^\circ) \times [0,T].$$

Moreover, we have

$$f_{\varepsilon,y}(y,0) = u_0(y) - \varepsilon, \qquad g_{\varepsilon,y}(y,0) = u_0(y) + \varepsilon \qquad \text{for } y \in \mathbb{R}^N$$

and

$$f_{\varepsilon,y}(x,t) \le u_0(x) \le g_{\varepsilon,y}(x,t).$$

We now set

$$f(x,t) = \max\{f_1(x,t), \sup\{f_{\varepsilon,y}(x,t) : 0 < \varepsilon < 1, |y| \le \tfrac{1}{\varepsilon}\}\},$$

$$g(x,t) = \min\{g_1(x,t), \inf\{g_{\varepsilon,y}(x,t) : 0 < \varepsilon < 1, |y| \le \tfrac{1}{\varepsilon}\}\}.$$

From the above observations we see that f and g are, resp., sub- and supersolutions of

$$u_t + H(x,t,u,Du) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T)$$

and satisfy

$$f(x,t) \le u_0(x) \le g(x,t) \qquad \text{for } (x,t) \in \mathbb{R}^N \times [0,T],$$

$$f(x,0) = g(x,0) = u_0(x) \qquad \text{for } x \in \mathbb{R}^N.$$

Also, we see that $f \in LSC(\mathbb{R}^N \times [0,T])$ and $g \in USC(\mathbb{R}^N \times [0,T])$. Now we apply Theorem 5.1, to get a solution u of (6.2) satisfying

$$f(x,t) \le u(x,t) \le g(x,t) \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,T).$$

Extend the domain of definition of u to $\mathbb{R}^N \times [0,T)$ by setting $u(x,0) = u_0(x)$, and apply Theorem 6.2, to obtain $u^* \le u_*$ in $\mathbb{R}^N \times [0,T)$. This proves that $u \in C(\mathbb{R}^N \times [0,T))$. Moreover, (6.11) implies that

$$\lim_{r \searrow 0} \sup \{|u(x,t) - u(y,t)| : x, y \in \mathbb{R}^N,\ 0 \le t < T,\ |x-y| \le r\} = 0.$$

If u_0 and $(x,t) \to H(x,t,0,0)$ are bounded, then $f(x,t) = -C(t+1)$ and $g(x,t) = C(t+1)$, with C large enough, are sub- and supersolution of (6.2), resp., and satisfy $f(x,0) \le u_0(x) \le g(x,0)$ for $x \in \mathbb{R}^N$. Hence, $|u(x,t)| \le C(t+1)$ for $\forall (x,t) \in \mathbb{R}^N \times [0,T)$ and some $C > 0$. ∎

§8. The Skorokhod problem

In this section we formulate the deterministic Skorokhod problem, show the unique existence of its solution and prepare for our discussion on the relation between the Neumann problem for H-J equations and optimal control problem where the Skorokhod problem governs the dynamics.

Let E be a closed subset of $\mathbb{R}^N$, and assume

(8.1) For each $x \in \partial E$, there is a (nonempty) set $N(x)$ of unit vectors in $\mathbb{R}^N$ such that

(i) there is a constant $C_0 > 0$ for which

$$\xi \cdot (x-y) + C_0 |x-y|^2 \geq 0 \qquad \forall x \in \partial E,\ \forall y \in E,\ \forall \xi \in N(x),$$

(ii) if $\xi \in \mathbb{R}^N$, $|\xi| = 1$, $C > 0$, $x \in \partial E$ and

$$\xi \cdot (x-y) + C|x-y|^2 \geq 0 \qquad \forall y \in E,$$

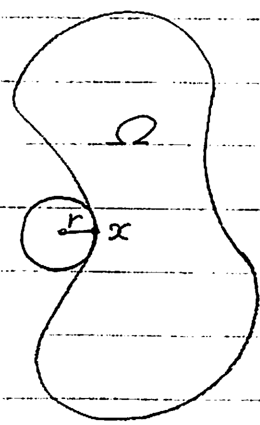
then $\xi \in N(x)$.

<u>Example</u> Let $\Omega \subset \mathbb{R}^N$ be an open subset with C^1 boundary. Assume Ω satisfies the uniform exterior sphere condition, i.e., there is an $r > 0$ such that

$$\Omega \cap B(x + r n(x), r) = \emptyset \qquad \forall x \in \partial\Omega,$$

where $n(x)$ denotes the outer unit normal of Ω at x.

Set $N(x) = \{n(x)\}$ for $x \in \partial\Omega$. Then $\overline{\Omega}$ satisfies (8.1) with $C_0 = \frac{1}{2r}$.



Indeed, if $x \in \partial\Omega$ and $y \in \overline{\Omega}$, then

$$|y - (x + r n(x))| \geq r,$$

and so

$$|y - x|^2 - 2r\, n(x) \cdot (y - x) + r^2 \geq r^2,$$

i.e.

$$n(x) \cdot (x - y) + \frac{1}{2r} |x - y|^2 \geq 0 .$$

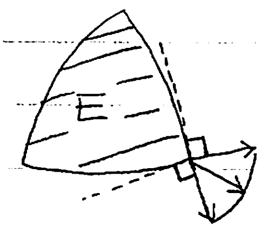
This shows that (8.1-i) is valid. The C^1

regularity assumption of $\partial\Omega$ guarantees that (7.1-ii) is also valid.

<u>Example</u> Let E be a closed convex subset of $\mathbb{R}^N$. For $x \in \partial E$ set

$$N(x) = \{\xi \in \mathbb{R}^N : \ \xi \cdot (x-y) \geq 0 \quad \forall y \in E, \ |\xi| = 1\}.$$

Then (8.1) is satisfied.



In what follows we assume (8.1). We define

$$NC(x) = \{\theta\xi : \theta \geq 0, \ \xi \in N(x)\} \qquad \text{for } x \in \partial E.$$

From (8.1-i) we have

(8.2) $$\xi \cdot (x-y) + C_0 |\xi| \, |x-y|^2 \geq 0 \quad \forall x \in \partial E, \ \forall y \in E, \ \forall \xi \in NC(x).$$

Note that (7.1-ii) is equivalent to that

(8.3) if $\xi \in \mathbb{R}^N$, $C \geq 0$, $x \in \partial E$ and

$$\xi \cdot (x-y) + C|x-y|^2 \geq 0 \qquad \forall y \in E,$$

then $\xi \in NC(x)$.

Lemma 8.1 For $x \in \partial E$, $NC(x)$ is a closed convex cone.

Proof Closedness Let $\{\xi_n\} \subset NC(x)$ converge to $\xi \in \mathbb{R}^N$.

By (8.2)

$$\xi_n \cdot (x-y) + |\xi_n| C_0 |x-y|^2 \geq 0 \qquad \forall y \in E.$$

Sending $n \to \infty$, we get

$$\xi \cdot (x-y) + |\xi| C_0 |x-y|^2 \geq 0 \qquad \forall y \in E.$$

This and (8.3) imply $\xi \in NC(x)$.

Being a convex cone Let $\xi, \eta \in NC(x)$ and $t, s \geq 0$.

By (8.2), we have

$$\xi \cdot (x-y) + |\xi| C_0 |x-y|^2, \quad \eta \cdot (x-y) + |\eta| C_0 |x-y|^2 \geq 0 \qquad \forall y \in E,$$

and hence

$$(t\xi + s\eta) \cdot (x-y) + (t|\xi| + s|\eta|) C_0 |x-y|^2 \geq 0 \qquad \forall y \in E.$$

Therefore, from (8.3) we have $t\xi + s\eta \in NC(x)$. ∎

Let $I : \mathbb{R}^N \to [0, +\infty]$ be the indicator function (of E), i.e.,

$$I(x) = \begin{cases} 0 & (x \in E) \\ +\infty & (x \notin E). \end{cases}$$

As E is closed, I is l.s.c. on $\mathbb{R}^N$. By definition we have

$$\xi \in D^- I(x) \iff I(y) - I(x) \geq \xi \cdot (y-x) + o(|y-x|) \quad \text{as } y \to x.$$

<u>Lemma 8.2</u> For $x \in \partial E$ we have

$$D^- I(x) = NC(x).$$

<u>Proof</u> Let $x \in \partial E$ and $\xi \in NC(x)$. Using (8.2), we have

$$I(y) - I(x) \geq 0 \geq \xi \cdot (y-x) - C_0 |\xi| \, |y-x|^2 \quad \text{for } y \in \mathbb{R}^N,$$

whence $\xi \in D^- I(x)$. Now, let $x \in \partial E$ and $\xi \in D^- I(x)$. We have

$$0 \geq \xi \cdot (y-x) + o(|y-x|) \quad \text{as } y \overset{\in E}{\to} x,$$

and so, setting $\eta = \xi / |\xi|$,

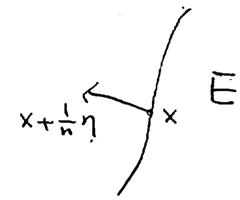
$$(8.4) \quad 0 \geq \eta \cdot (y-x) + o(|y-x|) \quad \text{as } y \to x.$$

(Notice that if $\xi = 0$, then $\xi \in NC(x)$, and hence we may assume $\xi \neq 0$.) For $n \in \mathbb{N}$ we choose

$x_n \in E$ so that

(8.5) $\quad |x + \frac{1}{n}\eta - x_n| = \operatorname{dist}(x + \frac{1}{n}\eta, E) \quad (\leq \frac{1}{n})$.

From this we have



$$|x - x_n|^2 + \frac{2}{n}\eta\cdot(x - x_n) \leq 0 .$$

This and (8.4) yield

$$\frac{n}{2}|x - x_n|^2 \leq \eta\cdot(x_n - x) \leq \omega(|x_n - x|)\,|x_n - x|,$$

where $\omega \in C([0,\infty))$ satisfies $\omega(0) = 0$. Clearly we have $|x_n - x| \leq \frac{2}{n}$ and so

(8.6) $\quad |x_n - x| \leq \frac{2}{n}\,\omega(\frac{2}{n})$.

Setting $z_n = x + \frac{1}{n}\eta$, from (8.5) we have

$$|z_n - x_n|^2 \leq |z_n - y|^2 \qquad \forall y \in E.$$

$|z_n - x_n + x_n - y|^2$

Hence,

(8.7) $\quad 2\,(z_n - x_n)\cdot(x_n - y) + |x_n - y|^2 \geq 0 \qquad \forall y \in E.$

Note that (8.6) guarantees that $z_n \neq x_n$ and hence $x_n \in \partial E$ for n enough large. Therefore, (8.7), (8.3) and (8.2) guarantee that

(8.8) $\quad (z_n - x_n)\cdot(x_n - y) + C_0 |z_n - x_n||x_n - y|^2 \geqslant 0 \quad \forall y \in E$

for large $n \in \mathbb{N}$. Using (8.6), we see that

$$n(z_n - x_n) = \eta + n(x - x_n) \to \eta \quad \text{as} \quad n \to \infty.$$

Hence, from (8.8) we deduce that

$$\eta\cdot(x-y) + C_0 |x-y|^2 \geqslant 0 \quad \forall y \in E.$$

Thus, $\xi = |\xi|\eta \in NC(x)$ in view of (8.3). ▨

Let us consider the Skorokhod problem

(8.9) $\quad \begin{cases} \dot{X} + D^- I(X) \ni g(X,t) & \text{in } (0,T) \\ X(0) = x \in E, \end{cases}$

where $g : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$ satisfies:

$$|g(x,t)| \leq M, \quad |g(x,t) - g(y,t)| \leq M|x-y| \quad \forall x,y \in \mathbb{R}^N, \quad 0 \leq t \leq T,$$

for some constant $M > 0$ and

$$t \to g(x,t) \quad \text{is measurable on } [0,T] \text{ for } x \in \mathbb{R}^N.$$

By a solution of (8.9) we mean a function X

$\in C([0,T]; \mathbb{R}^N)$ which is absolutely continuous on $[0,T]$ and satisfies

$$g(X(t),t) - \dot{X}(t) \in D^- I(X(t)) \quad \text{a.e. in } (0,T).$$

<u>Theorem 8.1</u> The problem (8.9) has a unique solution.

<u>Lemma 8.3</u> We have

$$(8.10) \quad (\xi - \eta)\cdot(x-y) + C_0(|\xi|+|\eta|)|x-y|^2 \geq 0$$

for $x, y \in \mathbb{R}^N$ and $\xi \in D^- I(x)$, $\eta \in D^- I(y)$.

<u>Proof</u> Note that $D^- I(x) = \emptyset$ for $x \notin E$ and $D^- I(x) = \{0\}$ for $x \in E^\circ$. From (8.2) together with Lemma 8.2 and the above observation we find that

$$\xi\cdot(x-y) + C_0|\xi||x-y|^2 \geq 0$$

for $x \in \mathbb{R}^N$, $\xi \in D^- I(x)$ and $y \in E$. Similarly, we have

$$\eta\cdot(y-x) + C_0|\eta||y-x|^2 \geq 0$$

for $y \in \mathbb{R}^N$, $\eta \in D^- I(y)$ and $x \in E$. Adding these two, we obtain (8.10). ∎

<u>Lemma 8.4</u> The function

$$y \to I(y) + \frac{1}{\varepsilon}|x-y|^2$$

has a unique minimum point if $\mathrm{dist}(x, E) < \frac{1}{2C_0}$.

<u>Proof</u> Let $x \in \mathbb{R}^N$ satisfy $\mathrm{dist}(x, E) < \frac{1}{2C_0}$ and y_1, y_2 be minimum points of the function $y \to I(y) + \frac{1}{\varepsilon}|x-y|^2$. Obviously, $y_1, y_2 \in E$,

$$|x-y_1| = |x-y_2| = \mathrm{dist}(x, E)$$

and

$$D^- I(y_i) + \frac{2}{\varepsilon}(y_i - x) \ni 0 \qquad \text{for } i = 1, 2.$$

By (8.10) we have

$$\frac{2}{\varepsilon}\{x-y_1 - (x-y_2)\}\cdot(y_1-y_2) + C_0 \frac{2}{\varepsilon}\left(|y_1-x| + |y_2-x|\right)|y_1-y_2|^2 \geq 0$$

That is,

$$|y_1-y_2|^2\{1 - 2C_0\,\mathrm{dist}(x, E)\} \geq 0.$$

Since $\mathrm{dist}(x, E) < \frac{1}{2C_0}$, we see that $y_1 = y_2$. ∎

For $x \in \mathbb{R}^N$ with $\mathrm{dist}(x, E) < \frac{1}{2C_0}$, we denote by $J(x)$ the unique minimum point of $y \to I(y) + \frac{1}{\varepsilon}|x-y|^2$. Note that $J(x)$ is the nearest point of E from x and independent of ε.

<u>Lemma 8.5</u> If $\mathrm{dist}(x_1, E) \vee \mathrm{dist}(x_2, E) < \frac{1}{2C_0}$, then

$$|J(x_1) - J(x_2)| \leq \frac{|x_1 - x_2|}{1 - C_0(\mathrm{dist}(x_1, E) + \mathrm{dist}(x_2, E))}.$$

<u>Proof</u> As

$$D^- I(J(x_i)) \ni \frac{2}{\varepsilon}(x_i - J(x_i)) \quad \text{for } i = 1, 2,$$

by (8.10) we have

$$\frac{2}{\varepsilon}\{(x_1 - J(x_1)) - (x_2 - J(x_2))\} \cdot (J(x_1) - J(x_2))$$

$$+ C_0 \frac{2}{\varepsilon}\left(|J(x_1) - x_1| + |J(x_2) - x_2|\right)|J(x_1) - J(x_2)|^2 \geq 0.$$

Hence

$$|J(x_1) - J(x_2)|^2 \{1 - C_0(\mathrm{dist}(x_1, E) + \mathrm{dist}(x_2, E))\}$$

$$\leq (x_1 - x_2) \cdot (J(x_1) - J(x_2)) \leq |x_1 - x_2|\,|J(x_1) - J(x_2)|,$$

from which

$$|J(x_1) - J(x_2)| \leq \frac{|x_1 - x_2|}{1 - C_0(\mathrm{dist}(x_1, E) + \mathrm{dist}(x_2, E))}.$$

We will write

$$\Omega = \{x \in \mathbb{R}^N : \ \mathrm{dist}(x, E) < \frac{1}{2C_0}\},$$

$$I_\varepsilon(x) = \min\{I(y) + \frac{2}{\varepsilon}|x-y|^2 : \ y \in \mathbb{R}^N\} \quad \text{for } x \in \mathbb{R}^N \text{ and } \varepsilon > 0.$$

<u>Lemma 8.6</u> $I_\varepsilon \in C^{1,1}(\Omega)$ and

$$DI_\varepsilon(x) = \frac{2}{\varepsilon}(x - J(x)) \in D^- I(J(x)) \quad \text{for } x \in \Omega.$$

<u>Proof</u> Fix $x, y \in \Omega$. We compute that

$$I_\varepsilon(y) - I_\varepsilon(x) = \frac{1}{\varepsilon}\left(|y - J(y)|^2 - |x - J(x)|^2\right)$$

$$= \frac{1}{\varepsilon}\left(y + x - J(y) - J(x)\right) \cdot \left(y - x - (J(y) - J(x))\right)$$

$$= \frac{2}{\varepsilon}(x - J(x)) \cdot (y - x - (J(y) - J(x))) + \frac{1}{\varepsilon}|y - x - (J(y) - J(x))|^2$$

$$\geqq \frac{2}{\varepsilon}(x - J(x)) \cdot (y - x) + \frac{2}{\varepsilon}(x - J(x)) \cdot (J(x) - J(y))$$

$$\geqslant \frac{2}{\varepsilon}(x - J(x)) \cdot (y - x) - \frac{2}{\varepsilon}|x - J(x)|\, C_0 |J(x) - J(y)|^2 = \quad \text{by (8.10)}$$

$$= \frac{2}{\varepsilon}(x - J(x)) \cdot (y - x) + O|x - y|^2 \quad \text{as } y \to x \text{ by Lemma 8.5.}$$

By symmetry we have

$$I_\varepsilon(x) - I_\varepsilon(y) \geqslant \frac{2}{\varepsilon}(y - J(y)) \cdot (x - y) - \frac{2}{\varepsilon}|y - J(y)|\, C_0 |J(x) - J(y)|^2 =$$

$$= \frac{2}{\varepsilon}(x - J(x)) \cdot (x - y) + O(|x-y|^2) \quad \text{as } y \to x.$$

Thus

$$I_\varepsilon(y) = I_\varepsilon(x) + \frac{2}{\varepsilon}(x - J(x)) \cdot (y - x) + O(|y-x|^2) \quad \text{as } y \to x. \quad ▮$$

<u>Proof of Theorem 8.1</u> <u>Uniqueness</u> Let X and Y be two solutions of (8.9). Define

$$\xi(t) = g(X(t), t) - \dot{X}(t) \quad \text{and} \quad \eta(t) = g(Y(t), t) - \dot{Y}(t).$$

By (8.10) we have

$$0 \leq (\xi(t) - \eta(t)) \cdot (X(t) - Y(t)) + C_0(|\xi(t)| + |\eta(t)|)\, |X(t) - Y(t)|^2$$

$$\leq -\frac{1}{2}\frac{d}{dt}|X(t) - Y(t)|^2 + \{M + C_0(|\xi(t)| + |\eta(t)|)\}\, |X(t) - Y(t)|^2.$$

By Gronwall's inequality we find that for $0 \leq t \leq T$,

$$|X(t) - Y(t)|^2 \leq |X(0) - Y(0)|^2 \exp 2\int_0^t \{M + C_0(|\xi(t)| + \eta(t))\}\, dt = 0.$$

<u>Existence</u> Fix $x \in E$. We solve

$$(8.11) \quad \begin{cases} \dot{X}_\varepsilon(t) + DI_\varepsilon(X(t)) = g(X(t), t) & \text{in } (0, T), \\ X_\varepsilon(0) = x. \end{cases}$$

This problem is solved as long as $X_\varepsilon(t) \in \Omega$.

Multiplying the equation by $\dot{X}_\varepsilon(t)$ and integrating over $[0,t]$, with $0 < t \leq T$, we get

$$\int_0^t |\dot{X}_\varepsilon(s)|^2 ds + I_\varepsilon(X_\varepsilon(t)) - I_\varepsilon(x) = \int_0^t g(X_\varepsilon(s), s)\cdot \dot{X}_\varepsilon(s)\, ds$$

$$\leq M\sqrt{T}\left(\int_0^t |\dot{X}_\varepsilon(s)|^2 ds\right)^{1/2}.$$

Therefore

(8.12) $\qquad \|\dot{X}_\varepsilon\|_{L^2(0,t)} \leq M\sqrt{T}$,

(8.13) $\qquad I_\varepsilon(X_\varepsilon(t)) \leq M^2 T$

for $0 \leq t \leq T$ as long as $X_\varepsilon(s) \in \Omega$ for $s \in [0,t]$. This last inequality implies that

(8.14) $\quad \operatorname{dist}(X_\varepsilon(t), E) \leq \sqrt{\varepsilon}\, M\sqrt{T}$

and hence $X_\varepsilon(t) \in \Omega$ for $0 \leq t \leq T$ if $0 < \varepsilon < \frac{1}{4M^2 T C_0^2}$.

That is, if $0 < \varepsilon < \frac{1}{4M^2 T C_0^2}$, then (8.11) has a solution X_ε on $[0,T]$. In view of (8.12) and (8.13) there is a sequence $\{\varepsilon_j\} \subset (0, \frac{1}{4M^2 T C_0^2})$ converging to zero such that

$$X_{\varepsilon_j} \to X \quad \text{in} \quad C([0,T]) \quad \text{and} \quad \text{weakly in} \quad H^1(0,T)$$

for some $X \in C([0,T]) \cap H^1(0,T)$.

What remains is to prove that

$$g(X(t),t) - \dot{X}(t) \in D^- I(X(t)) \quad \text{a.e. in} \quad (0,T).$$

Since

$$g(X_\varepsilon(t),t) - \dot{X}_\varepsilon(t) = DI_\varepsilon(X_\varepsilon(t)) \in D^- I(J(X_\varepsilon(t))) \qquad \text{a.e.,}$$

by (8.2) we have

$$(g(X_\varepsilon(t),t) - \dot{X}_\varepsilon(t)) \cdot (J(X_\varepsilon(t)) - y) + C_0 |g(X_\varepsilon(t),t) - \dot{X}_\varepsilon(t)| \, |J(X_\varepsilon(t)) - y|^2 \geq 0$$

for $y \in E$. Hence, passing to the limit along

$\boxed{\text{why ??}}$

$\varepsilon = \varepsilon_j \to 0$, we have

$$\int_0^T \{(g(X(t),t) - \dot{X}(t)) \cdot (J(X(t)) - y) + C_0 |g(X(t),t) - \dot{X}(t)| \, |J(X(t)) - y|^2\} \varphi(t)\, dt \geq 0$$

for $\varphi \in C([0,T])$ (satisfying $\varphi \geq 0$) and $y \in E$. Therefore, choosing any countable dense subset $\hat{E}$ of E, we have

$$(g(X(t),t) - \dot{X}(t)) \cdot (J(X(t)) - y) + C_0 |g(X(t),t) - \dot{X}(t)| \, |J(X(t)) - y|^2 \geq 0$$

for $t \in [0,T] \setminus Z$ and $y \in \hat{E}$, where meas $Z = 0$.

Note that (8.14) implies that $X(t) \in E$ and so $J(X(t)) = X(t)$ for $0 \leq t \leq T$. Thus

$$(g(X(t),t) - \dot{X}(t)) \cdot (X(t) - y) + C_0 |g(X(t),t) - \dot{X}(t)| |X(t) - y|^2 \geq 0$$

for $t \in [0, T] \setminus Z$ and $y \in E$, and (8.3) implies that

$$g(X(t),t) - \dot{X}(t) \in D^- I(X(t)) \quad \text{a.e. in } (0,T). \quad \blacksquare$$

We now assume that there is a $N \times N$ symmetric matrix $a(x)$ for each $x \in \mathbb{R}^N$ such that

(8.15) $\quad \lambda I \leq a(x) \leq \lambda^{-1} I \quad \forall x \in \mathbb{R}^N$

for some constant $\lambda > 0$ and

(8.16) $\quad a : \mathbb{R}^N \to \mathbb{R}^{N \times N}$ is Lipschitz continuous and of class C^1.

We consider the <u>Skorokhod problem with oblique reflection</u>

$$(8.17)\quad \begin{cases} \dot X(t) + a(X(t))D^-I(X(t)) \ni g(X(t),t) & \text{a.e. in } (0,T), \\ X(0) = x \in E. \end{cases}$$

<u>Theorem 8.2</u> The problem (8.17) has a unique solution on $[0,T]$.

<u>Proof</u> <u>Uniqueness</u> Let X, Y be two solutions of (8.17). Set

$$\xi(t) = g(X(t),t) - \dot X(t) \quad \text{and} \quad \eta(t) = g(Y(t),t) - \dot Y(t).$$

By using (8.10) we have

$$\left(a^{-1}(X(t))\xi(t) - a^{-1}(Y(t))\eta(t)\right)\cdot(X(t)-Y(t)) + C_1\left(|\xi(t)| + |\eta(t)|\right)|X(t)-Y(t)|^2 \geq 0 \quad \text{a.e.}$$

for some $C_1 > 0$. Noting that

$$\frac{d}{dt} a^{-1}(X(t))(X(t)-Y(t))\cdot(X(t)-Y(t)) = \frac{d}{dt}\sum_{i,j} a^{-1}_{ij}(X(t))(X_i(t)-Y_i(t))(X_j(t)-Y_j(t))$$

$$= \sum_{i,j,k} a^{-1}_{ij,x_k}(X(t))(X_i(t)-Y_i(t))(X_j(t)-Y_j(t))\dot X_k(t)$$

$$+ 2a^{-1}(X(t))(\dot X(t) - \dot Y(t))\cdot(X(t)-Y(t)) \quad \text{a.e.},$$

we get

$$\frac{d}{dt}\, a^{-1}(X(t))\,(X(t)-Y(t))\cdot(X(t)-Y(t))$$

$$\leq\ 2\,a^{-1}(X(t))\,(\dot X(t)-\dot Y(t))\cdot(X(t)-Y(t)) + C\,|\dot X(t)|\,|X(t)-Y(t)|^2 =$$

$$= 2\left(a^{-1}(X(t))\,\dot X(t) - a^{-1}(Y(t))\,\dot Y(t)\right)\cdot(X(t)-Y(t))$$

$$+ 2\left(a^{-1}(Y(t)) - a^{-1}(X(t))\right)\dot Y(t)\cdot(X(t)-Y(t)) + C|\dot X(t)|\,|X(t)-Y(t)|^2 \leq$$

$$\leq -2\left(a^{-1}(X(t))\,\xi(t) - a^{-1}(Y(t))\,\eta(t)\right)\cdot(X(t)-Y(t)) + C\left(|\dot X(t)|+|\dot Y(t)|\right)|X(t)-Y(t)|^2$$

$$+ 2\left(a^{-1}(X(t))\,g(X(t),t) - a^{-1}(Y(t))\,g(Y(t),t)\right)\cdot(X(t)-Y(t))$$

$$\leq \left\{C_1\left(|\xi(t)|+|\eta(t)|\right) + C\left(|\dot X(t)|+|\dot Y(t)|\right) + CM\right\}|X(t)-Y(t)|^2 \quad \text{a.e.}$$

for some constant $C > 0$. Gronwall's inequality now proves that $X(t) = Y(t)$ for $0 \leq t \leq T$.

<u>Existence</u> Let $X_\varepsilon(t)$ be the unique solution of

$$\begin{cases} \dot X_\varepsilon(t) + a(X_\varepsilon(t))\,DI_\varepsilon(X_\varepsilon(t)) = g(X_\varepsilon(t),t) & \text{a.e. in } (0,T), \\ X_\varepsilon(0) = x, \end{cases}$$

where $x \in E$. We multiply this by $a^{-1}(X_\varepsilon(t))$ from the left and then take the inner product with $X_\varepsilon(t)$, to obtain

$$a^{-1}(X_\varepsilon(t))\,\dot{X}_\varepsilon(t)\cdot\dot{X}_\varepsilon(t) + DI_\varepsilon(X_\varepsilon(t))\cdot\dot{X}_\varepsilon(t) = a^{-1}(X_\varepsilon(t))\,g(X_\varepsilon(t),t)\cdot X_\varepsilon(t).$$

Integrating this, we get

$$\int_0^t a^{-1}(X_\varepsilon(s))\,\dot{X}_\varepsilon(s)\cdot\dot{X}_\varepsilon(s)\,ds + I_\varepsilon(X_\varepsilon(t)) = \int_0^t a^{-1}(X_\varepsilon(s))\,g(X_\varepsilon(s),s)\cdot X_\varepsilon(s)\,ds$$

for $0 \leq t \leq T$ as far as $X_\varepsilon(s) \in \Omega$ for $0 \leq s \leq t$.

From this

$$\|\dot{X}_\varepsilon\|_{L^2(0,t)} \leq C,$$

$$dist(X_\varepsilon(t), E) \leq C\sqrt{\varepsilon}$$

for $0 \leq t \leq T$ as far as $X_\varepsilon(s) \in \Omega$ for $0 \leq s \leq T$, where C is a constant independent of ε and t.

The remainder of the proof is quite similar to the corresponding part of Theorem 8.1 and we omit it here. ∎

§9. The Neumann problem for H-J equations

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary. Let $n(x)$ denote the outer unit normal vector of Ω at $x \in \partial\Omega$. We consider the Neumann problem

$$(9.1) \qquad \begin{cases} H(x,u,Du) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

We look at this problem this way: Define $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$F(x,r,p) = \begin{cases} H(x,r,p) & \text{if } x \in \Omega, \\ n(x)\cdot p & \text{otherwise}. \end{cases}$$

Assuming $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$, we see that

$$F^*(x,r,p) = F_*(x,r,p) = H(x,r,p) \qquad \text{if } x \in \Omega,$$

$$F^*(x,r,p) = n(x)\cdot p \vee H(x,r,p) \qquad \text{if } x \in \partial\Omega,$$

$$F_*(x,r,p) = n(x)\cdot p \wedge H(x,r,p) \qquad \text{if } x \in \partial\Omega.$$

Then we have the notion of viscosity solution of

$$F(x, u, Du) = 0 \qquad \text{in } \overline{\Omega}.$$

In other words, for instance, a function $u : \overline{\Omega} \to \mathbb{R}$ is said to be a <u>subsolution</u> of (9.1) if u is bounded on $\overline{\Omega}$ and whenever $\varphi \in C^1(\overline{\Omega})$, $y \in \overline{\Omega}$ and $\max(u^* - \varphi) = (u^* - \varphi)(y)$, then

$$H(y, u^*(y), D\varphi(y)) \le 0 \qquad \text{if } y \in \Omega,$$

$$n(y) \cdot D\varphi(y) \equiv \frac{\partial \varphi}{\partial n}(y) \le 0 \quad \text{or} \quad H(y, u^*(y), D\varphi(y)) \le 0$$
$$\text{if } y \in \partial\Omega.$$

The last requirement is equivalent to:

$$\begin{cases} H(y, u^*(y), p) \le 0 & \text{for } y \in \Omega,\ p \in D^+u^*(y), \\ n(y) \cdot p \le 0 \quad \text{or} \quad H(y, u^*(y), p) \le 0 & \text{for } y \in \partial\Omega,\ p \in D^+u^*(y). \end{cases}$$

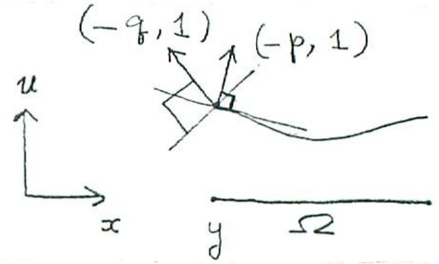
Notice the fact that

$$u \in C^1(\overline{\Omega}), \quad H(x, u(x), Du(x)) \le 0 \quad \forall x \in \overline{\Omega}$$
$$\not\Rightarrow \quad u \text{ a subsolution of } (9.1).$$

In fact, we have

<u>Proposition 9.1</u> If $u : \overline{\Omega} \to \mathbb{R}$ is bounded, $y \in \partial\Omega$ and $p \in D^+ u^*(y)$, then $p - \mathbb{R}_+ n(y) \subset D^+ u^*(y)$.

We omit the proof here.



$\exists t > 0 : \; -q = -p + t\,n(y)$

<u>Proposition 9.2.</u> (Let $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$.) For $\varepsilon > 0$, let $u^\varepsilon \in C^2(\overline{\Omega})$ be a classical subsolution of

$$\begin{cases} -\varepsilon \Delta u^\varepsilon + H(x, u^\varepsilon, Du^\varepsilon) = 0 & \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose $u^\varepsilon(x) \to u(x)$ uniformly on $\overline{\Omega}$ for some $u \in C(\overline{\Omega})$ as $\varepsilon \searrow 0$. Then u is a subsolution of (9.1).

<u>Proof.</u> We already know (Prop. 2.3) that u is a subsolution of $H(x, u, Du) = 0$ in Ω. Therefore,

we assume that $\varphi \in C^2(\overline{\Omega})$, $y \in \partial\Omega$ and

$\max(u-\varphi) = (u-\varphi)(y)$, and will prove

$$n(y)\cdot D\varphi(y) \wedge H(y, u(y), D\varphi(y)) \le 0.$$

To this end, we suppose $n(y)\cdot D\varphi(y) > 0$. We may also assume that y is a strict maximum of $u-\varphi$. We choose $\delta > 0$ so that $n(x)\cdot D\varphi(x) > 0$ for $x \in \partial\Omega \cap B(y,\delta)$. Obviously, we have

$$n(x)\cdot D(u^\varepsilon - \varphi)(x) < 0 \qquad \text{for } x \in \partial\Omega \cap B(y,\delta).$$

Let y_ε be a maximum point of $u^\varepsilon - \varphi$ on $\overline{\Omega} \cap B(y,\delta)$. From the above observation, we see that $y_\varepsilon \notin \partial\Omega$.
(Compute $\frac{d}{dt}(u^\varepsilon-\varphi)(y_\varepsilon + t n(y_\varepsilon))\big|_{t=0} \ge 0$) assuming $y_\varepsilon \in \partial\Omega$.
As usual, we know that $y_\varepsilon \to y$ as $\varepsilon \searrow 0$. Thus, for ε small enough, we have

$$y_\varepsilon \in \Omega,$$

and so $\quad D(u^\varepsilon - \varphi)(y_\varepsilon) = 0,$

$$-\Delta(u^\varepsilon - \varphi)(y_\varepsilon) \ge 0, \qquad -\varepsilon\Delta u^\varepsilon(y_\varepsilon) + H(y_\varepsilon, u^\varepsilon(y_\varepsilon), Du^\varepsilon(y_\varepsilon)) \le 0.$$

From these we obtain

$$-\varepsilon \Delta \varphi(y_\varepsilon) + H(y^\varepsilon, u^\varepsilon(y_\varepsilon), D\varphi(y_\varepsilon)) \le 0,$$

and, sending $\varepsilon \downarrow 0$, we conclude

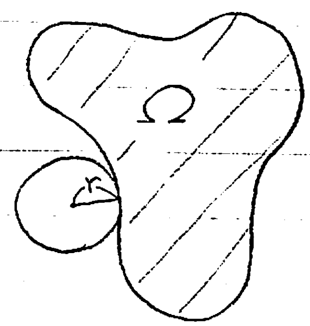
$$H(y, u(y), D\varphi(y)) \le 0. \qquad \blacksquare$$

Our comparison result is stated as follows.

<u>Theorem 9.1.</u> Let Ω be a bounded open subset of $\mathbb{R}^N$ with C^1 boundary. Assume the uniform exterior sphere condition, i.e.

(9.2) there is $r > 0$ such that

$$B(x + rn(x), r)^\circ \cap \Omega = \emptyset \quad \forall x \in \partial\Omega.$$



Let H be a function satisfying (H0) – (H2) and that

$(H3)_\partial$ there is a neighborhood U of $\partial\Omega$ in $\overline{\Omega}$ and a function $\sigma \in C([0,\infty))$ with $\sigma(0) = 0$ such that

$$|H(x,r,p)-H(x,r,q)| \le \sigma(|p-q|) \quad \forall x \in U,\ r \in \mathbb{R},\ p,q \in \mathbb{R}^N.$$

Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., a subsolution of (9.1) and a supersolution of

(9.3) $$\begin{cases} H(x,u,Du) = a & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a > 0$ is a constant. Then $u \le v$ on $\overline{\Omega}$.

<u>Lemma 9.1</u> Under the assumption (9.2) we have

(9.4) $$n(x)\cdot(x-y) + C|x-y|^2 \ge 0 \quad \forall x \in \partial\Omega,\ \forall y \in \overline{\Omega},$$

where C is a nonnegative constant.

<u>Proof</u> See 99. In fact, (9.2) and (9.4) are equivalent each other. ▨

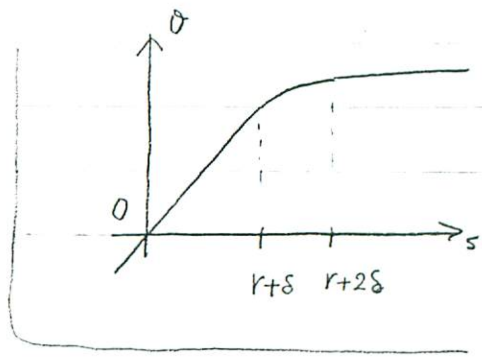
<u>Lemma 9.2</u> Under the assumptions on Ω of Theorem 9.1 there is a function $d \in C^\infty(\overline{\Omega})$ such that $n(x)\cdot Dd(x) < 0$ for $x \in \partial\Omega$.

<u>Proof</u> Let $r > 0$ be as in (9.2). Fix $z \in \partial\Omega$.

For $\varepsilon > 0$ we choose $\delta > 0$ so that

$$(9.5) \quad B(z + r n(z), r + 2\delta) \cap \partial\Omega \subset B(z, 2\varepsilon r).$$

We then choose $\theta \in C^\infty(\mathbb{R})$ so that $\theta(s) = s$ for $s \leq r + \delta$, $\theta(s) =$ constant for $s \geq r + 2\delta$ and $\theta' \geq 0$.



Define $d_z \in C^\infty(\overline{\Omega})$ by

$$d_z(x) = \theta(|x - (z + r n(z))|).$$

Then, $Dd_z(x) = \theta'(|\cdots|) \dfrac{x - (z + r n(z))}{|\cdots|}$,

and so

$$\begin{cases} n(x) \cdot Dd(x) = \theta'(|\cdots|) \dfrac{n(x)\cdot(x - z) - r\, n(x)\cdot n(z)}{|\cdots\cdots|} \\ \qquad \leq r\theta'(|\cdots|) \dfrac{2\varepsilon - n(x)\cdot n(z)}{|\cdots|} \quad \text{if } x \in B(z + r n(z), r + 2\delta) \cap \partial\Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\subset B(z, 2\varepsilon r)) \\ n(x) \cdot Dd(x) = 0 \quad \text{if } x \in \partial\Omega \setminus B(z + r n(z), r + 2\delta), \\ n(z) \cdot Dd(z) = -1. \end{cases}$$

We now fix $\varepsilon > 0$ small enough so that

$$n(x) \cdot n(z) \geq 2\varepsilon \quad \text{for } x \in B(z, \cancel{2}\,\varepsilon r).$$

Then

$$n(x)\cdot Dd(x) \le 0 \qquad \text{for } x \in \partial\Omega.$$

By a compactness argument, we see that there are points $z_1, \dots, z_k \in \partial\Omega$, positive numbers $\varepsilon_1, \dots, \varepsilon_k$ and functions $d_1, \dots, d_k \in C^\infty(\overline{\Omega})$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{k} B(z_i, \varepsilon_i),$$

$$n(x)\cdot Dd_i(x) \le 0 \qquad \text{for } x \in \partial\Omega, \quad i = 1, \dots, k,$$

$$n(x)\cdot Dd_i(x) < 0 \qquad \text{for } x \in \partial\Omega \cap B(z_i, \varepsilon_i), \quad i = 1, \dots, k.$$

Clearly, $d(x) = \sum_{i=1}^{k} d_i(x)$ has all the required properties. ▨

⟹ Insert here 123+1, 123+2, 123+3.

Proof of Theorem 9.1. By Lemma 9.2 there is a function $d \in C^1(\overline{\Omega})$ such that

$$n(x)\cdot Dd(x) \le -1 \quad \text{for } x \in \partial\Omega, \qquad d \le 0 \quad \text{on } \overline{\Omega},$$

and $d(x) = \text{constant}$ for $x \in \Omega \setminus U$,

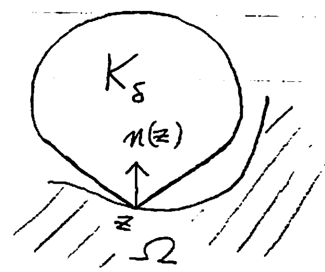
where U is from $(H3)_\partial$. For $\varepsilon > 0$ we set

$$u_\varepsilon = u + \varepsilon d \quad \text{and} \quad v_\varepsilon = v - \varepsilon d \quad \text{on } \overline{\Omega}.$$

<u>Remark</u> The conclusion of Lemma 9.2 is true under the assumptions that $\partial\Omega$ is of class C^1.

<u>Proof</u> For simplicity we assume that Ω is bounded. Fix $z \in \partial\Omega$. We observe that for each $0 < \theta < 1$ there is $\delta > 0$ such that

$$K_\delta \equiv \bigcup_{0<r\leq\delta} B(z + rn(z), \theta r) \subset (\overline{\Omega})^c.$$



This is a simple consequence of the fact that $\partial\Omega$ is the graph of a C^1 function locally. Fix $0 < \theta < 1$ and choose $\delta > 0$ so that

$$K_\delta \cap \overline{\Omega} = \emptyset.$$

We will prove that

$$0 < r \leq \tfrac{2}{3}\delta \Rightarrow B(z + rn(z), r) \cap \partial\Omega \subset B\left(z, 3\sqrt{\tfrac{1-\theta^2}{2}}\, r\right).$$

To see this, let $0 < r \leq \frac{2}{3}\delta$ and

$$x \in B(z + rn(z), r) \cap \partial\Omega.$$

Then, $x \in \partial\Omega \subset \overline{\Omega} \subset (K_\delta)^c$. Hence

$$|x - (z + t\,n(z))|^2 > \theta^2 t^2 \qquad 0 < {}^\forall t \leq \delta.$$

We also have

$$|x - (z + r\,n(z))|^2 \leq r^2.$$

Therefore,

$$|x - z|^2 - 2r\,n(z)\cdot(x - z) + r^2 \leq r^2,$$

$$|x - z|^2 - 2t\,n(z)\cdot(x - z) + t^2 > \theta^2 t^2 \qquad 0 < {}^\forall t \leq \delta,$$

and so

$$t|x - z|^2 \leq 2rt\,n(z)\cdot(x - z) \leq r|x - z|^2 + (1-\theta^2)t^2 r.$$

Plugging $t = \frac{3}{2}r \left(\leq \frac{3}{2}\cdot\frac{2}{3}\delta = \delta\right)$, we have

$$\frac{1}{2}r|x - z|^2 \leq (1-\theta^2)\,\frac{9}{4}r^3$$

i.e.

$$|x - z|^2 \leq \frac{9}{2}(1-\theta^2)\,r^2.$$

Thus, $x \in B(z, 3\sqrt{\frac{1-\theta^2}{2}}\,r)$.

Thus we conclude that for any $0 < \varepsilon < 1$ there is $r > 0$ such that

$$B(z + rn(z), r) \cap \partial\Omega \subset B(z, \varepsilon r) .$$

By a compactness argument we find that

$$(*) \quad B(z + rn(z), r + 2\delta) \cap \partial\Omega \subset B(z, 2\varepsilon r)$$

for some $\delta > 0$. Indeed, if

$$B(z + rn(z), r + \tfrac{1}{j}) \cap \partial\Omega \not\subset B(z, 2\varepsilon r) \qquad \forall j \in \mathbb{N},$$

then $\exists\, \xi_j \in B(z + rn(z), r + \frac{1}{j}) \cap \partial\Omega \setminus B(z, 2\varepsilon r)$. We may assume $\xi_j \to \xi \in \mathbb{R}^N$ for some ξ as $j \to \infty$. Obviously, we have

$$\xi \in B(z + rn(z), r) \cap \partial\Omega \setminus B(z, 2\varepsilon r)^\circ ;$$

this is a contradiction.

Using $(*)$, we proceed the remainder of the proof as that of Lemma 9.2. ▨

Then we compute formally that

$$H(x, u_\varepsilon, Du_\varepsilon) \equiv H(x, u, Du + \varepsilon Dd(x)) \leq H(x, u, Du) + \sigma(\varepsilon |Dd(x)|),$$

$$H(x, v_\varepsilon, Dv_\varepsilon) \geq H(x, v, Dv) - \sigma(\varepsilon |Dd(x)|),$$

$$n(x) \cdot Du_\varepsilon = n(x) \cdot Du + \varepsilon n(x) \cdot Dd(x) \qquad \text{for } x \in \partial\Omega,$$

$$n(x) \cdot Dv_\varepsilon = n(x) \cdot Dv - \varepsilon n(x) \cdot Dd(x) \qquad \text{for } x \in \partial\Omega.$$

Indeed, choosing $\varepsilon > 0$ so that

$$\sigma(\varepsilon \max_{\overline{\Omega}} |Dd(x)|) \leq \frac{a}{4},$$

we find that u_ε and v_ε are, resp., a subsolution of

$$(9.6) \qquad \begin{cases} H(x, u, Du) = \frac{a}{4} & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = -\varepsilon & \text{on } \partial\Omega \end{cases}$$

and

$$(9.7) \qquad \begin{cases} H(x, u, Du) = \frac{3}{4} a & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \varepsilon & \text{on } \partial\Omega. \end{cases}$$

Now, we consider the function $\Phi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ defined by

$$\Phi(x,y) = u_\varepsilon(x) - v_\varepsilon(y) - \frac{1}{\delta}|x-y|^2, \quad \text{where} \quad \delta > 0.$$

Let (x_δ, y_δ) be a maximum point of Φ. As usual, it is easy to see that

$$\begin{cases} \frac{1}{\delta}|x_\delta - y_\delta|^2 \to 0 & \text{as } \delta \searrow 0, \\ x_\delta, y_\delta \to \bar{x} \quad \text{for some } \bar{x} \in \bar{\Omega} & \text{as } \varepsilon \searrow 0 \\ \qquad \text{(taking a subsequence if necessary)}, & \\ \bar{x} \text{ is a maximum point of } u_\varepsilon - v_\varepsilon, & \\ u_\varepsilon(x_\delta) \to u_\varepsilon(\bar{x}), \quad v_\varepsilon(y_\delta) \to v_\varepsilon(\bar{x}) & \text{as } \varepsilon \searrow 0. \end{cases}$$

Assume that $x_\delta \in \partial\Omega$, and use (9.4) to find

$$n(x_\delta) \cdot \frac{2}{\delta}(x_\delta - y_\delta) \geq -\frac{2}{\delta} C |x_\delta - y_\delta|^2.$$

If δ is sufficiently small, then the right hand side is larger than $-\varepsilon$. Therefore, by the definiction of subsolution we always have

$$H(x_\delta, u_\varepsilon(x_\delta), \tfrac{2}{\delta}(x_\delta - y_\delta)) \leq \frac{a}{4}$$

for $\delta > 0$ small enough. Similarly we have

$$H(y_\delta, v_\varepsilon(y_\delta), \tfrac{2}{\delta}(x_\delta - y_\delta)) \geq \tfrac{3}{4} a$$

for $\delta > 0$ small enough. If we suppose $\max_{\overline{\Omega}} (u_\varepsilon - v_\varepsilon) > 0$, then $u_\varepsilon(x_\delta) \geq v_\varepsilon(y_\delta)$ for $\delta > 0$ small enough and hence the above two inequalities are contradictory. This proves that $u_\varepsilon \leq v_\varepsilon$ on $\overline{\Omega}$ and moreover $u \leq v$ on $\overline{\Omega}$. ∎

<u>Remark</u> In Theorem 9.1, if $p \to H(x,r,p)$ is convex for $x \in \overline{\Omega}$ (and $r \in \mathbb{R}$), then the conclusion holds without $(H3)_\partial$.

<u>Corollary 9.1</u> Under the hypotheses of Theorem 9.1 except that we now assume that u and v are, resp., sub- and supersolutions of

$$(9.8) \quad \begin{cases} u + H(x, u, Du) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

we have the same conclusion; $u \leq v$ on $\overline{\Omega}$.

<u>Proof.</u> Compare u and $v + a$, with $a > 0$, by

using Theorem 9.1. ▮

<u>Corollary 9.2.</u> Assume (H0) – (H2) and (9.2). Suppose there is a function $f \in C^1(\overline{\Omega})$ such that

$$\begin{cases} \frac{\partial f}{\partial n} \le 0 & \text{on } \partial\Omega, \\ \sup\{H(x, r, Df(x)) : x \in \overline{\Omega}, \ r \in \mathbb{R}\} < 0. \end{cases}$$

Assume that $p \to H(x, r, p)$ is convex for $x \in \overline{\Omega}$ and $r \in \mathbb{R}$. Let u and v be, resp., sub- and supersolutions of (9.1). Then $u \le v$ on $\overline{\Omega}$.

<u>Proof</u> For $\theta \in (0, 1)$ set

$$u_\theta = \theta u + (1-\theta) f \qquad \text{on } \overline{\Omega},$$

and

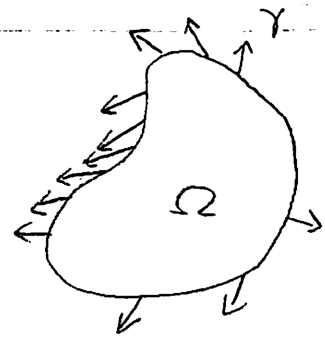
$$a = -\sup\{H(x, r, Df(x)) : x \in \overline{\Omega}, \ r \in \mathbb{R}\}.$$

Then u_θ is a subsolution of

$$\begin{cases} H(x, u, Du) \le -a(1-\theta) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} \le 0. \end{cases}$$

Then we use Theorem 9.1 together with Remark, to conclude $u_\theta \le v$ on $\overline{\Omega}$ for $0 < \theta < 1$. ■

Now we consider the Neumann problem with oblique derivatives. Let $\gamma : \overline{\Omega} \to \mathbb{R}^N$ be a vector field which is supposed to satisfy



(9.9) $\quad \gamma \in C(\overline{\Omega})$ and $n \cdot \gamma > 0$ on $\partial\Omega$.

Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy

(9.10) $\quad f \in C(\overline{\Omega} \times \mathbb{R})$ and $r \to f(x, r)$ is nondecreasing on $\mathbb{R}$ for $x \in \overline{\Omega}$.

<u>Theorem 9.2.</u> Let Ω and H be as in Theorem 9.1. Let γ and f satisfy (9.9) and (9.10), respectively. (Assume that $\gamma \in C^1(\overline{\Omega})$ and $\partial\Omega$ is of class C^2.) Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, respectively, a subsolution of

$$
(9.11) \qquad \begin{cases} H(x, u, Du) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} + f(x, u) = 0 & \text{on } \partial\Omega \end{cases}
$$

and a supersolution of

$$
(9.12) \qquad \begin{cases} H(x, v, Dv) = a & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \gamma} + f(x, v) = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $a > 0$ is a constant. Then $u \le v$ on $\overline{\Omega}$.

<u>Lemma 9.3.</u> If Ω is an open subset of $\mathbb{R}^N$ with $\partial\Omega$ of class C^1 and γ satisfies (9.9), then there is a function $d \in C^\infty(\overline{\Omega})$ such that $\gamma \cdot Dd < 0$ on $\partial\Omega$.

<u>Proof</u> The method of construction of functions in the proof of Lemma 9.1 together with the observation (*) in the proof of Remark after Lemma 9.1 applies to build a function d with the properties required here. ∎

Lemma 9.4. Let γ be a vector fied on $\overline{\Omega}$ which satisfies (9.9). Then there is an $n\times n$ real symmetric matrix valued function A on $\overline{\Omega}$ such that

(9.13) $$A(x)\gamma(x) = n(x) \qquad \forall x \in \partial\Omega,$$

(9.14) $$A(x) \geq \lambda I \qquad \forall x \in \Omega \quad \text{for some } \lambda > 0.$$

Moreover, if $\gamma \in C^k(\overline{\Omega})$ and $\partial\Omega \in C^{k+1}$, then $A \in C^k(\overline{\Omega}, \mathbb{R}^{n\times n})$.

Proof For some neighborhood U of $\partial\Omega$, we may assume that $n \in C^k(U)$, $n\cdot\gamma > 0$ in U and $|n| = |\gamma| = 1$ in U. We define the $n\times n$ matrix $B(x)$ by

$$B(x)\xi = (\xi\cdot n(x))\,n(x) + \xi - (\xi\cdot\gamma(x))\,\gamma(x) \quad \text{for } \xi \in \mathbb{R}^N \text{ and } x \in U$$

Then

$$B(x)\gamma(x) = (\gamma(x)\cdot n(x))\,n(x),$$

$$B(x)\xi\cdot\eta = (\xi\cdot n(x))(n(x)\cdot\eta) + \xi\cdot\eta - (\xi\cdot\gamma(x))(\gamma(x)\cdot\eta) = \xi\cdot B(x)\eta,$$

and

$$0 = B(x)\xi\cdot\xi = (\xi\cdot n)^2 + |\xi|^2 - (\xi\cdot\gamma)^2 = (\xi\cdot n)^2 + |(\xi\cdot\gamma)\gamma + \xi - (\xi\cdot\gamma)\gamma|^2$$

$$- (\xi\cdot\gamma)^2 = (\xi\cdot n)^2 + |\xi - (\xi\cdot\gamma)\gamma|^2 \iff \xi\cdot n = 0 \ \& \ \xi = (\xi\cdot\gamma)\gamma$$

$$\iff \xi = 0 .$$

We choose $\theta \in C_0^\infty(U)$ so that $\theta(x) = 1$ near $\partial\Omega$ and $0 \le \theta \le 1$, and set

$$A(x) = \theta(x)\frac{1}{\gamma(x)\cdot n(x)} B(x) + (1-\theta(x)) I \quad \text{for } x \in \overline{\Omega}.$$

Then A satisfies all the required properties. ▨

<u>Proof of Theorem 9.2.</u> Using $(H3)_\partial$, (9.9), (9.10) and Lemma 9.3, we can reduce the situation to the case where u and v are, resp., a subsolution of

$$(9.11)' \qquad \begin{cases} H(x,u,Du) = -a & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} + f(x,u) = -a & \text{on } \partial\Omega, \end{cases}$$

and a supersolution of

$$(9.12)' \qquad \begin{cases} H(x, v, Dv) = a & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \gamma} + f(x, v) = a & \text{on } \partial\Omega, \end{cases}$$

where a is a positive constant.

In order to prove that $u \leq v$ on $\overline{\Omega}$, we suppose that $\max_{\overline{\Omega}} (u - v) > 0$, and will conclude a contradiction. From Theorem 6.0 we see that $\max_{\partial\Omega} (u - v) = \max_{\overline{\Omega}} (u - v)$. (Compare $u - \max_{\partial\Omega} (u-v)^+$ and v by Theorem 6.0.) Let $z \in \partial\Omega$ be a maximum point of $u - v$. Put $M = f(z, u(z))$. In view of Lemma 9.4 we choose an $n \times n$ real symmetric matrix-valued function A so that $A(x)\gamma(x) = n(x)$ for $x \in \partial\Omega$, $A(x) \geq \lambda I$ for all $x \in \Omega$ and some $\lambda > 0$, and $A \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n})$. We may assume $|\gamma(z)| = 1$. For $\varepsilon > 0$ we consider the function Φ on $\overline{\Omega} \times \overline{\Omega}$ defined by

$$\Phi(x, y) = u(x) - v(y) - \frac{1}{\varepsilon} A(x)(x-y)\cdot(x-y) + M\gamma(z)\cdot(x-y) - |x - z|^2.$$

Let $(x_\varepsilon, y_\varepsilon)$ be a maximum point of Φ. Since

$$x \to \Phi(x, x) = u(x) - v(y) - |x - z|^2$$

takes its strict maximum at z, we see as usual that as $\varepsilon \searrow 0$,

$$x_\varepsilon, y_\varepsilon \to z, \qquad \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \leq \frac{1}{\lambda\varepsilon} A(x_\varepsilon)(x_\varepsilon - y_\varepsilon)\cdot(x_\varepsilon - y_\varepsilon) \to 0,$$

$$u(x_\varepsilon) \to u(z), \qquad v(y_\varepsilon) \to v(z).$$

Suppose that $x_\varepsilon \in \partial\Omega$ and $H(x_\varepsilon, u(x_\varepsilon), D_x\varphi(x_\varepsilon, y_\varepsilon)) > -a$, where $\varphi(x,y) = \frac{1}{\varepsilon} A(x)(x-y)\cdot(x-y) - M\gamma(z)\cdot(x-y) + |x-z|^2$.

Then we have

$$\gamma(x_\varepsilon)\cdot D_x\varphi(x_\varepsilon, y_\varepsilon) + f(x_\varepsilon, u(x_\varepsilon)) \leq -a$$

since u is a subsolution of $(9.11)'$. Hence, using (9.4), we get

$$-a \geq \gamma(x_\varepsilon)\cdot\left\{\frac{1}{\varepsilon} DA(x_\varepsilon)(x_\varepsilon - y_\varepsilon)\cdot(x_\varepsilon - y_\varepsilon) + \frac{2}{\varepsilon} A(x_\varepsilon)(x_\varepsilon - y_\varepsilon) - M\gamma(z) + 2(x_\varepsilon - z)\right\}$$

$$+ f(x_\varepsilon, u(x_\varepsilon)) \geq \frac{2}{\varepsilon} A(x_\varepsilon)\gamma(x_\varepsilon)\cdot(x_\varepsilon - y_\varepsilon) + f(x_\varepsilon, u(x_\varepsilon)) - M\gamma(z)\cdot\gamma(x_\varepsilon)$$
$$- O\left(\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} + |x_\varepsilon - z|\right),$$

(where $DA(x)\xi\cdot\eta = \left(\sum_{i,j} a_{ij,x_k}(x)\xi_i\eta_j\right)$ if $A(x) = (a_{ij}(x))$,)

$$\geq \frac{2}{\varepsilon} n(x_\varepsilon)\cdot(x_\varepsilon - y_\varepsilon) - o(1),$$

$$\geq -\frac{2}{\varepsilon} C |x_\varepsilon - y_\varepsilon|^2 - o(1) \qquad \text{by (9.4)},$$

$$\geq -o(1) \qquad \text{as } \varepsilon \searrow 0.$$

Therefore, if ε is small enough, then we always have

$$H(x_\varepsilon, u(x_\varepsilon), D_x\varphi(x_\varepsilon, y_\varepsilon)) \leq -a.$$

Similarly, we have

$$H(y_\varepsilon, v(y_\varepsilon), -D_y\varphi(x_\varepsilon, y_\varepsilon)) \geq a$$

provided ε is small enough. Subtracting one from the other, we then have

$$-2a \geq H\big(x_\varepsilon, u(x_\varepsilon), \tfrac{1}{\varepsilon} DA(x_\varepsilon)(x_\varepsilon - y_\varepsilon)\cdot(x_\varepsilon - y_\varepsilon) + \tfrac{2}{\varepsilon}A(x_\varepsilon)(x_\varepsilon - y_\varepsilon) - M\gamma(z) + 2(x_\varepsilon - z)\big)$$

$$- H\big(y_\varepsilon, v(y_\varepsilon), \tfrac{2}{\varepsilon}A(x_\varepsilon)(x_\varepsilon - y_\varepsilon) - M\gamma(z)\big)$$

$$\geq -m\left(|x_\varepsilon - y_\varepsilon|\left(\tfrac{2}{\varepsilon}\|A(x_\varepsilon)\|\,|x_\varepsilon - y_\varepsilon| + 1\right)\right) - \sigma\left(\left[\sum_{i,j,k}\left(a_{ij,x_k}(x_\varepsilon)\right)^2\right]^{1/2}\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 + 2|x_\varepsilon - z|\right)$$

if $u(x_\varepsilon) \geq v(y_\varepsilon)$, which is valid for ε small enough. Thus we obtain a contradiction taking $\varepsilon > 0$ small enough. ▨

<u>Remarks</u> (i) Theorem 9.2 does not include Theorem 9.1 because of the additional assumptions that $\gamma \in C^1(\overline{\Omega})$ and $\partial\Omega$ is of class C^2. Are these assumptions really necessary?

(ii) Obviously, assertions (corresponding to Theorem 9.2) similar to Corollaries 9.1 and 9.2 are valid.

(iii) In Theorem 9.2, if one of u or v is Lipschitz continuous on $\overline{\Omega}$, we do not need to assume that $\gamma \in C^1(\overline{\Omega})$ and $\partial\Omega$ is of class C^2. The proof is similar but we choose

$$(x,y) \to u(x) - v(y) - \frac{1}{\varepsilon} A(z)(x-y)\cdot(x-y) + M\gamma(z)\cdot(x-y) - |x-z|^2$$

instead of Φ.

Now we consider the initial value problem for

$$(9.13)\quad \begin{cases} u_t + H(x,t,u,Du) = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \gamma} + f(x,t,u) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

with initial data

$$(9.14)\qquad u(x,0) = h(x) \qquad \text{for } x \in \overline{\Omega},$$

where f and h are given functions. We assum

$$(9.15)\qquad f \in C(\partial\Omega \times [0,T] \times \mathbb{R}) \text{ and } r \to f(x,t,r) \text{ is}$$

$$\text{nondecreasing on } \mathbb{R} \text{ for } (x,t) \in \partial\Omega \times [0,T].$$

The following comparison result holds:

<u>Theorem 9.3.</u> Let Ω and H be as in Theorem 9.1. Let $u \in USC(\overline{\Omega} \times [0,T))$ and $v \in LSC(\overline{\Omega} \times [0,T))$ be, resp., sub- and supersolutions of (9.13). Assume that $u(x,0) \le v(x,0)$ for $x \in \overline{\Omega}$, f satisfies (9.15) and either (i) $\gamma = n$ on $\partial\Omega$, or (ii) $\partial\Omega$ is of class C^2 and $\gamma \in C^1(\overline{\Omega})$ satisfies (9.9). Then $u \le v$

on $\overline{\Omega} \times [0, T)$.

<u>Lemma 9.5.</u> Let u and v be as in the above theorem. Set $w(x,y,t) = u(x,t) - v(y,t)$ for $x, y \in \overline{\Omega}$ and $0 \le t < T$. Then w is a subsolution of

$$(9.16)\begin{cases} w_t + \tilde{H}(x, y, t, w, Dw) = 0 & \text{in } \Omega \times \Omega \times (0,T), \\ \gamma(x) \cdot D_x w + f(x, t, v(y,t) + w) = 0 & \text{on } \partial\Omega \times \overline{\Omega} \times (0,T), \\ \gamma(y) \cdot D_y w - f(y, t, u(x,t) - w) = 0 & \text{on } \overline{\Omega} \times \partial\Omega \times (0,T), \end{cases}$$

where

$$\tilde{H}(x, y, t, r, p, q) = H(x, t, r + v(y,t), p) - H(y, t, u(x,t) - r, -q).$$

<u>Proof</u> Review the proof of Lemma 6.2 and notice that it works even for the case when Ω is locally compact and H is locally bounded (and so possibly discontinuous) if we state the conclusion

this way: w is a subsolution of

$$[w_t + H_*(x,t, w+v(y,t), D_x w)] \vee [-w_t - H^*(y,t, u(x,t)-w, -D_y w)] = 0$$

in $\Omega \times \Omega \times (0,T)$. Then the above assertion is a corollary of this observation (take $\overline{\Omega}$ as Ω and F as H, where $F(x,t,r,p) = H(x,t,r,p)$ if $x \in \Omega$ and $= \gamma(x)\cdot p + f(x,t,r)$ if $x \in \partial\Omega$). ▨

<u>Outline</u> ~~of proof of~~ <u>Theorem 9.3.</u> Using Lemmas 9.5 and 9.3, we see that, in order to prove Theorem 9.3, it is enough to show

$$u \le v \quad \text{on } \overline{\Omega} \times (0,T),$$

provided $w = u(x,t) - v(y,t)$ is a subsolution of (9.16) with 0's on the right hand sides replaced by a positive constant and $w(x,x,0) < 0$ for $x \in \overline{\Omega}$.

Now, suppose $\sup\limits_{\overline{\Omega}\times(0,T)} (u-v) > 0$. Choose $\varepsilon > 0$

so that $\sup_{\overline{\Omega}\times(0,T)} \left(w(x,x,t) - \frac{\varepsilon}{T-t} \right) > 0$ and $\max_{\overline{\Omega}} w(x,x,0) \leq \frac{\varepsilon}{T}$. If

$$\sup_{\overline{\Omega}\times(0,T)} \left(w(x,x,t) - \frac{\varepsilon}{T-t} \right) > \sup_{\partial\Omega\times(0,T)} \left(w(x,x,t) - \frac{\varepsilon}{T-t} \right),$$

then we take

$$\Phi(x,y,t) = w(x,y,t) - \frac{1}{\delta}|x-y|^2 - \frac{\varepsilon}{T-t}, \quad \text{with } \delta > 0.$$

Otherwise, we choose $(z,\tau) \in \partial\Omega \times (0,T)$ so that

$$\sup_{\overline{\Omega}\times(0,T)} \left(w(x,x,t) - \frac{\varepsilon}{T-t} \right) = w(z,z,\tau) - \frac{\varepsilon}{T-\tau},$$

and define

$$\Phi(x,y,t) = w(x,y,t) - \frac{1}{\delta} A(x)(x-y)\cdot(x-y) + M\,\gamma(z)\cdot(x-y) - |x-z|^2 - (t-\tau)^2,$$

where $A(x)$ is from Lemma 9.4 (or $A(x) = I$ in the case (i)) and $M = f(z,\tau,u(z,\tau))/|\gamma(z)|$. Then, the remainder of arguments is now standard. ∎

Let us formulate existence results.

Theorem 9.4. Under the hypotheses on Ω and H (resp., Ω, H, f and γ) of Theorem 9.1 (resp., Theorem 9.2), there is a (unique) solution of

$$\begin{cases} u + H(x, u, Du) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \quad \left(\text{resp., } \frac{\partial u}{\partial \gamma} + f(x, u) = 0 \text{ on } \partial\Omega\right). \end{cases}$$

Moreover, it is continuous on $\overline{\Omega}$.

Outline of proof. We use Perron's method. Notice that if $u \in C^1(\overline{\Omega})$ satisfies

$$(9.17) \quad \begin{cases} u + H(x, u, Du) \le 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \gamma} + f(x, u) \le 0 & \text{on } \partial\Omega, \end{cases}$$

in the classical sense, then u is a subsolution of (9.17) in the viscosity sense. (Cf. Prop. 9.1.) We show how to construct a supersolution of (9.17). Let $d \in C^1(\overline{\Omega})$ be a function such that $\gamma \cdot Dd \le -1$ on $\partial\Omega$ and $d =$ constant on $\Omega \setminus U$, where U is from $(H3)_\partial$.

Let $A > 0$ and set $g_1(x) = -Ad(x)$. Then

$$\frac{\partial g_1}{\partial \gamma} + f(x,0) = -A\gamma \cdot Dd + f(x,0) \geqslant A + f(x,0) \quad \text{on } \partial\Omega.$$

We choose $A \geqslant \max_{\partial\Omega} |f(x,0)|$ so that

$$\frac{\partial}{\partial \gamma} g_1 + f(x,0) \geqslant 0 \quad \text{on } \partial\Omega.$$

Let $B > 0$ and set $g = g_1 + \max_{\overline{\Omega}} |g_1| + B$. Then

$$\frac{\partial}{\partial \gamma} g + f(x,g) \geqslant \frac{\partial}{\partial \gamma} g_1 + f(x,0) \geq 0 \quad \text{on } \partial\Omega,$$

and

$$g + H(x,g,Dg) \geqslant B + H(x,0,Dg_1) \quad \text{in } \Omega.$$

We now fix $B \geqslant \max_{\overline{\Omega}} |H(x,0,Dg_1(x))|$ so that

$$g + H(x,g,Dg) \geqslant 0 \quad \text{in } \Omega.$$

Thus we have constructed a supersolution of (9.17). Also, $-g$ is a subsolution of (9.17).

We apply Perron's method, to obtain a solution of (9.17). The uniqueness and continuity of the solution follows from comparison results. ∎

We have an existence result related to Corollary 9.2.

<u>Theorem 9.5.</u> Consider the problem

$$(9.18)\begin{cases} H(x,u,Du)=0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}=0 \quad \left(\text{or} \quad \dfrac{\partial u}{\partial \gamma}+f(x,u)=0\right) & \text{on } \partial\Omega. \end{cases}$$

In addition to the assumptions of Theorem 9.4, suppose that there is a supersolution w of (9.18) and a function $v \in C'(\bar{\Omega})$ satisfying

$$\begin{cases} H(x,v(x),Dv(x)) < 0 & \text{on } \bar{\Omega}, \\ \dfrac{\partial v}{\partial n} \le 0 \quad \left(\text{resp.,} \quad \dfrac{\partial v}{\partial \gamma}+f(x,v) \le 0\right) & \text{on } \partial\Omega. \end{cases}$$

Then there is a (unique) continuous solution of (9.18).

We skip the proof of this theorem.

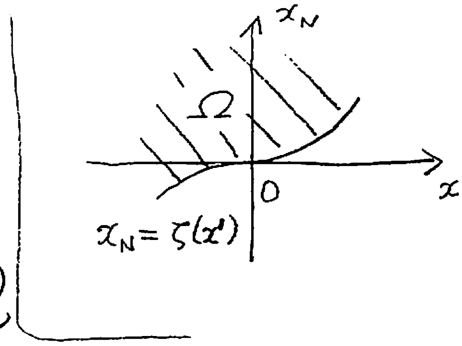
Theorem 9.6. Let $h \in C(\overline{\Omega})$. Assume that $\partial\Omega$ is of class C^2. Under the assumptions of Theorem 9.3, there is a (unique) continuous solution of (9.13) and (9.14).

Lemma 9.6. If $\partial\Omega$ is of class C^2, then the function $x \to \mathrm{dist}(x, \Omega^c)$ on $\overline{\Omega}$ is continuously differentiable in a neighborhood of $\partial\Omega$.

Proof Let $z \in \partial\Omega$. By using an orthogonal change of variables, we may assume that $z = 0$ and

$$B(0,r) \cap \overline{\Omega} = \{(x', x_N) \in B(0,r) : x_N \geq \zeta(x')\}, \text{ with } r > 0,$$

where $\zeta \in C^2(\{x' \in \mathbb{R}^{N-1} : |x'| \leq r\})$ and $\zeta(0) = D\zeta(0) = 0$.



Consider the map

$$\Phi : (x', t) \to (x', \zeta(x')) - t\, n(x', \zeta(x'))$$
$$= (x', \zeta(x')) - \frac{t}{\sqrt{|D\zeta(x')|^2 + 1}} (D\zeta(x'), -1),$$

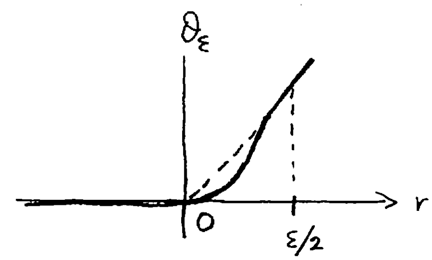
and compute its Jacobian ; $\det D\Phi(0,0) = 1$.

By the inverse mapping theorem, we see that Φ is a C^1 diffeomorphism between (some) neighborhoods of the origin in $\mathbb{R}^N$. Clearly, we have

$$\Phi^{-1}(x) = (\xi_x, \operatorname{dist}(x, \Omega^c)) \quad \text{with some } \xi_x \in \mathbb{R}^{N-1}$$

if $x \in \overline{\Omega}$. ∎

<u>Outline of proof of Theorem 9.6.</u> We show just how to build a supersolution $g \in LSC(\overline{\Omega} \times [0,T))$ of (9.13) satisfying $g(x,0) = h(x)$ for $x \in \overline{\Omega}$ and $g(x,t) \geq h(x)$ for $(x,t) \in \overline{\Omega} \times [0,T)$. Fix $\varepsilon > 0$, and choose a C^1 function h_ε on $\overline{\Omega}$ so that $h \leq h_\varepsilon \leq h + \varepsilon$ on $\overline{\Omega}$. Choose a C^1 function θ_ε on $\mathbb{R}$ so that ($\theta_\varepsilon \geq 0$ on $\mathbb{R}$, $\theta_\varepsilon(r) = r$ for $r \geq \frac{\varepsilon}{2}$ and $\theta_\varepsilon(r) = 0$ for $r \leq 0$. Let $\zeta(x) = \operatorname{dist}(x, \Omega^c)$.



For $A > 0$ we put

$$k_\varepsilon(x) = h_\varepsilon(x) + \theta_\varepsilon(\varepsilon - A\zeta(x)).$$

Clearly, $h \le k_\varepsilon \le h + 2\varepsilon$ on $\overline{\Omega}$, and if $\theta_\varepsilon(\varepsilon - A\zeta(x)) > 0$ then $\zeta(x) = \text{dist}(x, \Omega^c) < \varepsilon/A$. Also, if $\text{dist}(x, \Omega^c) \le \frac{\varepsilon}{2A}$, then $k_\varepsilon(x) = h_\varepsilon(x) + \varepsilon - A\zeta(x)$. In particular, we have

$$\gamma(x) \cdot Dk_\varepsilon(x) = \gamma(x) \cdot Dh_\varepsilon(x) - A\gamma(x) \cdot D\zeta(x) = \gamma(x) \cdot Dh_\varepsilon(x) + A\, n(x) \cdot \gamma(x)$$

for $x \in \partial\Omega$. Choosing A large enough, we may assume that

$$\gamma(x) \cdot Dk_\varepsilon(x) + f(x, t, k_\varepsilon(x)) \ge 0 \qquad \text{on } \partial\Omega \times (0,T),$$

and $k_\varepsilon \in C^1(\overline{\Omega})$. For $B > 0$, consider the function

$$g_\varepsilon(x,t) = k_\varepsilon(x) + Bt.$$

We compute:

$$g_{\varepsilon,t} + H(x, t, g_\varepsilon, Dg_\varepsilon) = B + H(x, t, k_\varepsilon(x), Dk_\varepsilon(x)).$$

We choose B large enough so that the right hand side of the last identity is nonnegative on $\overline{\Omega} \times [0,T]$. Then g_ε is a supersolution of

(9.13) and satisfies that $g_\varepsilon(x,t) \geq h(x)$ for $(x,t) \in \overline{\Omega} \times [0,T)$ and $g_\varepsilon(x,0) \leq h(x) + 2\varepsilon$ for $x \in \overline{\Omega}$.

Define

$$g(x,t) = \inf_{\varepsilon > 0} g_\varepsilon(x,t) \qquad \text{for } (x,t) \in \overline{\Omega} \times [0,T).$$

This has all the required properties. ∎

§ 10. Optimal control of the Skorokhod problem

As in § 3 we let A be a compact subset of $\mathbb{R}^m$ and $g : \mathbb{R}^N \times A \to \mathbb{R}^N$ and $f : \mathbb{R}^N \times A \to \mathbb{R}$ be continuous functions. Also, let

$$\mathcal{A} = \{ \alpha : [0,\infty) \to A \ \text{ measurable} \}.$$

We assume that Ω has a C^1 boundary and satisfies the uniform exterior sphere condition (9.2). For $\alpha \in \mathcal{A}$ and $x \in \overline{\Omega}$ we consider the initial value problem

$$(10.1) \qquad \begin{cases} \dot{X}(t) + D^- I(X(t)) \ni g(X(t), \alpha(t)) & (t > 0), \\ X(0) = x, \end{cases}$$

where I is the indicator function of $\overline{\Omega}$, i.e.

$$I(x) = \begin{cases} 0 & \text{if } x \in \overline{\Omega}, \\ +\infty & \text{otherwise}. \end{cases}$$

We assume:

(10.2) There is a constant $M > 0$ such that

$$|f(x,a)| \le M, \quad |g(x,a)| \le M, \quad |f(x,a) - f(y,a)| \le M|x-y|$$

$$|g(x,a) - g(y,a)| \le M|x-y| \quad \text{for } x, y \in \mathbb{R}^N, \ a \in A.$$

By Theorem 8.1 there is a unique solution of (10.1) which we denote by $X(t)$ or $X(t; x, \alpha)$.

We define the cost functional and the value function, resp., by

(10.3) $\quad J(x,\alpha) = \int_0^\infty e^{-\lambda t} f(X(t), \alpha(t))\, dt \quad$, with $\lambda > 0$,

and by

(10.4) $\quad V(x) = \inf\{J(x,\alpha) : \ \alpha \in \mathcal{A}\}$.

Clearly, $|V(x)| \le M$ for $x \in \overline{\Omega}$.

~~Definition We call a map $\tau : \mathcal{A} \to [0,\infty]$ a stopping time if whenever $\alpha, \beta \in \mathcal{A}$ and $\alpha(t) = \beta(t)$ a.e. on $[0, \tau(\alpha))$, then $\tau(\alpha) = \tau(\beta)$.~~

Lemma 10.1. (Dynamic programming principle) For any $x \in \overline{\Omega}$ and any map $\tau : \mathcal{A} \to [0,\infty]$ ~~stopping time τ~~, we have

$$(10.5) \qquad V(x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))\,dt + e^{-\lambda\tau} V(X(\tau)) \right\}.$$

Proof Repeat the proof of Lemma 3.1. ∎

Proposition 10.1. Under the assumptions above, V is a solution of

$$(10.6) \qquad \begin{cases} \lambda u + H(x, Du) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $H(x,p) = \max_{a \in A} \{ -g(x,a) \cdot p - f(x,a) \}$. Moreover, $V \in C(\overline{\Omega})$.

Remark. H satisfies

$$|H(x,p) - H(y,p)| \le M|x-y|(|p|+1), \quad |H(x,p) - H(x,q)| \le M|p-q|$$

and hence (H0)–(H2), $(H3)_\partial$. Thus, by Theorem 9.1, V is a unique solution of (10.6).

Proof. Once we know that V is a solution of (10.6), it is immediate from Theorem 9.1 that $V \in C(\overline{\Omega})$.

We first check that V is a subsolution of (10.6). Let $\varphi \in C^1(\overline{\Omega})$ and $y \in \overline{\Omega}$, and suppose that

$$\max_{\overline{\Omega}} (V^* - \varphi) = (V^* - \varphi)(y) = 0.$$

We may assume moreover that $(V^* - \varphi)(x) \leq -|x-y|^2$ for $x \in \overline{\Omega}$. We may also assume $y \in \partial\Omega$; otherwise the same argument as in the proof of Proposition 3.1 applies. We suppose

$$n(y) \cdot D\varphi(y) > 0 \quad \& \quad \lambda\varphi(y) + H(y, D\varphi(y)) > 0,$$

and will get a contradiction. By the definition of H and by continuity, for some $a \in A$, $\delta > 0$ and $r > 0$ we have

$$n(x) \cdot D\varphi(x) > 0 \quad \& \quad \lambda\varphi(x) - g(x,a) \cdot D\varphi(x) - f(x,a) > \delta \quad \forall x \in \overline{\Omega} \cap B(y,r)$$

Fix $z \in \overline{\Omega} \cap B(y,r)$ and define

$$\tau = \tau(\alpha) = \inf\{t \geq 0 ;\ X(t; z, \alpha) \in \partial B(y,r)\}.$$

~~It is easy to see that τ is a stopping time.~~

We set $\alpha(t) \equiv a$ and $X(t) = X(t; z, \alpha)$. Of course,

$X(t) \in \overline{\Omega} \cap B(y,r)$ for $0 \leq t \leq \tau$, and hence

$$n(X(t)) \cdot D\varphi(X(t)) > 0 \quad \&$$

$$\lambda\varphi(X(t)) - g(X(t),a)\cdot D\varphi(X(t)) - f(X(t),a) > \delta$$

for $0 \le t \le \tau$. Hence,

$$\delta\int_0^\tau e^{-\lambda t}dt \le \int_0^\tau e^{-\lambda t}\{\lambda\varphi(X(t)) - g(X(t),a)\cdot D\varphi(X(t)) - f(X(t),a)\}\,dt =$$

$$= \int_0^\tau e^{-\lambda t}\{\lambda\varphi(X(t)) - \dot X(t)\cdot D\varphi(X(t)) - \xi(t)\cdot D\varphi(X(t)) - f(X(t),a)\}\,dt,$$

(where $\xi \in L^2(0,\tau)$ satisfies $\xi(t) \in D^- I(X(t))$,)

$$\le \int_0^\tau e^{-\lambda t}\{\lambda\varphi(X(t)) - D\varphi(X(t))\cdot\dot X(t) - f(X(t),a)\}\,dt$$

$$= \int_0^\tau \{-\tfrac{d}{dt}(e^{-\lambda t}\varphi(X(t))) - e^{-\lambda t}f(X(t),a)\}\,dt$$

$$= \varphi(z) - e^{-\lambda\tau}\varphi(X(\tau)) - \int_0^\tau e^{-\lambda t}f(X(t),a)\,dt$$

$$\le \varphi(z) - e^{-\lambda\tau}[V^*(X(\tau)) + r^2] - \int_0^\tau e^{-\lambda t}f(X(t),a)\,dt$$

$$\le \varphi(z) - r^2e^{-\lambda\tau} - V(z), \quad \text{by Lemma 3.1.}$$

Thus,

$$V(z) \le \varphi(z) - r^2e^{-\lambda\tau} - \tfrac{\delta}{\lambda}(1-e^{-\lambda\tau})$$
$$\le \varphi(z) - r^2\wedge\left(\tfrac{\delta}{\lambda}\right),$$

and so, $\quad V^*(x) \le \varphi(x) - r^2\wedge\left(\frac{\delta}{\lambda}\right)$; a contradiction.

The proof of the fact that V is a supersolution of (10.6) is quite similar to the above, and we omit giving it here. ▯

<u>Remark</u>. Let $A(x)$ be a real symmetric $N \times N$ matrix for each $x \in \partial\Omega$. Assume that $A(x) \geqslant \gamma I$ for all $x \in \partial\Omega$ and some $\gamma > 0$ and the function $x \to A(x)$ is Lipschitz continuous on $\partial\Omega$. Then we have the same relation as above between the value function

$$V(x) = \inf_{\alpha \in \mathcal{A}} \int_0^\infty e^{-\lambda t} f(X(t), \alpha(t))\, dt,$$

where $\lambda > 0$ and $X(t)$ is the solution of

$$\begin{cases} \dot{X}(t) + A(X(t))\, D^- I(X(t)) \ni g(X(t), \alpha(t)) & (t > 0), \\ X(0) = x, \end{cases}$$

and the boundary problem

$$\begin{cases} \lambda u + \max\limits_{a \in A} \{-g(x,a)\cdot Du - f(x,a)\} = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\gamma(x) = A(x)\, n(x)$ for $x \in \partial\Omega$.

§ 11. The Dirichlet problem for H-J equations

Here we look at the Dirichlet problem

$$(11.1)\qquad \begin{cases} H(x, u, Du) = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases}$$

in the viscosity sense, where $\Omega \subset \mathbb{R}^N$ is open and bounded and $h \in C(\partial\Omega)$. That is, we consider the function F on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ defined by

$$F(x, r, p) = \begin{cases} H(x, r, p) & \text{if } x \in \Omega, \\ h(x) & \text{otherwise}, \end{cases}$$

and regard solutions of $F(x, u, Du) = 0$ in $\overline{\Omega}$ as solutions of (11.1). Assuming $H \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$, a function u on $\overline{\Omega}$ is a subsolution of (11.1) if whenever $\varphi \in C^1(\overline{\Omega})$, $y \in \overline{\Omega}$ and $\max_{\overline{\Omega}} (u^* - \varphi) = (u^* - \varphi)(y)$ then

$$H(y, u^*(y), D\varphi(y)) \le 0 \qquad \text{if } y \in \Omega,$$

$$h(y) \wedge H(y, u^*(y), D\varphi(y)) \le 0 \qquad \text{if } y \in \partial\Omega.$$

We begin with this comparison result:

<u>Theorem 11.1.</u> Assume Ω is bounded and that (H0)–(H2), $(H3)_\partial$ hold. Assume moreover that

(11.2) for each $z \in \partial\Omega$ there is $\eta \in \mathbb{R}^N$ and $b > 0$ such that

$$B(x + t\eta, bt) \subset \Omega \quad \text{for } x \in B(z, b) \cap \overline{\Omega} \text{ and } 0 < t \le b.$$

Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., a subsolution of (11.1) and a supersolution of

$$(11.3) \qquad \begin{cases} H(x, v, Dv) = a & \text{in } \Omega, \\ v = h & \text{on } \partial\Omega, \end{cases}$$

where $a > 0$ is a constant. Then, if u and v are continuous at points of $\partial\Omega$, then $u \le v$ on $\overline{\Omega}$.

<u>Proof</u> We may assume $\max_{\partial\Omega}(u - v) = \max_{\overline{\Omega}}(u - v)$; otherwise the conclusion follows from Theorem 6.0.
We also assume $\max_{\overline{\Omega}}(u - v) > 0$.
Pick $z \in \partial\Omega$ so that $(u - v)(z) = \max_{\overline{\Omega}}(u - v)$.

Case 1: $v(z) < h(z)$. Choose $b > 0$ and $\eta \in \mathbb{R}^N$ so that $B(z + t\eta, tb) \subset \Omega$ for $0 < t \leq b$. For $0 < \varepsilon < b$ consider the function

$$\Phi(x, y) = u(x) - v(y) - \left| \frac{x - y}{\varepsilon} - \eta \right|^2 - |y - z|^2 \quad \text{on } \bar{\Omega} \times \bar{\Omega}.$$

Let $(\bar{x}, \bar{y})$ be a maximum point of Φ. By the choice of η, b, we see that $z + \varepsilon\eta \in \Omega$ and so

$$\Phi(\bar{x}, \bar{y}) \geq \Phi(z + \varepsilon\eta, z).$$

This yields

$$(11.4) \quad \left| \frac{\bar{x} - \bar{y}}{\varepsilon} - \eta \right|^2 + |\bar{y} - z|^2 \leq u(\bar{x}) - v(\bar{y}) - (u(z + \varepsilon\eta) - v(z))$$
$$\leq u(\bar{x}) - v(\bar{y}) - (u - v)(z) + \omega(\varepsilon)$$

by the continuity of u at z, where $\omega(r) \to 0$ as $r \to 0$. Therefore, $\bar{x} - \bar{y} \to 0$ as $\varepsilon \searrow 0$, and hence

$$\overline{\lim_{\varepsilon \searrow 0}} \, [u(\bar{x}) - v(\bar{y})] \leq \max_{\bar{\Omega}} (u - v) = (u - v)(z)$$

by the semicontinuity. Thus

$$\frac{\bar{x} - \bar{y}}{\varepsilon} \to \eta \quad \& \quad \bar{y} \to z \quad \text{as } \varepsilon \searrow 0.$$

Observe that $\bar{x} = \bar{y} + \varepsilon\eta + o(\varepsilon) \in B(\bar{y}+\varepsilon\eta, o(\varepsilon))$ as $\varepsilon \searrow 0$ and so $\bar{x} \in \Omega$ for ε small enough by (11.2). Also, by (11.4) we have

$$v(\bar{y}) = v(z) + u(\bar{x}) - u(z) + \omega(\varepsilon)$$

and hence

$$\overline{\lim_{\varepsilon \searrow 0}}\ v(\bar{y}) \le v(z) \quad (< h(z))$$

by the semicontinuity of u. Thus, we have

$$H\left(\bar{x}, u(\bar{x}), \tfrac{2}{\varepsilon}\left(\tfrac{\bar{x}-\bar{y}}{\varepsilon} - \eta\right)\right) \le 0$$

and

$$H\left(\bar{y}, v(\bar{y}), \tfrac{2}{\varepsilon}\left(\tfrac{\bar{x}-\bar{y}}{\varepsilon} - \eta\right) + 2(z-\bar{y})\right) \geqslant a$$

for ε small enough. Hence

$$-a \geqslant H\left(\bar{x}, u(\bar{x}), \tfrac{2}{\varepsilon}\left(\tfrac{\bar{x}-\bar{y}}{\varepsilon} - \eta\right)\right) - H\left(\bar{y}, v(\bar{y}), \tfrac{2}{\varepsilon}\left(\tfrac{\bar{x}-\bar{y}}{\varepsilon} - \eta\right) + 2(z-\bar{y})\right)$$

$$\geqslant -\sigma(2|z-\bar{y}|) - m\left(\tfrac{2}{\varepsilon}|\bar{x}-\bar{y}|\left|\tfrac{\bar{x}-\bar{y}}{\varepsilon} - \eta\right| + |\bar{x}-\bar{y}|\right)$$

by (H1), (H2), (H3)$_2$ for ε small enough. This is a contradiction.

Case 2: $v(z) \geq h(z)$. As $u(z) > v(z)$, we now have $u(z) > h(z)$. We argue as above with the function

$$(x,y) \to u(x) - v(y) - \left|\frac{x-y}{\varepsilon} + \eta\right|^2 - |x-z|^2 \quad \text{on } \bar{\Omega}\times\bar{\Omega}$$

instead of Φ and get a contradiction. ■

Remarks. (i) Replacing Φ by

$$(x,y) \to u(x) - v(y) - \left|\frac{x-y}{\varepsilon} - \eta\right|^2 - |x-z|^2$$

does not cause any trouble in the argument above for Case 1.

(ii) If we know that u is Lipschitz continuous on $\bar{\Omega}$ and $u \leq h$ on $\partial\Omega$, then we may replace, in the assertion of Theorem 11.1, (H2) by (H2)$'$ (see 75) and $(H3)_\partial$ by the condition that H satisfies (H3) on $U \times \mathbb{R} \times \mathbb{R}^N$ for some neighborhood of $\partial\Omega$. Indeed, in this case we have just one possibility, i.e., Case 1 in the proof of Theorem 11.1 and in the argument for Case 1 we have

$$\Phi(\bar{x},\bar{y}) \geq \Phi(\bar{y}+\varepsilon\eta,\bar{y}),$$

which yields

$$\left|\frac{\bar{x}-\bar{y}}{\varepsilon}-\eta\right|^2 \leq u(\bar{x})-u(\bar{y}+\varepsilon\eta_\circ) \leq L|\bar{x}-\bar{y}-\varepsilon\eta| = \varepsilon L\left|\frac{\bar{x}-\bar{y}}{\varepsilon}-\eta\right|$$

for some constant $L>0$ provided ε is small enough. This estimate

$$\frac{1}{\varepsilon}\left|\frac{\bar{x}-\bar{y}}{\varepsilon}-\eta\right| \leq L$$

takes care of replacing (H2) and (H3)$_\partial$ by the weaker conditions.

(iii) We did not use the continity of v in Case 1 in the above proof. Moreover, the following assumption, instead of the continnity of u at points on $\partial\Omega$, is sufficient to carry through the argument above in Case 1:

(11.5) For each $z\in\partial\Omega$, where $v(z)<h(z)$, there is

$b > 0$, a bounded sequence $\{\eta_n\} \subset \mathbb{R}^N$ and a sequence $\{t_n\} \subset (0, \infty)$ converging to 0 such that

$$\begin{cases} B(x + t_n \eta_n, b t_n) \subset \Omega & \text{for} \quad x \in B(z, b) \cap \overline{\Omega}, \\ \lim_{n \to \infty} u(z + t_n \eta_n) = u(z). \end{cases}$$

We then use

$$\Phi_n(x, y) = u(x) - v(y) - \left| \frac{x - y}{t_n} - \eta_n \right|^2 - |y - z|^2$$

instead of Φ.

As usual we have these corollaries:

Corollary 11.1. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., sub- and supersolutions of

$$(11.6) \quad \begin{cases} u + H(x, u, Du) = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases},$$

Under the same assumptions of Theorem 11.1 on Ω and H, if u and v are continuous at points of $\partial\Omega$, then $u \le v$ on $\overline{\Omega}$.

<u>Corollary 11.2.</u> Let $u \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$ be, resp., sub- and supersolutions of (11.1). In addition to the assumptions of Theorem 11.1 on Ω and H, suppose that $p \to H(x,r,p)$ is convex on $\mathbb{R}^N$ for $(x,r) \in \bar{\Omega} \times \mathbb{R}$ and there is a C^1 function ψ on $\bar{\Omega}$ such that

$$\sup\{H(x,r,D\psi(x)) ; (x,r) \in \bar{\Omega} \times \mathbb{R}\} < 0.$$

Then, if u and v are continuous at points of $\partial\Omega$, then $u \le v$ on $\bar{\Omega}$.

Now let us turn to optimal control problems related to (11.1). Let $A \subset \mathbb{R}^m$, $f : \mathbb{R}^N \times A \to \mathbb{R}$ and $g : \mathbb{R}^N \times A \to \mathbb{R}^N$ be as in §3 and so satisfy

$$(11.7) \quad \begin{cases} |f(x,a)| \le M, & |f(x,a) - f(y,a)| \le M|x-y|, \\ |g(x,a)| \le M, & |g(x,a) - g(y,a)| \le M|x-y| \end{cases}$$

for some constant $M > 0$. We consider again the initial value problem

$$(11.8) \quad \begin{cases} \dot{X} = g(X(t), \alpha(t)) & (t > 0), \\ X(0) = x \end{cases}$$

for $x \in \mathbb{R}^N$ and $\alpha \in \mathcal{A} = \{\alpha : [0, \infty) \to A \text{ measurable}\}$, the solution of which will be denoted by $X(t)$ or $X(t; x, \alpha)$. We define the first exit times τ from Ω and ($\bar{\tau}$ from) $\bar{\Omega}$, resp., by

$$\tau = \tau(x, \alpha) = \inf\{t \geq 0 : X(t; x, \alpha) \in \Omega^c\},$$

$$\bar{\tau} = \bar{\tau}(x, \alpha) = \inf\{t \geq 0 : X(t; x, \alpha) \in \bar{\Omega}^c\}$$

for $x \in \bar{\Omega}$ and $\alpha \in \mathcal{A}$. We consider the problem of minimizing one of these cost functionals

$$J(x, \alpha) = \int_0^\tau e^{-t} f(X(t), \alpha(t))\,dt + e^{-\tau} h(X(\tau)),$$

$$\bar{J}(x, \alpha) = \int_0^{\bar{\tau}} e^{-t} f(X(t), \alpha(t))\,dt + e^{-\bar{\tau}} h(X(\bar{\tau}))$$

for $\alpha \in \mathcal{A}$ and $x \in \bar{\Omega}$. Thus our value functions

(The problem of finding a control which minimizes J or $\bar{J}$ is called the <u>exit time problem</u> in the theory of optimal control.) (163)

are

$$V(x) = \inf_{\alpha \in \mathcal{A}} J(x, \alpha) \quad \& \quad \bar{V}(x) = \inf_{\alpha \in \mathcal{A}} \bar{J}(x, \alpha) \qquad \forall x \in \bar{\Omega}.$$

Clearly, $V = h$ on $\partial\Omega$.

<u>Proposition 11.1.</u> Let (11.7) hold. The value functions V and $\bar{V}$ are solutions of

$$(11.9) \quad \begin{cases} u + H(x, Du) = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases}$$

where $H(x,p) = \max_{a \in A} \{-g(x,a)\cdot p - f(x,a)\}$.

<u>Lemma 11.1.</u> Assume (11.7). Then, for any $\sigma : \mathcal{A} \to [0, \infty]$ and $x \in \bar{\Omega}$ we have

$$V(x) = \inf_{\alpha \in \mathcal{A}} \Big\{ \int_0^{\tau \wedge \sigma} e^{-t} f(X(t), \alpha(t))\, dt + 1_{\{\sigma < \tau\}} e^{-\sigma} V(X(\sigma)) + 1_{\{\sigma \geq \tau\}} e^{-\tau} h(X(\tau)) \Big\},$$

$$\bar{V}(x) = \inf_{\alpha \in \mathcal{A}} \Big\{ \int_0^{\bar{\tau} \wedge \sigma} e^{-t} f(X(t), \alpha(t))\, dt + 1_{\{\sigma < \bar{\tau}\}} e^{-\sigma} V(X(\sigma)) + 1_{\{\sigma \geq \bar{\tau}\}} e^{-\bar{\tau}} h(X(\bar{\tau})) \Big\}.$$

<u>Proof.</u> Although the proof is similar to that of

Lemma 3.1, let us check the latter formula. We denote the right hand side of it by $\overline{V}_\sigma(x)$. Fix $x \in \overline{\Omega}$ and $\alpha \in \mathcal{A}$. If $\sigma(\alpha) < \overline{\tau}(\alpha)$, then

$$\overline{J}(x,\alpha) = \int_0^\sigma e^{-t} f(X(t),\alpha(t))\,dt + \int_\sigma^{\overline{\tau}} e^{-t} f(X(t),\alpha(t))\,dt + e^{-\overline{\tau}} h(X(\overline{\tau}))$$

$$= \int_0^\sigma e^{-t} f(X(t),\alpha(t))\,dt + \int_0^{\overline{\tau}-\sigma} e^{-(t+\sigma)} f(X(t+\sigma),\alpha(t+\sigma))\,dt + e^{-\overline{\tau}} h(X(\overline{\tau})).$$

Set $Y(t) = X(t+\sigma(\alpha))$ and $\beta(t) = \alpha(t+\sigma(\alpha))$ for $t \geq 0$.

Clearly, $Y(t) = X(t;\, X(\sigma(\alpha)),\, \beta)$ and so $\overline{\tau}(X(\sigma),\beta) = \overline{\tau}(x,\alpha) - \sigma(\alpha)$

Thus, if $\sigma(\alpha) < \overline{\tau}(\alpha)$, then

$$\overline{J}(x,\alpha) = \int_0^\sigma e^{-t} f(X(t),\alpha(t))\,dt + e^{-\sigma}\int_0^{\overline{\tau}(X(\sigma),\beta)} e^{-t} f(Y(t),\beta(t))\,dt$$

$$+ e^{-\sigma - \overline{\tau}(X(\sigma),\beta)} h(Y(\overline{\tau}(X(\sigma),\beta))).$$

$$\geq \int_0^\sigma e^{-t} f(X(t),\alpha(t))\,dt + e^{-\sigma}\,\overline{V}(X(\sigma)).$$

On the other hand, if $\sigma(\alpha) \geq \overline{\tau}(\alpha)$, then it is clear that

$$\overline{J}(x,\alpha) = \int_0^{\overline{\tau}} e^{-t} f(X(t),\alpha(t))\,dt + e^{-\overline{\tau}} h(X(\overline{\tau})).$$

These together yield

$$\bar{J}(x,\alpha) \geqslant \bar{V}_\sigma(x) \qquad \forall \alpha \in \mathcal{A}$$

and hence $\bar{V}(x) \geqslant \bar{V}_\sigma(x)$. Now, fix $\alpha, \beta \in \mathcal{A}$.

Writing $X(t) = X(t;x,\alpha)$ and $Y(t) = X(t; X(\sigma(\alpha)), \beta)$, (if $\sigma(\alpha) < \bar{\tau}(x,\alpha)$, then) we have

$$\int_0^{\sigma(\alpha)} e^{-t} f(X(t),\alpha(t))\,dt + e^{-\sigma(\alpha)} \left\{ \int_0^{\bar{\tau}(X(\sigma),\beta)} e^{-t} f(Y(t),\beta(t))\,dt + e^{-\bar{\tau}(X(\sigma),\beta)} h(Y(\bar{\tau}(X(\sigma),\beta))) \right\}$$

$$= \int_0^{\sigma(\alpha)} e^{-t} f(X(t),\alpha(t))\,dt + \int_{\sigma(\alpha)}^{\sigma(\alpha)+\bar{\tau}(X(\sigma),\beta)} e^{-t} f(Y(t-\sigma(\alpha)), \beta(t-\sigma(\alpha)))\,dt + e^{-[\sigma(\alpha)+\bar{\tau}(X(\sigma),\beta)]}\, h(Y(\bar{\tau}(X(\sigma),\beta))),$$

(setting

$$Z(t) = \begin{cases} X(t) & (0 \leqslant t < \sigma(\alpha)) \\ Y(t-\sigma(\alpha)) & (\sigma(\alpha) \leqslant t < \infty) \end{cases},$$

$$\gamma(t) = \begin{cases} \alpha(t) & (0 \leqslant t < \sigma(\alpha)) \\ \beta(t-\sigma(\alpha)) & (\sigma(\alpha) \leqslant t < \infty) \end{cases}$$

and noting that $Z(t) = X(t;x,\gamma)$ and $\bar{\tau}(x,\gamma) = \sigma(\alpha) + \bar{\tau}(X(\sigma),\beta)$,

we continue)

$$= \int_0^{\bar{\tau}(x,\gamma)} e^{-t} f(Z(t),\gamma(t))\,dt + e^{-\bar{\tau}(x,\gamma)} h(Z(\bar{\tau}(x,\gamma)))$$

$$\geqslant \bar{V}(x).$$

Thus, if $\sigma(\alpha) < \bar{\tau}(x, \alpha)$, then

$$\bar{V}(x) \leq \int_0^{\sigma(\alpha)} e^{-t} f(X(t), \alpha(t))\,dt + e^{-\sigma(\alpha)} \bar{V}(X(\sigma)) \qquad \forall \alpha \in \mathcal{A}.$$

Therefore,

$$\bar{V}(x) \leq \int_0^{\bar{\tau}(\alpha) \wedge \sigma(\alpha)} e^{-t} f(X(t), \alpha(t))\,dt + 1_{\{\sigma < \bar{\tau}\}} e^{-\sigma} \bar{V}(X(\sigma))$$
$$+ 1_{\{\sigma \geq \bar{\tau}\}} e^{-\bar{\tau}} h(X(\bar{\tau})) \qquad \forall \alpha \in \mathcal{A},$$

and so $\bar{V}(x) \leq \bar{V}_\sigma(x)$. ∎

<u>Proof of Proposition 11.1</u> We will only check that $\bar{V}$ is a supersolution of (11.9). Let $\varphi \in C^1(\bar{\Omega})$ and assume $\min_{\bar{\Omega}} (\bar{V}_* - \varphi) = (\bar{V}_* - \varphi)(y)$ for some $y \in \bar{\Omega}$. If $y \in \Omega$, then the previous argument applies. We may therefore assume $y \in \partial\Omega$. We argue by contradiction, and so suppose

$$\bar{V}_*(y) < h(y) \quad \& \quad \bar{V}_*(y) + H(y, D\varphi(y)) < 0.$$

We may also assume $\bar{V}_*(y) = \varphi(y)$ (and that $(\bar{V}_* - \varphi)(x) \geq |x-y|^2$ for $x \in \bar{\Omega}$). By continuity there is $\delta > 0$ such that

$$(\varphi - h)(x) \leq -\delta \qquad \text{for} \quad x \in B(y,\delta) \cap \partial\Omega,$$

$$\varphi(x) + H(x, D\varphi(x)) \leq -\delta \qquad \text{for} \quad x \in B(y,\delta) \cap \overline{\Omega}.$$

Fix $z \in \overline{\Omega} \cap B(y, \frac{\delta}{2})$.

Define $\sigma(\alpha) = \inf\{t \geqslant 0 : X(t; z, \alpha) \notin B(y,\delta) \cap \overline{\Omega}\}$ for $\alpha \in \mathcal{A}$.

Clearly, $\sigma \leq \bar{\tau}$. Fix any $\alpha \in \mathcal{A}$. By the above inequality, for $X(t) = X(t; z, \alpha)$ we have

$$-\delta \int_0^\sigma e^{-t} dt \geqslant \int_0^\sigma e^{-t}\{\varphi(X(t)) - g(X(t), \alpha(t)) \cdot D\varphi(X(t)) - f(X(t), \alpha(t))\} dt =$$

$$= \int_0^\sigma \left\{-\frac{d}{dt} e^{-t}\varphi(X(t)) - e^{-t} f(X(t), \alpha(t))\right\} dt$$

$$= \varphi(z) - e^{-\sigma}\varphi(X(\sigma)) - \int_0^\sigma e^{-t} f(X(t), \alpha(t))\, dt$$

$$\leq \begin{cases} \varphi(z) + e^{-\sigma}\left[\frac{\delta^2}{4} - \overline{V}_*(X(\sigma))\right] - \int_0^\sigma e^{-t} f(X(t), \alpha(t))\, dt & (\sigma < \bar{\tau}) \\ \varphi(z) + e^{-\sigma}\left[\delta - h(X(\sigma))\right] - \int_0^\sigma e^{-t} f(X(t), \alpha(t))\, dt & (\sigma = \bar{\tau}). \end{cases}$$

Thus,

$$\varphi(z) \leqslant -\delta\left\{\int_0^\sigma e^{-t} dt + \left(\tfrac{\delta}{4} \wedge 1\right) e^{-\sigma}\right\} + \int_0^{\sigma\wedge\bar{\tau}} e^{-t} f(X(t), \alpha(t))\, dt$$

$$+ 1_{\{\sigma < \bar{\tau}\}}\, e^{-\sigma}\, \overline{V}(X(\sigma)) + 1_{\{\sigma \geqslant \bar{\tau}\}}\, e^{-\bar{\tau}} h(X(\bar{\tau})).$$

Hence, by Lemma 11.1 we conclude

$$\varphi(z) \leqslant -\varepsilon + \overline{V}(z) \qquad \text{for } z \in B(y, \tfrac{\delta}{2}) \cap \overline{\Omega} \text{ and some } \varepsilon > 0$$

whence $\bar{V}_*(y) > \varphi(y)$, a contradiction. ▩

Next we consider the existence question of a continuous solution to problem (11.1). It is not easy to answer this question unlike in the Neumann problem case since our comparison theorem (Theorem 11.1) does not applies to (possibly) discontinuous solutions. In fact, we can not expect such generality in comparison assertions as the following examples show.

<u>Example</u>. Let $N = 2$.

Let Ω be a domain of $\mathbb{R}^2$ like in Fig. a. Let $h \in C(\partial\Omega)$ satisfy

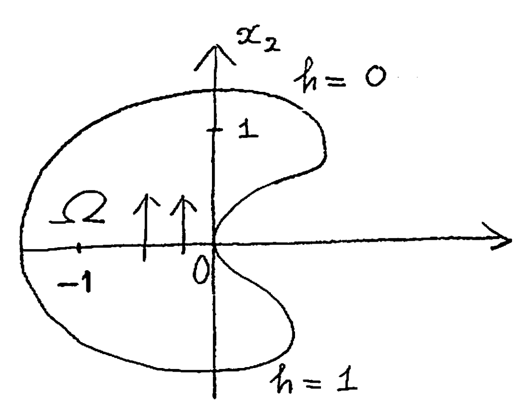


Fig. a

$$\begin{cases} h(x) = 0 & \text{if} \quad x_1 \geq -1 \quad \text{and} \quad x_2 \geq 1, \\ h(x) = 1 & \text{if} \quad x_2 \leq 0. \end{cases}$$

Consider the Dirichlet problem

$$(11.10)\qquad \begin{cases} u - u_{x_2} = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases}$$

The following exit time problem is associated with this problem: Our state equation is given by

$$\dot{X}(t) = (0,1), \quad X(0) = x \in \Omega \quad (\text{i.e. } g(x) = (0,1)),$$

and the value function is given by

$$V(x) = e^{-\tau} h(X(\tau)) \qquad (\text{i.e. } f \equiv 0),$$

where $\tau = \inf\{t \geq 0 : X(t) \notin \Omega\}$. By Prop. 11.1 we see that V is a solution of (11.10). Also, it is easy to see that ,

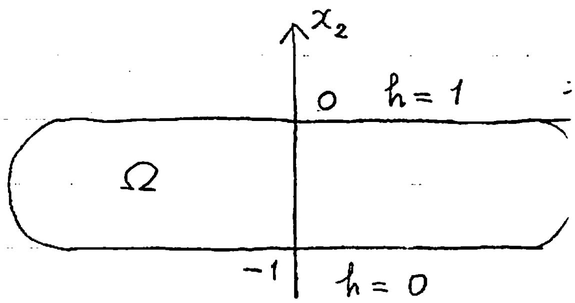
$$V(x) = 0 \quad \text{if} \quad x_1 < 0 \quad \text{and} \quad x_1 \sim 0,$$

$$V(x) \sim 1 \quad \text{if} \quad x_1 \geq 0, \quad x_2 \leq 0 \quad \text{and} \quad x \sim 0,$$

and $V(x)$ has discontinuities along the line $x_1 = 0$,

$x_2 < 0$. If Theorem 11.1 were true without the continuity assumption on sub- and supersolutions then it (together with Perron's method) would tell us that $V^* \leq V_*$, which is obviously impossible.

Example. Let $N = 2$. Let Ω be a domain of $\mathbb{R}^2$ as in Fig. 6.



We consider the exit time problem with $A = \{1, 2\}$,

$$g(x,1) = (0, x_1), \qquad g(x,2) = (0, |x_1|),$$

$$f(x,1) = f(x,2) = 0, \quad \text{and} \quad h \in C(\partial\Omega) \text{ satisfying}$$

$$h(x) = \begin{cases} 1 & \text{if } x_2 = 0 \\ 0 & \text{if } x_2 = -1. \end{cases}$$

Then, it is easy to see that the value $\overline{V}(x)$ is calculated in a neighborhood of the line $x_1 = 0$ as follows:

$$\overline{V}(x) = 0 \quad \text{if } x_1 \leq 0,$$

$$\overline{V}(x) = e^{x_2/x_1} \qquad \text{if } x_1 > 0.$$

Thus, $\overline{V}$ is not continuous at 0 but continuous in Ω near $x = 0$.

We will give a sufficient condition under which we have the existence of a continuous solution of (11.1). Before that, let us recall the following nice situation: For each $L > 0$

(11.11) $$\lim_{R\to\infty} \sup\{H(x,r,p) : (x,r,p) \in \overline{\Omega}\times[-L,L]\times\mathbb{R}^N,\ |p| \geq R\} > 0.$$

In this case, if a subsolution u of (11.1) is u.s.c. on $\overline{\Omega}$ and $u \leq h$, then the argument in the proof of Example on p. 55 is employed to show that $u \in C(\overline{\Omega})$. Our argument below refines this observation. Our assumption is:

(A.1) For each $z \in \partial\Omega$ there are closed convex cones K_i and K_e in $\mathbb{R}^N$ with vertex at the origin, and a constant $b > 0$ such that

$K_i^\circ \neq \emptyset$, $K_e^\circ \neq \emptyset$,

$\xi \cdot p_i < 0$ for $\xi \in K_i \setminus \{0\}$ and some $p_i \in \mathbb{R}^N$,

$\xi \cdot p_e < 0$ for $\xi \in K_e \setminus \{0\}$ and some $p_e \in \mathbb{R}^N$,

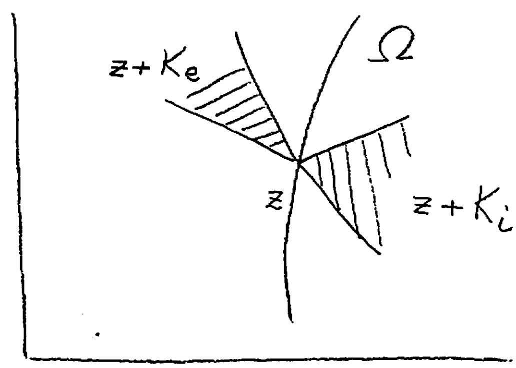
$B(z,b) \cap (x + K_i) \subseteq \Omega$ for $x \in B(z,b) \cap \Omega$,

$B(z,b) \cap (z + K_e) \cap \overline{\Omega} = \{z\}$,

and for each $L > 0$ there are constants R, $C_L > 0$ for which

$$(11.12) \quad H(x,r,p) \geq -C_L + \min_{\xi \in K_e \cap S^{N-1}} (-\xi \cdot p) \vee \min_{\xi \in K_i \cap S^{N-1}} (-\xi \cdot p)$$

for $(x,r) \in \overline{\Omega} \cap B(z,b) \times [-L,L]$ and $p \in \mathbb{R}^N$ with $|p| \geq R$.

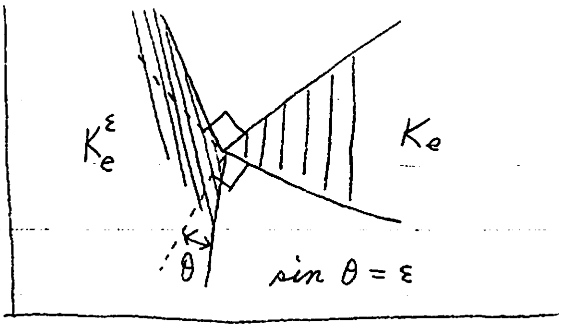


An equivalent way to state that (11.12) holds for some $C_L > 0$ and $R > 0$ is this:

$$(11.13) \quad \limsup_{R \to \infty} \{ H(x,r,p)/R : (x,r) \in \overline{\Omega} \cap B(z,b) \times [-L,L],\ p \in K_e^\varepsilon \cup K_i^\varepsilon,$$

$$|p| \geq R\} \geq \varepsilon,$$

for small $\varepsilon > 0$, where $K_e^\varepsilon = \{p \in \mathbb{R}^N : \xi \cdot p \leq -\varepsilon|\xi||p|, \forall \xi \in K_e\}$ and $K_i^\varepsilon = \{p \in \mathbb{R}^N : \xi \cdot p \leq -\varepsilon|\xi||p|, \forall \xi \in K_i\}$.



Observe also that if u is a subsolution of (11.1), then it is also a subsolution of (11.1) with H replaced by $(x, r, p) \to |p| H^+(x, r, p)$. That is, as far as subsolutions are concerned, the requirement (11.11) on H is essentially the same as assuming

$$\lim_{R\to\infty} \sup\{H(x,r,p)/R : (x,r,p) \in \overline{\Omega} \times [-L, L] \times \mathbb{R}^N, |p| \geq R\} > 0.$$

This argument shows that (A.1) is essentially a weaker condition than (11.11).

<u>Lemma 11.2.</u> Assume $h \in C(\partial\Omega)$ and (H0), (H1), (A.1). Let u be a subsolution of (11.1). Then $u^* \leq h$ on $\partial\Omega$.

<u>**Proof.**</u> Fix $z \in \partial\Omega$. Choose $b > 0$ and a closed

cone K_e as in (A.1), and $p_0 \in \mathbb{R}^N$ so that $\xi \cdot p_0 \leq -1$ for $\xi \in K_e \cap S^{N-1}$. We set $d(x) = \{\mathrm{dist}(x, z + K_e)\}^2$. Then, $d \in C^{1,1}(\mathbb{R}^N)$ and $Dd(x) \subset \{p \in \mathbb{R}^N : p \cdot \xi \leq 0, \forall \xi \in K_e\}$ by Lemma 8.6 on p. 108. Fix $\varepsilon > 0$. For $A, B > 0$ we set

$$v(x) = h(z) + \varepsilon + A p_0 \cdot (x - z) + B d(x).$$

Let $L = |h(z)| \vee \max\{|u^*(x)| : x \in \overline{\Omega} \cap B(z, r)\}$. Then we have

$$H(x, v(x), Dv(x)) \geq H(x, -L, Dv(x)) \geq -C_L + \min_{\xi \in K_e \cap S^{N-1}} [-\xi \cdot (A p_0 + B Dd(x))]$$

$$\geq -C_L + \min_{\xi \in K_e \cap S^{N-1}} (-A \xi \cdot p_0) \geq -C_L + A$$

if $\varepsilon + A p_0 \cdot (x - z) + B d(x) \geq 0$ and $|A p_0 + B Dd(x)| \geq R$, where R is from (A.1). Note that $A \leq -\xi \cdot (A p_0 + B Dd(x)) \leq |A p_0 + B Dd(x)|$ for $x \in \mathbb{R}^N$ and $\xi \in K_e \cap S^{N-1}$. We fix $A = R \vee (C_L + 1)$, and choose B so large that $h(x) \leq v(x)$ for $x \in B(z, \delta) \cap \partial\Omega$, $u^*(x) \leq v(x)$ for $x \in \Omega \cap \partial B(z, \delta)$ and $\varepsilon + A p_0 \cdot (x - z) + B d(x) \geq 0$ for $x \in B(z, \delta) \cap \Omega$. Set

$\omega = B(z, b)^\circ \cap \Omega$. Then we have

$$H(x, v(x), Dv(x)) \geq 1 \quad \text{in } \omega \quad \& \quad |Dv(x)| \geq R \quad \text{on } \overline{\omega}.$$

Define $\tilde{h} \in C(\partial\omega)$ by $\tilde{h} = v$. Clearly, v is a supersolution of (11.3) with $\tilde{h}$, 1 and ω in place of h, a and Ω, and u^* is a subsolution of (11.1) with $\tilde{h}$ and ω in place of h and Ω. Arguing by contradiction and taking advantage of the C^1 regularity of v, we easily conclude that $u^* \leq v$ on $\overline{\omega}$. Thus we see that $u^*(z) \leq h(z) + \varepsilon$ for $\varepsilon > 0$. ∎

<u>Lemma 11.3.</u> Assume (H0), (H1), (A.1). Let u be a subsolution of (11.1). Let $z \in \partial\Omega$. Then

$$(11.14) \qquad u^*(x) \leq \max\{u^*(y) : y \in (x + K_i) \cap B(x, r)\} + Cr$$

for $x \in \Omega \cap B(z, \frac{b}{2})$ and $0 < r < \frac{b}{2}$ and some constant $C > 0$, where K_i and b is from (A.1).

<u>Proof.</u> Choose $\xi_0 \in K_i \cap S^{N-1}$, and $p_0 \in \mathbb{R}^N$ so that

$p_0 \cdot \xi \le -1$ for $\xi \in K_i$. Fix $y \in \Omega \cap B(z, \frac{b}{2})$ and $0<r<\frac{b}{2}$. Choose $\varepsilon > 0$ so small that $y - \varepsilon \xi_0 \in \Omega \cap B(y, \frac{b}{2} - r)$. We write $y_\varepsilon = y - \varepsilon \xi_0$. For $0 < \delta < 1$ we set $K_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, y_\varepsilon + K_i) < \delta\}$. We fix δ small enough so that $K_\delta \cap B(z, b) \subset \Omega$, and set $\omega = K_\delta \cap B(y_\varepsilon, r)$.

For $A, B > 0$ we set

$$d(x) = \{\text{dist}(x, y_\varepsilon + K_i)\}^2,$$

$$m_\delta = \max\{u^*(x) : x \in K_\delta \wedge \partial B(y_\varepsilon, r)\},$$

$$v(x) = m_\delta + A p_0 \cdot (x - y_\varepsilon) + B d(x) + A|p_0| r.$$

We have

$$H(x, v(x), Dv(x)) \geq H(x, -|m_\delta|, Dv(x))$$

$$\geq -C_L + \min_{\xi \in K_i \cap S^{N-1}} \left[-\xi \cdot (A p_0 + B Dd(x))\right] \geq -C_L + A,$$

for $x \in \omega$, with $L = \sup_{0<\delta<1} |m_\delta|$, if $|Ap_0 + BDd(x)| \geq R$, where R is from (A.1). We fix $A = (C_L + 1) \vee R$. (Notice that $|Dv(x)| \geq A$ for $x \in \mathbb{R}^N$ and $A, B > 0$.)

Fix $B > 0$ so large that $u^*(x) \le v(x)$ for $x \in \partial K_\delta \cap B(y_\varepsilon, r)$. Clearly, $u^*(x) \le v(x)$ for $x \in K_\delta \cap \partial B(y_\varepsilon, r)$. Therefore, $u^* \le v$ on $\partial\omega$. By using the standard comparison argument, we see that $u^* \le v$ in ω. This shows in particular that

$$u^*(y) \le m_\delta + \varepsilon A|p_0||\xi_0| + A|p_0|\, r.$$

(Notice that $y = y_\varepsilon + \varepsilon\xi_0 \in y_\varepsilon + K_i$.) Sending $\delta \searrow 0$ and $\varepsilon \searrow 0$, we obtain

$$u^*(y) \le \max\{u^*(x) : x \in (y + K_i) \cap \partial B(y, r)\} + A|p_0|\, r. \qquad \blacksquare$$

Using these two lemmas, we have the following.

<u>Theorem 11.2.</u> Assume $h \in C(\partial\Omega)$ and (H0), (H1), (A.1). Let u be a subsolution of (11.1). Define

$$\bar{u}(x) = \lim_{r \searrow 0} \sup\{u(y) : y \in \Omega \cap B(x, r)\} \quad \text{for } x \in \bar{\Omega}. \tag{11.15}$$

Then $\bar{u}$ is a subsolution of (11.1), $\bar{u} \in USC(\bar{\Omega})$ and $\bar{u} \le h$ on $\partial\Omega$. Moreover, for each $z \in \partial\Omega$ there are constants b, $C > 0$ and a closed cone K of $\mathbb{R}^N$ with vertex at 0 such that

(11.16) $\bar{u}(z) \le \max\{\bar{u}(y) : y \in (z+K) \cap B(z,r)\} + Cr$

for $0 < r \le b$, and

$$(x+K) \cap B(z,b) \subset \Omega \quad \text{for } x \in \Omega \cap B(z,b).$$

Proof. Define u^* as usual, i.e.

$$u^*(x) = \lim_{r \searrow 0} \sup\{u(y) : y \in B(x,r) \cap \bar{\Omega}\} \quad \text{for } x \in \bar{\Omega}.$$

Clearly, we have

$$\bar{u} = u^* \text{ in } \Omega \quad \text{and} \quad \bar{u} \le u^* \text{ on } \partial\Omega.$$

Hence, $\bar{u} \le h$ on $\partial\Omega$ by Lemma 11.2 and so $\bar{u}$ is a subsolution of (11.1). By the definition, we have $\bar{u} \in USC(\bar{\Omega})$. Fix $z \in \partial\Omega$. Choose $\{y_n\} \subset \Omega$ so that $y_n \to z$ and $\bar{u}(y_n) \to \bar{u}(z)$ as $n \to \infty$. Let K_i and b be as in (A.1). By Lemma 11.3 we have

$\bar{u}(y_n) \le \max\{\bar{u}(y) : y \in (y_n + K_i) \cap B(y_n, r)\} + Cr$

for $0 < r < \frac{b}{2}$ if n is large enough. Send $n \to \infty$, to obtain

$$\bar{u}(z) \leq \max\{\bar{u}(y) : y \in (z + K_i) \cap B(z, r)\} + Cr$$

for $0 < r < \frac{b}{2}$. We also have

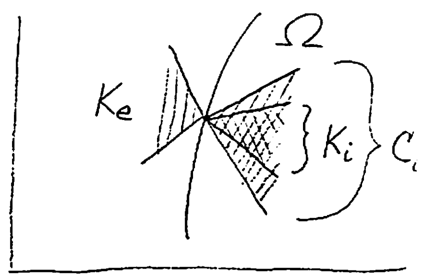
$$(x + K_i) \cap B(z, b) \subset \Omega \qquad \text{for } x \in \Omega \cap B(z, b).$$

We now introduce a condition which is slightly stronger than (A.1). We, in addition, assume in condition (A.1) that there is an open convex cone C_i with vertex at the origin such that

$$K_i \subset C_i \cup \{0\} \qquad \text{and}$$

$$(x + C_i) \cap B(z, b) \subset \Omega \qquad \text{for } x \in \overline{\Omega} \cap B(z, b).$$

We refer this new condition to (A.2).



<u>Theorem 11.3</u>. Assume $h \in C(\partial\Omega)$, (H0) – (H2), $(H3)_\partial$ and (A.2), and that Ω is bounded. Then there is a continuous solution of (11.6).

Proof. For $A>0$ large enough, $g(x)\equiv A$ and $f(x)\equiv -A$ are, resp., sub- and supersolution of (11.6). By Perron's method, we have a solution u of (11.6).

Let u_* be the l.s.c. envelope of u and $\bar{u}$ be defined by (11.15). Clearly, we have $u_* \leq \bar{u}$ on $\bar{\Omega}$.

If we replace (A.1) by (A.2) in Theorem 11.2, then we have a stronger conclusion; i.e. we may assume there is an open cone $\hat{K}$ of $\mathbb{R}^N$ with vertex at the origin such that $K \subset \hat{K} \cup \{0\}$ and $(x+\hat{K}) \cap B(z,b) \subset \Omega$ for $x \in \bar{\Omega} \cap B(z,b)$. This guarantees that for $n \in \mathbb{N}$ small enough (and $z \in \partial\Omega$) there is $\eta_n \in S^{N-1} \cap (z+K)$ such that

$$\bar{u}(z) = \lim_{n\to\infty} \bar{u}(z+\tfrac{1}{n}\eta_n),,$$

$$B(x+\tfrac{1}{n}\eta_n, \tfrac{b}{n}) \subset \Omega \qquad \text{for all } x \in \bar{\Omega} \cap B(z,b)$$

and some $b>0$. By Remark (iii) after Theorem 11.1 we conclude that $\bar{u} \leq u_*$ on $\bar{\Omega}$. Thus $\bar{u} = u_*$ on

$\overline{\Omega}$, and hence $\overline{u}$ is continuous on $\overline{\Omega}$ and it is a supersolution of (11.6).

<u>Remark</u>. (i) We probably need not to assume (H3), in Theorem 11.3. See Remark (ii) after Theorem 11.1 and the estimate (11.16). (ii) The above proof shows that any solution v of (11.6) satisfies

$$v = \overline{u} \quad \text{in } \Omega \quad \text{and} \quad \overline{u} \leq v \leq h \quad \text{on } \partial\Omega$$

and any function v satisfying this condition is a solution of (11.6).

Let us go back to the exit time problems.

We assume:

(11.17) For each $z \in \partial\Omega$ there are $a_1, a_2 \in A$ and $b > 0$ such that $g(z, a_1) \neq 0$, $g(z, a_2) \neq 0$,

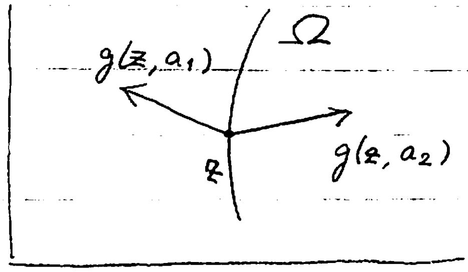
$$B(z + t g(z, a_1), bt) \subset \Omega^c \quad \text{for } 0 < t \leq b,$$

$$B(x + t g(z, a_2), bt) \subset \Omega \quad \text{for } 0 < t \leq b, \ x \in B(z, b) \cap \overline{\Omega}.$$

Then, under the assumption (11.7), we have

$$r + \max_{a \in A}\left(-g(x,a)\cdot p - f(x,a)\right) \geqslant -(L + M)$$
$$+ \min\{-\xi\cdot p : \xi \in B(g(z,a_1), \cancel{M}\varepsilon)\} \vee \min\{-\xi\cdot p \mid \xi \in B(g(z,a_2), \cancel{M}\varepsilon)\}$$

for $(x, r) \in \overline{\Omega} \cap B(z, \varepsilon) \times [-L, L]$, with $L > 0$. Therefore, (11.7) and (11.17) imply (A.2).



<u>Theorem 11.4</u>. Assume (11.7) and (11.17). (i) V is the unique solution of (11.9) which satisfies $V = h$ on $\partial\Omega$. (ii) $\overline{V}$ is the unique solution of (11.9) in $C(\overline{\Omega})$. (iii) V is the maximum and $\overline{V}$ is the minimum among solutions of (11.9).

<u>Proof</u>. It will be enough to show $\overline{V} \in C(\overline{\Omega})$. To do this, it suffices to prove $u \leq \bar{u}$ on $\partial\Omega$, where $u = \overline{V}$ and $\bar{u}$ is defined by (11.15). Fix any $z \in \partial\Omega$. We suppose $\bar{u}(z) < u(z)$. By continuity

there is $\delta > 0$ such that $\bar{u}(x) \leq u(z) - \delta$ for $x \in \bar{\Omega} \cap B(z, \delta)$. Let $\alpha_2(t) \equiv a_2$ for $t \geq 0$ and $\varepsilon > 0$ be sufficiently small. Then, by Lemma 11.1, we have

$$u(z) \leq \int_0^{\bar{\tau} \wedge \varepsilon} e^{-t} f(X(t), \alpha_2(t))\,dt + 1_{\{\varepsilon < \bar{\tau}\}}\, e^{-\varepsilon}\, u(X(\sigma))$$

$$+ 1_{\{\varepsilon \geq \bar{\tau}\}}\, e^{-\bar{\tau}}\, h(X(\bar{\tau})),$$

where $X(t) = X(t, z, \alpha_2)$ and $\bar{\tau} = \inf\{t \geq 0 : X(t) \notin \bar{\Omega}\}$

We may assume $X(t) \in \Omega$ for $0 < t \leq 2\varepsilon$, and so $\bar{\tau} > \varepsilon$ and $X(\bar{\tau}) \in \Omega$. Thus

$$u(z) \leq \int_0^{\varepsilon} e^{-t} f(X(t), \alpha_2(t))\,dt + e^{-\varepsilon}(u(z) - \delta);$$

this is a contradiction. That is, we have $\bar{u}(z) \geq u(z)$.