Part II Second order elliptic PDE's

§1. Degenerate elliptic PDE's and viscosity solutions

In what follows we study equations of the form

(1.1) $$F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$, $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$ ($\mathbb{S}^N$ denotes the space of $N \times N$ real symmetric matrices and $D^2u = (\partial^2 u/\partial x_i \partial x_j)_{1 \le i, j \le N}$.

First of all we recall the definition of degenerate ellipticity. Equation (1.1), the functio[n] F or operator F are called (degenerate) elliptic if whenever $B \in \mathbb{S}^N$ satisfies $B \geq 0$ (i.e., $B\xi \cdot \xi \geq 0$ $\forall \xi \in \mathbb{R}^N$), then

(1.2) $$F(x, r, p, A+B) \leq F(x, r, p, A)$$

holds for $(x, r, p, A) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$. Equations

$$-\Delta u = f(x), \qquad u_t - \Delta u = 0, \qquad F(x, u, Du) = 0$$

are examples of elliptic equations.

The notion of viscosity solution is readily generalized to 2nd order PDE's (1.1). Let $u: \Omega \to \mathbb{R}$ and $y \in \Omega$. The 2nd order superdifferential $D_2^+ u(y)$ and subdifferential $D_2^- u(y)$ of u at y are defined by

$$D_2^+ u(y) = \{ (p, A) \in \mathbb{R}^N \times \mathbb{S}^N : \quad u(x) \leq u(y) + p \cdot (x-y) + \tfrac{1}{2} A(x-y)\cdot(x-y) + o(|x-y|^2) \quad \text{as } x \in \Omega \to y \},$$

$$D_2^- u(y) = \{ (p, A) \in \mathbb{R}^N \times \mathbb{S}^N : \quad u(x) \geq u(y) + p \cdot (x-y) + \tfrac{1}{2} A(x-y)\cdot(x-y) + o(|x-y|^2) \quad \text{as } x \in \Omega \to y \}.$$

A function $u : \Omega \to \mathbb{R}$ is called a (viscosity) subsolution of (1.1) if u is locally bounded in Ω and

$$F_*(y, u^*(y), p, A) \leq 0 \qquad \forall (p, A) \in D_2^+ u^*(y).$$

Similarly, $u : \Omega \to \mathbb{R}$ is called a (viscosity) supersolution of (1.1) if u is locally bounded in Ω and

$$F^*(y, u_*(y), p, A) \geq 0 \qquad \forall (p, A) \in D_2^- u_*(y).$$

A (viscosity) solution of (1.1) is a function in Ω which is both a sub- and supersolution of (1.1).

We have another equivalent definition of viscosity solution as before. For instance, a locally bounded $u : \Omega \to \mathbb{R}$ is a subsolution of (1.1) if and only if whenever $y \in \Omega$, $\varphi \in C^2(\{y\})$ and $\max(u^* - \varphi) = (u^* - \varphi)(y)$, then

$$F_*(y, u^*(y), D\varphi(y), D^2\varphi(y)) \leq 0.$$

This is proved as in the proof of Prop. 2.1 with the following observation: Let $\omega \in C[0, \infty)$ be non-decreasing and satisfy $\omega(0) = 0$. If we set

$$\rho(r) = \int_0^r dt \int_0^t \omega(s)\,ds \qquad \text{for} \quad r \geq 0$$

and $\quad \varphi(x) = \rho(4|x|) \quad$ for $x \in \mathbb{R}^N$, then $\varphi \in C^2(\mathbb{R}^N)$

and $\quad \varphi(0) = 0, \quad D\varphi(0) = 0, \quad D^2\varphi(0) = 0 \quad$ and

$$\varphi(x) \geq \int_{2|x|}^{4|x|} dt \int_{t/2}^{t} \omega(s)\,ds \geq 2|x| \int_{|x|}^{2|x|} \omega(s)\,ds \geq 2|x|^2 \omega(|x|).$$

Basic properties of viscosity solutions for 1st order PDE's are preserved for those for 2nd order PDE's. Especially, Propositions 4.2 (the stability of subsolutions under taking a pointwise supremum) and 4.3 (the stability of a nonincreasing sequence of u.s.c. subsolutions under the pointwise limit) and Theorem 5.1 (Perron's method) are valid for viscosity solutions of 2nd order PDE's (1.1). The proofs of those assertions for 1st order PDE's are easily adapted to yield the corresponding assertions for 2nd order PDE's. The only minor change needed is that we now assume

$$(u^* - \varphi)(x) \le -|x-y|^4$$

instead of assuming $(u^* - \varphi)(x) \le -|x-y|^2$, which comes from the fact that if φ is a C^2 function near y, $\varphi_1(x) = \varphi(x) + |x-y|^2$ and $\varphi_2(x) = \varphi(x) + |x-y|^4$, then $D^2\varphi(y) = D^2\varphi_2(y) \neq D^2\varphi_1(y)$. With this observation the existence question reduces to the question of comparison of solutions.

The generalization of the notion of viscosity solution to 2nd order PDE's is due to P. L. Lions, who also obtained general comparison results for solutions of (1.1) with $F(x, r, p, A)$ being convex in (r, p, A), i.e. the so-called Bellman equations. The extensions of the comparison results to equations (1.1) with nonconvex functions F were done quite recently. The key observation in the proof by P.L. Lions is that subsolutions of (1.1) with convex F are approximated by C^2 subsolutions of approximate equations, which seems the characteristic of equations having the convexity. On the other hand, the arguments in the proof of comparison results in the first order case seemed to break down because of the classical maximum principle. To see this, let us take the equation

$$u - \Delta u = f(x) \quad \text{in } \Omega,$$

and let u and v be, resp., sub- and super-solutions of the above equation. Let $w(x,y) = u(x) - v(y)$. Taking the difference of the equations for u and v, we get

$$w - \Delta_{x,y} w = f(x) - f(y) \quad \text{in } \Omega \times \Omega,$$

where $\Delta_{x,y} = \sum_{i=1}^{N} \partial^2/\partial x_i^2 + \sum_{i=1}^{N} \partial^2/\partial y_i^2$. Our argument then requires the existence of a C^2 function z on $\overline{\Omega} \times \overline{\Omega}$ which satisfies

$$z - \Delta_{x,y} z \geq C|x-y| \quad \text{in } \Omega \times \Omega,$$

$$z(x,x) \leq \varepsilon \quad \text{for } x \in \Omega,$$

$$z(x,y) \geq C|x-y| \quad \text{for } (x,y) \in \partial(\Omega \times \Omega),$$

where $\varepsilon > 0$ is an arbitrary number and $C > 0$ is a constant for which $f(x) - f(y) \leq C|x-y|$ for $x, y \in \Omega$ and $u(x) - v(y) \leq C|x-y|$ for $(x,y) \in \partial(\Omega \times \Omega)$ (assuming these inequalities hold for some $C > 0$). We can not expect the existence of such a function z in general in view of the classical maximum principle.

The breakthrough was brought by R. Jensen in this difficult situation. He observed that the classical maximum principle (the Aleksandrov - Bakelman - Bony - Pucci theorem) applies to semi-convex solutions and also showed how to approximate a subsolution (resp. supersolution) of (1.1) by a semi-convex subsolution (resp. semi-concave supersolution) of an approximate equation.

§2 Sup- and inf convolutions

We recall here the definition of sup- and infconvolutions of a function and their properties. Let $E \subset \mathbb{R}^N$ and $u : E \to \mathbb{R}$ be a bounded function (for simplicity). For $\varepsilon > 0$ we set

$$(2.1)\quad u^\varepsilon(x) = \sup\{u(y) - \tfrac{1}{2\varepsilon}|x-y|^2 : y \in E\} \qquad \forall x \in \mathbb{R}^N,$$

$$(2.2)\quad u_\varepsilon(x) = \inf\{u(y) + \tfrac{1}{2\varepsilon}|x-y|^2 : y \in E\} \qquad \forall x \in \mathbb{R}^N.$$

The functions u^ε and u_ε are called, resp., the <u>sup</u>- and <u>infconvolutions</u> of u. They are also called the <u>Moreau-Yosida</u> <u>approximations</u>. The functions

$$x \to u(y) \mp \tfrac{1}{2\varepsilon}|x-y|^2 \pm \tfrac{1}{2\varepsilon}|x|^2$$

are affine functions in $\mathbb{R}^N$, and hence

$$x \to u^\varepsilon(x) + \tfrac{1}{2\varepsilon}|x|^2 \quad \text{is convex in } \mathbb{R}^N \text{ and}$$

$$x \to u_\varepsilon(x) - \tfrac{1}{2\varepsilon}|x|^2 \quad \text{is concave in } \mathbb{R}^N.$$

Notice also that $u_\varepsilon = -(-u)^\varepsilon$; this formula translates properties of supconvolutions into those of inf-convolutions. The above convexity, i.e. the semi-convexity of u^ε, implies that u^ε is locally Lipschitz continuous in $\mathbb{R}^N$. Observe that

$$(2.3)\quad u^\varepsilon(x) = \sup\{\bar{u}(y) - \tfrac{1}{2\varepsilon}|x-y|^2 : y \in \bar{E}\} \quad \text{for } x \in \mathbb{R}^N,$$

where $\bar{u}(x) = \lim_{r \searrow 0} \sup\{u(y) : y \in E, |x-y| \le r\}$ for $x \in \bar{E}$.

This allows us to assume that E is closed in $\mathbb{R}^N$

<u>Lemma 2.1</u>. Let u be a subsolution (resp., super-solution) of (1.1). Assume u is bounded on Ω. Then, if $y \in \mathbb{R}^N$, $\varphi \in C^2(\{y\})$, $\max(u^\varepsilon - \varphi) = (u^\varepsilon - \varphi)(y)$ (resp., $\min(u_\varepsilon - \varphi) = (u_\varepsilon - \varphi)(y)$) and $y + \varepsilon D\varphi(y) \in \Omega$ (resp., $y - \varepsilon D\varphi(y) \in \Omega$), we have

$$(2.4)\quad F_*(y + \varepsilon D\varphi(y), u^\varepsilon(y) + \tfrac{\varepsilon}{2}|D\varphi(y)|^2, D\varphi(y), D^2\varphi(y)) \le 0$$

$$(\text{resp.,}\quad F^*(y - \varepsilon D\varphi(y), u_\varepsilon(y) - \tfrac{\varepsilon}{2}|D\varphi(y)|^2, D\varphi(y), D^2\varphi(y)) \ge 0\ ).$$

<u>Proof</u>. We only treat the case when u is a

subsolution. Let $y \in \mathbb{R}^N$ and $\varphi \in C^2(U)$, where U is a neighborhood of y. Assume $y + \varepsilon D\varphi(y) \in \Omega$ and $\max_U (u^\varepsilon - \varphi) = (u^\varepsilon - \varphi)(y)$. Define $\bar{u} \in USC(\bar{\Omega})$ as in (2.3). The boundedness of u guarantees that

$$u^\varepsilon(y) = \bar{u}(\eta) - \frac{1}{2\varepsilon}|y-\eta|^2 \quad \text{for some } \eta \in \bar{\Omega}.$$

The function $(x, \xi) \to \bar{u}(\xi) - \frac{1}{2\varepsilon}|x-\xi|^2 - \varphi(x)$ on $U \times \bar{\Omega}$ attains a maximum at (y, η). Therefore, $x \to \bar{u}(\eta) - \frac{1}{2\varepsilon}|x-\eta|^2 - \varphi(x)$ attains a maximum (over U) at y, and so

$$D\varphi(y) + \frac{1}{\varepsilon}(y-\eta) = 0, \quad \text{i.e.} \quad \eta = y + \varepsilon D\varphi(y) \; (\in \Omega).$$

Also, $\xi \to \bar{u}(\xi) - \frac{1}{2\varepsilon}|y-\eta|^2 - \varphi(\xi + y - \eta)$ has a maximum over $\bar{\Omega} \cap (\eta - y + U)$ at η, and hence, using the definition of subsolution, we have

$$F_x(\eta, \bar{u}(\eta), D\varphi(y), D^2\varphi(y)) \le 0.$$

Thus

$$F_x(y + \varepsilon D\varphi(y), u^\varepsilon(y) + \frac{\varepsilon}{2}|D\varphi(y)|^2, D\varphi(y), D^2\varphi(y)) \le 0. \quad \blacksquare$$

Lemma 2.2. Let $u : E \to \mathbb{R}$ be bounded. Then:

(i) $u^\varepsilon(x) \searrow u^*(x)$ for $x \in \bar{E}$ as $\varepsilon \searrow 0$. (ii) $u^\varepsilon(x) \le \sup_{\bar{E}} u - \frac{1}{2\varepsilon} \operatorname{dist}(x,E)^2$ for $x \in \mathbb{R}^N$.

The proof is easy and so we skip it.

We now recall Sard's lemma.

Proposition 2.1 (Sard). Let $\Omega \subset \mathbb{R}^N$ be open and $f : \Omega \to \mathbb{R}^N$ be a C^1 function. Then

(2.5) $$\operatorname{meas} f(E) \le \int_E |\det Df(x)|\, dx$$

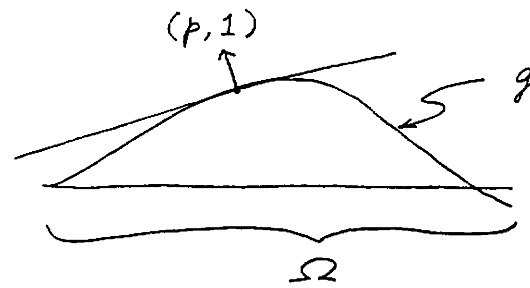
for any measurable $E \subset \Omega$.

See J. T. Schwartz "Nonlinear Functional Analysis".

Proposition 2.2 (R. Jensen). Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $u \in C(\bar{\Omega})$ satisfy $\max_{\bar{\Omega}} u > \max_{\partial\Omega} u$. Assume that for some $M > 0$, the function $x \to u(x) + M|x|^2$ is convex in Ω. Then there are constants $\delta_0 > 0$ and $c_0 > 0$ such that

(2.6) $$\text{meas}\{y \in \Omega : \exists p \in B(0,\delta) \text{ such that } u(x) \le u(y) + p\cdot(x-y) \text{ for } x \in \Omega\} \geq c_0 \delta^N \quad \forall \delta \in (0,\delta_0).$$

Proof Set $\omega_\delta = \{y \in \Omega : \exists p \in B(0,\delta) \text{ s.t. } u(x) \le u(y) + p\cdot(x-y) \ \forall x \in \Omega\}$.



Case 1 : $u \in C^2(\overline{\Omega})$. Let $d = \text{diam}\,\Omega$ (>0), $h = \max_{\Omega} u - \max_{\partial\Omega} u$ (>0), and $\delta_0 = h/d$. Let $0 < \delta < \delta_0$ and $p \in B(0,\delta)$. Then the function $x \to u(x) - p\cdot x$ attains a maximum at a point in Ω. Indeed, otherwise we have

$$u(x) - p\cdot x \le u(x_0) - p\cdot x_0 \quad \text{for } x \in \overline{\Omega} \text{ and some } x_0 \in \partial\Omega$$

Hence, choosing $x_1 \in \Omega$ so that $u(x_1) = \max_{\overline{\Omega}} u$, we have

$$\max_{\overline{\Omega}} u - p\cdot x_1 \le u(x_0) - p\cdot x_0 \le \max_{\partial\Omega} u - p\cdot x_0,$$

whence

$$h \le p\cdot(x_1 - x_0) \le \delta d < \delta_0 d \quad ; \text{ a contradiction.}$$

Let $y \in \Omega$ be a maximum point of $x \to u(x) - p\cdot x$. Of course, we have

$$u(x) \le u(y) + p\cdot(x-y) \quad \forall x \in \Omega \quad \& \quad p = -Du(y).$$

This means that if $p \in B(0,\delta)$, then $-p \in Du(\omega_\delta)$. That is, $B(0,\delta) \subset Du(\omega_\delta)$. By Sard's lemma (Prop. 2.1) we now conclude

$$\text{(2.7)} \quad \text{meas } B(0,\delta) \le \int_{Du(\omega_\delta)} dp \le \int_{\omega_\delta} |\det D^2u|\, dx.$$

By the convexity of u, we have

$$D^2u(x) + 2MI \ge 0.$$

If $y \in \omega_\delta$, then $x \longrightarrow u(x) + Du(y)\cdot x$ takes a maximum at y and hence $D^2u(y) \le 0$. Thus

$$-2MI \le D^2u(y) \le 0 \qquad \text{for } y \in \omega_\delta,$$

and so

$$|\det D^2u(y)| = \prod_{i=1}^{N} |\lambda_i| \le (2M)^N \qquad \text{for } y \in \omega_\delta,$$

where the λ_i are the eigenvalues of $D^2u(y)$. Going back to (2.7), we obtain

$$\gamma_N \delta^N \leq (2M)^N \operatorname{meas}(\omega_\delta),$$

where γ_N denotes the volume of the unit sphere of $\mathbb{R}^N$. This proves (2.6) in the case when $u \in C^2(\Omega)$.

<u>Case 2</u>: <u>general u</u>. Replacing Ω by a smaller one if necessary, we may assume that u is continuous on $\Omega_\varepsilon \equiv \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < \varepsilon\}$ for some $\varepsilon > 0$ and $x \to u(x) + M|x|^2$ is convex in Ω_ε. Choose $\psi \in C_0^\infty(\mathbb{R}$ so that $\psi \geq 0$, $\operatorname{supp} \psi \subset B(0,1)$ and $\int \psi(x)\,dx = 1$. For $0 < \eta < \varepsilon$, we set

$$\psi_\eta(x) = \eta^{-N}\psi(x/\eta) \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad u_\eta(x) = \int u(y)\psi_\eta(x-y)\,dy \quad \forall x \in \bar{\Omega}$$

It is standard that

$$u_\eta(x) \to u(x) \quad \text{uniformly on } \bar{\Omega} \quad \text{as } \eta \searrow 0,$$

$$u_\eta \in C^\infty(\bar{\Omega}).$$

We see also that

$$x \to u_\eta(x) + M|x|^2 = \int (u(x-y) + M|x|^2)\,\psi_\eta(y)\,dy$$

$$= \int [u(x-y) + M|x-y|^2 + 2M(x-y)\cdot y + M|y|^2]\,\psi_\eta(y)\,dy$$

is a convex function on Ω. We may assume

$$\max_{\Omega} u_\eta - \max_{\partial\Omega} u_\eta \geqq h \quad \text{for } 0 < {}^\forall\eta < \varepsilon \text{ and } \exists\, h > 0.$$

For $\delta > 0$, $\eta > 0$ we define

$$\omega_\delta(\eta) = \{ y \in \Omega : \exists p \in B(0,\delta) \text{ s.t. } u_\eta(x) \le u_\eta(y) + p\cdot(x-y) \ {}^\forall x \in \Omega \}$$

As we saw in Case 1, we have

(2.8) $$\text{meas } \omega_\delta(\eta) \geqq C_0 \delta^N \quad \text{for } 0 < {}^\forall\eta < \varepsilon \ \& \ 0 < {}^\forall\delta < \delta_0,$$

where $C_0 > 0$ and $\delta_0 > 0$ are constants.

Let $0 < \delta < \delta_0$, and we show that

(2.9) $$\bigcap_{0 < \frac{1}{k} < \varepsilon} \ \bigcup_{0 < \eta < \frac{1}{k}} \omega_\delta(\eta) \subset \omega_\delta .$$

We put $\Gamma_k = \bigcup_{0<\eta<\frac{1}{k}} \omega_\delta(\eta)$. Fix $y \in \bigcap_{k > 1/\varepsilon} \Gamma_k$. Then for any $k > \frac{1}{\varepsilon}$ there are $0 < \eta_k < \frac{1}{k}$ and $p_k \in B(0,\delta)$ such that $u_{\eta_k}(x) \le u_{\eta_k}(y) + p_k\cdot(x-y)$ ${}^\forall x \in \Omega$. Sending

$k \to \infty$, we find that $u(x) \le u(y) + p \cdot (x - y)$ for $\forall x \in \Omega$ and $\exists p \in B(0, \delta)$. Thus, $y \in \omega_\delta$; this proves (2.9).

Note that $\Gamma_k \supset \Gamma_{k+1} \supset \cdots$, and hence

$$\Gamma_k \setminus \bigcap_\ell \Gamma_\ell = \Gamma_k \setminus \Gamma_{k+1} \cup \Gamma_{k+1} \setminus \Gamma_{k+2} \cup \cdots .$$

Therefore,

$$\operatorname{meas}\Big(\Gamma_k \setminus \bigcap_\ell \Gamma_\ell\Big) = \sum_{j=k}^{\infty} \operatorname{meas}(\Gamma_j \setminus \Gamma_{j+1}) < \infty ,$$

from which

$$\sum_{j=k}^{\infty} \operatorname{meas}(\Gamma_j \setminus \Gamma_{j+1}) \to 0 \quad \text{as} \quad k \to \infty .$$

Hence,

$$\operatorname{meas} \Gamma_k \to \operatorname{meas}\Big(\bigcap_\ell \Gamma_\ell\Big) \quad \text{as} \quad k \to \infty .$$

This together with (2.8) and (2.9) yields

$$\operatorname{meas} \omega_\delta \geq C_0 \delta^N \qquad \forall \delta \in (0, \delta_0). \qquad \blacksquare$$

We need the following observation due to Aleksandrov.

<u>Proposition 2.3.</u> Let Ω be an open subset of $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ be a convex function. Then, u has the 2nd differential a.e. More precisely, for a.e. $y \in \Omega$ there are $p \in \mathbb{R}^N$ and $A \in \mathbb{S}^N$ such that

$$(2.10) \quad u(x) = u(y) + p\cdot(x-y) + A(x-y)\cdot(x-y) + o(|x-y|^2) \quad \text{as } x \to y.$$

Combining the above observations, we obtain

<u>Proposition 2.4.</u> Let u and v be, resp., sub- and supersolutions of (1.1). Assume Ω is locally compact and that u and v are bounded on Ω. Let $\bar{x}, \bar{y} \in \mathbb{R}^N$ and $\varphi \in C^2(U)$, where U is a neighborhood of $(\bar{x}, \bar{y}) \in \mathbb{R}^N \times \mathbb{R}^N$. Assume

$$\max_U \left(u^\varepsilon(x) - v_\varepsilon(y) - \varphi(x,y) \right) = u^\varepsilon(\bar{x}) - v_\varepsilon(\bar{y}) - \varphi(\bar{x},\bar{y}) \quad \text{and}$$

$$(\bar{x},\bar{y}) + \varepsilon\left(D_x\varphi(\bar{x},\bar{y}), +D_y\varphi(\bar{x},\bar{y})\right) \in \Omega\times\Omega.$$

Then there are $X, Y \in \mathbb{S}^N$ such that

$$-\frac{1}{\varepsilon} I \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le D^2\varphi(\bar{x},\bar{y}),$$

$$F_*\left(\bar{x} + \varepsilon D_x\varphi(\bar{x},\bar{y}),\ u^\varepsilon(\bar{x}) + \frac{\varepsilon}{2}|D_x\varphi(\bar{x},\bar{y})|^2,\ D_x\varphi(\bar{x},\bar{y}),\ X\right) \le 0,$$

$$F^*(\bar{y}+\varepsilon D_y\varphi(\bar{x},\bar{y}),\ v_\varepsilon(\bar{y})-\tfrac{\varepsilon}{2}|D_y\varphi(\bar{x},\bar{y})|^2,\ -D_y\varphi(\bar{x},\bar{y}),\ -Y) \geq 0.$$

<u>Proof</u> By replacing φ by $(x,y)\to\varphi(x,y)+|x-\bar{x}|^4+|y-\bar{y}|^4$ if necessary, we may assume that $(\bar{x},\bar{y})$ is a strict maximum point of $(x,y)\longrightarrow u^\varepsilon(x)-v_\varepsilon(y)-\varphi(x,y)$. By Prop. 2.2 and 2.3 there are $p_k, q_k \in \mathbb{R}^N$ and $(x_k,y_k)\in U$ for $k\in\mathbb{N}$ such that

$p_k, q_k \to 0$ & $(x_k,y_k)\to(\bar{x},\bar{y})$ as $k\to\infty$,

$(x,y)\to u^\varepsilon(x)-v_\varepsilon(y)-\varphi(x,y)-p_k\cdot x-q_k\cdot y$ attains a local maximum at (x_k,y_k),

$(x,y)\to u^\varepsilon(x)-v_\varepsilon(y)$ has the 2nd differential at (x_k,y_k).

We may assume also that $u^\varepsilon(x_k)-v_\varepsilon(y_k)\to u^\varepsilon(\bar{x})-v_\varepsilon(\bar{y})$.

Therefore,

$$(2.11)\quad Du^\varepsilon(x_k)=D_x\varphi(x_k,y_k)+p_k,\qquad Dv_\varepsilon(y_k)=-D_y\varphi(x_k,y_k)-q_k,$$

$$(2.12)\quad \begin{pmatrix} D^2u^\varepsilon(x_k) & 0 \\ 0 & -D^2v_\varepsilon(y_k)\end{pmatrix} \leq D^2\varphi(x_k,y_k).$$

Also, the differentiability implies that for some C^2

function ψ_k defined near x_k, $u^\varepsilon - \psi_k$ attains a maximum at x_k and $Du^\varepsilon(x_k) = D\psi_k(x_k)$, $D^2u^\varepsilon(x_k) = D^2\psi(x_k)$.

Since $x_k + \varepsilon D\psi(x_k) = x_k + \varepsilon D_x\varphi(x_k, y_k) + \varepsilon p_k \to \bar{x} + \varepsilon D_x\varphi(\bar{x}, \bar{y}) \in \Omega$ as $k \to \infty$ and $x_k + \varepsilon D\psi_k(x_k) \in \overline{\Omega}$ (see the proof of Lemma 2.1), we see in view of the local compactness of Ω that $x_k + \varepsilon D\psi_k(x_k) \in \Omega$ for large k. We may assume $x_k + \varepsilon D\psi_k(x_k) \in \Omega$ for $\forall k \in \mathbb{N}$. Applying Lemma 2.1, we have

$$F_*(x_k + \varepsilon Du^\varepsilon(x_k), u^\varepsilon(x_k) + \frac{\varepsilon}{2}|Du^\varepsilon(x_k)|^2, Du^\varepsilon(x_k), D^2u^\varepsilon(x_k)) \leq 0.$$

Similarly, we have

$$F^*(y_k - \varepsilon Dv_\varepsilon(y_k), v_\varepsilon(y_k) - \frac{\varepsilon}{2}|Dv_\varepsilon(y_k)|^2, Dv_\varepsilon(y_k), D^2v_\varepsilon(y_k)) \geq 0.$$

Since $x \to u^\varepsilon(x) + \frac{1}{2\varepsilon}|x|^2$ is convex in $\mathbb{R}^N$, we find that

$$D^2u^\varepsilon(x_k) + \frac{1}{\varepsilon}I \geq 0.$$

Also,

$$D^2v_\varepsilon(y_k) - \frac{1}{\varepsilon}I \leq 0.$$

That is,

$$\text{(2.13)} \qquad \begin{pmatrix} D^2 u^\varepsilon(x_k) & 0 \\ 0 & -D^2 v_\varepsilon(y_k) \end{pmatrix} \geq \frac{1}{\varepsilon} I .$$

Now (2.12) and (2.13) together imply the compactne of the sequences $\{Du^\varepsilon(x_k)\}$, $\{Dv_\varepsilon(y_k)\}$. Thus, sending $k \to \infty$, we get

$$-\frac{1}{\varepsilon} I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2\varphi(\bar{x},\bar{y}),$$

$$F_*(\bar{x} + \varepsilon D_x\varphi(\bar{x},\bar{y}),\ u^\varepsilon(\bar{x}) + \frac{\varepsilon}{2}|D_x\varphi(\bar{x},\bar{y})|^2,\ D_x\varphi(\bar{x},\bar{y}),\ X) \leq 0,$$

$$F^*(\bar{y} + \varepsilon D_y\varphi(\bar{x},\bar{y}),\ v_\varepsilon(\bar{y}) - \frac{\varepsilon}{2}|D_y\varphi(\bar{x},\bar{y})|^2,\ -D_y\varphi(\bar{x},\bar{y}),\ -Y) \geq 0,$$

for some $X, Y \in \mathbb{S}^N$. ∎

§3. Comparison criteria

Let Ω be an open bounded subset of $\mathbb{R}^N$. We begin with a simple comparison result. Let $F : \mathbb{S}^N \to \mathbb{R}$ and $f : \overline{\Omega} \to \mathbb{R}$ be continuous, and consider the equation

(3.1) $\quad u + F(D^2 u) = f(x) \quad$ in Ω.

Theorem 3.1. Assume F is elliptic. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., sub- and super-solutions of (3.1). Suppose $u \leq v$ on $\partial\Omega$. Then, $u \leq v$ in Ω.

Proof Let $\varepsilon > 0$, and define u^ε, v_ε by

$$u^\varepsilon(x) = \sup\{u(y) - \tfrac{1}{2\varepsilon}|x-y|^2 : y \in \overline{\Omega}\} \qquad \forall x \in \mathbb{R}^N,$$

$$v_\varepsilon(x) = \inf\{v(y) + \tfrac{1}{2\varepsilon}|x-y|^2 : y \in \overline{\Omega}\} \qquad \forall x \in \mathbb{R}^N.$$

By the assumptions we see that (for each $\delta > 0$) there is a function $\gamma : (0,\infty) \to (\delta, \infty)$ and a neighborhood U of

$\partial(\Omega \times \Omega)$ such that

$$u(x) - v(y) \le \frac{1}{2}\gamma(\varepsilon) + \frac{1}{2\varepsilon}|x-y|^2 \qquad \forall (x,y) \in U,$$

$$\gamma(\varepsilon) \to \delta \quad \text{as} \quad \varepsilon \searrow 0.$$

As $\left(u^\varepsilon(x) - v_\varepsilon(y) - \frac{1}{2}\gamma(\varepsilon) - \frac{1}{2\varepsilon}|x-y|^2\right)^+ \searrow 0 \quad \forall (x,y) \in U$

as $\varepsilon \searrow 0$, in view of Dini's lemma we may assume

that

$$u^\varepsilon(x) - v_\varepsilon(y) - \frac{1}{2}\gamma(\varepsilon) - \frac{1}{2\varepsilon}|x-y|^2 \le \frac{1}{2}\gamma(\varepsilon) \quad \forall \varepsilon > 0, \ \forall (x,y) \in U.$$

Now we consider the function $\Phi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ defined

by

$$\Phi(x,y) = u^\varepsilon(x) - v_\varepsilon(y) - \gamma(\varepsilon) - \frac{1}{2\varepsilon}|x-y|^2.$$

We suppose

$$\max_{\overline{\Omega}} (u - v) > \delta,$$

and will get a contradiction. Since $v^\varepsilon \ge u$ and

$v \ge v_\varepsilon$ on $\overline{\Omega}$, we find that

$$\max_{\overline{\Omega} \times \overline{\Omega}} \Phi \ge \max_{\overline{\Omega}} (u - v) - \gamma(\varepsilon) > 0. \tag{3.2}$$

Let $(\bar{x}, \bar{y})$ be a maximum point of Φ. Clearly, we have $(\bar{x}, \bar{y}) \in (\Omega \times \Omega) \setminus U$. As usual, (3.2) guarantees that as $\varepsilon \searrow 0$,

$$\frac{1}{2\varepsilon}|\bar{x} - \bar{y}|^2 \to 0, \qquad u^\varepsilon(\bar{x}) - v_\varepsilon(\bar{y}) \to \max_{\overline{\Omega}} (u - v).$$

We may also assume that $\bar{x}, \bar{y} \to {}^\exists z \in \Omega$ as $\varepsilon \searrow 0$. We are going to use Prop. 2.4. Restricting our attention to small $\varepsilon > 0$ and noting that

$$\bar{x} + \varepsilon\left(\frac{1}{\varepsilon}(\bar{x} - \bar{y})\right), \quad \bar{y} + \varepsilon\left(\frac{1}{\varepsilon}(\bar{y} - \bar{x})\right) \to z \quad \text{as} \quad \varepsilon \searrow 0,$$

we may apply Prop. 2.4, to find that there are matrices $X, Y \in \mathbb{S}^N$ such that

$$(3.3) \qquad \frac{1}{\varepsilon} I \le \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

$$(3.4) \qquad u^\varepsilon(\bar{x}) + F(X) - f(2\bar{x} - \bar{y}) \le 0 \le v_\varepsilon(\bar{y}) + F(-Y) - f(2\bar{y} - \bar{x}).$$

The second inequality in (3.3) implies $X + Y \le 0$. Indeed, multiplying $\binom{\xi}{\xi}$ with $\xi \in \mathbb{R}^N$ from the right and taking the inner product with $\binom{\xi}{\xi}$, we get

$$X\xi\cdot\xi + Y\xi\cdot\xi \le \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}\cdot\begin{pmatrix}\xi\\ \xi\end{pmatrix} = 0 \qquad \forall \xi \in \mathbb{R}^N.$$

Using the ellipticity of F, we find

$$F(-Y) \le F(X)$$

and hence

$$u^\varepsilon(\bar{x}) + F(X) - f(2\bar{x}-\bar{y}) \le v_\varepsilon(\bar{y}) + F(X) - f(2\bar{y}-\bar{x})$$

Thus, $\quad u^\varepsilon(\bar{x}) - v_\varepsilon(\bar{y}) \le o(1)$ as $\varepsilon \searrow 0$, and so

$$\max(u-v) - \gamma(\varepsilon) \le \max \Phi \le o(1) \qquad \text{as} \quad \varepsilon \searrow 0.$$

This gives a contradiction by sending $\varepsilon \searrow 0$. ∎

We now give a (too) general criterion which guarantees comparison of solutions. For a given F we define

(3.4+1)
$$\ell_F(\varepsilon,\lambda,R) = \sup\Big\{ F(2y-x, r, \tfrac{x-y}{\varepsilon}, -Y) - F(2x-y, r, \tfrac{x-y}{\varepsilon}, X) :$$
$$x, y \in \mathbb{R}^N,\ 2x-y,\ 2y-x \in \Omega,\ |x-y| \le \lambda,\ |r| \le R,$$
$$X, Y \in \mathcal{S}^N,\quad \tfrac{1}{\varepsilon} I \le \begin{pmatrix} X & O \\ O & Y\end{pmatrix} \le \tfrac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I\end{pmatrix} \Big\} \quad \text{for } \varepsilon, \lambda, R >$$

<u>Theorem 3.2.</u> Assume that

(F0) $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$,

(F1) F is elliptic,

(F2) for each $R > 0$ there is $\gamma_R > 0$ such that

$$F(x, r, p, A) - F(x, s, p, A) \geq \gamma_R (r - s)$$

for $x \in \Omega$, $-R \leq s \leq r \leq R$, $p \in \mathbb{R}^N$ and $A \in \mathbb{S}^N$,

and

(F3) $\lim_{\rho \searrow 0} \lim_{\varepsilon \searrow 0} l_F(\varepsilon, \sqrt{\varepsilon}\rho, R) = 0 \qquad \forall R > 0.$

Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., sub- and supersolutions of

(3.5) $F(x, u, Du, D^2 u) = 0$ in Ω.

Assume $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.

<u>Proof</u> Noting that if $\varphi(x,y) = \frac{1}{2\varepsilon}|x-y|^2$, then

$$D_x \varphi(x,y) = \frac{1}{\varepsilon}(x-y), \quad -D_y \varphi(x,y) = \frac{1}{\varepsilon}(x-y),$$

we have only to follow the proof of Theorem 3.1. ∎

It is not obvious which kind of functions satisfies (F3). We give some of sufficient conditions for F to satisfy (F3).

Example 1. Assume (F1) and that for each $R>0$ there is a function $m_R \in C[0,\infty)$ such that $m_R(0)=0$ and

$$(3.6)\quad \sup_{|r|\le R} |F(x,r,p,X) - F(y,r,p,X)| \le m_R(|x-y|(|p|+1))$$

for $(x,p,X) \in \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^N$. Then (F3) holds.

For example, if $F(x,r,p,X) = H(x,r,p) + G(X)$, G satisfies (F1) and H satisfies (H2) in Part I, then (3.6) is satisfied. In particular, Theorem 3.2 includes the assertion of Theorem 3.1.

Proof Let $x, y \in \Omega$ and $X, Y \in \mathbb{S}^N$ satisfy

$$2x-y,\ 2y-x \in \Omega, \quad |x-y| \le \sqrt{\varepsilon}\,\rho, \quad \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Let $|r| \le R$. Then $X+Y \le 0$, and hence

$$F\left(2y-x,\, r,\, \tfrac{x-y}{\varepsilon},\, -Y\right) - F\left(2x-y,\, r,\, \tfrac{x-y}{\varepsilon},\, X\right)$$

$$\leq F\left(2y-x,\, r,\, \tfrac{x-y}{\varepsilon},\, X\right) - F\left(2x-y,\, r,\, \tfrac{x-y}{\varepsilon},\, X\right)$$

$$\leq m_R\left(\overset{3}{\cancel{2}}|x-y|\left(\tfrac{|x-y|}{\varepsilon}+1\right)\right) \leq m_R\left(2\sqrt{\varepsilon}\,\rho\left(\tfrac{\rho}{\sqrt{\varepsilon}}+1\right)\right) =$$

(we are assuming that m_R is nondecreasing)

$$= m_R\left(\overset{3}{\cancel{2}}(\rho+\sqrt{\varepsilon})\rho\right) \xrightarrow{\varepsilon \downarrow 0} m_R(\overset{3}{\cancel{2}}\rho^2) \xrightarrow{\rho \downarrow 0} 0 .$$

Thus, $\displaystyle\lim_{\rho\downarrow 0}\ \overline{\lim_{\varepsilon\downarrow 0}}\ \ell_F(\varepsilon, \sqrt{\varepsilon}\rho, R) = 0 \quad \forall R > 0.$ ∎

<u>Example 2</u> (quasi-linear elliptic PDE's). Let

$\sigma : \overline{\Omega} \times \mathbb{R}^N \to M(N, N')$ satisfy

(3.7) $$\|\sigma(x,p) - \sigma(y,p)\| \leq C|x-y| \qquad \forall x, y \in \Omega,\ \forall p \in \mathbb{R}^N$$

for some constant $C > 0$, where $M(N, N')$ denotes the set of $N \times N'$ real matrices. Let

(3.8) ~~(3.7)~~ $$F(x,p,X) = -\mathrm{Tr}\, A(x,p)X, \text{ where } A = \sigma^t\sigma .$$

Then (F3) holds.

<u>Proof</u> Let $x, y \in \Omega$ and $X, Y \in \mathcal{S}^N$ satisfy

$$2x-y,\ 2y-x \in \Omega, \quad |x-y| \leq \sqrt{\varepsilon}\,\rho, \quad \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Recall that if $B, C \in \mathcal{S}^N$ and $B, C \geq 0$, then $\operatorname{Tr} BC \geq 0$. Let $B, C \in M(N, N')$. Then

$$0 \leqslant \begin{pmatrix} B \\ C \end{pmatrix}^t \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix} ({}^tB \quad {}^tC) = \begin{pmatrix} B^tB & B^tC \\ C^tB & C^tC \end{pmatrix}.$$

Using these observations, we get

$$0 \geqslant \operatorname{Tr} \begin{pmatrix} B \\ C \end{pmatrix}^t \begin{pmatrix} B \\ C \end{pmatrix} \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} - \frac{1}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right\} =$$

$$= \operatorname{Tr}\left(B^tBX + C^tCY\right) - \frac{1}{\varepsilon} \operatorname{Tr}\left(B^tB - B^tC - C^tB + C^tC\right)$$

$$= \operatorname{Tr}\left(B^tBX + C^tCY\right) - \frac{1}{\varepsilon} \operatorname{Tr} (B-C)^t(B-C).$$

Apply this to $B = \sigma(2x-y, p)$ and $C = \sigma(2y-x, p)$, with $p = \frac{x-y}{\varepsilon}$, to get

$$F\left(2y-x, \frac{x-y}{\varepsilon}, -Y\right) - F\left(2x-y, \frac{x-y}{\varepsilon}, X\right)$$

$$= \operatorname{Tr}\left(A\left(2x-y, \frac{x-y}{\varepsilon}\right) X + A\left(2y-x, \frac{x-y}{\varepsilon}\right) Y\right)$$

$$\leqslant \frac{1}{\varepsilon} \operatorname{Tr} \left(\sigma\left(2x-y, \frac{x-y}{\varepsilon}\right) - \sigma\left(2y-x, \frac{x-y}{\varepsilon}\right)\right)^t \left(\qquad '' \qquad \right)$$

$$\leqslant \frac{C_1}{\varepsilon} |x-y|^2 \leqslant C_1 \rho^2 \qquad \text{for some constant } C_1 > 0.$$

Thus, $\ell_F(\varepsilon, \sqrt{\varepsilon}\rho, R) \leq C_1 \rho^2$ for $R > 0$ and hence

$$\lim_{\rho \searrow 0} \overline{\lim_{\varepsilon \searrow 0}} \ \ell_F(\varepsilon, \sqrt{\varepsilon}\rho, R) = 0 \qquad \forall R > 0. \quad ▨$$

The following example shows that the Lipschitz assumption (3.7) on σ is optimal in a sense in order to have a comparison assertion for (3.8). We consider a linear elliptic PDE

(3.9) $\quad u - \mathrm{Tr}\, A(x) D^2 u = 0 \quad$ in $\mathbb{R}^N$,

where $A(x) = \dfrac{|x|^{2-\alpha}}{(N-1)\alpha}\left(I - \dfrac{1}{|x|^2} x \otimes x\right)$ for some $\alpha \in (0,2)$

Here we have used the notation : $x \otimes x = (x_i x_j)_{1 \le i,j \le N}$

Define $\sigma(x)$ by

$$\sigma(x) = \frac{|x|^{1-\frac{\alpha}{2}}}{\sqrt{N-1}\,\sqrt{\alpha}}\left(I - \frac{1}{|x|^2} x \otimes x\right) \qquad \text{for } x \in \mathbb{R}^N.$$

It is easy to check that $\sigma(x) \in \mathbb{S}^N$ and $\sigma^2(x) = A(x)$

Notice that $\sigma \in C^{0,1-\frac{\alpha}{2}}(\mathbb{R}^N)$. Define $u, v : \mathbb{R}^N \to \mathbb{R}$ by

$$u(x) = e^{|x|^\alpha} \qquad \text{and} \qquad v(x) = \begin{cases} e^{|x|^\alpha} & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

A simple calculation shows that

$$u(x) - \operatorname{Tr} A(x) D^2 u(x) = v(x) - \operatorname{Tr} A(x) D^2 v(x) = 0 \qquad \forall x \neq 0.$$

Notice also that $u \in C(\mathbb{R}^N)$ and $v \in LSC(\mathbb{R}^N)$. Noting that $D_+^2 u(0) = \emptyset$ and $A(0) = 0$, we conclude that u and v are both solutions of (3.9).

Thus we know that the conclusion of Theorem 3.2 does not hold for (3.8) with σ satisfying

$$|\sigma(x,p) - \sigma(y,p)| \le C|x-y|^\alpha \quad \text{for some } 0<\alpha<1$$

and $C > 0$ in general.

Example 3. We call F strictly elliptic if for each $R > 0$ there is $\nu_R > 0$ such that

(3.10) $\quad F(x,r,p,A+B) \le F(x,r,p,A) - \nu_R \operatorname{Tr} B$

if $(x,r,p,A) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$, $B \in \mathbb{S}^N$, $B \ge 0$ and $|r| \le R$. Notice that $\operatorname{Tr} B = \|B\|$ for $B \in \mathbb{S}^N$ with $B \ge 0$. We now assume that F is strictly elliptic and that

(3.11) $\quad$ for each $R > 0$ there is $m_R \in C[0,\infty)$ such that $m_R(0)=0$.

$$\overline{\lim_{r\to\infty}} \frac{m_R(r)}{r} < \infty .$$

$$\sup_{|r|\le R} |F(x,r,p,A) - F(y,r,p,A)| \le m_R\big(|x-y|(1+|p|+\|A\|)\big)$$

for $x, y \in \overline{\Omega}$, $p \in \mathbb{R}^N$, and $A \in \mathbb{S}^N$.

Then (F3) holds for F.

<u>Proof.</u> Let $X, Y \in \mathbb{S}^N$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \text{ with } \varepsilon > 0.$$

Multiply by $\begin{pmatrix} I & I \\ I & -I \end{pmatrix} \in \mathbb{S}^{2N}$ from the right and (also from) the left,

to obtain

$$(3.12)\quad \begin{pmatrix} X+Y & X-Y \\ X-Y & X+Y \end{pmatrix} \le \frac{4}{\varepsilon}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Note that this implies $X+Y \le 0$.
Fix $\xi \in \mathbb{R}^N$ and $t \in \mathbb{R}$, and we have

$$\begin{pmatrix} X+Y & X-Y \\ X-Y & X+Y \end{pmatrix}\begin{pmatrix} t\xi \\ \xi \end{pmatrix}\cdot\begin{pmatrix} t\xi \\ \xi \end{pmatrix} \le \frac{4}{\varepsilon}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} t\xi \\ \xi \end{pmatrix}\cdot\begin{pmatrix} t\xi \\ \xi \end{pmatrix}.$$

We can rewrite this as follows:

$$(X+Y)\xi\cdot\xi\, t^2 + 2(X-Y)\xi\cdot\xi\, t + (X+Y)\xi\cdot\xi - \frac{4}{\varepsilon}|\xi|^2 \le 0.$$

This yields

$$|(X-Y)\xi\cdot\xi|^2 \leqslant |(X+Y)\xi\cdot\xi|\,\Big[|(X+Y)\xi\cdot\xi| + \tfrac{1}{\varepsilon}|\xi|^2\Big] \qquad \forall \xi \in \mathbb{R}^N.$$

That is,

$$(3.13) \qquad \|X-Y\| \leqslant \|X+Y\|^{\frac{1}{2}}\Big(\|X+Y\| + \tfrac{1}{\varepsilon}\Big)^{1/2}.$$

Hence

$$(3.14) \qquad \|X\|,\ \|Y\| \leqslant \|X+Y\|^{\frac{1}{2}}\Big(\|X+Y\| + \tfrac{1}{\varepsilon}\Big)^{1/2}.$$

Now, as usual, we let $x, y \in \Omega$ and $\rho > 0$, and assume $2x-y,\ 2y-x \in \Omega$ and $|x-y| \leq \rho\sqrt{\varepsilon}$.

Then, for $R > 0$, we have

$$F\Big(2y-x, r, \tfrac{x-y}{\varepsilon}, \overset{X-(X+Y)}{\overset{\shortparallel}{-Y}}\Big) - F\Big(2x-y, r, \tfrac{x-y}{\varepsilon}, X\Big)$$

$$\leq F\Big(2y-x, r, \tfrac{x-y}{\varepsilon}, X\Big) - F\Big(2x-y, r, \tfrac{x-y}{\varepsilon}, X\Big) - \nu_R \|X+Y\|$$

$$\leqslant m_R\Big(3|x-y|\Big(1 + \tfrac{|x-y|}{\varepsilon} + \|X\|\Big)\Big) - \nu_R\|X+Y\|$$

$$\leqslant \sup_{t \geqslant 0}\Big\{ m_R\Big[3\rho\sqrt{\varepsilon} + 3\rho^2 + 3\sqrt{\varepsilon}\,\rho\, t^{\frac{1}{2}}\big(t + \tfrac{1}{\varepsilon}\big)^{1/2}\Big] - \nu_R t\Big\} =$$

$$= \sup_{t \geqslant 0}\Big\{ m_R\Big[3\rho\sqrt{\varepsilon} + 3\rho^2 + \rho\, t^{\frac{1}{2}}(\varepsilon t + 1)^{\frac{1}{2}}\Big] - \nu_R t\Big\}$$

if $|r| \leq R$. Thus,

$$\lim_{\rho \searrow 0} \overline{\lim_{\varepsilon \searrow 0}} \; l_F(\varepsilon, \rho\sqrt{\varepsilon}, R) = 0 \qquad \forall R > 0. \qquad \blacksquare$$

Let Λ be any set, and $l^\infty(\Lambda; \mathbb{R})$ denote the set of mappings $\xi : \Lambda \ni \alpha \to \xi_\alpha \in \mathbb{R}$ such that $\sup_{\alpha \in \Lambda} |\xi_\alpha| < \infty$. For $\xi, \eta \in l^\infty(\Lambda : \mathbb{R})$ we write

$$\|\xi - \eta\| = \sup_{\alpha \in \Lambda} |\xi_\alpha - \eta_\alpha| .$$

<u>Proposition 3.1.</u> Let $\Phi : l^\infty(\Lambda : \mathbb{R}) \to \mathbb{R}$ be a uniformly continuous mapping. For each $\alpha \in \Lambda$ let $F_\alpha : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$ and l_α be the function defined by (3.4+1) with $F = F_\alpha$. Assume $\{F_\alpha(x, r, p, A)\} \in l^\infty(\Lambda : \mathbb{R})$ for $(x, r, p, A) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N$ and that

$$\lim_{\rho \searrow 0} \overline{\lim_{\varepsilon \searrow 0}} \sup_{\alpha \in \Lambda} l_\alpha(\varepsilon, \rho\sqrt{\varepsilon}, R) = 0 \qquad \forall R > 0.$$

Let l_Φ be the function defined by (3.4+1) with $F = \Phi(\{F_\alpha\})$. Then

$$\lim_{\rho \searrow 0} \overline{\lim_{\varepsilon \searrow 0}} \; l_\Phi(\varepsilon, \rho\sqrt{\varepsilon}, R) = 0 \qquad \forall R > 0.$$

<u>Proof</u> We have

$$l_\Phi(\varepsilon, \lambda, R) \le \omega_\Phi\left(\sup_\alpha l_\alpha(\varepsilon, \lambda, R)\right),$$

where ω_Φ is the modulus of continuity of Φ. ∎

<u>Example 4</u>. Let $F_i : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$ for $i = 1, 2, 3$

Assume that for each $i = 1, 2, 3$, F_i satisfies the conditions in Example i. Then

(3.15) $$\lim_{\rho \searrow 0} \overline{\lim_{\varepsilon \searrow 0}}\, l_F(\varepsilon, \rho\sqrt{\varepsilon}, R) = 0 \quad \forall R > 0,$$

for $F = F_1 + F_2 + F_3$.

Let A and B be any (finite) sets. For $(\alpha, \beta) \in A \times B$, (let) $F^{\alpha\beta} : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}$ have the form

$$F^{\alpha\beta} = F_1^{\alpha\beta} + F_2^{\alpha\beta} + F_3^{\alpha\beta},$$

where $F_i^{\alpha\beta}$ satisfies the conditions of Example i for $i = 1, 2, 3$. Then (3.15) holds for

$$F = \sup_{\alpha \in A} \inf_{\beta \in B} F^{\alpha\beta}.$$

When we deal with 2nd order elliptic equations, there will be many chances to find the Lipschitz continuity of solutions.

Let $u : \overline{\Omega} \to \mathbb{R}$ be a Lipschitz continuous function. Let $\delta > 0$, and define $\Omega_\delta = \{x \in \mathbb{R}^N : \ \mathrm{dist}(x, \Omega^c) > \delta\}$. Then $\{u^\varepsilon\}_{\varepsilon > 0}$ is an equi-Lipschitz continuous family of functions on Ω_δ. To see this, let $L > 0$ be a constant so that

$$|u(x) - u(y)| \leq L|x-y| \qquad \forall x, y \in \overline{\Omega}.$$

Fix $x \in \Omega_\delta$. For $y \in \overline{\Omega}$, we have

$$u^\varepsilon(x) \geq u(x) \geq u(y) - L|x-y| = u(y) - \tfrac{1}{2\varepsilon}|x-y|^2 + \left(\tfrac{1}{2\varepsilon}|x-y| - L\right)|x-y|,$$

and so

$$u^\varepsilon(x) > u(y) - \tfrac{1}{2\varepsilon}|x-y|^2 \qquad \text{if } |x-y| > 2\varepsilon L.$$

We now assume $2\varepsilon L < \delta$. From the above we see

$$u^\varepsilon(x) = \max\{u(y) - \tfrac{1}{2\varepsilon}|x-y|^2 : \ y \in B(x, 2\varepsilon L)\}$$

$$= \max \{ u(x-z) - \frac{1}{2\varepsilon}|z|^2 ; \ z \in B(0, 2\varepsilon L) \} .$$

The last formula tells us that

$$|u^\varepsilon(x) - u^\varepsilon(y)| \leq L|x-y| \qquad \forall x, y \in \Omega_\delta ,$$

if $2\varepsilon L < \delta$.

If we use the above observation in the proof of Theorem 3.1 assuming that either u or v is Lipschitz continuous, then we have a bound of $\frac{1}{\varepsilon}|\bar{x} - \bar{y}|$ independent of $\varepsilon > 0$. This is, of course, also true in the proof of Theorem 3.2, and we have this conclusion:

<u>Theorem 3.3</u>. Assume (F0)-(F2) and

(F4) $$\lim_{\varepsilon \searrow 0} l_F(\varepsilon, R\varepsilon, R) = 0 \qquad \forall R > 0.$$

Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., sub- and supersolutions of (3.5). Assume that u or v is Lipschitz continuous on $\overline{\Omega}$ and that $u \leq v$

on $\partial\Omega$. Then $u \le v$ in Ω.

Example 1'. If we replace (3.6) by a weaker condition (3.6)' in Example 1, then

$$(3.15) \quad \overline{\lim_{\varepsilon \searrow 0}}\ l_F(\varepsilon, R\varepsilon, R) = 0 \qquad \forall R > 0.$$

And the condition is:

$$(3.6)' \quad \sup_{|r|, |p| \le R} |F(x, r, p, X) - F(y, r, p, X)| \le m_R(|x-y|)$$

for $x, y \in \overline{\Omega}$, $X \in \mathcal{S}^N$ and $R > 0$, where $m_R \in C[0, \infty)$ satisfies $m_R(0) = 0$.

Example 2'. We replace (3.7) by

$$(3.7)' \quad \sup_{|p| \le R} \| \sigma(x, p) - \sigma(y, p) \| \le C_R |x-y|^\theta \qquad \forall x, y \in \overline{\Omega}, \ \forall R > 0,$$

where $C_R > 0$ and $\theta > \frac{1}{2}$, in Example 2. Then we have (3.15) for F defined by (3.8).

Proof. Reviewing the proof of Example 2, we see that

$$l_F(\varepsilon, R\varepsilon, R) \le \frac{1}{\varepsilon}\{C_R (3R\varepsilon)^\theta\}^2 = C_R^2 (3R)^{2\theta} \varepsilon^{2\theta - 1}.$$

This proves (3.15). ∎

The example after Example 2 shows the optimality of the requirement that $\theta > \frac{1}{2}$ in (3.7)$'$

<u>Example 3$'$.</u> We replace (3.11) by (3.11)$'$ below in Example 3.

$$(3.11)' \quad \sup_{|p|,|r|\le R} |F(x,r,p,A) - F(y,r,p,A)| \le m_R\big(|x-y|^\theta (1+\|A\|)\big)$$

for $x, y \in \overline{\Omega}$, $A \in \mathbb{S}^N$ and $R>0$, where $\theta > \frac{1}{2}$ and $m_R \in C[0,\infty)$ satisfies $m_R(0) = 0$ and $\overline{\lim\limits_{r\to\infty}} \frac{m_R(r)}{r} < \infty$.

Then (3.15) holds.

<u>Proof</u> We review the proof of Example 3, to find

$$\ell_F(\varepsilon, R\varepsilon, R) \le \sup_{t\ge 0} \left\{ m_R\Big[(3R\varepsilon)^\theta\big(1 + t^{\frac{1}{2}}(t+\tfrac{1}{\varepsilon})^{\frac{1}{2}}\big)\Big] - \nu_R t \right\} =$$

$$= \sup_{t\ge 0} \left\{ m_R\big((3R\varepsilon)^\theta + (3R)^\theta \varepsilon^{\theta-\frac{1}{2}} t^{\frac{1}{2}}(\varepsilon t+1)^{\frac{1}{2}}\big) - \nu_R t \right\},$$

which proves (3.15). ∎

<u>Remark</u> One may replace $|x-y|^\theta$ in (3.11)$'$ by $|x-y|^{\frac{1}{2}}\omega(|x-y|)$, where $\omega \in C[0,\infty)$ satisfies $\omega(0)=0$.

So far our comparison results do not give the uniqueness of solutions of the Dirichlet problem for

$$-\Delta u = f(x) \quad \text{in } \Omega.$$

In this respect the following obvious generalization of Theorems 3.2 and 3.3 is useful.

Theorem 3.4. Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., a subsolution of (3.5) and a supersolution of

(3.16) $$F(x, u, Du, D^2u) = a \quad \text{in } \Omega,$$

where $a > 0$ is a constant. Assume $u \leq v$ on $\partial\Omega$. Assume (F0), (F1) and

(F5) $r \to F(x, r, p, A)$ is nondecreasing for $(x, p, A) \in \overline{\Omega} \times \mathbb{R} \times$

(i) If (F3) holds, then $u \leq v$ in Ω. (ii) If (F4) holds and either u or v are Lipschitz continuous in Ω, then $u \leq v$ in Ω.

The general idea to apply Theorem 3.4 to obtaining comparison results for (3.5) is to find a sequence $\{u_n\}$ $(\subset USC(\overline{\Omega}))$ of subsolutions of

$$F(x, u_n, Du_n, D^2u_n) = -\frac{1}{n} \quad \text{in } \Omega \tag{3.17}$$

for a given subsolution u of (3.5) such that $u_n(x) \nearrow u(x)$ for $x \in \overline{\Omega}$ as $n \to \infty$, or a sequence $\{v_n\} \subset LSC(\overline{\Omega})$ of supersolutions of

$$F(x, v_n, Dv_n, D^2v_n) = \frac{1}{n} \quad \text{in } \Omega \tag{3.18}$$

for a given supersolution v such that $v_n(x) \searrow v(x)$ for $x \in \overline{\Omega}$ as $n \to \infty$.

We give two examples where (, for a given subsolution u of (3.5) we can find a sequence $\{u_n\}$ with the properties described above. We will seek u_n in the form $u_n = \alpha_n u + \psi_n$ where the α_n are constants, the ψ_n are functions on $\overline{\Omega}$ and $\alpha_n \to 1$,

$\psi_n(x) \to 0$ as $n \to \infty$.

The first example requires a convexity of F, which is the situation we have already met in the case of first order PDE's. Our assumptions are : (i) $(p, A) \to F(x, r, p, A)$ is convex for $(x, r) \in \overline{\Omega} \times \mathbb{R}$, and (ii) (for each $R > 0$) there is a C^2 function ψ_R on $\overline{\Omega}$ such that $\psi_R(x) \le -R$ on $\overline{\Omega}$ and

$$\max\{F(x, R, D\psi_R(x), D^2\psi_R(x)) : x \in \overline{\Omega}\} < 0.$$

Let u be a subsolution of (3.5), and put $R = \sup_{\overline{\Omega}} |u|$. If we assume (F5) together with (i) and (ii) and set $u_\theta = \theta u + (1-\theta)\psi_R$ on $\overline{\Omega}$ for $0 < \theta < 1$, then we formally have

$$F(x, u_\theta, Du_\theta, D^2u_\theta) \le F(x, u, Du_\theta, D^2u_\theta) \le$$

$$\le \theta F(x, u, Du, D^2u) + (1-\theta) F(x, R, D\psi_R(x), D^2\psi_R(x)) \le -(1-\theta)a,$$

where $a = -\max\{F(x, R, D\psi_R(x), D^2\psi_R(x)) : x \in \overline{\Omega}\}$ (>0),

and also $u_\theta(x) \nearrow u(x)$ for $x \in \overline{\Omega}$ as $\theta \nearrow 1$.

The second example is an adaptation of a standard technique (in the theory of 2nd order PDEs). We now assume that for each $R > 0$ there are ν_R, $C_R > 0$ such that

(3.19) $$F(x, r, p+q, A+B) \leq F(x, r, p, A) - \nu_R \operatorname{Tr} B + C_R |q|$$

for $x \in \overline{\Omega}$, $|r| \leq R$, $p \in \mathbb{R}^N$, $q \in B(0, R)$ and $A, B \in \mathbb{S}^N$ with $B \geq 0$. Let u be a subsolution of (3.5).

Choose $M > 0$ so that $\overline{\Omega} \subset B(0, M)$, and define

$$\psi(x) = e^{\lambda(|x|^2 - M^2)} - 1 \quad \text{for } x \in \overline{\Omega} \text{ with } \lambda > 0.$$

Observe that $\psi \leq 0$ on $\overline{\Omega}$, $D\psi(x) = 2\lambda x \, e^{\lambda(|x|^2 - M^2)}$ and $D^2\psi(x) = 2\lambda e^{\lambda(|x|^2 - M^2)}(I + 2\lambda\, x \otimes x) \geq 0$ for $x \in \overline{\Omega}$.

Define

$$u_n(x) = u(x) + \frac{1}{n}\psi(x) \quad \text{for } x \in \overline{\Omega} \text{ and } n \in \mathbb{N}.$$

Then we formally have

$$F(x, u_n, Du_n, D^2u) \leq F(x, u, Du + \tfrac{2\lambda}{n} x e^{\lambda(|x|^2 - M^2)}, D^2u$$

$$+ 2\lambda e^{\lambda(|x|^2 - M^2)}(I + 2\lambda\, x \otimes x))$$

$$\leq F(x, u, Du, D^2u) + e^{\lambda(|x|^2 - M^2)} \left\{ C_R \frac{2\lambda|x|}{n} - \nu_R (2N\lambda + 4\lambda^2|x|^2) \right\},$$

where $R = \sup_{\overline{\Omega}} |u|$. Hence, choosing λ so large that

$$\nu_R (4\lambda^2|x|^2 + N\lambda) \geq 2C_R \lambda |x| \qquad \text{for } x \in \overline{\Omega},$$

we see that u_n is a subsolution of

$$F(x, u, Du, D^2u) = -\nu_R N \lambda e^{-\lambda M^2} \qquad \text{in } \Omega.$$

Also, we see that $u_n(x) \nearrow u(x)$ for $x \in \overline{\Omega}$ as $n \to \infty$.

The strict ellipticity yields an estimate, under a mild regularity assumption on F, which takes the place of the Lipschitz continuity on u or v in Theorem 3.3:

<u>Proposition 3.1.</u> Assume (F0), (F5), (3.10) and that for each $R > 0$ there are $C_R > 0$, $m_R \in C[0,\infty)$ satisfying $m_R(0) = 0$, and $\mu_R \in C[0,\infty)$ satisfying $\mu_R \geq 0$ and $\int_{+0} \frac{\mu_R(r)}{r}\,dr < \infty$ such that

$$(3.20)\quad |F(x,r,p,A) - F(y,r,p,A)| \leq C_R + m_R(|x-y|)\,|x-y|^{\tau}\,|p|^{2+\tau} + \mu_R(|x-y|)\,\|A\|$$

for $x, y \in \overline{\Omega}$, $|r| \leq R$, $p \in \mathbb{R}^N$, and $A \in \mathbb{S}^N$.

Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., sub- and supersolutions of (3.5). Suppose that $\max_{\overline{\Omega}}(u - v) > 0 \geq \max_{\partial\Omega}(u - v)$. Then there is $C > 0$ such that

$$(3.21)\qquad u^{\varepsilon}(x) - v_{\varepsilon}(y) \leq \max_{\overline{\Omega}}(u^{\varepsilon} - v_{\varepsilon}) + C\,|x-y|$$

for $x, y \in \overline{\Omega}$ and $\varepsilon > 0$ small enough.

If we have the estimate (3.21) in the proof of Theorem 3.1, then we have

$$\max_{\overline{\Omega}} (u^\varepsilon - v_\varepsilon) - \gamma(\varepsilon) \leq \Phi(\bar{x}, \bar{y}) = u^\varepsilon(\bar{x}) - v_\varepsilon(\bar{y}) - \gamma(\varepsilon) - \frac{1}{2\varepsilon}|\bar{x} - \bar{y}|^2$$

$$\leq \max_{\overline{\Omega}} (u^\varepsilon - v_\varepsilon) + C|\bar{x} - \bar{y}| - \gamma(\varepsilon) - \frac{1}{2\varepsilon}|\bar{x} - \bar{y}|^2,$$

whence

$$\frac{1}{2\varepsilon}|\bar{x} - \bar{y}| \leq C .$$

This observation together with Theorem 3.3 and Example 3′ proves:

<u>Corollary 3.1.</u> Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ be, resp., sub- and supersolutions of (3.5). Assume (F0), (F2), (3.10), (3.11)′ and (3.20). Assume also that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.

Proof of Proposition 3.1. Let $R_0 = \max_{\overline{\Omega}} (|u| + |v|)$, and we will write

$$\nu = \nu_{R_0}, \quad m = m_{R_0}, \quad \mu = \mu_{R_0} \quad \text{and} \quad C_0 = C_{R_0}.$$

We may assume that $\mu(r) \geq r$. Define

$$\ell(r) = \int_0^r ds \int_0^s \frac{\mu(\sigma)}{\sigma}\, d\sigma \qquad \text{for } r \geq 0,$$

$$d = \max_{\overline{\Omega}} (u^\varepsilon - v_\varepsilon), \qquad \Phi(x) = d + MK|x| - M\ell(K|x|)$$

$$\text{and} \quad \varphi(x,y) = \Phi(x-y) \qquad \text{for } x, y \in \mathbb{R}^N,$$

where M, K are positive constants. Choose $r_0 > 0$ so that $\ell'(r_0) = \frac{1}{2}$, and observe that $\ell(r) \leq r\ell'(r)$, and hence if $K|x| \leq r_0$, then $\Phi(x) \geq d + \frac{M}{2}K|x|$.

We set

$$\Delta_K = \{(x,y) \in \Omega \times \Omega \;:\; |x-y| < \frac{r_0}{K}\},$$

and fix M so that $\sup_{\substack{x,y \in \overline{\Omega} \\ 0<\varepsilon\leq 1}} (u^\varepsilon(x) - v_\varepsilon(y)) \leq \frac{M}{2} r_0$.

We assume $0 < \varepsilon \leq 1$. Moreover, by assuming ε small enough we may suppose that

$$\sup_{U} \, (u^\varepsilon - v_\varepsilon) \leq \frac{d}{2}$$

for some neighborhood U of $\partial\Omega$, and therefore that

$$u^\varepsilon(x) - v_\varepsilon(y) \leq d + C|x-y| \quad \text{for } (x,y) \in V,$$

where V is a compact neighborhood of $\partial(\Omega\times\Omega)$ and $C > 0$ is a constant. Thus, restricting K so that $MK \geq 2C$, we deduce that

$$(3.22) \qquad u^\varepsilon(x) - v_\varepsilon(y) - \varphi(x,y) \leq 0 \quad \text{for } (x,y) \in V \cap \partial\Delta_K.$$

We now claim that

$$(3.23) \qquad u^\varepsilon(x) - v_\varepsilon(y) - \varphi(x,y) \leq 0 \quad \text{for } (x,y) \in \Delta_K$$

for ε small enough and K large enough, which proves (3.21). To this end, we suppose

$$\sup_{(x,y)\in\Delta_K} \left(u^\varepsilon(x) - v_\varepsilon(y) - \varphi(x,y) \right) > 0,$$

and will get a contradiction. There is a max. point $(\bar{x},\bar{y}) \in \Delta_K$ of $u^\varepsilon(x) - v_\varepsilon(y) - \varphi(x,y)$ and matrices $X, Y \in \mathcal{S}^N$ such that

$$(3.24)\qquad \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}, \quad \text{with} \quad B = D^2\Phi(\bar{x}-\bar{y}),$$

$$(3.25)\qquad F(2\bar{x}-\bar{y},\, u^\varepsilon(\bar{x}) + \tfrac{\varepsilon}{2}|D\Phi(\bar{x}-\bar{y})|^2,\, D\Phi(\bar{x}-\bar{y}),\, X) \leq 0 \leq$$

$$\leq F(2\bar{y}-\bar{x},\, v_\varepsilon(\bar{y}) - \tfrac{\varepsilon}{2}|D\Phi(\bar{x}-\bar{y})|^2,\, D\Phi(\bar{x}-\bar{y}),\, -Y),$$

$$(3.26)\qquad u^\varepsilon(\bar{x}) \geq v_\varepsilon(\bar{y}).$$

Using the argument in the proof of Example 3, we see

$$X + Y \leq 0, \qquad X + Y \leq 4B, \quad \text{and}$$

$$\|X\| \leq \|X+Y\|^{\frac{1}{2}} \left(\|X+Y\| + \tfrac{1}{\varepsilon}\|B\|\right)^{\frac{1}{2}}.$$

Let $P \in \mathcal{S}^N$ satisfy $0 \leq P \leq I$. Then

$$\mathrm{Tr}\, P(X+Y) \leq 4\,\mathrm{Tr}\, PB,$$

and also

$$\mathrm{Tr}\, (I-P)(X+Y) \leq 0,$$

from which it follows that $\mathrm{Tr}\,(X+Y) \leq \mathrm{Tr}\, P(X+Y)$.

Hence

$$Tr\,(X+Y) \leqslant 4\,Tr\, PB.$$

We apply this inequality to $P = \frac{1}{|\bar{x}-\bar{y}|^2}(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})$,

to obtain

$$(3.27)\quad Tr\,(X+Y) \leqslant -4MK^2 \ell''(K|\bar{x}-\bar{y}|) = -4MK\frac{\mu(K|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|}.$$

Now, from (3.25) we have

$$0 \geqslant F(2\bar{x}-\bar{y},\, v_\varepsilon(\bar{y}) - \tfrac{\varepsilon}{2}|D\Phi(\bar{x}-\bar{y})|^2,\, D\Phi(\bar{x}-\bar{y}),\, X)$$
$$- F(2\bar{y}-\bar{x},\, v_\varepsilon(\bar{y}) - \tfrac{\varepsilon}{2}|D\Phi(\bar{x}-\bar{y})|^2,\, D\Phi(\bar{x}-\bar{y}),\, -Y)$$
$$\geqslant \nu|Tr\,(X+Y)| - C_0 - m(3|\bar{x}-\bar{y}|)(3|\bar{x}-\bar{y}|)^{\tau}|D\Phi(\bar{x}-\bar{y})|^{2+\tau}$$
$$-\mu(3|\bar{x}-\bar{y}|)\,\|X+Y\|^{\frac{1}{2}}\left(\|X+Y\| + \tfrac{1}{\varepsilon}\|D^2\Phi(\bar{x}-\bar{y})\|\right)^{\frac{1}{2}}.$$

From this, observing that

$$|D\Phi(\bar{x}-\bar{y})| \leqslant MK\,(1-\ell'(K|\bar{x}-\bar{y}|)) \leqslant MK(1-\ell'(r_0)) = \frac{MK}{2},$$

$$\|D^2\Phi(\bar{x}-\bar{y})\| \leqslant C_1\, MK\,(1+\mu(r_0))\,/\,|\bar{x}-\bar{y}|$$

for some constant $C_1 > 0$,
and using (3.27), we obtain

$$\frac{\nu}{2}\|X+Y\| + 2\nu MK^2\frac{\mu(K|\bar{x}-\bar{y}|)}{K|\bar{x}-\bar{y}|} \leqslant C_0 + \left(\frac{MK}{2}\right)^2\left(\frac{3}{2}MK|\bar{x}-\bar{y}|\right)^{\tau}$$

$$\times\, m(3|\bar{x}-\bar{y}|) + \mu(3|\bar{x}-\bar{y}|)\,\|X+Y\| + \mu(3|\bar{x}-\bar{y}|) > \frac{C_1 M K^2(1+\mu(r_0))}{K|\bar{x}-\bar{y}|}.$$

This yields a contradiction for K large enough. $\blacksquare$

§4. Boundary value problems

Let Ω be an open bounded subset of $\mathbb{R}^N$ with $\partial\Omega$ of class C^2. Let us consider the Neumann problem

$$(4.1)\qquad \begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega; \end{cases}$$

where $n = n(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$. We need the following assumption:

$$(4.2)\qquad |F(x, r, p, A) - F(x, r, q, B)| \le m_R(|p-q| + \|A-B\|)$$

holds if $x \in U_R$, $|r| \le R$, $p, q \in \mathbb{R}^N$, $A, B \in S^N$ and $R > 0$, where U_R is a neighborhood of $\partial\Omega$ and $m_R \in C[0, \infty)$ satisfies $m_R(0) = 0$.

Theorem 4.1. Assume $\partial\Omega \in C^2$ (F0) – (F3) and (4.2). Let $u \in USC(\bar\Omega)$ and $v \in LSC(\bar\Omega)$ be, resp., sub- and super-solutions of (4.1). Then $u \le v$ on $\partial\Omega$.

Proof As in the 1st order case, we may assume that u and v are, resp., sub- and supersolutions of

$$F(x, u, Du, D^2u) = -\alpha \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = -\alpha \quad \text{on } \partial\Omega$$

and

$$F(x, v, Dv, D^2v) = \alpha \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial n} = \alpha \quad \text{on } \partial\Omega$$

for some $\alpha > 0$.

Notice that $\lim_{|x| \to \infty} u^\varepsilon(x) = -\infty$ and $\lim_{|x|\to\infty} v_\varepsilon(x) = \infty$ for $\varepsilon > 0$. Applying Prop. 2.4 to $\varphi(x,y) = \frac{1}{2\varepsilon}|x-y|^2$ (with $\overline{\Omega}$ in place of Ω), we find $X, Y \in \mathbb{S}^N$ such that

(4.3) $$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2\varphi(\bar{x}, \bar{y}) = \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

(4.4) $$F_*^\alpha(2\bar{x} - \bar{y}, u^\varepsilon(\bar{x}), \tfrac{1}{\varepsilon}(\bar{x}-\bar{y}), X) \leq 0 \leq F^*_\alpha(2\bar{y} - \bar{x}, v_\varepsilon(\bar{y}), \tfrac{1}{\varepsilon}(\bar{x}-\bar{y}), -Y),$$

where $(\bar{x}, \bar{y})$ is a max. point of $(x,y) \to u^\varepsilon(x) - v_\varepsilon(y) - \varphi(x,y)$ over $\mathbb{R}^N \times \mathbb{R}^N$,

$$F_*^\alpha(x, r, p, X) = \begin{cases} F(x, r, p, X) + \alpha & \text{if } x \in \Omega, \\ n(x)\cdot p \wedge F(x, r, p, X) + \alpha & \text{if } x \in \partial\Omega \end{cases}$$

and

$$F_\alpha^*(x, r, p, X) = \begin{cases} F(x, r, p, X) - \alpha & \text{if } x \in \Omega \\ (n(x)\cdot p) \vee F(x, r, p, X) - \alpha & \text{if } x \in \partial\Omega . \end{cases}$$

As usual we have

$$(4.5) \qquad \frac{1}{\varepsilon}|\bar{x} - \bar{y}|^2 \to 0 \qquad \text{as} \quad \varepsilon \searrow 0 .$$

For example, if $2\bar{x} - \bar{y} \in \partial\Omega$ and

$$F_*^\alpha(2\bar{x} - \bar{y},\ u^\varepsilon(\bar{x}),\ \tfrac{1}{\varepsilon}\bar{x} - \bar{y},\ X) = n(2\bar{x} - \bar{y}) \cdot \tfrac{1}{\varepsilon}(\bar{x} - \bar{y}) + \alpha ,$$

then, using the estimate

$$n(x)\cdot(x - y) + C|x - y|^2 \geqslant 0 \qquad \forall x \in \partial\Omega,\ \forall y \in \bar{\Omega}$$

for some constant $C > 0$, we see that

$$n(2\bar{x} - \bar{y}) \cdot 3(\bar{x} - \bar{y}) + C\left(3|\bar{x} - \bar{y}|\right)^2 \geqslant 0$$

and hence

$$n(2\bar{x} - \bar{y}) \cdot \tfrac{1}{\varepsilon}(\bar{x} - \bar{y}) + \alpha \geqslant \alpha - \frac{3C}{\varepsilon}|\bar{x} - \bar{y}|^2 .$$

From this and (4.5) we see that this last term is positive if $\varepsilon > 0$ is small enough. Thus, assuming ε is small enough, we conclude from (4.4) that

$$F(2\bar{x}-\bar{y},\ u^\varepsilon(\bar{x}),\ \tfrac{1}{\varepsilon}(\bar{x}-\bar{y}),\ X) + \alpha \leq 0 .$$

Similarly, we have

$$F(2\bar{y}-\bar{x},\ v_\varepsilon(\bar{y}),\ \tfrac{1}{\varepsilon}(\bar{x}-\bar{y}),\ -Y) - \alpha \geq 0 .$$

If $u^\varepsilon(\bar{x}) \geq v_\varepsilon(\bar{y})$, then

$$2\alpha \leq F(2\bar{y}-\bar{x},\ v^\varepsilon(\bar{x}),\ \tfrac{1}{\varepsilon}(\bar{x}-\bar{y}),\ -Y) - F(2\bar{x}-\bar{y},\ u^\varepsilon(\bar{x}),\ \tfrac{1}{\varepsilon}(\bar{x}-\bar{y}),\ X) \leq$$

$$\leq \ell_F(\varepsilon,\ |\bar{x}-\bar{y}|,\ |u^\varepsilon(\bar{x})|) \leq \ell_F(\varepsilon,\ \rho(\varepsilon)\sqrt{\varepsilon},\ |u^\varepsilon(\bar{x})|),$$

where $\rho(\varepsilon) = \frac{1}{\varepsilon}|\bar{x}-\bar{y}|^2$. Noting that $u^\varepsilon(\bar{x}) - v_\varepsilon(\bar{y}) \to \max_{\bar{\Omega}}(u-v)$ as $\varepsilon \searrow 0$, we see from the above and (F3) that $\max_{\bar{\Omega}}(u-v) \leq 0$. ∎

<u>Theorem 4.2.</u> Assume $\partial\Omega \in C^3$ and that $\gamma \in C^2(\mathbb{R}^N;\mathbb{R}^N)$ satisfies $\gamma \cdot n > 0$ on $\partial\Omega$. Assume (F0) – (F3) and (4.2).

Insert 56, a here.

Let $f \in C(\partial\Omega \times \mathbb{R})$ and $r \to f(x,r)$ be nondecreasing.

Let $u \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$ be, resp., sub- and supersolutions of

56, 0

In addition, assume

(F6) $$\lim_{\rho\searrow 0}\ \lim_{\varepsilon\searrow 0}\ \tilde{\ell}_F(\varepsilon, \rho\sqrt{\varepsilon}, R) = 0 \qquad \forall R > 0,$$

where

$$\tilde{\ell}_F(\varepsilon, \lambda, R) = \sup\Big\{F\big(2y-x, r, \tfrac{x-y}{\varepsilon}, -Y\big) - F\big(2x-y, r, \tfrac{x-y}{\varepsilon}, X\big) :$$

$$2y-x,\ 2x-y \in U_R,\quad |r| \leq R,\quad |x-y| \leq \lambda,\quad -\tfrac{1}{\varepsilon} I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq$$

$$\leq \frac{1}{\varepsilon}\begin{pmatrix} A+B+{}^tB & -A-B \\ -A-{}^tB & A \end{pmatrix}, \quad A \in \mathbb{S}^N,\ \|A\| \leq R, \quad B \text{ real } N\times N$$

matrix, $\|B\| \leq R\lambda \Big\}$ for $\varepsilon, \lambda, R > 0$.

(4.6) $$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} + f(x, u) = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \leq v$ on $\overline{\Omega}$.

Application

$b : \mathbb{R}^N \to \mathbb{R}^N$ Lipschitz, W_t ; N-dim Brownian motion.
$\varepsilon > 0$.
SDE

(1) $$\begin{cases} dX_s = b(X_s)\,ds + \varepsilon\, dW_s & (t < s < T), \\ X_t = x \end{cases}$$

$G \subset \mathbb{R}^N$ bdd open set

$u^\varepsilon(x,t) = P(\exists s \in [t,T] \text{ s.t. } X_s \in G^c)$

(2) $$\begin{cases} -u_t^\varepsilon - \frac{\varepsilon^2}{2}\Delta u^\varepsilon - b(x)\cdot Du^\varepsilon = 0 & \text{in } G \times (0,T), \\ u^\varepsilon(x,t) = \begin{cases} 0 & (x \in G,\ t = T) \\ 1 & (x \in \partial G,\ 0 \leq t \leq T) \end{cases} \end{cases}$$

Question $\quad u^\varepsilon(x,t) \to ? \quad (\varepsilon \searrow 0)$

strong max. principle $\Rightarrow$ $0 < u^\varepsilon < 1$ in $G \times (0,T)$.

logarithmic transform

$$u^\varepsilon = e^{-\frac{v^\varepsilon}{\varepsilon^2}} \quad (v^\varepsilon = -\varepsilon^2 \log u^\varepsilon)$$

$$u_t^\varepsilon = -\frac{1}{\varepsilon^2} v_t^\varepsilon u^\varepsilon, \qquad Du^\varepsilon = -\frac{1}{\varepsilon^2} Dv^\varepsilon u^\varepsilon,$$

$$D^2 u^\varepsilon = \left[-\frac{1}{\varepsilon^2} D^2 v^\varepsilon + \frac{1}{\varepsilon^4} Dv^\varepsilon \otimes Dv^\varepsilon\right] u^\varepsilon.$$

$$u^\varepsilon \left\{ \frac{1}{\varepsilon^2} v_t^\varepsilon + \frac{\varepsilon^2}{2}\left[\frac{1}{\varepsilon^2}\Delta v^\varepsilon - \frac{1}{\varepsilon^4}|Dv^\varepsilon|^2\right] + b \cdot \frac{1}{\varepsilon^2} Dv^\varepsilon \right\} = 0$$

in $Q \equiv G \times (0,T)$

$$\therefore \quad -v_t^\varepsilon - \frac{\varepsilon^2}{2}\Delta v^\varepsilon + \frac{1}{2}|Dv^\varepsilon|^2 - b\, Dv^\varepsilon = 0 \qquad \text{in } Q.$$

i)'

<u>Theorem</u> $u^{\varepsilon}(x,t) = e^{-\frac{I(x,t)+o(1)}{\varepsilon^2}}$ as $\varepsilon \searrow 0$, where

$$I(x,t) = \inf\left\{ \frac{1}{2}\int_t^{\tau} |\dot{\xi}(s) - b(\xi(s))|^2\, ds \ : \quad t \leq \tau \leq T, \quad \xi \in C[t,\tau], \right.$$

$$\left. \xi(t) = x, \quad \xi(\tau) \in \partial G, \quad \dot{\xi} \in L^2(0,\tau) \right\}.$$

ε

$$w^\varepsilon(x,t) = v^\varepsilon(x, T-t) ,$$

$$(3) \quad \begin{cases} w^\varepsilon_t - \dfrac{\varepsilon^2}{2}\Delta w^\varepsilon + \dfrac{1}{2}|Dw^\varepsilon|^2 - b\cdot Dw^\varepsilon = 0 & \text{in } Q, \\ w^\varepsilon(x,t) = \begin{cases} +\infty & (x\in G,\ t=0), \\ 0 & (x\in\partial G,\ 0\le t\le T). \end{cases} \end{cases}$$

$$z^\varepsilon(x,t) = -e^{-w^\varepsilon} \left(= -[u^\varepsilon(x,T-t)]^{\varepsilon^2} \right), \qquad z^\varepsilon_t = (-w^\varepsilon_t)z^\varepsilon, \qquad Dz^\varepsilon = (-Dw^\varepsilon)z^\varepsilon,$$

$$D^2 z^\varepsilon = (-D^2 w^\varepsilon + Dw^\varepsilon \otimes Dw^\varepsilon)z^\varepsilon,$$

$$0 = w^\varepsilon_t - \frac{\varepsilon^2}{2}\Delta w^\varepsilon \cdots = -\frac{z^\varepsilon_t}{z^\varepsilon} - \frac{\varepsilon^2}{2}\left(|Dw^\varepsilon|^2 - \frac{\Delta z^\varepsilon}{z^\varepsilon}\right) + \frac{1}{2}\left|\frac{Dz^\varepsilon}{z^\varepsilon}\right|^2 - b\cdot\left(-\frac{Dz^\varepsilon}{z^\varepsilon}\right)$$

$$= -\frac{1}{z^\varepsilon}\left\{ z^\varepsilon_t - \frac{\varepsilon^2}{2}\Delta z^\varepsilon + \frac{\varepsilon^2}{2} z^\varepsilon \left|\frac{Dz^\varepsilon}{z^\varepsilon}\right|^2 - \frac{1}{2}\frac{1}{z^\varepsilon}|Dz^\varepsilon|^2 - b\cdot Dz^\varepsilon \right\}.$$

$$\therefore \quad (4) \quad \begin{cases} z^\varepsilon_t - \dfrac{\varepsilon^2}{2}\Delta z^\varepsilon - \dfrac{1-\varepsilon^2}{2z^\varepsilon}|Dz^\varepsilon|^2 - b\cdot Dz^\varepsilon = 0 & \text{in } Q, \\ z^\varepsilon(x,t) = \begin{cases} 0 & (x\in G,\ t=0), \\ -1 & (x\in\partial G,\ 0\le t\le T) \end{cases} \end{cases}$$

<u>Lemma 1</u> $+\infty \ge w^\varepsilon \ge 0$, $\quad -1 \le z^\varepsilon \le 0 \quad$ on $\overline{Q}$.

<u>Proof</u> $\quad 0 \le u^\varepsilon \le 1$ on $\overline{Q}$. ∎

<u>Lemma 2</u> $\quad \exists\, C_1 > 0$ s.t. $\quad w^\varepsilon \le \dfrac{C_1}{t}$ (for $0<\varepsilon\le 1$ on $\overline{G}\times(0,T]$).

<u>Proof</u> Fix $x_0 \notin \overline{G}$. By translation, we may assume $x_0 = 0$. Let $g(x,t) = K\dfrac{|x|}{t}$ with $K>0$. Then

$$g_t = -\frac{K|x|}{t^2}, \qquad Dg = \frac{K}{t}\frac{x}{|x|}, \qquad D^2 g = \frac{K}{t|x|}I - \frac{K}{t}\frac{x\otimes x}{|x|^3}.$$

ξ

So,

$$g_t - \frac{\varepsilon^2}{2}\Delta g + \frac{1}{2}|Dg|^2 - b\cdot Dg = -\frac{K}{t^2}|x| - \frac{\varepsilon^2}{2}\frac{K(N-1)}{t|x|} + \frac{1}{2}\frac{K^2}{t^2} - b\cdot\frac{Kx}{t|x|}$$

$$\geq + \frac{K}{t^2}\left\{\frac{K}{2} - |x| - \frac{\varepsilon}{2}\frac{t}{|x|}(N-1) - \|b\|_\infty t\right\}.$$

Choose $K > 0$ big enough so that

$$K > 2\left(|x| + \frac{1}{2}\frac{t}{|x|}(N-1) + \|b\|_\infty t\right) \quad \text{for } (x,t) \in \overline{G}\times[0,T]$$

Then, for $0 \leq \varepsilon \leq 1$,

$$\begin{cases} g_t - \frac{\varepsilon^2}{2}\Delta g + \frac{1}{2}|Dg|^2 - b\cdot Dg > 0 & \text{in } Q, \\ g(x,t)\begin{cases} = +\infty & (x\in\overline{G},\ t=0), \\ > 0 & (x\in\partial G,\ 0\leq t\leq T). \end{cases} \end{cases}$$

Compare $w^\varepsilon(x,t)$ and $g(x,t-\delta)$ on $\overline{G}\times[\delta,T]$, to conclude

$$w^\varepsilon(x,t) \leq g(x,t) \quad \text{on } \overline{G}\times(0,T] \text{ for } 0<\varepsilon\leq 1$$

■

Define

$$\overline{w}(x,t) = \lim_{\varepsilon\searrow 0}\left(\sup_{0<\delta\leq\varepsilon} w^\delta(x,t)\right)^* = \inf_{\varepsilon>0}\left(\sup_{0<\delta\leq\varepsilon} w^\delta(x,t)\right)^*;$$

$$\overline{z}(x,t) = \lim_{\varepsilon\searrow 0}\left(\sup_{0<\delta\leq\varepsilon} z^\delta(x,t)\right)^*,$$

$$\underline{w}(x,t) = \lim_{\varepsilon\searrow 0}\left(\inf_{0<\delta\leq\varepsilon} w^\delta(x,t)\right)_* = \sup_{\varepsilon>0}\left(\inf_{0<\delta\leq\varepsilon} w^\delta(x,t)\right)_*,$$

$$\underline{z}(x,t) = \lim_{\varepsilon\searrow 0}\left(\inf_{0<\delta<\varepsilon} z^\delta(x,t)\right)_*.$$

By definition,

$$\underline{w}(x,t) \leq \overline{w}(x,t) \qquad \text{on } \overline{Q},$$

$$\underline{z}(x,t) \leq \overline{z}(x,t) \qquad \text{on } \overline{Q}.$$

Note also that $\overline{z}(x,t) = -e^{-\overline{w}(x,t)}$, ...

Lemma 3. $\quad -1 \leq \underline{z} \leq \overline{z} \leq 0 \quad$ on $\overline{Q}$,

$$0 \leq \underline{w} \leq \overline{w} \leq \frac{C_1}{t} \qquad \text{on } \overline{G} \times (0,T].$$

Generalizations of Prop. 4.2 and 4.3 gu(a)rantee:

Lemma 4. (i) $\overline{z}$ and $\underline{z}$ are, resp., sub- and super-solutions of the Dirichlet (in the viscosity sense) problem

$$(5) \qquad \begin{cases} -z(z_t - b\cdot Dz) + \frac{1}{2}|Dz|^2 = 0 & \text{in } Q, \\ z(x,t) = 0 \quad (x \in G,\ t=0), \\ \qquad\ \ = -1 \quad (x \in \partial G,\ 0 \leq t < T). \end{cases}$$

(ii) $\overline{w}$ and $\underline{w}$ are, resp., sub- and supersolutions of

$$(6) \qquad \begin{cases} w_t + \frac{1}{2}|Dw|^2 - b\cdot Dw = 0 & \text{in } Q, \\ w = 0 \quad (x \in \partial G,\ 0 < t < T). \end{cases}$$

Remark. Notice that the sets $\overline{G} \times [0,T)$ and $\overline{G} \times (0,T)$ are locally compact.

<u>Lemma 5</u> $\quad \bar{w}(x,t) \leq C_2 \dfrac{\mathrm{dist}(x, \partial G)}{t} \qquad \forall (x,t) \in \bar{G} \times (0,T)$

for some $C_2 > 0$.

<u>Proof</u> Fix $0<\delta<1$, (and $x_0 \in \mathbb{R}^N$ so that $\mathrm{dist}(x_0, G) = \delta$.) By translation, we may assume $x_0 = 0$. Define $g(x,t) = K\dfrac{|x|}{t}$ with $K > 0$. Then

$$g_t + \frac{1}{2}|Dg|^2 - b\cdot Dg = -\frac{K|x|}{t^2} + \frac{1}{2}\left(\frac{K}{t}\right)^2 - b\cdot\frac{Kx}{t|x|} \geq$$

$$\geq \frac{K}{t^2}\left\{\frac{K}{2} - |x| - \|b\|_\infty t\right\} \geq \frac{K}{t^2}\left\{\frac{K}{2} - \mathrm{diam}\, G - 1 - \|b\|_\infty T\right\}.$$

Choose $K > 0$ so large that $K > 2(\mathrm{diam}\, G + 1 + \|b\|_\infty T)$.

Now, set

$$f(x,t) = g(x, t-\delta) + \frac{\delta}{T-t}.$$

Then, $f \in C^1(\bar{G} \times (\delta, T))$ and satisfies

$$f_t + \frac{1}{2}|Df|^2 - b(x)\cdot Df > 0 \quad \text{in } \bar{G}\times(\delta,T)$$

$$f(x,t) > 0 \quad \text{for } (x,t) \in \partial G \times (\delta, T)$$

$$(\bar{w} - f)(x,t) \to -\infty \quad \text{uniformly in } x \in \bar{G} \text{ as } t \begin{cases} \searrow \delta, \\ \nearrow T. \end{cases}$$

This implies :

$$\bar{w} - f \leq 0 \quad \text{in } \bar{G} \times (\delta, T).$$

$$\therefore \quad \bar{w}(x,t) \leq K\, \mathrm{dist}(x, \partial G)/t. \qquad \blacksquare$$

(E)

Lemma 6. $\quad \underline{Z}(x,0) = 0 \quad$ for $x \in G$.

Proof. Fix $x_0 \in G$ and $\delta > 0$. We show that $\underline{Z}(x_0, 0) \geq -\delta$. By translation, we may assume $x_0 = 0$.

Define

$$f(x,t) = -\delta - K|x|^2 - Lt \quad , \text{with } K, L > 0.$$

Choose K so large that

$$(f(x,t) \leq) \; -\delta - K|x|^2 < -1 \quad \text{for } x \in \partial G.$$

Compute

$$-f\,(f_t - b\cdot Df) + \tfrac{1}{2}|Df|^2 \leq (\delta + K|x|^2 + Lt)(-L + \|b\|_\infty 2K|x|) + \tfrac{1}{2}(2K|x|)^2$$

$$\leq -\frac{\delta L}{2} + \frac{1}{2}(2K|x|)^2$$

Choose L so large that

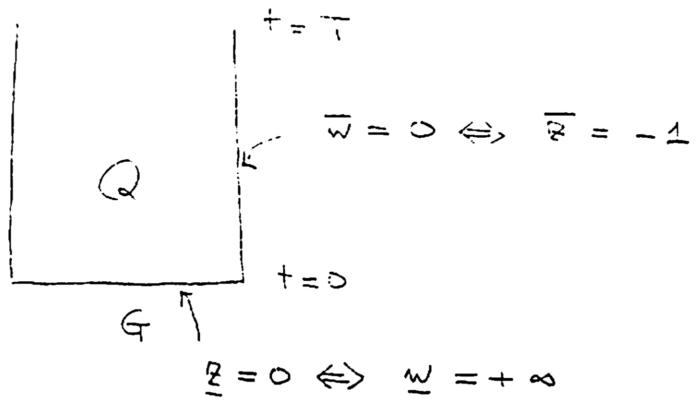
$$\frac{L}{2} \geq 2K|x|\,\|b\|_\infty, \qquad L > \frac{4K^2}{\delta}|x|^2 \qquad \text{for } x \in \overline{G}.$$

Then

$$\begin{cases} -f(f_t - b\cdot Df) + \frac{1}{2}|Df|^2 < 0 \quad (, f<0,\ f_t - b\cdot Df < 0) \quad \text{in } \overline{G}\times[0,T), \\ f(x,t) < \begin{cases} 0 & (x \in G,\ t=0), \\ -1 & (x \in \partial G,\ 0 \leq t < T). \end{cases} \end{cases}$$

Define $g(x,t) = f(x,t) - \frac{\varepsilon}{T-t}$, with $\varepsilon > 0$.

We easily conclude that $\underline{z} \geq g$ on $\overline{G} \times [0,T)$, from which $\underline{z}(0,\cdot) \geq -\delta$. ■



<u>Theorem 1</u>. Let u (resp. v) : $G \times (0,T) \to \mathbb{R}$ be a subsolution (resp. supersolution) of

(7) $\quad u_t + \frac{1}{2}|Du|^2 - b(x) \cdot Du = 0 \quad$ in $\quad G \times (0,T)$.

Assume that

(8) $\quad u^*(x,t) \leq 0 \quad$ (resp. $\quad v_*(x,t) \geq 0$) $\quad \forall (x,t) \in \partial G \times (0,T)$,

(9) $\quad v_*(x,0) = +\infty \quad (x \in G)$.

Then, $u_* \leq v_*$ and $u^* \leq v^*$ on $\overline{G} \times [0,T)$.

<u>Proof</u> Fix $\delta > 0$, and define $w(x,t) = v_*(x, t-\delta)$ for $(x,t) \in \overline{G} \times [\delta, T)$. Then

$$(u^* - w)(x,t) \leq 0 \quad \text{for } (x,t) \in G \times [\delta, T) \cup \partial G \times [\delta, T).$$

By the standard comparison results,

$$u^*(x,t) \leq w(x,t) = v_*(x,t-\delta) \qquad \forall (x,t) \in \overline{G} \times [\delta, T).$$

This can be stated as

$$u^*(x,t+\delta) \leq v_*(x,t) \qquad \forall (x,t) \in \overline{G} \times [0, T-\delta).$$

From these we conclude

$$u^*(x,t) \leq \varliminf_{\delta \searrow 0} v_*(x,t-\delta) \leq v^*(x,t) \qquad \forall (x,t) \in \overline{G} \times (0,T),$$

$$u_*(x,t) \leq \varlimsup_{\delta \searrow 0} u^*(x,t+\delta) \leq v_*(x,t) \qquad \forall (x,t) \in \overline{G} \times [0,T).$$

Clearly we have

$$u^*(x,0) \leq +\infty = v^*(x,0) \qquad \text{for} \quad x \in \overline{G}.$$

$\Rightarrow$ ~~Insert ③'.~~

For $n \in \mathbb{N}$, choose $\varphi_n \in C(G \times \{T\} \cup \partial G \times [0,T])$ so that $\varphi_n(x,t) = 0$ for $(x,t) \in \partial G \times [0,T]$ and $\varphi_n(x,T) \nearrow +\infty$ for $x \in G$ as $n \to \infty$. Define

$$I_n(x,t) = \inf \Big\{ \frac{1}{2} \int_t^\tau |\dot\xi(s) - b(\xi(s))|^2 \, ds + \varphi_n(\xi(\tau)) : \ t \leq \tau \leq T,$$

$$\xi \in C[t,\tau], \ \xi(t) = x, \ \xi(\tau) \in \partial G, \ \dot\xi \in L^2(t,\tau) \Big\}.$$

We note that

$$I(x,t) = \inf\left\{ \frac{1}{2}\int_t^{\tau} |\dot{\xi}(s) - b(\xi(s))|^2 ds : t \le \tau \le T, \quad \xi \in C[t,\tau], \right.$$

$$\left. \xi(t) = x, \quad \xi(\tau) \in \partial G, \quad \dot{\xi} \in L^2(t,\tau) \right\}$$

$$= \inf\left\{ \frac{1}{2}\int_t^{\tau\wedge T} |\alpha(s) - b(X(s))|^2 ds + \chi(X(\tau\wedge T)) : \alpha \in L^2(t,T), \right.$$

$$\left. \dot{X}(s) = \alpha(s) \quad (t \le s \le T), \quad X(t) = x \right\},$$

where $\tau = \inf\{s \ge t : X(s) \in \partial G\} \left(\in [t,+\infty]\right)$ and

$$\chi(x) = \begin{cases} 0 & (x \in \partial G) \\ +\infty & (x \in G). \end{cases}$$

<u>Lemma 7</u> (i) $I(x,t) \ge 0 \quad \forall (x,t) \in \overline{G} \times [0,T)$.

(ii) $\forall \delta > 0, \exists C_\delta > 0$ s.t. $I(x,t) \le C_\delta \quad \forall (x,t) \in \overline{G} \times [\delta, T)$.

(iii) $I(y,s) \to 0$ as $(y,s) \to (x,t) \in \partial G \times (0,T)$.

(iv) $I(y,s) \to +\infty$ as $s \searrow 0$, $y \to x \in G$.

(v) I is a solution of

$$-I_t + \max_{q \in \mathbb{R}^N}\left\{ -q \cdot DI - \frac{1}{2}|q - b(x)|^2 \right\} = 0 \quad \text{in } G \times (0,T)$$

$$\|$$

$$-I_t + \frac{1}{2}|DI|^2 - b \cdot DI$$

Define $J(x,t) = I(x, T-t)$.

<u>Theorem 2</u> $\quad J_* \le \underline{w} \le \overline{w} \le J^*$ on $\overline{G}\times[0,T)$.

<u>Proof</u> $\quad J$ (resp. $\overline{w}$ and $\underline{w}$) is a solution (resp. sub- and supersolution) of

$$u_t + \frac{1}{2}|Du|^2 - b\cdot Du = 0 \quad \text{in } G\times(0,T),$$

and satisfies

$$u^*(x,t) = u_*(x,t) = 0 \qquad (x\in\partial G,\ 0<t<T),$$

$$u_*(x,0) = +\infty \qquad (x\in G),$$

with $u = J$, $\overline{w}$ and $\underline{w}$. From Theorem 1,

$$J_* \le \underline{w}, \qquad \overline{w} \le J^* \quad \text{on } \overline{G}\times[0,T).$$

Also, we have $\underline{w} \le \overline{w}$ on $\overline{G}\times[0,T)$. ∎

<u>Corollary 1</u> $\quad$ At any point (x,t) of continuity of I, we have

$$v^\varepsilon(x,t) \to I(x,t)$$

$$\left(\text{or} \quad u^\varepsilon(x,t) = e^{-\frac{I(x,t)+o(1)}{\varepsilon^2}}\right).$$

Theorem 3 $\quad I \in C(\overline{G} \times (0,T))$.

Proof $\quad$ Look at the definition of I.

Remark $\quad \forall \delta > 0, \quad I \in \mathrm{Lip}(\overline{G} \times (0, T-\delta))$.

Proof of Remark $\quad$ Define $\quad H(x,p) = \frac{1}{2}|p|^2 - b(x)\cdot p$.

We show $u = J$ is Lip on $\overline{G} \times (\delta, T)$ for $\delta > 0$.

From the proof of Theorem 1, we see

$$(10) \qquad u^*(x, t_2) \leq u_*(x, t_1) \quad \text{if } x \in \overline{G} \ \& \ 0 \leq t_1 < t_2 < T.$$

Fix $\delta \in (0, T)$ and $\delta \leq t_1 < t_2 < T$. We want to prove

$$(11) \qquad u^*(x, t_1) \leq u_*(x, t_2) + L(t_2 - t_1) \qquad \forall x \in \overline{G}$$

for some $L > 0$ independent of t_1 and t_2.

We note :

$$(12) \qquad \exists A > 1, \ \exists B > 0 \quad \text{s.t.}$$

$$(1 + Ar) H(x, \tfrac{p}{1+r}) \leq H(x,p) + Br \qquad 0 < \forall r \leq 1.$$

Let $-C_1$ be a lower bound of H, i.e. $H(x,p) + C_1 \geq 0$

Define $\quad H_1 = H(x,p) + C_1$. $\quad$ Then

E

$$(1+Ar)\, H_1\!\left(x, \frac{p}{1+r}\right) \leqslant H(x,p) + Br + AC_1 r \qquad \forall\, 0 < r \le 1.$$
$$= H(x,p) + B_1 r \qquad \forall\, 0 < r \le 1,$$

where $B_1 = B + AC_1$. We set

$$\alpha = \frac{t_2 - t_1}{\delta} \ (>0), \qquad a = \frac{A-1}{2} \ (>0), \qquad \beta = \frac{1}{a}\alpha \ (>0).$$

We assume $\beta \leqslant 1$ i.e. $t_2 - t_1 \leqslant \frac{\delta(A-1)}{2}$. Note that $2a+1 = A$, $\alpha = a\beta$. Define $v : \overline{G} \times [0, \frac{T}{1+\alpha}) \to [0, \infty]$ by

$$v(x,t) = (1+\beta)\, u_*(x, (1+\alpha)t) + B_1 \beta t \quad .$$

Formally we have

$$v_t = (1+\alpha)(1+\beta)\, u_{*,t} + B_1\beta, \qquad Dv = (1+\beta) Du_*, \quad \text{and}$$
$$0 \leqslant u_{*,t} + H(x, Du_*) = \frac{v_t - B_1\beta}{(1+\alpha)(1+\beta)} + H\!\left(x, \frac{Dv}{1+\beta}\right).$$

Therefore,

$$0 \leqslant v_t - B_1\beta + (1+\alpha)(1+\beta)\, H_1\!\left(x, \frac{Dv}{1+\beta}\right)$$
$$\leqslant v_t - B_1\beta + (1+a\beta)(1+\beta)\, H_1(\ ) \leqslant v_t - B_1\beta + [1 + \overset{A}{\overset{\shortparallel}{(2a+1)}}\beta]\, H_1(\ )$$
$$\leqslant v_t - B_1\beta + H(x, Dv) + B_1\beta = v_t + H(x, Dv) \qquad \text{in } G \times \left(0, \frac{T}{1+\alpha}\right).$$

By Theorem 1,

$$u^*(x,t) \leqslant v(x, t-\varepsilon) \qquad \forall \varepsilon > 0, \quad \forall (x,t) \in \overline{G} \times \left(\varepsilon, \frac{T}{1+\alpha}\right).$$

(3)

We take $t = t_1$ in this formula:

$$u^*(x,t_1) \le v(x,t_1-\varepsilon) = (1+\beta)\,u_*(x,(1+\alpha)(t_1-\varepsilon)) - B_1\beta t_1$$

Since $(1+\alpha)t_1 = t_1 + \frac{t_1}{\delta}(t_2-t_1) > t_2$, choosing ε small enough we have

$$(1+\alpha)(t_1-\varepsilon) \ge t_2,$$

and so, by (10)

$$u^*(x,t_1) \le (1+\beta)\,u_*(x,t_2) + B_1\beta T$$

$$\le u_*(x,t_2) + (M_\delta + B_1 T)\frac{t_2-t_1}{a\delta},$$

where $M_\delta = \sup_{\overline{G}\times[\delta,T]} u_*$, which proves (11).

(10) and (11) show that $u \in C(\overline{G}\times(\delta,T))$ and $t \to u(x,t)$ is Lipschitz continuous on (δ,T) with Lipschitz constant L. Thus

$$H(x,Du) \le L \quad \text{in} \quad G\times(\delta,T).$$

This and the fact that $H(x,p)\to+\infty$ uniformly for $x\in\overline{G}$ as $|p|\to\infty$ yield the Lipschitz continuity of

u in the x variable. $\blacksquare$

<u>Check of (12)</u>

$$(1+Ar)\, H_1\left(x, \frac{p}{1+r}\right) = \frac{1+Ar}{(1+r)^2}\left\{\frac{1}{2}|p|^2 - b\cdot p\,(1+r) + C_1(1+r)^2\right\}$$

$$\leq \begin{cases} \dfrac{1+Ar}{(1+r)^2}\Big[H_1(x,p) + r(r+2)\,C_1\Big] & \text{if } b\cdot p \geq 0, \\[2ex] \dfrac{1+Ar}{(1+r)^2}\Big[H_1(x,p) - b\cdot p\, r + r(r+2)\,C_1\Big] \leq \end{cases}$$

$$\leq \frac{1+Ar}{(1+r)^2}\left[H_1(x,p) + \left(\frac{|p|^2}{4} + \|b\|_\infty^2\right) r + r(r+2)\,C_1\right] \leq$$

$$\leq \frac{1+Ar}{(1+r)^2}\left[\left(1+\frac{r}{2}\right) H_1(x,p) + \left(3C_1 + \|b\|_\infty^2\right) r\right] \qquad \text{if } b\cdot p < 0.$$

Choose $A = \frac{3}{2}$. Then

$$1 + Ar \leq (1+Ar)\left(1+\frac{r}{2}\right) = 1 + 2r + \frac{3}{4}r^2 \leq (1+r)^2.$$

Hence

$$(1+Ar)\, H_1\left(x, \frac{p}{1+r}\right) \leq H_1(x,p) + C_2\, r \qquad 0 < r \leq 1$$

(12) follows easily from this.

①

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Differential games

$$(1) \quad \begin{cases} \dot{X}(s) = g(X(s), s, \alpha(s), \beta(s)) & t \le s \le T, \\ X(t) = x . \end{cases}$$

$T > 0$, $x \in \mathbb{R}^N$, $\alpha : [t, T] \to A$, $\beta : [t, T] \to B$,
$A \subseteq \mathbb{R}^m$, $B \subset \mathbb{R}^n$ compact

$\mathcal{C}_I(t) = \{ \alpha : [t, T] \to A \text{ measurable} \}$, controls for I,
$\mathcal{C}_{II}(t) = \{ \beta : [t, T] \to B \text{ measurable} \}$. controls for II.

$$(2) \quad J(x, t, \alpha, \beta) = \int_t^T f(X(s), s, \alpha(s), \beta(s))\, ds + h(X(T))$$

cost, payoff

(A) $\exists M > 0$ s.t.

$|f(x,t,a,b)|$, $|g(x,t,a,b)|$, $|h(x)| \le M$

$|f(x,t,a,b) - f(y,t,a,b)| \le M|x-y|$,

$|g(x,t,a,b) - g(y,t,a,b)| \le M|x-y|$,

$|h(x) - h(y)| \le M|x-y|$

$\forall x, y \in \mathbb{R}^N$, $0 \le t \le T$, $a \in A$, $b \in B$

$\xi : \mathcal{C}_{II}(t) \to \mathcal{C}_I(t)$ a strategy for I

$\Longleftrightarrow$ $\beta(s) = \hat{\beta}(s)$ a.e. on $[t, \tau)$ implies $\xi(\beta)(s) = \xi(\hat{\beta})(s)$ a.e. on $[t, \tau)$

$\gamma : \mathcal{C}_I(t) \to \mathcal{C}_{II}(t)$ a strategy for II

$\Leftrightarrow$ $\alpha(s) = \hat{\alpha}(s)$ a.e. on $[t, \tau)$ implies $\gamma(\alpha)(s) = \gamma(\hat{\alpha})(s)$ a.e. on $[t, \tau)$

$\mathcal{S}_I(t)$, $\mathcal{S}_{II}(t)$ the sets of strategies for I and II

$\Rightarrow V^-(x,t) = \inf_{\gamma \in \mathcal{S}_{II}} \sup_{\alpha \in \mathcal{C}_I} J(x,t,\alpha,\gamma(\alpha))$ the lower value

$\Rightarrow V^+(x,t) = \sup_{\xi \in \mathcal{S}_I} \inf_{\beta \in \mathcal{C}_{II}} J(x,t,\xi(\beta),\beta)$ the upper value

__Theorem 1__ V^+ and V^- are bounded and Lipschitz continuous on $\mathbb{R}^N \times [0,T]$

__Theorem 2__ (DPP) For any $t < \sigma < T$ we have

(5) $$V^-(x,t) = \inf_{\gamma \in \mathcal{S}_{II}} \sup_{\alpha \in \mathcal{C}_I} \left\{ \int_t^\sigma f(X(s), s, \alpha(s), \gamma(\alpha)(s))\,ds + V^-(X(\sigma), \sigma) \right\},$$

(6) $$V^+(x,t) = \sup_{\xi \in \mathcal{S}_I} \inf_{\beta \in \mathcal{C}_{II}} \left\{ \int_t^\sigma f(X(s), s, \xi(\beta)(s), \beta(s))\,ds + V^+(X(\sigma), \sigma) \right\}.$$

__Proof__ We only prove (5). We set

$$W(x,t) = \inf_{\gamma \in \mathcal{S}_{II}} \sup_{\alpha \in \mathcal{C}_I} \left\{ \int_t^\sigma f(X(s), s, \alpha(s), \gamma(\alpha)(s))\,ds + V^-(X(\sigma), \sigma) \right\}$$

Fix $\varepsilon > 0$ and choose $\tilde{\gamma} \in \mathcal{S}_{II}$ so that

③

$$W(x,t)+\varepsilon > \sup_{\alpha \in \mathcal{C}_I} \left\{ \int_t^\sigma f(X(s), s, \alpha(s), \tilde{\eta}(\alpha)(s))\, ds + V^-(X(\sigma), \sigma) \right\}.$$

For each $(y,\tau) \in \mathbb{R}^N \times [0,T)$ we choose $\zeta_{y,\tau} \in \mathcal{S}_{II}(\tau)$

so that for $X(s) = X(s; y, \tau, \alpha, \zeta_{y,\tau}(\alpha))$,

$$V^-(y,\tau) + \varepsilon > \sup_{\alpha \in \mathcal{C}_I} \left\{ \int_\tau^T f(X(s), s, \alpha(s), \zeta_{y,\tau}(\alpha)(s))\, ds + h(X(T)) \right\}$$

Define $\eta \in \mathcal{S}_{II}(t)$ by

$$\eta(\alpha)(s) = \begin{cases} \tilde{\eta}(\alpha)(s) & (t \leq s < \sigma) \\ \zeta_{X(\sigma),\sigma}(\bar{\alpha})(s) & (\sigma \leq s \leq T), \end{cases}$$

where $\bar{\alpha} \in \mathcal{C}_I(\sigma)$ is defined by $\bar{\alpha} = \alpha|_{[\sigma,T]}$.

Then we have

$$W(x,t) + 2\varepsilon > \int_t^\sigma f(X(s), s, \alpha(s), \eta(\alpha)(s))\, ds + V^-(X(\sigma), \sigma) + \varepsilon$$

$$> \int_t^\sigma f(X(s), s, \alpha(s), \eta(\alpha)(s))\, ds + \int_\sigma^T f(X(s), s, \alpha(s), \eta(\alpha)(s))\, ds + h(X(T)) =$$

$$= \int_t^T f(X(s), s, \alpha(s), \eta(\alpha)(s))\, ds + h(X(T)) \qquad \forall \alpha \in \mathcal{C}_I(t).$$

Hence

$$W(x,t) + 2\varepsilon \geq \sup_{\alpha \in \mathcal{C}_I(t)} \left\{ \int_t^T f(X(s), s, \alpha(s), \gamma(\alpha)(s))\, ds + h(X(T)) \right\} \geq V^-(x,t),$$

and so $W(x,t) \geq V^-(x,t)$.

Next, fix $\varepsilon > 0$ and choose $\tilde{\gamma} \in \mathcal{S}_{II}(t)$ so that

$$V^-(x,t) + \varepsilon \geq \sup_{\alpha \in \mathcal{C}_I(t)} \left\{ \int_t^T f(X(s), s, \alpha(s), \tilde{\gamma}(\alpha)(s))\, ds + h(X(T)) \right\}.$$

Then

$$W(x,t) \leq \sup_{\alpha \in \mathcal{C}_I(t)} \left\{ \int_t^\sigma f(X(s), s, \alpha(s), \tilde{\gamma}(\alpha)(s))\, ds + V^-(X(\sigma), \sigma) \right\}.$$

So, there is $\bar{\alpha} \in \mathcal{C}_I(t)$ so that

$$W(x,t) < \varepsilon + \int_t^\sigma f(X(s), s, \bar{\alpha}(s), \tilde{\gamma}(\bar{\alpha})(s))\, ds + V^-(X(\sigma), \sigma).$$

Define $\gamma \in \mathcal{S}_{II}(\sigma)$ by $\gamma(\alpha) = \tilde{\gamma}(\tilde{\alpha})\big|_{[\sigma,T]}$, where

$$\tilde{\alpha}(s) = \begin{cases} \bar{\alpha}(s) & (t \leq s < \sigma) \\ \alpha(s) & (\sigma \leq s \leq T). \end{cases}$$

Now,

$$V^-(X(\sigma), \sigma) \leq \sup_{\alpha \in \mathcal{C}_I(\sigma)} \left\{ \int_\sigma^T f(X(s), s, \alpha(s), \gamma(\alpha)(s))\, ds + h(X(T)) \right\},$$

and so, there is $\hat{\alpha} \in \mathcal{C}_I(\sigma)$ so that

$$V^-(X(\sigma),\sigma) < \varepsilon + \int_\sigma^T f(X(s), s, \hat{\alpha}(s), \gamma(\hat{\alpha})(s))\,ds + h(X(T)).$$

Define $\alpha \in \mathcal{C}_I(t)$ by

$$\alpha(s) = \begin{cases} \bar{\alpha}(s) & (t \leq s < \sigma) \\ \hat{\alpha}(s) & (\sigma \leq s \leq T) \end{cases}$$

Then

$$W(x,t) < 2\varepsilon + \int_t^T f(X(s), s, \alpha(s), \gamma(\alpha)(s))\,ds + h(X(T))$$

$$\leq 2\varepsilon + \sup_{\alpha \in \mathcal{C}_I(t)} \left\{ \int_t^T f(X(s), s, \alpha(s), \tilde{\gamma}(\alpha)(s))\,ds + h(X(T)) \right\} <$$

$$< 3\varepsilon + V^-(x,t).$$

Thus, $W(x,t) \leq V^-(x,t)$. $\blacksquare$

Theorem 3 (i) V^- is a solution of the lower Isaacs equation

$$(7) \qquad -u_t + \min_{a \in A} \max_{b \in B} \{-g(x,t,a,b)\cdot Du - f(x,t,a,b)\} = 0 \quad \text{in } \mathbb{R}^N \times (0,T)$$

(ii) V^+ is a solution of the upper Isaacs equation

$$(8) \qquad -u_t + \max_{b \in B} \min_{a \in A} \{-g(x,t,a,b)\cdot Du - f(x,t,a,b)\} = 0 \quad \text{in } \mathbb{R}^N \times (0,T).$$

6

Define

$$H^-(x,t,p) = \min_{a\in A} \max_{b\in B} \{-g(x,t,a,b) \cdot p - f(x,t,a,b)\},$$

$$H^+(x,t,p) = \max_{b\in B} \min_{a\in A} \{ \qquad \}.$$

We have

$$H^- \geq H^+,$$

$$|H^\pm(x,t,p) - H^\pm(y,s,p)| \leq M(|x-y| + |t-s|)(|p|+1),$$

$$|H^\pm(x,t,p) - H^\pm(x,t,q)| \leq M|p-q|$$

Corollary 1 (i) $V^- \leq V^+$ in $\mathbb{R}^N \times (0,T]$

(ii) If the Isaacs condition holds, i.e. $H^+ = H^-$,

then $V^+ = V^-$ (the game has value).

Proof. We only prove (ii). Let $\varphi \in C^1(\mathbb{R}^N \times (0,T))$, $(\bar{x}, \bar{t}) \in \mathbb{R}^N \times (0,T)$, and assume

$$V^- \le \varphi, \qquad (V^- - \varphi)(\bar{x}, \bar{t}) = 0,$$
$$-\varphi_t(\bar{x}, \bar{t}) + H^-(\bar{x}, \bar{t}, D\varphi(\bar{x}, \bar{t})) > 2\delta$$

for some $\delta > 0$. For each $\bar{a} \in A$ there is $\bar{b} = \bar{b}(\bar{a}) \in B$ so that

$$-\varphi_t(\bar{x}, \bar{t}) - g(\bar{x}, \bar{t}, \bar{a}, \bar{b}) \cdot D\varphi(\bar{x}, \bar{t}) - f(\bar{x}, \bar{t}, \bar{a}, \bar{b}) > 2\delta.$$

By continuity there is $r(\bar{a}) > 0$ so that

$$-\varphi_t(x, t) - g(x, t, a, \bar{b}) \cdot D\varphi(x, t) - f(x, t, a, \bar{b}) > \delta$$

if $|x - \bar{x}| \le r(\bar{a})$, $|t - \bar{t}| \le r(\bar{a})$, $|a - \bar{a}| \le r(\bar{a})$. By compactness there are $a_1, \dots, a_n \in A$ so that

$$A \subset \bigcup_{i=1}^{n} B(a_i, r(a_i))$$

Set

(c)

$$r_0 = \min_{1 \leq i \leq n} r(a_i),$$

$$A_1 = B(a_1, r(a_1)), \ \dots \ A_i = B(a_i, r(a_i)) \setminus (A_1 \cup \dots \cup A_{i-1}), \dots$$

Define $b : A \to B$ by

$$b(a) = \bar{b}(a_i) \quad \text{if } a \in A_i.$$

Then b is Borel measurable, and

$$-\varphi_t(x,t) - g(x,t,a,b(a)) \cdot D\varphi(x,t) - f(x,t,a,b(a)) > \delta$$

if $|x-\bar{x}| \leq r_0$ and $|t-\bar{t}| \leq r_0$.

Define $\gamma \in S_{II}(\bar{t})$ by $\gamma(\alpha)(t) = b(\alpha(t))$. Set $\sigma = \bar{t} + \min\{r_0, \frac{r_0}{M}\}$ $(> \bar{t})$. Fix any $\alpha \in \mathcal{C}_I(\bar{t})$. Let $X(t) = X(t : \bar{x}, \bar{t}, \alpha, \gamma(\alpha))$. Then,

$$|X(t) - \bar{x}| \leq M|t-\bar{t}| \leq M(\sigma - \bar{t}) \leq r_0 \quad \text{if } \bar{t} \leq t \leq \sigma,$$

and hence

$$-\varphi_t(X(t), t) - g(X(t), t, \alpha(t), \gamma(\alpha)(t)) \cdot D\varphi(X(t), t)$$

$$-f(X(t), t, \alpha(t), \eta(\alpha)(t)) > \delta \quad \text{if} \quad \bar{t} \le t \le \sigma .$$

Integrating this,

$$\varphi(X(\sigma), \sigma) - \varphi(\bar{x}, \bar{t}) + \int_{\bar{t}}^{\sigma} f(X(t), t, \alpha(t), \eta(\alpha)(t))\, dt < -\delta(\sigma - \bar{t})$$

and so

$$V^-(\bar{x}, \bar{t}) > +\delta(\sigma - \bar{t}) + \int_{\bar{t}}^{\sigma} f(X(t), t, \alpha(t), \eta(\alpha)(t))\, dt + V^-(X(\sigma), \sigma)$$

Thus

$$V^-(\bar{x}, \bar{t}) \geq +\delta(\sigma - \bar{t}) + \sup_{\alpha \in \mathcal{C}_I(\bar{t})} \left\{ \int_{\bar{t}}^{\sigma} f(\ )\, dt + V^-(X(\sigma), \sigma) \right\} \geq$$

$$\geq +\delta(\sigma - \bar{t}) + \inf_{\eta \in \mathcal{C}_{II}(\bar{t})} \ \sup_{\alpha \in \mathcal{C}_I(\bar{t})} \left\{ \quad \right\},$$

which contradicts Theorem 2. Hence, V^- is a subsolution of (7).

Representation of solutions

$$(9)\quad \begin{cases} -u_t + H(x,t,Du) = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ u(x,T) = h(x) & x \in \mathbb{R}^N \end{cases}$$

(A2) $\exists\, M > 0$ s.t.

$$\begin{cases} |h(x)| \le M, \quad |h(x) - h(y)| \le M|x-y|, \\ |H(x,t,0)| \le M, \\ |H(x,t,p) - H(y,s,p)| \le M(|x-y| + |t-s|)(|p|+1), \\ |H(x,t,p) - H(x,t,q)| \le M|p-q|, \end{cases}$$

Theorem 4 (9) has a unique bounded solution.

Moreover, there is a constant $L > 0$ such

that $|u(x,t) - u(y,t)| \le L|x-y|$.

Proof Lipschitz continuity Define

$$v(x,y,t) = u^*(x,t) - u_*(y,t)$$

Then

$$-v_t + H(x,t,D_x v) - H(y,t,-D_y v) \le 0 \quad \text{in } \mathbb{R}^{2N} \times (0,T).$$

For $K>0$, we set

$$w(x,y,t) = M|x-y|\, e^{K(T-t)}$$

Then, $w(x,y,T) \geq v(x,y,T)$ and

$$-w_t + H(x,t,D_x w) - H(y,t,-D_y w)$$

$$\geq KM|x-y|\, e^{K(T-t)} - M|x-y| \left(M e^{K(T-t)} + 1 \right)$$

$$= |x-y| \left(M\left(\tfrac{K}{2} - M\right) e^{K(T-t)} + \left(\tfrac{MK}{2} - 1\right) \right) > 0$$

if K is large enough. Thus

$$u(x,t) - u(y,t) \leq M|x-y|\, e^{KT}$$

if K is large enough. ∎

<u>Lemma</u> For any $L > 0$ we have

(10) $$H(x,t,p) = \min_{a\in A} \max_{b\in B} \{-g(x,t,a,b)\cdot p - f(x,t,a,b)\}$$

for $(x,t) \in \mathbb{R}^n \times [0,T]$ and $p \in B(0,L)$, where

$$\begin{cases} A = B(0,L) \subset \mathbb{R}^n, & B = B(0,1) \subset \mathbb{R}^n, \\ g(x,t,a,b) = -Mb, & f(x,t,a,b) = Ma\cdot b - H(x,t,a). \end{cases}$$

<u>Proof</u> We have

~~$H(x,t,p) \leq H(x,t,q) + M|p-q|$~~

and so, if $p \in B(0,L)$, then

$$H(x,t,p) = \min_{q\in B(0,L)} \left(H(x,t,q) + M|p-q| \right)$$

$$= \min_{q\in B(0,L)} \max_{b\in B(0,1)} \left(H(x,t,q) + Mb\cdot(p-q) \right) \quad \blacksquare$$

<u>Define</u> $\hat{H}$ by

$$\hat{H}(x,t,p) = \min_{a\in A} \max_{b\in B} \{-g(x,t,a,b)\cdot p - f(x,t,a,b)\}.$$

Consider

(13)

$$(11)\quad \begin{cases} -u_t + \hat{H}(x,t,Du) = 0 & \text{in } \mathbb{R}^N \times (0,T) \\ u(x,T) = h(x) & \forall x \in \mathbb{R}^N. \end{cases}$$

Theorem 5 The unique sol'n u of (9) is also a (unique) sol'n of (11), and so

$$(12)\quad u(x,t) = \inf_{\gamma \in \mathcal{S}_{II}(t)} \sup_{\alpha \in \mathcal{C}_I(t)} \left\{ \int_t^T f(X(s), s, \alpha(s), \gamma[\alpha](s))\, ds + h(X(T)) \right\}.$$