A CONVERGENCE RESULT FOR THE ERGODIC PROBLEM FOR HAMILTON-JACOBI EQUATIONS WITH NEUMANN TYPE BOUNDARY CONDITIONS

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Abstract. We consider the ergodic (or additive eigenvalue) problem for the Neumann type boundary value problem for Hamilton-Jacobi equations and the corresponding discounted problems. When denoting by u^λ the solution of the discounted problem with discount factor $\lambda > 0$, we establish the convergence of the whole family $\{u^\lambda\}_{\lambda>0}$ to a solution of the ergodic problem, as $\lambda \to 0$, and give a representation formula for the limit function via the Mather measures and Peierls function. As an interesting byproduct, we introduce Mather measures associated with Hamilton-Jacobi equations with the Neumann type boundary condtitions. These results are variants of the main results in the paper "Convergence of the solutions of the discounted equations" by A. Davini, A. Fathi, R. Iturriaga and M. Zavidovique, where they study the same convergence problem on smooth compact manifolds without boundary.

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1. Introduction

We consider the ergodic problem (or additive eigenvalue problem) with Neumann type boundary condition

$$\begin{cases} H(x, Du(x)) = c & \text{in } \Omega, \\ \gamma(x) \cdot Du(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here the problem is to seek for a pair $(u, c) \in C(\overline{\Omega}) \times \mathbb{R}$ such that the above two conditions hold in the viscosity sense. Throughout this article we assume that Ω is a given subset of $\mathbb{R}^n$ and H, γ and g are given functions on $\overline{\Omega} \times \mathbb{R}^n$, $\partial\Omega$ and $\partial\Omega$, respectively, and moreover,

(i) Ω is a bounded open connected subset of $\mathbb{R}^n$, with C^1 boundary,
(ii) $H \in C(\overline{\Omega} \times \mathbb{R}^n, \mathbb{R})$, $g \in C(\partial\Omega, \mathbb{R})$,
(iii) $\gamma \in C(\partial\Omega, \mathbb{R}^n)$ is a vector field oblique to the boundary $\partial\Omega$, that is, $\gamma(x) \cdot \nu(x) > 0$ for all $x \in \partial\Omega$, where $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$,
(iv) H is a convex Hamiltonian, that is, the function $p \mapsto H(x, p)$ is convex on $\mathbb{R}^n$ for every $x \in \overline{\Omega}$,

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(v) H is a coercive Hamiltonian, that is,

$$\lim_{R\to\infty} \inf\{H(x,p) \,:\, (x,p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_R)\} = \infty, \tag{2}$$

where B_R denotes the open ball of $\mathbb{R}^n$ with radius R and center at the origin.

Furthermore, solutions, sub- and super-solutions of Hamilton-Jacobi equations here mean those in the viscosity sense (see [3, 1, 5]).

The following existence and uniqueness result has been established.

Theorem 1. (i) *There exists a solution* $(u,c) \in \mathrm{Lip}\,(\overline{\Omega}) \times \mathbb{R}$ *of* (1) *and* (ii) *the constant* c *is unique in the sense that if* $(v,d) \in C(\overline{\Omega}) \times \mathbb{R}$ *is another solution of* (1)*, then* $d = c$.

The above theorem is valid without the convexity assumption on the Hamiltonian H. We refer to [12, Corollary 3.7, Theorem 6.1], [4, Theorem 1.2, (i)] for a proof of the above theorem.

We call the constant c, given by the above theorem, the *critical value* (or additive eigenvalue) for (1) and denote it by c_H.

It is a classical observation (see [16] for the case of periodic settings) that the discounted problem with discount factor $\lambda > 0$

$$\begin{cases} \lambda u^\lambda(x) + H(x, Du^\lambda(x)) = 0 & \text{in } \ \Omega, \\ \gamma(x) \cdot Du^\lambda(x) = g(x) & \text{on } \ \partial\Omega \end{cases} \tag{3}$$

gives an efficient approach to solving problem (1). This problem (3) is a standard boundary value problem of the Neumann type for the Hamilton-Jacobi equation $\lambda u^\lambda + H(x, Du^\lambda) = 0$ in Ω. As in [16, 4], one way to establish the existence result of Theorem 1 is the following. We solve (3), to get a unique solution u^λ for every $\lambda > 0$, next show that the families $\{\lambda u^\lambda\}_{\lambda>0}$ and $\{u^\lambda\}_{\lambda>0}$ are respectively uniformly bounded and equi-Lipschitz continuous on $\overline{\Omega}$, and then define $c \in \mathbb{R}$ and $u \in \mathrm{Lip}\,(\overline{\Omega})$ by taking the limit (uniform limit on $\overline{\Omega}$), along a suitable sequence $\lambda = \lambda_j \to 0$,

$$\begin{cases} c = -\lim \lambda u^\lambda(x), \\ u(x) = \lim (u^\lambda(x) - \min_{\overline{\Omega}} u^\lambda), \end{cases}$$

to find a solution (u,c) of (1).

Recently there has been much interest in the question if for the solution $u^\lambda \in \mathrm{Lip}\,(\overline{\Omega})$, the following convergence holds or not:

$$\lim_{\lambda\to 0} (u^\lambda(x) + \lambda^{-1} c_H) = u(x) \quad \text{ uniformly on } \ \overline{\Omega} \tag{4}$$

for some function $u \in \mathrm{Lip}\,(\overline{\Omega})$. See for this [10, 15, 6]. It is quite recent that Davini, Fathi, Iturriaga and Zavidovique [6] have given a positive and decisive answer to this question when H is a convex and coercive Hamiltonian and Ω is a compact smooth manifold without boundary.

We show in this article that the method of [6], together with the recent developments [12, 13, 14] of weak KAM theory for Hamilton-Jacobi equations with Neumann type boundary conditions, is adapted and modified to the Neumann type boundary value problem, to establish the following two theorems. As is seen later, an interesting outcome of this study is that notion of Mather measures is naturally generalized to Hamilton-Jacobi equations with Neumann type boundary conditions.

Theorem 2. *For each $\lambda > 0$ let $u^\lambda \in C(\overline{\Omega})$ be a (unique) solution of* (3)*. Then there exists a solution $u \in \mathrm{Lip}\,(\overline{\Omega})$ of* (1)*, with $c = c_H$, such that*

$$u(x) = \lim_{\lambda\to 0}(u^\lambda(x) + \lambda^{-1}c_H) \quad \textit{uniformly on } \ \overline{\Omega}. \tag{5}$$

The theorem above will be proved in the next section. Before stating the second theorem, some preparations are needed. Henceforth in this introduction we assume that Theorem 2 is valid, and we write u_0 for the limit function given by (5).

Let L denote the Lagrangian of H. That is, L is the function on $\overline{\Omega}\times\mathbb{R}^n$ given by

$$L(x,\xi) = \sup_{p\in\mathbb{R}^n}(\xi\cdot p - H(x,p)).$$

We note (see also [14]) that L is lower semicontinuous and bounded below by the constant $-\min_{x\in\overline{\Omega}} H(x,0)$ on $\overline{\Omega}\times\mathbb{R}^n$, for any $x\in\overline{\Omega}$ the function $\xi\mapsto L(x,\xi)$ is convex, and L is bounded on $\overline{\Omega}\times B_\delta$ for some $\delta>0$ while it takes possibly the value $+\infty$.

We introduce Mather measures (μ_1,μ_2) associated with (1). We call a pair of finite Borel measures μ_1 and μ_2, with compact support, on $\overline{\Omega}\times\mathbb{R}^n$ and on $\partial\Omega$, respectively, a *Mather measure* associated with (1) if the following three conditions hold:

$$\begin{cases} \displaystyle\int_{\overline{\Omega}\times\mathbb{R}^n}(L(x,\xi)+c_H)\mu_1(\mathrm{d}x\mathrm{d}\xi) + \int_{\partial\Omega} g(x)\mu_2(\mathrm{d}x) \le 0, \\ \displaystyle\int_{\overline{\Omega}\times\mathbb{R}^n} D\phi(x)\cdot\xi\,\mu_1(\mathrm{d}x\mathrm{d}\xi) + \int_{\partial\Omega}\gamma(x)\cdot D\phi(x)\mu_2(\mathrm{d}x) = 0 \quad \text{for all } \ \phi\in C^1(\overline{\Omega}), \\ \mu_1 \ \text{ is a probability measure on } \ \overline{\Omega}\times\mathbb{R}^n. \end{cases} \tag{6}$$

Our definition of Mather measures above is an adaptation of the standard closed Mather measures (see [6]) to the Neumann type boundary value problem for Hamilton-Jacobi equations. The introduction of Mather measures, associated with Hamilton-Jacobi equations with the Neumann type boundary conditions, seems to be new and original. See also [18, 9, 8] for general scopes of Mather measures. The second condition in the above list of conditions corresponds to the requirement that the measure (μ_1,μ_2) be closed. In this note, we denote by $\mathcal{M}$ the collection of all Mather measures (μ_1,μ_2) associated with (1).

For any Borel measure on $\overline{\Omega}\times\mathbb{R}^n$, we denote by $\tilde\mu_1$ the projection of μ_1 on $\overline{\Omega}$. That is, we define $\tilde\mu_1$ by setting $\tilde\mu_1(B)=\mu_1(B\times\mathbb{R}^n)$ for every Borel subset B of $\overline{\Omega}$.

Next we introduce briefly the Skorokhod problem. That is the problem, for given $v\in L^\infty([0,\,\infty),\mathbb{R}^n)$, to look for a pair $(\eta,l)\in \mathrm{Lip}\,([0,\,\infty),\overline{\Omega})\times L^\infty([0,\,\infty),\mathbb{R})$ such that

$$\begin{cases} \eta(0)=x, \\ \dot\eta(t)+l(t)\gamma(\eta(t)) = v(t) \quad \text{a.e. } \ t\ge 0, \\ l(t)\ge 0 \quad \text{and, if } \ \eta(t)\in\Omega, \ \ l(t)=0 \quad \text{a.e. } \ t\ge 0. \end{cases} \tag{7}$$

We denote by $\mathrm{SP}(x)$ the set of all triplets $(\eta,v,l)\in \mathrm{Lip}\,([0,\,\infty),\overline{\Omega})\times L^\infty([0,\,\infty),\mathbb{R}^n)\times L^\infty([0,\,\infty),\mathbb{R})$ which satisfy (7). For more details of the Skorokhod problem, we refer to [12, 14] and the references therein.

For given $t>0$ we define $p_t\,:\,\overline{\Omega}\times\overline{\Omega}\to\mathbb{R}\cup\{\infty\}$ by

$$p_t(x,y) = \inf\int_0^t [L(\eta(s),-v(s)) + l(s)g(\eta(s)) + c_H]\mathrm{d}s, \tag{8}$$

where the infimum is taken over all $(\eta, v, l) \in \mathrm{SP}(x)$ such that $\eta(t) = y$. We understand here, in view of the boundedness of L from below, that if the function $s \mapsto L(\eta(s), -v(s))$ is not integrable on $[0,\, t]$, then

$$\int_0^t L(\eta(s), -v(s))\mathrm{d}s = +\infty.$$

It is easily seen that

$$p_{t+s}(x,y) = \inf_{z \in \overline{\Omega}} (p_t(x,z) + p_s(z,y)) \quad \text{for all} \;\; x, y \in \overline{\Omega} \;\text{ and }\; t, s \in [0,\, \infty). \tag{9}$$

We define as well a function p on $\overline{\Omega} \times \overline{\Omega}$ by

$$p(x,y) = \liminf_{t \to \infty} p_t(x,y). \tag{10}$$

This function p is called the *Peierls function* and is Lipschitz continuous on $\overline{\Omega} \times \overline{\Omega}$, which is a consequence of Lemma 12 below.

We are ready to state the second main result, which will be proved in Section 3.

Theorem 3. *We have*

$$u_0(x) = \min_{(\mu_1, \mu_2) \in \mathcal{M}} \int_{\overline{\Omega}} p(x,y) \tilde{\mu}_1(\mathrm{d}y) \quad \textit{for all} \;\; x \in \overline{\Omega}. \tag{11}$$

2. Proof of Theorem 2

The main arguments of the proof of Theorem 2 are stated in the following several lemmas.

Lemma 4. *For the solution u^λ of* (3)*, the formula*

$$u^\lambda(x) = \inf \int_0^\infty \mathrm{e}^{-\lambda t}[L(\eta(t), -v(t)) + l(t) g(\eta(t))]\mathrm{d}t \tag{12}$$

holds for all $x \in \overline{\Omega}$, where the infimum is taken over all triplets $(\eta, v, l) \in \mathrm{SP}(x)$.

As before, we understand in the above formula that if the function $t \mapsto \mathrm{e}^{-\lambda t} L(\eta(t), -v(t))$ is not integrable on $[0,\, \infty)$, then

$$\int_0^\infty \mathrm{e}^{-\lambda t}[L(\eta(t), -v(t)) + l(t) g(\eta(t))]\mathrm{d}t = \infty.$$

The above lemma is well-known (see [17]), for instance, if L is uniformly continuous on $\overline{\Omega} \times \mathbb{R}^n$. Also, in [12, Theorem 5.1] or [14, Theorem 5.5], a similar formula has been established, in the generality of the above lemma, for solutions of the initial-boundary value problem for Hamilton-Jacobi equations with the Neumann type boundary conditions, the proof of which can be easily adapted to the case of the above lemma. However, for completeness and the reader's convenience, we give a proof after the following lemma.

Lemma 5. *For the solution u^λ of* (3) *and any $x \in \overline{\Omega}$, there exists $(\eta, v, l) \in \mathrm{SP}(x)$ such that*

$$u^\lambda(x) = \int_0^\infty \mathrm{e}^{-\lambda t}[L(\eta(t), -v(t)) + l(t) g(\eta(t))]\mathrm{d}t.$$

Proof. We prove the lemma as a corollary of [12, Theorem 7.3]. Set $G(x,p) = H(x,p) + \lambda u^\lambda(x)$. Then G is a continuous, convex, coercive Hamiltonian and its Lagrangian is given by $L(x,\xi) - \lambda u^\lambda(x)$. Fix any $x \in \overline{\Omega}$. Thanks to [12, Theorem 7.3], there exists $(\eta, v, l) \in \mathrm{SP}(x)$ such that

$$u^\lambda(x) = \int_0^t [L(\eta(s), -v(s)) + l(s)g(\eta(s)) - \lambda u^\lambda(\eta(s))]\mathrm{d}s + u^\lambda(\eta(t)) \quad \text{for all } t > 0.$$

Differentiating this, we get

$$0 = L(\eta(t), -v(t)) + l(t)g(\eta(t)) - \lambda u^\lambda(\eta(t)) + \frac{\mathrm{d}}{\mathrm{d}t} u^\lambda(\eta(t)) \quad \text{a.e } t > 0,$$

and furthermore, multiplying this by $\mathrm{e}^{-\lambda t}$ and integrating over $[0, T]$, with $T > 0$,

$$0 = \int_0^T \mathrm{e}^{-\lambda t}[L(\eta(t), -v(t)) + l(t)g(\eta(t))]\mathrm{d}t + \mathrm{e}^{-\lambda T} u^\lambda(\eta(T)) - u^\lambda(x). \tag{13}$$

Sending $T \to \infty$ yields the desired identity. □

Proof of Lemma 4. In view of Lemma 5, we only need to prove that, for any $x \in \overline{\Omega}$ and $(\eta, v, l) \in \mathrm{SP}(x)$, the following inequality holds:

$$u^\lambda(x) \le \int_0^\infty \mathrm{e}^{-\lambda t}[L(\eta(r), -v(r)) + l(r)g(\eta(r))]\mathrm{d}t. \tag{14}$$

We prove this based on [12, Theorem 5.1]. Fix any $x \in \overline{\Omega}$ and $(\eta, v, l) \in \mathrm{SP}(x)$. We may assume that the function $t \mapsto \mathrm{e}^{-\lambda t} L(\eta(t), -v(t))$ is integrable on $[0, \infty)$. According to [12, Theorem 5.1], with Hamiltonian $H(x,p)$ replaced by $H(x,p) + \lambda u^\lambda(x)$, we have

$$u^\lambda(x) \le \int_0^t [L(\eta(r), -v(r)) + l(r)g(\eta(r)) - \lambda u^\lambda(\eta(r))]\mathrm{d}r + u^\lambda(\eta(t)) \quad \text{for all } \ t > 0. \tag{15}$$

We set

$$f(t) := -u^\lambda(x) + \int_0^t [L(\eta(r), -v(r)) + l(r)g(\eta(r)) - \lambda u^\lambda(\eta(r))]\mathrm{d}r \quad \text{for } \ t \ge 0,$$

and observe by using (15) that for a.e. $t > 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f(t) &= L(\eta(t), -v(t)) + l(t)g(\eta(t)) - \lambda u^\lambda(\eta(t)) \\ &\le L(\eta(t), -v(t)) + l(t)g(\eta(t)) + \lambda f(t),\end{aligned}$$

and, hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda t} f(t)\right) \le \mathrm{e}^{\lambda t}[L(\eta(t), -v(t)) + l(t)g(\eta(t))].$$

Integrating this over $[0, T]$, with $T > 0$, we get

$$\mathrm{e}^{-\lambda T} f(T) - f(0) \le \int_0^T \mathrm{e}^{\lambda t}[L(\eta(t), -v(t)) + l(t)g(\eta(t))]\mathrm{d}t.$$

This combined with (15) yields

$$u^\lambda(x) \le \int_0^T \mathrm{e}^{\lambda t}[L(\eta(t), -v(t)) + l(t)g(\eta(t))]\mathrm{d}t + \mathrm{e}^{-\lambda T} u^\lambda(\eta(T)),$$

which shows after sending $T \to \infty$ that

$$u^\lambda(x) \le \int_0^\infty \mathrm{e}^{\lambda t}[L(\eta(t), -v(t)) + l(t)g(\eta(t))]\mathrm{d}t.$$

Thus, (14) holds. □

Lemma 6. *Assume that $c_H = 0$. For each $\lambda > 0$ let $u^\lambda \in \mathrm{Lip}\,(\overline{\Omega})$ be the solution of* (3). *Then the family $\{u^\lambda\}_{\lambda\in(0,1)}$ is uniformly bounded and equi-Lipschitz continuous on $\overline{\Omega}$.*

The observations in the above lemma should have be done when establishing Theorem 1 in the approach described briefly after the statement of Theorem 1. But, for completeness, we give a proof of the above lemma.

Proof. Since $c_H = 0$, by Theorem 1 there exists a solution $u \in \mathrm{Lip}\,(\overline{\Omega}$ of (1), with $c = 0$. We choose a constant $M > 0$ so that $\|u\|_{\infty,\Omega} \le M$ and set $u_M^\pm(x) := u(x) \pm M$ for $x \in \overline{\Omega}$. Observe that if $\lambda \in (0,\, 1)$, then u_M^+ and u_M^- are a supersolution and subsolution of (3), respectively. Hence, by comparison (see [2, Remark (i) of Theorem 1] or [7, Remark 2.2]), we see that, for any $\lambda \in (0,\, 1)$, $u_M^- \le u^\lambda \le u_M^+$ on $\overline{\Omega}$, which shows that $\|u^\lambda\|_{\infty,\Omega} \le 2M$. That is, the collection $\{u^\lambda\}_{\lambda\in(0,\,1)}$ is uniformly bounded on $\overline{\Omega}$.

Next we choose a constant $K > 0$ so that $H(x,p) > 2M$ for all $(x,p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_K)$. Let $\lambda \in (0,\, 1)$. If $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$ satisfies $\lambda u^\lambda(x) + H(x,p) \le 0$, then $H(x,p) \le 2M$ and, hence, $|p| \le K$. This observation shows that u^λ is a subsolution of $|Du^\lambda(x)| \le K$ in Ω, which implies (see [14, Lemma 2.2]) that u^λ is Lipschitz continuous on $\overline{\Omega}$ with a Lipschitz bound $C_\Omega K$ for some constant $C_\Omega > 0$, depending only on Ω. Thus, the collection $\{u^\lambda\}_{\lambda\in(0,\,1)}$ is equi-Lipschitz continuous on $\overline{\Omega}$. □

We denote by $\mathcal{U}$ the set of all limit functions u obtained as

$$u(x) = \lim_{j\to\infty} u^{\lambda_j}(x) \quad \text{uniformly on } \ \overline{\Omega},$$

where $\{\lambda_j\}_{j\in\mathbb{N}}$ is a sequence of positive numbers λ_j converging to zero. Observe by Lemma 6 and the Ascoli-Arzela theorem that if $c_H = 0$, then $\mathcal{U} \ne \emptyset$. Also, note by the stability of the viscosity property under uniform convergence that every $u \in \mathcal{U}$ is a solution of (1), with $c = 0$.

Lemma 7. *For each $A > 0$ there exists a constant $C_A > 0$ such that*

$$L(x,\xi) \ge A|\xi| - C_A \quad \textit{for all } \ (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

For a proof of Lemma 7 above, we refer to [11, Lemma 6.4] or [14, Lemma 5.1].

Lemma 8. *There is a constant $C > 0$, depending only on Ω and γ, such that for any $x \in \overline{\Omega}$ and $(\eta,\, v,\, l) \in \mathrm{SP}(x)$,*

$$|\dot{\eta}(s)| \vee l(s) \le C|v(s)| \quad \textit{a.e. } s \ge 0.$$

See [11, Proposition 4.1] or [14, Proposition 5.2] for a proof of the above lemma.

Lemma 9. *Let $u \in C(\overline{\Omega})$ be a subsolution of* (1), *with a given $c \in \mathbb{R}$, and $\varepsilon > 0$ a constant. Then there exists a function $w \in C^1(\overline{\Omega})$ such that*

$$\begin{cases} H(x, Dw(x)) \le c + \varepsilon & \text{in } \ \Omega, \\ \gamma(x)\cdot Dw(x) \le g(x) \ \text{ on } \ \partial\Omega, \\ \|u - w\|_{\infty,\Omega} < \varepsilon. \end{cases} \tag{16}$$

See [14, Theorem 4.2] for a proof of the above lemma.

Lemma 10. *Assume that* $c_H = 0$. *We have*

$$\int_{\overline{\Omega}} u(x)\tilde{\mu}_1(\mathrm{d}x) \le 0 \quad \textit{for all} \;\; u \in \mathcal{U} \;\; \textit{and} \;\; (\mu_1, \mu_2) \in \mathcal{M}.$$

Proof. Fix any $u \in \mathcal{U}$ and $(\mu_1, \mu_2) \in \mathcal{M}$, and choose $\{\lambda_j\}_{j\in\mathbb{N}}$ so that $0 < \lambda_j < 1$ for all j, $\lim_{j\to\infty} \lambda_j = 0$ and

$$u(x) = \lim_{j\to\infty} u^{\lambda_j}(x) \quad \text{uniformly on } \overline{\Omega}. \tag{17}$$

Fix $\varepsilon > 0$ and $j \in \mathbb{N}$. We agree to write λ for λ_j for the moment. By applying Lemma 9 with the function $\lambda u^\lambda(x) + H(x, p)$ in place of the function $H(x, p)$, we obtain a function $w \in C^1(\overline{\Omega})$ such that

$$\begin{cases} \lambda u^\lambda(x) + H(x, Dw(x)) \le \varepsilon & \text{in } \Omega, \\ \gamma(x) \cdot Dw(x) \le g(x) & \text{on } \partial\Omega, \\ \|u^\lambda - w\|_{\infty,\Omega} < \varepsilon. \end{cases} \tag{18}$$

Since $H(x, p) \ge \xi \cdot p - L(x, \xi)$ for all $x \in \overline{\Omega}$ and $p, \xi \in \mathbb{R}^n$, we get from (18)

$$\lambda u^\lambda(x) \le \varepsilon + L(x, \xi) - Dw(x) \cdot \xi \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Integrating this with respect to μ_1 and setting $T = \overline{\Omega} \times \mathbb{R}^n$, we get

$$\lambda \int_{\overline{\Omega}} u^\lambda(x)\tilde{\mu}_1(\mathrm{d}x) \le \varepsilon\mu_1(T) + \int_T L(x, \xi)\mu_1(\mathrm{d}x\mathrm{d}\xi) - \int_T Dw(x) \cdot \xi \, \mu_1(\mathrm{d}x\mathrm{d}\xi).$$

Furthermore, using the second equality and inequality, respectively, of (6) and (18), we get

$$\begin{aligned} \lambda \int_{\overline{\Omega}} u^\lambda(x)\tilde{\mu}_1(\mathrm{d}x) &\le \varepsilon\mu_1(T) + \int_T L(x, \xi)\mu_1(\mathrm{d}x\mathrm{d}\xi) + \int_{\partial\Omega} \gamma(x) \cdot Dw(x)\mu_2(\mathrm{d}x) \\ &\le \varepsilon\mu_1(T) + \int_T L(x, \xi)\mu_1(\mathrm{d}x\mathrm{d}\xi) + \int_{\partial\Omega} g(x)\mu_2(\mathrm{d}x). \end{aligned}$$

Now, this combined with the first inequality of (6) yields

$$\lambda \int_{\overline{\Omega}} u^\lambda(x)\tilde{\mu}_1(\mathrm{d}x) \le \varepsilon\mu_1(T),$$

which implies that

$$\int_{\overline{\Omega}} u^\lambda(x)\tilde{\mu}_1(\mathrm{d}x) \le 0 \quad \text{for all } \lambda = \lambda_j, \; j \in \mathbb{N}.$$

Sending $j \to \infty$, we get the desired inequality

$$\int_{\overline{\Omega}} u(x)\tilde{\mu}_1(\mathrm{d}x) \le 0. \qquad \square$$

Lemma 11. *Assume that* $c_H = 0$. *Let* $w \in C(\overline{\Omega})$ *be a subsolution of* (1), $u \in \mathcal{U}$ *and* $x \in \overline{\Omega}$. *Then there exists* $(\mu_1, \mu_2) \in \mathcal{M}$ *such that*

$$u(x) \ge w(x) - \int_{\overline{\Omega}} w(y)\tilde{\mu}_1(\mathrm{d}y). \tag{19}$$

Proof. We choose $\{\lambda_j\}_{j\in\mathbb{N}}$ so that $0<\lambda_j<1$ for all $j\in\mathbb{N}$, $\lim_{j\to\infty}\lambda_j=0$ and

$$u(x)=\lim_{j\to\infty}u^{\lambda_j}(x)\quad \text{ uniformly on } \ \overline{\Omega}.$$

Thanks to Lemma 6, we may choose a Lipschitz bound $M>0$ for the functions u^λ, with $\lambda\in(0,\,1)$, so that $|u^\lambda(x)|\leq M$ for all $x\in\overline{\Omega}$ and $\lambda\in(0,\,1)$.

Fix any $x\in\overline{\Omega}$ and $j\in\mathbb{N}$, and write λ for λ_j. By Lemma 5, there exists $(\eta,v,l)=(\eta_j,v_j,l_j)\in\mathrm{SP}(x)$ such that

$$u^\lambda(x)=\int_0^\infty \mathrm{e}^{-\lambda t}[L(\eta(t),-v(t))+l(t)g(\eta(t))]\mathrm{d}t. \tag{20}$$

From this, in view of the dynamic programming principle (or the proof of Lemma 5), we see that

$$u^\lambda(x)=\int_0^t \mathrm{e}^{-\lambda s}[L(\eta(s),-v(s))+l(s)g(\eta(s))]\mathrm{d}s+\mathrm{e}^{-\lambda t}u^\lambda(\eta(t))\quad \text{ for all } \ t>0. \tag{21}$$

Next, observe that the function $u^\lambda\circ\eta$ is Lipschitz continuous with $M\|\dot{\eta}\|_{L^\infty(0,\,\infty)}$ as a Lipschitz bound and, hence,

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}u^\lambda\circ\eta(t)\Big|\leq M\|\dot{\eta}\|_{L^\infty(0,\,\infty)}\quad \text{ a.e. } \ t>0.$$

Using this and differentiating (21) in t, we get

$$0=\mathrm{e}^{-\lambda t}\Big[L(\eta(t),-v(t))+l(t)g(\eta(t))-\lambda u^\lambda(\eta(t))+\frac{\mathrm{d}}{\mathrm{d}t}u^\lambda\circ\eta(t)\Big]\quad \text{ a.e. } \ t>0.$$

That is, we have

$$0=L(\eta(t),-v(t))+l(t)g(\eta(t))-\lambda u^\lambda(\eta(t))+\frac{\mathrm{d}}{\mathrm{d}t}u^\lambda\circ\eta(t)\quad \text{ a.e. } \ t>0,$$

and hence,

$$0\geq L(\eta(t),-v(t))-l(t)\|g\|_{\infty,\partial\Omega}-\|u^\lambda\|_{\infty,\Omega}-M\|\dot{\eta}\|_{L^\infty(0,\infty)}\quad \text{ a.e. } \ t>0.$$

Furthermore, using Lemma 8, we obtain

$$0\geq L(\eta(t),-v(t))-C\|g\|_{\infty,\partial\Omega}\|v\|_{L^\infty(0,\infty)}-M-CM\|v\|_{L^\infty(0,\infty)}\quad \text{ a.e. } \ t>0.$$

Here C is the constant from Lemma 8. We now use Lemma 7, with $A=1+C(\|g\|_{\infty,\partial\Omega}+M)$, to get

$$A|v(t)|\leq C_A+M+(A-1)\|v\|_{L^\infty(0,\infty)}\quad \text{ a.e. } \ t>0,$$

where C_A is the constant from Lemma 7, with the above choice of A. Hence, we get

$$\|v\|_{L^\infty(0,\infty)}\leq C_1, \tag{22}$$

where $C_1=C_A+M$, and furthermore, by Lemma 8,

$$l(t)\vee|\dot{\eta}(t)|\leq CC_1\quad \text{ a.e. } \ t>0. \tag{23}$$

Now, we are going to take the limit as $\lambda=\lambda_j$ and $j\to\infty$, and recall that $(\eta,v,l)=(\eta_j,v_j,l_j)$ depends on j. For each j we define the functionals F_j and G_j on $\mathrm{BUC}(\overline{\Omega}\times\mathbb{R}^n)$ and on $C(\partial\Omega)$, respectively, by

$$F_j(\phi)=\lambda_j\int_0^\infty \mathrm{e}^{-\lambda_j t}\phi(\eta(t),-v(t))\mathrm{d}t\quad \text{ and } \quad G_j(\psi)=\lambda_j\int_0^\infty \mathrm{e}^{-\lambda_j t}l_j(t)\psi(\eta_j(t))\mathrm{d}t.$$

It is clear that

$$\begin{cases} |F_j(\phi)| \le \|\phi\|_{\infty,\overline{\Omega}\times\mathbb{R}^n} & \text{for all } \ \phi \in \mathrm{BUC}(\overline{\Omega}\times\mathbb{R}^n), \\ |G_j(\psi)| \le CC_1\|\psi\|_{\infty,\partial\Omega} & \text{for all } \ \psi \in C(\partial\Omega). \end{cases}$$

By (22), we see that the supports of the functionals F_j are contained in the compact set $\overline{\Omega}\times\overline{B}_{C_1}$. Thus, we may choose an increasing sequence $\{j_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$ such that

$$\mu_1 = \lim_{k\to\infty} F_{j_k} \quad \text{and} \quad \mu_2 = \lim_{k\to\infty} G_{j_k},$$

in the weak convergence of measures, for some finite Borel measures μ_1 and μ_2 on $\overline{\Omega}\times\mathbb{R}^n$ and on $\partial\Omega$, respectively. Furthermore, if $\phi(x,\xi)\equiv 1$, then $F_j(\phi)=1$ and, hence, $\mu_1(\phi)=1$, which ensures that μ_1 is a probability measure.

We fix any $\varepsilon>0$. Thanks to Lemma 9, we may choose a function $z\in\mathrm{Lip}\,(\overline{\Omega})$ so that

$$\begin{cases} H(x,Dz(x)) \le \varepsilon & \text{in } \ \Omega, \\ \gamma(x)\cdot Dz(x) \le g(x) & \text{on } \ \partial\Omega, \\ \|z-w\|_{\infty,\Omega} < \varepsilon. \end{cases} \tag{24}$$

From (20), using (24), we obtain

$$\begin{aligned} u^{\lambda_j}(x) &= \int_0^\infty \mathrm{e}^{-\lambda_j t}[L(\eta_j(t),-v_j(t)) + l_j(t)g(\eta_j(t))]\mathrm{d}t \\ &\ge \int_0^\infty \mathrm{e}^{-\lambda_j t}[(-v_j(t))\cdot Dz(\eta_j(t)) - H(\eta_j(t),Dz(\eta_j(t))) + l_j(t)g(\eta_j(t))]\mathrm{d}t \\ &\ge \int_0^\infty \mathrm{e}^{-\lambda_j t}[(-v_j(t))\cdot Dz(\eta_j(t)) - \varepsilon + l_j(t)\gamma(\eta_j(t))\cdot Dz(\eta_j(t))]\mathrm{d}t \\ &= -\lambda_j^{-1}\varepsilon - \int_0^\infty \mathrm{e}^{-\lambda_j t} Dz(\eta_j(t))\cdot\dot{\eta}_j(t)\mathrm{d}t \\ &= -\lambda_j^{-1}\varepsilon + z(x) - \lambda_j\int_0^\infty \mathrm{e}^{-\lambda_j t} z(\eta_j(t))\mathrm{d}t. \end{aligned}$$

Here, noting that z depends on ε and sending $\varepsilon\to 0$, we get

$$u^{\lambda_j}(x) \ge w(x) - \lambda_j\int_0^\infty \mathrm{e}^{-\lambda_j t} w(\eta_j(t))\mathrm{d}t,$$

and then we send $j\to\infty$ along the subsequence j_k, to obtain

$$u(x) \ge w(x) - \int_{\overline{\Omega}} w(y)\tilde{\mu}_1(\mathrm{d}y).$$

It remains to show that (μ_1,μ_2) is a Mather measure. We may choose a sequence $\{L_m\}_{m\in\mathbb{N}}$ of functions $L_m\in\mathrm{BUC}(\overline{\Omega}\times\mathbb{R}^n)$ such that

$$\begin{cases} L_m(x,\xi) \le L_{m+1}(x,\xi) & \text{for all } \ (m,x,\xi)\in\mathbb{N}\times\overline{\Omega}\times\mathbb{R}^n, \\ L(x,\xi) = \lim_{m\to\infty} L_m(x,\xi) & \text{pointwise on } \ \overline{\Omega}\times\mathbb{R}^n. \end{cases}$$

For any $j,m\in\mathbb{N}$, by (20), we get

$$\lambda_j u^{\lambda_j}(x) \ge \lambda_j\int_0^\infty \mathrm{e}^{-\lambda_j t}[L_m(\eta_j(t),-v_j(t)) + l_j(t)g(\eta_j(t))]\mathrm{d}t.$$

Hence, sending $j \to \infty$ along the subsequence $j = j_k$, we get

$$0 \geq \int_{\overline{\Omega}\times\mathbb{R}^n} L_m(x,\xi)\mu_1(\mathrm{d}x\mathrm{d}\xi) + \int_{\partial\Omega} g(x)\mu_2(\mathrm{d}x).$$

By the monotone convergence theorem, in the limit as $m \to \infty$, we get

$$\int_{\overline{\Omega}\times\mathbb{R}^n} L(x,\xi)\mu_1(\mathrm{d}x\mathrm{d}\xi) + \int_{\partial\Omega} g(x)\mu_2(\mathrm{d}x) \leq 0.$$

Next, let $\phi \in C^1(\overline{\Omega})$ and observe that for any $j \in \mathbb{N}$,

$$\begin{aligned} -\phi(x) &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda_j t}\phi(\eta_j(t))\right)\mathrm{d}t \\ &= -\lambda_j\int_0^\infty \mathrm{e}^{-\lambda_j t}\phi(\eta_j(t))\mathrm{d}t + \int_0^\infty \mathrm{e}^{-\lambda_j t}D\phi(\eta_j(t))\cdot\dot{\eta}_j(t)\mathrm{d}t \\ &= -\lambda_j\int_0^\infty \mathrm{e}^{-\lambda_j t}\phi(\eta_j(t))\mathrm{d}t + \int_0^\infty \mathrm{e}^{-\lambda_j t}D\phi(\eta_j(t))\cdot(v_j(t) - l_j(t)\gamma(\eta_j(t)))\mathrm{d}t. \end{aligned}$$

Multiplying the above by λ_j and sending $j \to \infty$ along the subsequence $j = j_k$, we get

$$\int_{\overline{\Omega}\times\mathbb{R}^n} D\phi(x)\cdot\xi\,\mu_1(\mathrm{d}x\mathrm{d}\xi) + \int_{\partial\Omega}\gamma(x)\cdot D\phi(x)\mu_2(\mathrm{d}x) = 0.$$

Hence, (μ_1,μ_2) is a Mather measure. This completes the proof. □

Proof of Theorem 2. We first assume that $c_H = 0$, and will come back to the general case. As noted after Lemma 6, the set $\mathcal{U}$ is non-empty.

To prove the desired uniform convergence, we need only to show that $\mathcal{U}$ is a singleton. Let $u, w \in \mathcal{U}$ and $x \in \overline{\Omega}$. By Lemma 11, there is a Mather measure $(\mu_1,\mu_2) \in \mathcal{M}$ such that

$$u(x) \geq w(x) - \int_{\overline{\Omega}} w(y)\tilde{\mu}_1(\mathrm{d}y).$$

Hence, by Lemma 10, we get

$$u(x) \geq w(x).$$

Since $x \in \overline{\Omega}$ is arbitrary, we see that $u(x) \geq w(x)$ for all $x \in \overline{\Omega}$. Also, by symmetry, we have $w(x) \geq u(x)$ for all $x \in \overline{\Omega}$. Thus, we conclude that $u = w$, which shows that $\mathcal{U}$ is a singleton.

Next, we consider the general case. We set $\widetilde{H}(x,p) = H(x,p) - c_H$ and $\tilde{u}^\lambda(x) = u^\lambda(x) + \lambda^{-1}c_H$, and note that the critical value for (1), with H replaced by $\widetilde{H}$, is zero and that $\tilde{u}^\lambda$ is the unique solution of (3), with H replaced by $\widetilde{H}$. The previous argument now yields

$$u(x) = \lim_{\lambda\to 0}\tilde{u}^\lambda(x) \quad \text{uniformly on } \overline{\Omega}$$

for some solution $u \in \mathrm{Lip}\,(\overline{\Omega})$ of (1), with $c = c_H$, and the proof is complete. □

3. Proof of Theorem 3

In this section, except otherwise stated, we assume that $c_H = 0$, and let $\mathcal{S}^-$ denotes the set of all subsolutions $w \in C(\overline{\Omega})$ of (1), with $c = 0$.

Let d be the function on $\overline{\Omega}\times\overline{\Omega}$ defined by

$$d(x,y) = \sup\{w(x) - w(y) \,:\, w \in \mathcal{S}^-\}.$$

We recall that, since $\mathcal{S}^-$ is equi-Lipschitz (see e.g. [14, Lemma 4.1]), d is Lipschitz continuous on $\overline{\Omega} \times \overline{\Omega}$, and the function $x \mapsto d(x, y)$ is in $\mathcal{S}^-$ for any $y \in \overline{\Omega}$. Furthermore, we have (see e.g. [14, Proposition 5.4])

$$d(x, y) = \inf_{t>0} p_t(x, y) \quad \text{for all } \ x, y \in \overline{\Omega}, \tag{25}$$

and

$$d(x, y) \le d(x, z) + d(z, y) \quad \text{for all } \ x, y, z \in \overline{\Omega}. \tag{26}$$

It is obvious that, if $w \in \mathcal{S}^-$,

$$w(x) - w(y) \le d(x, y) \le p_t(x, y) \quad \text{for all } \ x, y \in \overline{\Omega},\ t > 0. \tag{27}$$

Now, we consider the initial-boundary value problem

$$\begin{cases} \partial_t u(x, t) + H(x, D_x u(x, t)) = 0 & \text{in } \Omega \times (0, \infty), \\ \gamma(x) \cdot D_x u(x, t) = g(x) & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{28}$$

$$u(x, 0) = f(x) \quad \text{for } \ x \in \overline{\Omega}, \tag{29}$$

where $f \in C(\overline{\Omega})$ is a given initial data. According to [12, Theorem 5.1] or [14, Theorem 3.3], there exists a unique solution $u \in \mathrm{BUC}(\overline{\Omega} \times [0, \infty))$ of (28), (29), the solution u is given by

$$u(x, t) = \inf_{y \in \overline{\Omega}} (p_t(x, y) + f(y)) \quad \text{for all } \ (x, t) \in \overline{\Omega} \times (0, \infty), \tag{30}$$

and moreover, $u \in \mathrm{Lip}\,(\overline{\Omega} \times [0, \infty))$ if $f \in \mathrm{Lip}\,(\overline{\Omega})$.

We introduce a function q on $\overline{\Omega} \times \overline{\Omega} \times [0, \infty)$ defined by

$$q(x, y, t) = \inf_{z \in \overline{\Omega}} (p_t(x, z) + d(z, y)) \quad \text{if } \ t > 0, \tag{31}$$

and $q(x, y, 0) = d(x, y)$. We note that, for any $y \in \overline{\Omega}$, the function $(x, t) \mapsto q(x, y, t)$ is a solution of (28) with the initial function $x \mapsto d(x, y)$.

We need the following lemma for the proof of Theorem 3.

Lemma 12. *We have:* $p \in \mathrm{Lip}\,(\overline{\Omega} \times \overline{\Omega})$, $\ q \in \mathrm{Lip}\,(\overline{\Omega} \times \overline{\Omega} \times [0, \infty))$,

$$d(x, y) \le q(x, y, t) \le \min\{p(x, y),\, p_t(x, y)\} \quad \textit{for all } \ x, y \in \overline{\Omega},\ t \ge 0,$$

and

$$p(x, y) = \lim_{t \to \infty} q(x, y, t) \quad \textit{uniformly on } \ \overline{\Omega} \times \overline{\Omega}.$$

Furthermore, for any $y \in \overline{\Omega}$, *the function* $x \mapsto p(x, y)$ *is a solution of* (1), *with* $c = 0$.

The following lemma is needed for the proof of the above lemma.

Lemma 13. *The function* q *is Lipschitz continuous on* $\overline{\Omega} \times \overline{\Omega} \times [0, \infty)$ *and, for any* $x, y \in \overline{\Omega}$, *the function* $t \mapsto q(x, y, t)$ *is nondecreasing on* $[0, \infty)$.

Proof. Observe that for any $x, y, z \in \overline{\Omega}$ and $t, s \in [0, \infty)$,

$$\begin{aligned} p_{t+s}(x, z) + d(z, y) &= \inf_{\xi \in \overline{\Omega}} (p_t(x, \xi) + p_s(\xi, z) + d(z, y)) \\ &\ge \inf_{\xi \in \overline{\Omega}} (p_t(x, \xi) + d(\xi, y) = q(x, y, t), \end{aligned}$$

and, hence,

$$q(x, y, t + s) \ge q(x, y, t) \quad \text{for all } \ x, y \in \overline{\Omega},$$

which shows that the function $t \mapsto q(x,y,t)$ is nondecreasing on $[0,\,\infty)$ for any $x, y \in \overline{\Omega}$.

Equation (28) for the function $(x,t) \mapsto q(x,y,t)$ and the monotonicity of $t \mapsto q(x,y,t)$ yield, for any $y \in \overline{\Omega}$,

$$\begin{aligned}|\partial_t q(x,y,t)| &+ H(x, D_x q(x,y,t)) \\ &= \partial_t q(x,y,t) + H(x, D_x q(x,y,t)) = 0 \quad \text{ a.e. in } \ \overline{\Omega} \times [0,\,\infty).\end{aligned}$$

Due to the coercivity of H, this ensures that the family of functions $(x,t) \mapsto q(x,y,t)$, parametrized by $y \in \overline{\Omega}$, is equi-Lipschitz continuous on $\overline{\Omega} \times [0,\,\infty)$. Moreover, we note by (31) that if $C > 0$ is a Lipschitz bound of d, then $|q(x,y,t) - q(x,z,t)| \le C|y - z|$ for all $y, z \in \overline{\Omega}$. Thus we deduce that $q \in \mathrm{Lip}\,(\overline{\Omega} \times \overline{\Omega} \times [0,\,\infty))$. □

Proof of Lemma 12. We first show that

$$p(x,y) = \lim_{t\to\infty} q(x,y,t) \quad \text{ for all } \ x, y \in \overline{\Omega}. \tag{32}$$

To this end, we fix any $y \in \overline{\Omega}$ and set $u(x,t) = q(x,y,t)$ for $(x,t) \in \overline{\Omega} \times [0,\,\infty)$. Since the function u is a solution of (28) with the initial function $x \mapsto d(x,y)$, we see that u is bounded and Lipschitz continuous on $\overline{\Omega} \times [0,\,\infty)$. Recalling the monotonicity of the function $u(x,t)$ in t and using the Ascoli-Arzela theorem, we deduce that

$$u^\infty(x) = \lim_{t\to\infty} u(x,t) \quad \text{ uniformly on } \ \overline{\Omega}, \tag{33}$$

for some function $u^\infty \in \mathrm{Lip}\,(\overline{\Omega})$. It follows that the function u^∞ is a solution of (1), with $c = 0$. By the monotonicity of the function $u(x,t)$ in t, we get

$$d(x,y) \le q(x,y,t) \le u^\infty(x) \quad \text{ for all } \ (x,t) \in \overline{\Omega} \times [0,\,\infty). \tag{34}$$

Observe by (9) and (27) that for all $x \in \overline{\Omega}$ and $t, s \in [0,\,\infty)$,

$$p_{t+s}(x,y) \ge \inf_{z\in\overline{\Omega}} (p_t(x,z) + d(z,y)) = q(x,y,t),$$

from which we get

$$p(x,y) \ge u^\infty(x) \quad \text{ for all } \ x \in \overline{\Omega}. \tag{35}$$

Now, we fix $x \in \overline{\Omega}$ as well. Also, fix any $T > 0$ and $\varepsilon > 0$. The function $u^\infty(x)$, regarded as a function of (x,t), is a solution of (28), and, by (30), we get

$$u^\infty(x) = \inf_{z\in\overline{\Omega}} (p_T(x,z) + u^\infty(z)) > -\varepsilon + p_T(x, z_\varepsilon) + u^\infty(z_\varepsilon)$$

for some $\hat{z} \in \overline{\Omega}$. Furthermore, by (34) and (25), we get

$$\begin{aligned} u^\infty(x) &> -\varepsilon + p_T(x,\hat{z}) + d(\hat{z},y) > -2\varepsilon + p_T(x,\hat{z}) + p_\tau(\hat{z},y) \\ &\ge -2\varepsilon + \inf_{z\in\overline{\Omega}} (p_T(x,z) + p_\tau(z,y)) = -2\varepsilon + p_{T+\tau}(x,y)\end{aligned}$$

for some $\tau > 0$, which shows that

$$u^\infty(x) \ge \liminf_{t\to\infty} p_t(x,y) = p(x,y).$$

This, (35) and (33) together ensure that

$$p(x,y) = u^\infty(x) = \lim_{t\to\infty} q(x,y,t).$$

Noting that the choice of $x, y \in \overline{\Omega}$ above is arbitrary, we conclude that (32) holds.

By Lemma 13, we know that $q \in \mathrm{Lip}\,(\overline{\Omega} \times \overline{\Omega} \times [0,\,\infty))$. Hence, we infer that, by the Ascoli-Arzela theorem, the convergence (1) is indeed uniform on $\overline{\Omega} \times \overline{\Omega}$ and that the limit function p is Lipschitz continuous on $\overline{\Omega}\times\overline{\Omega}$. Moreover, it is now clear that, for any $y\in\overline{\Omega}$, the function $x\mapsto p(x,y)$ is a solution of (1), with $c=0$. and that $q(x,y,t)\leq p_t(x,y)$ and $d(x,y)\leq q(x,y,t)\leq p(x,y)$ for all $x,y\in\overline{\Omega}$ and $t\geq 0$. The proof is complete. □

Remark 3.1. It is worth noting that the above proof is easily modified to show that if $f\in C(\overline{\Omega})$ (and $c_H=0$), then, for the solution $u\in \mathrm{BUC}(\overline{\Omega}\times[0,\infty))$ of (28), (29), we have

$$\liminf_{t\to\infty} u(x,t)=\inf_{y\in\overline{\Omega}}\,(p(x,y)+f(y))\quad \text{ for all }\ x\in\overline{\Omega}.$$

In other words, if $f\in C(\overline{\Omega})$, then the function $f^\infty\in \mathrm{Lip}\,(\overline{\Omega})$ defined by

$$f^\infty(x)=\inf_{y\in\overline{\Omega}}\,(p(x,y)+f(y))$$

is the minimal solution of (1), with $c_H=0$, among those solutions ϕ satisfying $\phi\geq f^-$ on $\overline{\Omega}$, where the function $f^-\in\mathrm{Lip}\,(\overline{\Omega})$ given by

$$f^-(x)=\inf_{y\in\overline{\Omega}}(d(x,y)+f(y)),$$

and this function f^- is the maximal one among those $\psi\in\mathcal{S}^-$ satisfying $\psi\leq f$ on $\overline{\Omega}$. See also [13, Proposition 4.1].

We continue to assume until the middle of the following proof that $c_H=0$.

Proof of Theorem 3. By (27), we have

$$u_0(x)\leq u_0(y)+p(x,y)\quad \text{ for all }\ x,y\in\overline{\Omega}.$$

Let $(\mu_1,\mu_2)\in\mathcal{M}$. We integrate the both sides of the above in y, then use Lemma 10, to get

$$u_0(x)\leq\int_{\overline{\Omega}}u_0(y)\tilde\mu_1(\mathrm{d}y)+\int_{\overline{\Omega}}p(x,y)\tilde\mu_1(\mathrm{d}y)\leq\int_{\overline{\Omega}}p(x,y)\tilde\mu_1(\mathrm{d}y)\quad \text{ for all }\ x\in\overline{\Omega},$$

and conclude that

$$u_0(x)\leq\inf_{(\mu_1,\mu_2)\in\mathcal{M}}\int_{\overline{\Omega}}p(x,y)\tilde\mu_1(\mathrm{d}y)\quad \text{ for all }\ x\in\overline{\Omega}. \tag{36}$$

Next, fix $x\in\overline{\Omega}$. Let $\lambda>0$ and $u^\lambda\in\mathrm{Lip}\,(\overline{\Omega})$ be the solution of (3). In view of Lemma 5, we may choose $(\eta,v,l)=(\eta_\lambda,v_\lambda,l_\lambda)\in\mathrm{SP}(x)$ so that

$$u^\lambda(x)=\int_0^\infty \mathrm{e}^{-\lambda t}[L(\eta(t),-v(t))+l(t)g(\eta(t))]\mathrm{d}t.$$

Similarly to the proof of Lemma 5, we deduce that for all $t\geq 0$,

$$\begin{aligned}u^\lambda(x)&=\int_0^t[L(\eta(s),-v(s))+l(s)g(\eta(s))-\lambda u^\lambda(\eta(s))]\mathrm{d}s+u^\lambda(\eta(t))\\&\geq p_t(x,\eta(t))-\lambda\int_0^t u^\lambda(\eta(s))\mathrm{d}s+u^\lambda(\eta(t))\end{aligned}$$

Moreover, multiplication by $\lambda e^{-\lambda t}$ and integration over $[0, \infty)$ yield

$$\begin{aligned} u^\lambda(x) &\geq \lambda \int_0^\infty e^{\lambda t} p_t(x, \eta(t)) dt + \lambda \int_0^\infty \frac{d}{dt} \left(e^{-\lambda t} \int_0^t u^\lambda(\eta(s)) ds \right) dt \\ &= \lambda \int_0^\infty e^{\lambda t} p_t(x, \eta(t)) dt \\ &= \lambda \int_0^t e^{-\lambda s} p_s(x, \eta(s)) ds + \int_t^\infty e^{-\lambda s} p_s(x, \eta(s)) ds \quad \text{for all } t > 0. \end{aligned}$$

We fix any $t > 0$, and, using Lemmas and 12 and 13, we observe from the above that

$$u^\lambda(x) \geq \lambda \int_0^t e^{-\lambda s} d(x, \eta(t)) ds + \lambda \int_t^\infty e^{-\lambda s} q(x, \eta(s), t) ds. \tag{37}$$

We argue exactly as in the proof of Lemma 11, to find a sequence $\{\lambda_j\} \in (0, 1)$, converging to zero, and a Mather measure (μ_1, μ_2) such that for any $\psi \in C(\overline{\Omega})$,

$$\int_{\overline{\Omega}} \psi(x) \tilde{\mu}_1(dx) = \lim_{j \to \infty} \lambda_j \int_0^\infty e^{-\lambda_j s} \psi(\eta_{\lambda_j}(s)) ds$$

Thus, sending $\lambda \to 0$ in (37) along the sequence $\{\lambda_j\}$ and using Theorem 2, we get

$$u_0(x) \geq \int_{\overline{\Omega}} q(x, y, t) \tilde{\mu}_1(dy).$$

Furthermore, sending $t \to \infty$ yields

$$u_0(x) \geq \int_{\overline{\Omega}} p(x, y) \tilde{\mu}_1(dy).$$

Since $x \in \overline{\Omega}$ is arbitrary in the above inequality, we combine this with (36), to conclude the proof of Theorem 3 in the case when $c_H = 0$.

Now, we consider the general case regarding c_H. Given a Hamiltonian H, as above, let L, c_H and p be the corresponding Lagrangian, critical value and Peierls function, respectively. Let $\mathcal{M}$ denote the corresponding set of Mather measures associated with (1). Moreover, let u_0 be the limit function given by (5). If we define the function $\widetilde{H}$ on $\overline{\Omega} \times \mathbb{R}^n$ by setting $\widetilde{H}(x, p) = H(x, p) - c_H$, then the critical value and Lagrangian, corresponding to the Hamiltonian $\widetilde{H}$, are zero and the function $L(x, \xi) + c_H$, respectively. On the other hand, the Peierls function p, the set $\mathcal{M}$ of Mather measures, and the limit function u_0 do not change under the above replacement between H and $\widetilde{H}$. Thus, by the previous result for $c_H = 0$, we get

$$u_0(x) = \min_{(\mu_1, \mu_2) \in \mathcal{M}} \int_{\overline{\Omega}} p(x, y) \tilde{\mu}_1(dx) \quad \text{for all } x \in \overline{\Omega},$$

which completes the proof. □

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