ASYMPTOTIC ANALYSIS FOR THE EIKONAL EQUATION WITH THE DYNAMICAL BOUNDARY CONDITIONS

EMAN S. AL-AIDAROUS, EBRAHEEM O. ALZAHRANI, HITOSHI ISHII,
AND ARSHAD M. M. YOUNAS

ABSTRACT. We study the dynamical boundary value problem for Hamilton-Jacobi equations of the eikonal type with a small parameter. We establish two results concerning the asymptotic behavior of solutions of the Hamilton-Jacobi equations: one concerns with the convergence of solutions as the parameter goes to zero and the other with the large-time asymptotics of solutions of the limit equation.

CONTENTS

1. INTRODUCTION AND THE MAIN RESULTS

We consider the initial-boundary value problem for the eikonal equation

(1)
$$\begin{cases} \varepsilon u_t^\varepsilon(x,t) + |D_x u^\varepsilon(x,t)| = 1 & \text{in } Q, \\ u_t^\varepsilon(x,t) + \nu(x) \cdot D_x u^\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{for } x \in \bar{\Omega}. \end{cases}$$

Here $\varepsilon \in (0, 1)$ is a parameter, Ω is a bounded open connected subset of $\mathbb{R}^n$, with C^1 boundary, $Q := \Omega \times (0, \infty)$, $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$, and u_0

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represents the initial data. We adapt the notion of viscosity solution as the notion of solution of eikonal equations in this article. Throughout this article we assume for simplicity that $u_0 \in \mathrm{Lip}\,(\bar{\Omega})$, i.e., u_0 is Lipschitz continuous on $\bar{\Omega}$.

The boundary condition in the above problem is called the *dynamical boundary condition*. If we set

$$\tilde{\nu}(x,t) = (\nu(x), 0) \quad \text{and} \quad \gamma(x,t) = (\nu(x), 1) \quad \text{for } (x,t) \in \partial\Omega \times (0, \infty),$$

then $\tilde{\nu}(x,t)$ is the outer unit normal vector of Q at $(x,t) \in \partial\Omega \times (0, \infty)$,

$$\tilde{\nu}(x,t) \cdot \gamma(x,t) = 1 \quad \text{for all } (x,t) \in \partial\Omega \times (0, \infty)$$

and

$$u_t(x,t) + \nu(x) \cdot D_x u(x,t) = \gamma(x,t) \cdot Du(x,t).$$

From this observation, we see that the above dynamical boundary condition is a kind of Neumann type boundary condition posed on the portion $\partial\Omega \times (0, \infty)$ of the boundary ∂Q of the domain Q.

Motivated with applications to superconductivity and surface evolution, Elliott-Giga-Goto [7] have studied the well-posedness of a Hamilton-Jacobi equation with a dynamical boundary condition, where the boundary condition is "tangential" to the lateral boundary and has the form $u_t(x,t) + g(x,t) = 0$ (see also (4) below). As far as the authors know, a general study of Hamilton-Jacobi equations with dynamical boundary conditions goes back to Barles [5], where the well-posedness of dynamical boundary problems has been established (see for instance [5, Theoremé 4.11]).

We are also motivated by the recent studies on the Laplace equation

$$\Delta_x u(x,t) = 0 \quad \text{in } \Omega \times (0, \infty)$$

with the nonlinear dynamical boundary condition of the type

$$u_t(x,t) + \nu(x) \cdot D_x u(x,t) = |u(x,t)|^q, \quad \text{with a constant } q > 1,$$

due to Amann-Fila [1], Fila-Ishige-Kawakami [10] and others, where the blow-up phenomena and large time behavior of solutions are investigated. The Laplace equation above is, of course, the limit equation of the heat equations $\varepsilon u_t(x,t) - \Delta_x u(x,t) = 0$ in $\Omega \times (0, \infty)$ as $\varepsilon \to 0+$. Here we replace these heat equations by the eikonal equations and the nonlinear dynamical boundary condition by the linear one as in (1).

We are thus concerned with the asymptotic behavior of the solution u^ε of (1) as $\varepsilon \to 0+$. Roughly speaking, if there is a limit function of u^ε as $\varepsilon \to 0+$, the limit function u should satisfy

$$(2) \qquad \begin{cases} |D_x u(x,t)| = 1 & \text{in } Q, \\ u_t(x,t) + \nu(x) \cdot D_x u(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

Regarding the initial condition for the limit function, as we will see in our main results, the solutions u^ε develop an initial layer and the original initial condition $u(\cdot, 0) = u_0$ does not make sense for the limit function u in general.

To overcome the difficulty of initial layer, we introduce a new (slower) time scale and, for the solutions u^ε of (1), we set

$$v^\varepsilon(x,t) = u^\varepsilon(x, \varepsilon t) \quad \text{for } (x,t) \in \bar{Q}.$$

Note that the v^ε satisfy

(3)
$$\begin{cases} v_t^\varepsilon(x,t) + |D_x v^\varepsilon(x,t)| = 1 & \text{in } Q, \\ v_t^\varepsilon(x,t) + \varepsilon\nu(x) \cdot D_x v^\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ v^\varepsilon(x,0) = u_0(x) & \text{for } x \in \bar{\Omega}. \end{cases}$$

In the informal level, by setting $\varepsilon = 0$ we get the problem for the limit function v of the u^ε as $\varepsilon \to 0+$:

(4)
$$\begin{cases} v_t(x,t) + |D_x v(x,t)| = 1 & \text{in } Q, \\ v_t(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ v(x,0) = u_0(x) & \text{for } x \in \bar{\Omega}. \end{cases}$$

The initial condition for the limit function u of the u^ε is then given as the limit function $v_\infty(x)$ of the solution $v(x,t)$ of (4) as $t \to \infty$. The recent developments concerning the large time asymptotics for solutions of Hamilton-Jacobi equations (see [12, 4]) suggest that the limit function v_∞ should be described as follows: define first the function v_0^- on $\bar{\Omega}$ as the maximal subsolution of the stationary eikonal equation

(5)
$$|Dv(x)| = 1 \quad \text{in } \Omega$$

among those v which satisfy $v \le u_0$ on $\bar{\Omega}$, and then v_∞ as the minimal solution of (5) among those v which satisfy $v \ge v_0^-$ on $\bar{\Omega}$. It is well-known (see the end of this section) that v_0^- and v_∞ are Lipschitz continuous on $\bar{\Omega}$ with a Lipschitz bound depending only on the domain Ω. See [12, Lemma 2.2] for this Lipschitz continuity.

The main purpose of this paper is twofold. First, we consider the convergence of u^ε as $\varepsilon \to 0+$ and, second, we study the large time asymptotics for solutions of (2). Our result on the convergence of u^ε is stated as follows:

Theorem 1.1. *Let $u^\varepsilon \in C(\bar{Q})$ be a solution of (1), with $\varepsilon \in (0, 1)$, and $u \in C(\bar{Q})$ a solution of (2) satisfying the initial condition $u(\cdot, 0) = v_\infty$ on $\bar{\Omega}$. Then*

$$\lim_{\varepsilon \to 0+} u^\varepsilon(x,t) = u(x,t) \qquad \text{uniformly on } \bar{\Omega} \times [T^{-1}, T]$$

for all $T > 1$.

The stationary problem corresponding to (2) is the following.

(6)
$$\begin{cases} |Du(x)| = 1 & \text{in } \Omega, \\ 1 + \nu(x) \cdot Du(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

As we will see, this problem has a solution in $C(\bar{\Omega})$ and v_∞ is a supersolution of this problem. We define the function u_∞ as the maximal solution of (6) among those u which satisfy $u \le v_\infty$ on $\bar{\Omega}$. Our result concerning the large time behavior of solutions of (2) is as follows.

Theorem 1.2. *Let $u \in C(\bar{Q})$ be a solution of (2) satisfying the initial condition $u(\cdot, 0) = v_\infty$. Then*

$$\lim_{t \to \infty} (u(x,t) - t) = u_\infty(x) \qquad \text{uniformly on } \bar{\Omega}.$$

We use the following notation: as above let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. By the implicit function theorem, there exists a function $\rho \in C^1(\mathbb{R}^n)$ such that

$$\begin{cases} \Omega = \{x \in \mathbb{R}^n \,:\, \rho(x) < 0\}, \\ D\rho(x) \neq 0 & \text{for all} \ \ x \in \partial\Omega. \end{cases}$$

Note that $D\rho(x) = |D\rho(x)|\nu(x)$ for all $x \in \partial\Omega$. We call such a function ρ a *defining function* of Ω. Let $\phi \in C(\bar{\Omega})$ be a (viscosity) subsolution of

$$|D\phi(x)| \leq 1 \ \ \text{in} \ \ \Omega.$$

It is well-known (see [12, Proposition 1.14] and [5, 2, 16]) that this property is equivalent to the following Lipschitz property: for any ball $B \subset \Omega$,

$$|\phi(x) - \phi(y)| \leq |x - y| \ \ \ \text{for all} \ \ x, y \in B.$$

Due to the C^1 regularity, connectedness and boundedness of Ω, any such function ϕ is Lipschitz continuous on $\bar{\Omega}$, with a uniform Lipschitz bound. (See [12, Lemma 2.2] for this.) In the following, the minimum of such uniform bounds will be denoted as L_Ω. It is obvious that $L_\Omega \geq 1$. For any bounded function f on a set A, $\|f\|_{\infty, A}$ denotes the sup-norm $\sup_{x \in A} |f(x)|$. For any $T > 0$, Q_T denotes the domain $\Omega \times (0, T)$.

2. Preliminaries

We begin with the following theorem.

Theorem 2.1. *Let $\varepsilon > 0$. There exists a unique solution $u^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$ of* (1).

Recall that $u_0 \in \mathrm{Lip}\,(\bar{\Omega})$ is assumed here, which is crucial to conclude the Lipschitz continuity of u^ε in the above theorem. On the other hand, for any continuous u_0, one can show the unique existence of a uniformly continuous solution of (1). The above result is known in the literature (see for instance [5, 3]), but we give a proof here for the reader's convenience.

Proof. We first note that the uniqueness of solution of (1) is a direct consequence of Theorem A.1 (comparison theorem) in the appendix.

We next show that there exists a solution of (1) which is continuous on $\bar{Q}$. Let $L > 0$ be a Lipschitz bound of u_0. Define the functions $U^\pm \in \mathrm{Lip}\,(\bar{Q})$ by

$$U^\pm(x, t) = u_0(x) \pm \max\{\varepsilon^{-1}, \, \varepsilon^{-1}L, \, L\}\, t.$$

It is easily checked that U^+ and U^- are, respectively, a supersolution and a subsolution of

$$(7) \qquad \begin{cases} \varepsilon u_t(x, t) + |D_x u(x, t)| = 1 & \text{in} \ \ Q, \\ u_t(x, t) + \nu(x) \cdot D_x u(x, t) = 0 & \text{on} \ \ \partial\Omega \times (0, \infty). \end{cases}$$

According to Perron's method (see [6, 5, 2, 12] for instance), if we denote by $\mathcal{S}$ the set of all subsolutions ϕ of (7) such that $U^- \leq \phi \leq U^+$ on $\bar{Q}$ and set

$$u^\varepsilon(x, t) = \sup\{\phi(x, t) \,:\, \phi \in \mathcal{S}\} \ \ \ \text{for} \ \ (x, t) \in \bar{Q},$$

then u^ε is a solution of (7) and $u^\varepsilon \in \mathcal{S}$. More precisely, u^ε is a solution of (7) in the sense that $u^\varepsilon \in \mathrm{USC}(\bar{Q})$, u^ε is a subsolution of (7) and the lower semicontinuous envelope u_*^ε of u^ε is a supersolution of (7). It is obvious that $U^- \leq u_*^\varepsilon \leq u^\varepsilon \leq U^+$ on $\bar{Q}$. We apply the

comparison theorem (Theorem A.1 in the appendix) to u^ε and u^ε_*, to obtain $u^\varepsilon \le u^\varepsilon_*$ on $\bar{Q}$. Thus we see that $u^\varepsilon = u^\varepsilon_*$ is continuous on $\bar{Q}$. It is now obvious that $u^\varepsilon(x, 0) = u_0(x)$ for all $x \in \bar{\Omega}$ and that u^ε is a solution of (1).

Finally we show that $u^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$. We set $M_\varepsilon = \max\{\varepsilon^{-1}, \varepsilon^{-1}L, L\}$ and observe that for any $h > 0$,

$$u^\varepsilon(x, h) - M_\varepsilon h \le U^+(x, h) - M_\varepsilon h \le u_0(x) \quad \text{for all } x \in \bar{\Omega}.$$

For any $h > 0$, by comparison between the solutions $u^\varepsilon(x, t+h) - M_\varepsilon h$ and $u^\varepsilon(x, t)$ of (1), we get

$$u^\varepsilon(x, t+h) - M_\varepsilon h \le u^\varepsilon(x, t) \quad \text{for all } (x, t) \in \bar{Q}.$$

Similarly, we get

$$u^\varepsilon(x, t+h) + M_\varepsilon h \ge u^\varepsilon(x, t) \quad \text{for all } (x, t) \in \bar{Q}.$$

Hence, we get

$$|u^\varepsilon(x, t) - u^\varepsilon(x, s)| \le M_\varepsilon |t - s| \quad \text{for all } t, s \ge 0, \ x \in \bar{\Omega}.$$

This Lipschitz estimate together with Lemma A.3 in the appendix guarantees that $u^\varepsilon \in \mathrm{Lip}\,(\bar{\Omega} \times (0, \infty))$. But, since $u^\varepsilon \in C(\bar{Q})$, we conclude that $u^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$. □

Given a function $u^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$, we define the function $v^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$ by

$$v^\varepsilon(x, t) = u^\varepsilon(x, \varepsilon t).$$

It is easy to check that u^ε is a solution of (1) if and only if v^ε is a solution of (3). Hence, Theorem 2.1 implies the following proposition.

Corollary 2.2. *There exists a unique solution $v^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$ of* (3).

We remark here on the definition of v_0^-, v_∞ and u_∞. By definition, the function $v_0^- : \bar{\Omega} \to \mathbb{R}$ is given by

$$v_0^-(x) = \sup\{\phi(x) : \phi \in \mathcal{S}_0\},$$

where $\mathcal{S}_0$ denotes the set of all subsolutions $\phi \in \mathrm{Lip}\,(\bar{\Omega})$ of (5) satisfying the inequality $\phi \le u_0$ on $\bar{\Omega}$. It is a classical observation that $\mathcal{S}_0 \ne \emptyset$ and the above formula gives a Lipschitz continuous subsolution of (5). The function $v_\infty : \bar{\Omega} \to \mathbb{R}$ is defined by

$$v_\infty(x) = \inf\{\phi(x) : \phi \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the set of all solutions $\phi \in \mathrm{Lip}\,(\bar{\Omega})$ of (5) satisfying $\phi \ge v_0^-$ on $\bar{\Omega}$. It is well-known (see also Proposition A.5 in the appendix or Perron's method as well) that $\mathcal{S} \ne \emptyset$ and the above formula gives a solution in $\mathrm{Lip}\,(\bar{\Omega})$ of (5).

The definition of u_∞ is related to the additive eigenvalue problem (or, ergodic problem): consider the problem of finding a pair $(c, v) \in \mathbb{R} \times \mathrm{Lip}\,(\bar{\Omega})$ such that v is a solution of

$$(8) \qquad \begin{cases} |Dv(x)| = 1 & \text{in } \Omega, \\ c + \nu \cdot Dv(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

It is clear that if (c, v) is a solution of the above additive eigenvalue problem, so is the pair $(c, v + A)$, with any constant $A \in \mathbb{R}$. On the other hand, the following theorem assures that the *additive eigenvalue* c is unique and, indeed, $c = 1$.

Theorem 2.3. *There exists a solution $v \in C(\bar{\Omega})$ of* (6). *Moreover, for any $c \neq 1$, there exists no solution of* (8).

Proof. 1. To prove the existence of a solution of (6), we show that the function

$$v(x) = \text{dist}\,(x, \partial\Omega)$$

is a solution of (6).

For any $z \in \partial\Omega$ the function $x \mapsto |x - z|$ is a classical solution of (5), and hence, the function

$$v(x) = \inf\{|x - z| \,:\, z \in \partial\Omega\}$$

is a solution of (5) (see Proposition A.5). Next let $\phi \in C^1(\bar{\Omega})$ and $x \in \partial\Omega$. We first assume that $v - \phi$ has a maximum at $x \in \partial\Omega$. We set $y = x - \varepsilon\nu$, where $\nu = \nu(x)$ and $\varepsilon > 0$. Note that if $\varepsilon > 0$ is sufficiently small, then $y \in \Omega$. The C^1 regularity of Ω ensures that

$$v(y) - v(x) = v(x - \varepsilon\nu) = \varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0 + .$$

Hence, as $\varepsilon \to 0+$, we get

$$\varepsilon + o(\varepsilon) = v(y) - v(x) \leq \phi(y) - \phi(x) = -\varepsilon\nu \cdot D\phi(x) + o(\varepsilon),$$

which yields

$$1 + \nu \cdot p \leq 0.$$

Next, we assume that $v - \phi$ has a minimum at $x \in \partial\Omega$. Set $y = x - \varepsilon\nu$, with $\varepsilon > 0$. Observe that as $\varepsilon \to 0+$,

$$\begin{aligned} \varepsilon + o(\varepsilon) &= v(y) - v(x) \geq \phi(y) - \phi(x) \\ &= D\phi(x) \cdot (y - x) + o(\varepsilon) = -\varepsilon\nu D\phi(x) + o(\varepsilon), \end{aligned}$$

from which we get

$$1 + \nu \cdot p \geq 0.$$

We thus conclude that v is a supersolution of (6) and moreover that v is a solution of (6).

2. We next show that $c = 1$ is the only possible choice for which (8) has a solution. We actually show that if there exist solutions $(c_1, u), (c_2, v) \in \mathbb{R} \times \text{Lip}\,(\bar{\Omega})$ of (8), then $c_1 = c_2$. By symmetry, we only need to show that $c_1 \leq c_2$. To this end, we argue by contradiction. Thus, we assume that the inequality $c_1 > c_2$ holds. Let $A > 0$, and define the functions $V, W \in \text{Lip}\,(\bar{Q})$ by

$$\begin{cases} V(x, t) = v(x) + c_1 t, \\ W(x, t) = w(x) + c_2 t + A. \end{cases}$$

It is easily seen that V and W are both solutions of (2). We select A sufficiently large so that $W(x, 0) \geq V(x, 0)$ for all $x \in \bar{\Omega}$. By the comparison principle (see Theorem A.1), we obtain

$$W(x, t) \geq V(x, t) \quad \text{for all } (x, t) \in \bar{Q}.$$

But this is a contradiction since $c_1 > c_2$. Thus we must have $c_1 \leq c_2$. □

3. Proof of the main results

Theorem 3.1. *For each $\varepsilon > 0$ let $v^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$ be the solution of* (3). *Then there exists a function $v \in \mathrm{Lip}\,(\bar{Q})$ such that*

$$\lim_{\varepsilon \to 0+} v^\varepsilon(x,t) = v(x,t) \quad \text{uniformly on} \ \ \bar{\Omega} \times [0, T)$$

for all $0 < T < \infty$. The function v is a solution of (4).

The existence and uniqueness of solution of (3) have been shown in Corollary 2.2. The following lemma is needed in our proof of the above theorem.

Lemma 3.2 (Comparison). *Let $v \in \mathrm{Lip}\,(\bar{Q})$ and $w \in \mathrm{Lip}\,(\bar{Q})$ be a subsolution and a super-solution of*

$$(9) \qquad \begin{cases} u_t(x,t) + |D_x u(x,t)| = 1 & in \ \ Q, \\ u_t(x,t) = 0 & on \ \ \partial\Omega \times (0, \infty), \end{cases}$$

respectively. Assume that $v(x,0) \le w(x,0)$ for all $x \in \bar{\Omega}$. Then $v \le w$ on $\bar{Q}$.

The above comparison principle does not hold in general if the Lipschitz regularity of the functions v, w is removed. For this see Example A.6 in the appendix.

Proof. Fix any $\varepsilon > 0$. Let $M > 0$ be a Lipschitz bound of the functions v and w. It is easily checked that the functions $v_\varepsilon(x,t) := v(x,t) - \varepsilon M t$ and $w_\varepsilon(x,t) := w(x,t) + \varepsilon M t$ are, respectively, a subsolution and a supersolution of

$$\begin{cases} u_t(x,t) + |D_x u(x,t)| = 1 & in \ \ Q, \\ u_t(x,t) + \varepsilon \nu(x) \cdot D_x u(x,t) = 0 & on \ \ \partial\Omega \times (0, \infty), \end{cases}$$

and that $v_\varepsilon(x,0) = v(x,0) \le w(x,0) = w_\varepsilon(x,0)$ for all $x \in \bar{\Omega}$. Applying a standard comparison theorem (for instance, Theorem A.1), we get

$$v_\varepsilon(x,t) = v(x,t) - \varepsilon M t \le w_\varepsilon(x,t) = w(x,t) + \varepsilon M t \quad \text{for all} \ \ (x,t) \in \bar{Q}.$$

Sending $\varepsilon \to 0$ yields the desired inequality. $\qquad \square$

The following proposition is an immediate consequence of Theorem 3.1 and Lemma 3.2.

Corollary 3.3. *There exists a unique solution of* (4) *in the class $\mathrm{Lip}\,(\bar{Q})$.*

Proof of Theorem 3.1. We show first that the family $\{v^\varepsilon\}_{0 < \varepsilon < 1}$ is equi-Lipschitz continuous on $\bar{Q}$. The argument is similar to the last part of the proof of Theorem 2.1.

Let $0 < \varepsilon < 1$. Let $M \ge 1$ be a Lipschitz bound of the function u_0. It is easily checked that the functions U^+, $U^- \in \mathrm{Lip}\,(\bar{Q})$ given by

$$U^\pm(x,t) = u_0(x) \pm Mt \quad \text{for} \ \ (x,t) \in \bar{Q}$$

are a supersolution and a subsolution of (3), respectively. By comparison (Theorem A.1), we get

$$U^-(x,t) \le v^\varepsilon(x,t) \le U^+(x,t) \quad \text{for all} \ \ (x,t) \in \bar{Q}.$$

Consequently, for any $h > 0$, we have

$$v^\varepsilon(x,h) - Mh \le u_0(x) = v^\varepsilon(x,0) \le v^\varepsilon(x,h) + Mh \quad \text{for all} \ \ x \in \bar{\Omega}.$$

Again, by comparison, we get

$$v^\varepsilon(x, t+h) - Mh \le v^\varepsilon(x, t) \le v^\varepsilon(x, t+h) + Mh \quad \text{for all} \ \ (x, t) \in \bar{Q}.$$

Hence, using Lemma A.3, we deduce that the collection $\{v^\varepsilon\}_{0 < \varepsilon < 1}$ is equi-Lipschitz continuous on $\bar{Q}$.

Thanks to the Ascoli-Arzela theorem, there are a sequence $\{\varepsilon_j\} \subset (0, 1)$ converging to zero and a function $v \in \mathrm{Lip}\,(\bar{Q})$ such that

$$(10) \qquad \lim_{j \to \infty} \|v^{\varepsilon_j} - v\|_{\infty, \bar\Omega \times [0, T]} = 0 \quad \text{for every} \ \ T > 0.$$

By the well-known stability property of viscosity solutions, we see that v is a solution of (4).

To complete the proof, we need to show that for any $T > 0$,

$$\lim_{\varepsilon \to 0+} \|v^\varepsilon - v\|_{\infty, \bar\Omega \times [0, T]} = 0.$$

For this, we argue by contradiction and suppose that there were a sequence $\{\varepsilon(j)\}_j \subset (0, 1)$ converging to zero and a constant $0 < S < \infty$ such that

$$(11) \qquad \limsup_{j \to \infty} \|v^{\varepsilon(j)} - v\|_{\infty, \bar\Omega \times [0, S]} > 0.$$

Passing to a subsequence and arguing as in the case of the sequence $\{\varepsilon_j\}$, we may assume that there is a solution $w \in \mathrm{Lip}\,(\bar{Q})$ of (4) such that for all $T > 0$,

$$(12) \qquad \lim_{j \to \infty} \|v^{\varepsilon(j)} - w\|_{\infty, \bar\Omega \times [0, T]} = 0.$$

But, by Lemma 3.2, we must have $v = w$ on $\bar{Q}$, and (11) contradicts (12). $\qquad \square$

Theorem 3.4. *Let $v \in \mathrm{Lip}\,(\bar{Q})$ be the solution of (4). Then*

$$\lim_{t \to \infty} v(x, t) = v_\infty(x) \quad \text{uniformly on} \ \ \bar\Omega.$$

Indeed, one can prove that there exists a constant $T > 0$ such that

$$v(x, t) = v_\infty(x) \qquad \text{for all} \ \ (x, t) \in \bar\Omega \times [T, \infty).$$

Proof of Theorem 3.4. 1. We show first that v is bounded on $\bar{Q}$. Fix an $e \in \mathbb{R}^n$ so that $|e| = 1$. For any $C \in \mathbb{R}$ the function $w_C(x, t) := e \cdot x + C$ is a solution of (9). Hence, choosing $C > 0$ so large that $|u_0(x) - e \cdot x| \le C$ for all $x \in \bar\Omega$, by comparison (Lemma 3.2), we get

$$w_{-C}(x, t) \le v(x, t) \le w_C(x, t) \quad \text{for all} \ \ (x, t) \in \bar{Q},$$

which shows that v is bounded on $\bar{Q}$.

2. We next show that for each $x \in \partial\Omega$ the function $t \mapsto v(x, t)$ is nonincreasing on $[0, \infty)$. We fix any $\hat{x} \in \partial\Omega$ and show that the function $t \mapsto v(\hat{x}, t)$ is nonincreasing on $[0, \infty)$. To this end, we assume, by contradiction, that there were two positive numbers $t_0 < t_1$ such that

$$v(\hat{x}, t_0) < v(\hat{x}, t_1).$$

We may choose an increasing function $\psi \in C^1([t_0, t_1])$ such that $v(\hat{x}, \cdot) - \psi$ attains a strict maximum at some point $\hat{t} \in (t_0, t_1)$ and $\inf_{r \ge 0} \psi'(r) > 0$.

For $\alpha \ge 0$ we introduce the function

$$\Phi_\alpha(x, t, y, s) := v(x, t) - \psi(t) - \alpha|x - \hat{x}|^2 + \alpha^2 \rho(x)$$

on $\bar{\Omega} \times [t_0, t_1]$, where ρ is a defining function of Ω. Note that

$$\Phi_\alpha(x,t) \leq \Phi_0(x,t) \quad \text{for all} \quad (x,t) \in \bar{\Omega} \times [t_0, t_1], \, \alpha > 0.$$

Let $\alpha > 0$ and let (x_α, t_α) be a maximum point of the function Φ_α over $\bar{\Omega} \times [t_0, t_1]$. Note that

$$\Phi_0(\hat{x}, \hat{t}) = \Phi_\alpha(\hat{x}, \hat{t}) \leq \Phi_\alpha(x_\alpha, t_\alpha) \leq \Phi_0(x_\alpha, t_\alpha) - \alpha|x_\alpha - \hat{x}|^2.$$

This sequence of inequalities guarantees that

$$\lim_{\alpha \to \infty} (x_\alpha, t_\alpha) = (\hat{x}, \hat{t}).$$

In particular, if α is sufficiently large, then $t_0 < t_\alpha < t_1$. For such a large α, by the viscosity property of v, we have either

$$\psi'(t_\alpha) \leq 0 \quad \text{or} \quad \psi'(t_\alpha) + |2\alpha(x_\alpha - \hat{x}) - \alpha^2 D\rho(x_\alpha)| \leq 1.$$

Noting that

$$\lim_{\alpha \to \infty} |2\alpha(x_\alpha - \hat{x}) - \alpha^2 D\rho(x_\alpha)| = \infty$$

and sending $\alpha \to \infty$, we get

$$\psi'(\hat{t}) \leq 0,$$

which contradicts our choice of ψ. Thus we see that for each $x \in \partial\Omega$ the function $t \mapsto v(x,t)$ is nonincreasing on $[0, \infty)$ and, therefore, the limit

$$\lim_{t \to \infty} v(x,t) = v_b(x) \quad \text{exists for any} \quad x \in \partial\Omega,$$

where v_b is a function on $\partial\Omega$. Noting that $v \in \mathrm{Lip}\,(\bar{Q})$, we see that $v_b \in \mathrm{Lip}\,(\partial\Omega)$.

3. We define $v^\pm \in \mathrm{Lip}\,(\bar{Q})$ by

$$\begin{cases} v^+(x,t) = \sup_{s \geq 0} v(x, t+s), \\ v^-(x,t) = \inf_{s \geq 0} v(x, t+s). \end{cases}$$

By the monotonicity of the function $t \mapsto v(x,t)$ for $x \in \partial\Omega$, we see that

(13) $$v^-(x,t) = v_b(x) \quad \text{for all} \quad (x,t) \in \partial\Omega \times [0, \infty).$$

It is a standard observation (see [12, Proposition 1.10], [6, 2, 5] for instance) that v^+ and v^- are a subsolution and a supersolution of (9), respectively. Because of the Lipschitz continuity and boundedness of $v^\pm$, we find that

$$\begin{cases} \lim_{s \to \infty} v^+(x, t+s) = V^+(x,t), \\ \lim_{s \to \infty} v^-(x, t+s) = V^-(x,t) \end{cases}$$

for some functions $V^\pm \in \mathrm{Lip}\,(\bar{Q})$, where the convergence is uniform for $(x,t) \in \bar{\Omega} \times [0, T]$, with every $T > 0$. By the stability of the viscosity property under uniform convergence, we see that V^+ and V^- are a subsolution and a supersolution of (9), respectively. It is easily seen that the functions $V^\pm(x,t)$ in fact do not depend on t. We may thus denote them respectively by $V^\pm(x)$, and we have for all $x \in \bar{\Omega}$,

$$\begin{cases} V^+(x) = \limsup_{t \to \infty} v(x,t), \\ V^-(x) = \liminf_{t \to \infty} v(x,t). \end{cases}$$

Obviously we have

$$\begin{cases} V^+(x) = V^-(x) = v_b(x) & \text{for all } x \in \partial\Omega, \\ V^+(x) \ge V^-(x) & \text{for all } x \in \bar\Omega, \end{cases}$$

and the functions V^+ and V^- are a subsolution and a supersolution of the eikonal equation

(5) $$|Du(x)| = 1 \quad \text{in } \Omega,$$

respectively. By the standard comparison result ([14, 5, 2, 16]), we get

$$V^+(x) \le V^-(x) \quad \text{for all } x \in \bar\Omega.$$

It is now clear that

$$\lim_{t\to\infty} v(x,t) = V^+(x) = V^-(x) \quad \text{uniformly on } \bar\Omega.$$

4. Set $V := V^+ = V^-$. We intend to identify V with v_∞. By the definition of v_0^-, we have

$$v_0^-(x) \le u_0(x) \quad \text{for all } x \in \bar\Omega,$$

and the function $v_0^-(x)$ as a function on $\bar Q$ is a subsolution of (9). By comparison, we get

$$v_0^-(x) \le v(x,t) \quad \text{for all } (x,t) \in \bar Q.$$

Consequently, we have

$$v_0^-(x) \le V(x) \quad \text{for all } x \in \bar\Omega.$$

Since V is a solution of (5), we see that

(14) $$v_\infty(x) \le V(x) \quad \text{for all } x \in \bar\Omega.$$

It is immediate to see by the definition of v^- that for each $x \in \bar\Omega$, the function $t \mapsto v^-(x,t)$ is nondecreasing on $[0, \infty)$. Since (9) is a convex Hamilton-Jacobi equation, we deduce (see Proposition A.5) that v^- is a solution of

$$u_t(x,t) + |D_x u(x,t)| = 1 \quad \text{in } Q.$$

From these observations, we infer that for each $t > 0$ the function $x \mapsto v^-(x,t)$ is a subsolution of (5). Noting that

$$u_0(x) \ge v^-(x,0) \quad \text{for all } x \in \bar\Omega$$

and that

$$v^-(x,0) = \lim_{t\to 0+} v^-(x,t) \quad \text{uniformly for all } x \in \bar\Omega,$$

which shows that the function $x \mapsto v^-(x,0)$ is a subsolution of (5), we see that

$$v^-(x,0) \le v_0^-(x) \le v_\infty(x) \quad \text{for all } x \in \bar\Omega.$$

By the constancy (13), it is now easy to check that v^- is a subsolution of (9). Also, the function $v_\infty(x)$, regarded as a function of (x,t), is a solution of (9). Hence, by comparison, we get

$$v^-(x,t) \le v_\infty(x) \quad \text{for all } (x,t) \in \bar Q.$$

Consequently, we get

$$V(x) \le v_\infty(x) \quad \text{for all } x \in \bar\Omega.$$

This together with (14) guarantees that $V = v_\infty$. $\qquad\square$

Proof of Theorem 1.1. We set $v^\varepsilon(x,t) := u^\varepsilon(x, \varepsilon t)$ for $(x,t) \in \bar{Q}$. Let $v \in \mathrm{Lip}\,(\bar{Q})$ be the solution of (4). Fix any $\delta > 0$. By Theorem 3.4, there is a constant $0 < T < \infty$ such that

$$|v(x,T) - v_\infty(x)| < \delta \quad \text{for all} \ \ x \in \bar{\Omega}.$$

By Theorem 3.1, there exists a constant $\varepsilon_0 \in (0,\,1)$ such that

$$|v^\varepsilon(x,T) - v(x,T)| < \delta \quad \text{for all} \ \ x \in \bar{\Omega} \ \ \text{and} \ \ 0 < \varepsilon < \varepsilon_0.$$

Thus, we have

$$|v^\varepsilon(x,T) - v_\infty(x)| < 2\delta \quad \text{for all} \ x \in \bar{\Omega} \ \ \text{and} \ \ 0 < \varepsilon < \varepsilon_0,$$

which reads

$$(15) \qquad |u^\varepsilon(x, \varepsilon T) - v_\infty(x)| < 2\delta \quad \text{for all} \ x \in \bar{\Omega} \ \ \text{and} \ \ 0 < \varepsilon < \varepsilon_0.$$

Since v_∞ is a solution of (5), the function v_∞ is Lipschitz continuous on $\bar{\Omega}$ with a Lipschitz bound L_Ω. Hence, the functions $v_\infty(x) \pm L_\Omega t$ are a supersolution and a subsolution of (1), respectively. Using (15), by comparison (Theorem A.1), we get

$$(16) \qquad |u^\varepsilon(x, t + \varepsilon T) - v_\infty(x)| < 2\delta + L_\Omega t \quad \text{for all} \ \ (x,t) \in \bar{Q}, \ 0 < \varepsilon < \varepsilon_0.$$

We define the functions $u^\pm$ on $\bar{\Omega} \times (0, \infty)$ by

$$\begin{cases} u^+(x,t) = \lim_{r \to 0+} \sup\{u^\varepsilon(y,s) \,:\, (y,s) \in \bar{\Omega} \times (0, \infty), \ 0 < \varepsilon < r, \ |y - x| + |s - t| < r\}, \\ u^-(x,t) = \lim_{r \to 0+} \inf\{u^\varepsilon(y,s) \,:\, (y,s) \in \bar{\Omega} \times (0, \infty), \ 0 < \varepsilon < r, \ |y - x| + |s - t| < r\}. \end{cases}$$

The functions u^+ and u^- are called the half-relaxed limits of the functions u^ε, and it is well-known (see [12, Theorem 1.3], [6, 2, 5]) that $u^+ \in \mathrm{USC}(\bar{\Omega} \times (0, \infty))$, $u^- \in \mathrm{LSC}(\bar{\Omega} \times (0, \infty))$, $u^- \le u^+$ in $\bar{\Omega} \times (0, \infty)$, and u^+ and u^- are a subsolution and a supersolution of (2), respectively. Due to estimate (16), we see that if we set

$$u^\pm(x,0) = v_\infty(x) \quad \text{for} \ \ x \in \bar{\Omega},$$

then $u^+ \in \mathrm{USC}(\bar{Q})$ and $u^- \in \mathrm{LSC}(\bar{Q})$. By comparison (Theorem A.1), we get $u^+ \le u \le u^-$ on $\bar{Q}$, which shows that $u^+ = u^- = u$ on $\bar{Q}$ and moreover that as $\varepsilon \to 0+$, $u^\varepsilon(x,t) \to u(x,t)$ uniformly for $(x,t) \in \bar{\Omega} \times [S^{-1},\, S]$ for every $S > 1$. $\qquad\square$

Proof of Theorem 1.2. We define the function $w \in \mathrm{Lip}\,(\bar{Q})$ by

$$w(x,t) = u(x,t) - t,$$

and observe that w is a solution of

$$(17) \qquad \begin{cases} |D_x w(x,t)| = 1 & \text{in} \ \ Q, \\ w_t(x,t) + 1 + \nu(x) \cdot D_x w(x,t) = 0 & \text{on} \ \ \partial\Omega \times (0, \infty). \end{cases}$$

The function v_∞ is a solution of (5) and hence it is a supersolution of (6). (Indeed, for any $x \in \partial\Omega$ and $p \in D^- v(x)$, we have two possibilities: either $|p| \ge 1$ or $|p| < 1$, and if $|p| < 1$, then $1 + \nu(x) \cdot p \ge 1 - |\nu(x)||p| > 0$.) Accordingly, the function $v_\infty(x)$, as a function of (x,t), is a supersolution of (17). By comparison (Theorem A.1), we get

$$w(x,t) \le v_\infty(x) \quad \text{for all} \ \ (x,t) \in \bar{Q}.$$

Again, by the comparison between $w(x,t)$ and $w(x, t+h)$, with $h > 0$, we get

$$w(x, t+h) \le w(x,t) \quad \text{for all} \ \ (x,t) \in \bar{Q} \ \ \text{and} \ \ h > 0.$$

That is, for each $x \in \bar{\Omega}$, the function $t \mapsto w(x, t)$ is nonincreasing on $[0, \infty)$.

Note that the function $u_\infty(x)$ is, as a function of (x, t), a solution of (17) and that $u_\infty(x) \leq v_\infty(x)$ for all $x \in \bar{\Omega}$. By comparison, we get

$$w(x, t) \geq u_\infty(x) \quad \text{for all} \ \ (x, t) \in \bar{Q}.$$

Noting also that w is bounded and Lipschitz continuous on $\bar{Q}$, we see that

$$\lim_{t \to \infty} w(x, t) = w_\infty(x) \quad \text{uniformly on} \ \ \bar{\Omega}$$

for some function $w_\infty \in \text{Lip}\,(\bar{\Omega})$. Clearly, w_∞ is a solution of (6) and satisfies

$$u_\infty(x) \leq w_\infty(x) \leq v_\infty(x) \quad \text{for all} \ \ x \in \bar{\Omega}.$$

Because of the maximality of u_∞, we conclude that $w_\infty = u_\infty$ and that

$$\lim_{t \to \infty} w(x, t) = u_\infty(x) \quad \text{uniformly on} \ \ \bar{\Omega}. \qquad \square$$

4. Initial value problem for (2)

We discuss here the well-posedness of the initial value problem for (2) and consider first the initial value problem

$$(18) \qquad \begin{cases} |D_x u(x, t)| = 1 & \text{in} \ \ Q, \\ u_t(x, t) + \nu(x) \cdot D_x u(x, t) = 0 & \text{on} \ \ \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{for} \ \ x \in \bar{\Omega}. \end{cases}$$

This problem has been studied in the previous sections, but it is overdetermined in its initial condition. Indeed, if u is a solution of (18) and continuous on $\bar{Q}$, then the function u_0 should be a solution of $|Du_0(x)| = 1$ in Ω and therefore it should be given by the boundary data $u_0|_{\partial\Omega}$. This suggests another formulation: let $u_0 \in C(\partial\Omega)$ and consider the initial value problem

$$(19) \qquad \begin{cases} |D_x u(x, t)| = 1 & \text{in} \ \ Q, \\ u_t(x, t) + \nu(x) \cdot D_x u(x, t) = 0 & \text{on} \ \ \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{for} \ \ x \in \partial\Omega. \end{cases}$$

Suppose that there is a solution $u \in C(Q \cup (\partial\Omega \times [0, \infty)))$ of (19). Observe that for each $t > 0$ the function $v(x) := u(x, t)$ is a solution of $|Dv(x)| = 1$ in Ω, which shows that the collection $\{u(\cdot, t) : t > 0\}$ is equi-Lipschitz continuous on $\bar{\Omega}$. Hence, we may choose a sequence $t_j \to 0+$ such that the limit

$$\bar{u}_0(x) = \lim_{j \to \infty} u(x, t_j)$$

exists for all $x \in \bar{\Omega}$ and the convergence is uniform on $\bar{\Omega}$. It is a standard observation that the limit function $\bar{u}_0$ is a unique solution of the Dirichlet problem

$$(20) \qquad \begin{cases} |Dv(x)| = 1 & \text{in} \ \ \Omega, \\ v(x) = u_0(x) & \text{for} \ \ x \in \partial\Omega. \end{cases}$$

Moreover, it follows that

$$\lim_{t\to 0+} u(x,t) = \bar{u}_0(x) \quad \text{uniformly for } x \in \bar{\Omega}.$$

We thus see that if $u \in C(Q \cup (\partial\Omega \times [0,\infty)))$ is a solution of (19), then u is extended uniquely to a function on $\bar{Q}$ and the resulting function solves (18), with u_0 replaced by the unique solution v of the Dirichlet problem (20). It is well-known that there exists a solution $v \in \text{Lip}(\bar{\Omega})$ of (20) if and only if there exists a subsolution $w \in \text{Lip}(\bar{\Omega})$ of (20).

Now, an existence result for (19) is stated as follows.

Theorem 4.1. *Let $u_0 \in \text{Lip}(\partial\Omega)$. Assume that there exists a subsolution $w \in \text{Lip}(\bar{\Omega})$ of (20). Then there exists a unique solution $u \in \text{Lip}(\bar{Q})$ of (19).*

Before giving a proof of the above theorem, we present a comparison principle.

Lemma 4.2. *Let $v \in \text{USC}(Q \cup (\partial\Omega \times [0,\infty)))$ and $w \in \text{LSC}(Q \cup (\partial\Omega \times [0,\infty)))$ be a subsolution and a supersolution of (2), respectively. Assume that $v(x,0) \le w(x,0)$ for all $x \in \partial\Omega$. Then $v \le w$ on $Q \cup (\partial\Omega \times [0,\infty))$.*

Proof. Fix any $\varepsilon > 0$. By the semicontinuity of v, w and the inequality $v(\cdot,0) \le w(\cdot,0)$ on $\partial\Omega$, we find that there exists a constant $\delta > 0$ such that

$$v(x,t) \le w(x,t) + \varepsilon \quad \text{for all } (x,t) \in \partial\Omega \times (0,\delta).$$

For each $0 < t < \delta$, since

$$|D_x v(x,t)| \le 1 \quad \text{and} \quad |D_x w(x,t)| \ge 1 \quad \text{in } \Omega$$

hold in the viscosity sense, by a standard comparison result (see Lemma A.4 or [14, 5, 2]), we see that $v(x,t) \le w(x,t) + \varepsilon$ for all $(x,t) \in \bar{\Omega} \times (0,\delta)$. We now apply Theorem A.1, with the interval $[t,\infty)$ in place of $[0,\infty)$, in the appendix, to conclude that

$$v(x,t+s) \le w(x,t+s) + \varepsilon \quad \text{for all } (x,s) \in \bar{Q}, \ t \in (0,\delta),$$

which implies that $v(x,t) \le w(x,t)$ for all $(x,t) \in Q \cup (\partial\Omega \times [0,\infty))$. $\square$

Proof of Theorem 4.1. The uniqueness of solution of (19) follows readily from Lemma 4.2.

To show the existence of a solution of (19), we consider problem (18), with u_0 replaced by the solution $w \in \text{Lip}(\bar{\Omega})$ of (20), and argue as in the proof of Theorem 2.1. Recall that the constant L_Ω is a Lipschitz bound of the function w. We define the functions $U^\pm \in \text{Lip}(\bar{Q})$ by

$$U^\pm(x,t) = w(x) \pm L_\Omega t,$$

and observe that U^+ and U^- are a supersolution and a subsolution of (2), respectively. By Perron's method, we may find a function $u \in \text{USC}(\bar{Q})$ such that u and the lower semicontinuous envelope u_* are a subsolution and a supersolution of (2), respectively, and $U^- \le u_* \le u \le U^+$ on $\bar{Q}$. By the comparison (see Theorem A.1) between u and u_*, we see that $u = u_* \in C(\bar{Q})$. Also, by the comparison between the functions $u(x,t+h)$ and $u(x,t) + L_\Omega h$ of (x,t), with $h > 0$, we get $u(x,t+h) \le u(x,t) + L_\Omega h$ for all $(x,t) \in \bar{Q}$ and $h > 0$. Similarly, we get $u(x,t+h) \ge u(x,t) - L_\Omega h$ for all $(x,t) \in \bar{Q}$ and $h > 0$. Consequently, we obtain

$$|u(x,t) - u(x,s)| \le L_\Omega|t-s| \quad \text{for all } x \in \bar{\Omega} \text{ and } t,s \in [0,\infty).$$

For each $t > 0$ the function $x \mapsto u(x, t)$ is a solution of $|D_x u(x, t)| = 1$ in Ω, which shows that $|u(x, t) - u(y, t)| \le L_\Omega |x - y|$ for all $x, y \in \bar{\Omega}$ and $t > 0$. Thus we see that $u \in \mathrm{Lip}\,(\bar{Q})$ and u is a solution of (18), with w in place of u_0.   □

5. Variational formulas

We have studied several Hamilton-Jacobi equations of the eikonal type. We discuss in this section the variational (or optimal control) formulas for solutions of such Hamilton-Jacobi equations.

We recall first the general principle (the Bellman principle). Let U be an open subset of $\mathbb{R}^m$ and Γ a closed subset of ∂U. Let $\gamma : \partial U \setminus \Gamma \to \mathbb{R}^m$ be a continuous vector field such that $\gamma(x) \cdot \nu(x) > 0$ (or $\gamma(x) \cdot \nu(x) \ge 0$) for all $x \in \partial U \setminus \Gamma$, where $\nu(x)$ denotes the outer unit normal vector of U at x and $p \cdot q$ denotes the inner product in $\mathbb{R}^m$. Given a function $v \in L^\infty([0, \infty), \mathbb{R}^n)$, the Skorokhod problem is then to seek for $\tau \in [0, \infty)$ and $(X, l) \in \mathrm{Lip}\,([0, \tau], \mathbb{R}^m) \times L^\infty([0, \tau], \mathbb{R})$ such that

$$
(21) \quad
\begin{cases}
\dot{X}(t) + l(t)\gamma(X(t)) = v(t) & \text{a.e. } t \in [0, \tau], \\
X(t) \in \bar{U} & \text{for all } t \in [0, \tau], \\
l(t) \ge 0 & \text{a.e. } t \in [0, \tau], \\
l(t) = 0 \ \text{ if } \ X(t) \in U & \text{a.e. } t \in [0, \tau], \\
X(\tau) \in \Gamma.
\end{cases}
$$

The Skorokhod problem has been investigated extensively in the literature (see [15]), and we refer to [12, Theorem 5.2], [13] for the existence results convenient for our discussion here. We denote by SP the set of all quadruples (X, l, τ, v) which satisfy (21). We consider the function

$$
V(x) = \inf \int_0^\tau [f(X(t), v(t)) + l(t)g(X(t))]\, \mathrm{d}t
$$

on $\bar{U}$, where f and g are given functions on $\bar{U} \times \mathbb{R}^m$ and ∂U, respectively, and the infimum is taken over all $(X, l, \tau, v) \in$ SP such that $X(0) = x$. This is an optimal control problem, where the function v plays the role of control and where the function V is called the *value function*. The dynamic programming principle leads to the boundary-value problem for the value function V:

$$
\begin{cases}
\sup_{v \in \mathbb{R}^m} \{-v \cdot DV(x) - f(x, v)\} = 0 & \text{in } U, \\
\gamma(x) \cdot DV(x) = g(x) & \text{on } \partial U \setminus \Gamma, \\
V(x) = V_0(x) & \text{for } x \in \Gamma,
\end{cases}
$$

where V_0 is a given function representing the Dirichlet (or initial) data on Γ.

We apply the above principle to find correct variational formulas for solutions of the Hamilton-Jacobi equations discussed in the previous sections.

We treat first the equation (1), where the Hamilton-Jacobi equation can be written as

$$
(22) \qquad \sup\{-(\eta, \sigma) \cdot Du(x, t) - \delta_{\bar{B}_1(0) \times \{\varepsilon\}}(-\eta, -\sigma) - 1 : (\eta, \sigma) \in \mathbb{R}^n \times \mathbb{R}\} = 0,
$$

where δ_A denotes the indicator function of the set A, i.e., the function δ_A is defined by $\delta_A(z) = 0$ if $z \in A$ and $\delta_A(z) = \infty$ otherwise. The sets Q and $\Omega \times \{0\}$ in (1) correspond

to U and Γ in the above, respectively, and the vector field $(\nu(x), 1)$ on $\partial\Omega \times (0, \infty)$, where $\nu(x)$ and ε are from (1), corresponds to the $\gamma(x, t)$ in the above. Given functions $(v, w) \in L^\infty([0, \infty), \mathbb{R}^n \times \mathbb{R})$, our current Skorokhod problem is stated as

$$(23) \quad \begin{cases} \dot{X}(t) + l(t)\gamma(X(t)) = v(t) & \text{a.e. } t \in [0, \tau], \\ \dot{T}(t) + l(t) = w(t) & \text{a.e. } t \in [0, \tau], \\ X(t) \in \bar{\Omega} \text{ and } T(t) \geq 0 & \text{for all } t \in [0, \tau], \\ l(t) \geq 0 & \text{a.e. } t \in [0, \tau], \\ l(t) = 0 \text{ if } (X(t), T(t)) \in Q & \text{a.e. } t \in [0, \tau], \\ T(\tau) = 0, \end{cases}$$

where $\tau \in [0, \infty)$ and $(X, T, l) \in \mathrm{Lip}\,([0, \tau], \mathbb{R}^n \times \mathbb{R}) \times L^\infty([0, \tau], \mathbb{R})$ are to be looked for. Accordingly, SP denotes the set of all sextuples (X, T, l, τ, v, w) satisfying (23) and the minimization problem at $(x, t) \in \bar{Q}$, i.e., the value at (x, t) of the optimal control problem associated with (23) or (22) is stated as

$$\inf \left\{ \int_0^\tau (\delta_{\bar{B}_1(0) \times \{\varepsilon\}}(-v(t), -w(t)) + 1)\, \mathrm{d}t \,:\, (X, T, l, \tau, v, w) \in \mathrm{SP}, \ X(0) = x, \ T(0) = t \right\}.$$

For any $(X, T, l, \tau, v, w) \in \mathrm{SP}$, with $(X(0), T(0)) = (x, t) \in \bar{Q}$, if the integral is finite in the above minimization formula, then $|v(t)| \leq 1$ and $w(t) = -\varepsilon$ for a.e. $t \in [0, \tau]$ and τ is characterized by

$$-t + \int_0^\tau l(t)\, \mathrm{d}t = -\varepsilon\tau.$$

These observations suggest a modification of SP and we introduce $\mathrm{SP}(1; x, t)$ as the set of all triples (X, l, τ) of $\tau \in [0, \infty)$, $X \in \mathrm{Lip}\,([0, \tau], \mathbb{R}^n)$ and $l \in L^\infty([0, \tau], \mathbb{R})$ such that

$$(24) \quad \begin{cases} \dot{X}(s) + l(s)\nu(X(s)) \in \bar{B}_1(0) & \text{a.e. } s \in [0, \tau], \\ t = \int_0^\tau l(s)\, \mathrm{d}s + \varepsilon\tau, \\ X(0) = x, \quad X(s) \in \bar{\Omega} & \text{for all } s \in [0, \tau], \\ l(s) = 0 \text{ if } X(s) \in \Omega, \quad l(s) \geq 0 & \text{a.e. } s \in [0, \tau]. \end{cases}$$

Theorem 5.1. *The solution $u^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$ of (1) is represented as*

$$(25) \quad u^\varepsilon(x, t) = \inf\{\tau + u_0(X(\tau)) \,:\, (X, l, \tau) \in \mathrm{SP}(1; x, t)\} \quad \text{for all } (x, t) \in \bar{Q}.$$

Proof. We write $V(x, t)$ for the right hand side of (25).

1. It is a standard observation that the dynamic programming principle holds: for any $(x, t) \in \bar{\Omega} \times (0, \infty)$ and $\delta \in (0, t)$,

$$(26) \quad V(x, t) = \inf\{\tau + V(X(\tau), t - \delta) \,:\, (X, l, \tau) \in \mathrm{SP}(1; x, \delta)\}.$$

Here we have for any $(X, l, \tau) \in \mathrm{SP}(1; x, \delta)$,

$$\delta = \varepsilon\tau + \int_0^\tau l(r)\, \mathrm{d}r,$$

and hence,

$$(27) \qquad \max_{s \in [0,\tau]} |X(s) - x| \le \int_0^\tau (l(r) + 1)\, \mathrm{d}r \le (\varepsilon^{-1} + 1)\delta.$$

2. We have shown in the proof of Theorem 2.1 that, for some constant $A > 0$, the function $U(x,t) := u_0(x) - At$ is a subsolution of (1). Fix such a function U on $\bar{Q}$. Fix any $(x,t) \in \bar{Q}$ and $(X, l, \tau) \in \mathrm{SP}(1; x, t)$. We set

$$T(s) := t - \varepsilon s - \int_0^s l(r)\, \mathrm{d}r \quad \text{and} \quad v(s) := \dot{X}(s) + l(s)\nu(X(s)) \quad \text{for} \quad s \in [0,\tau],$$

and compute informally that

$$
\begin{aligned}
U(x,t) &= U(X(0), T(0)) = U(X(\tau), T(\tau)) - \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}s} U(X(s), T(s))\, \mathrm{d}s \\
&= u_0(X(\tau)) + \int_0^\tau \left[-U_t(X(s), T(s))\dot{T}(s) - D_x U(X(s), T(s)) \cdot \dot{X}(s) \right] \mathrm{d}s \\
&= u_0(X(\tau)) + \int_0^\tau \left[\varepsilon U_t(X(s), T(s)) - D_x U(X(s), T(s)) \cdot v(s) \right] \mathrm{d}s \\
&\quad + \int_0^\tau l(s) \left[U_t(X(s), T(s)) + D_x U(X(s), T(s)) \cdot \nu(X(s)) \right] \mathrm{d}s \\
&\le u_0(X(\tau)) + \int_0^\tau \left[\varepsilon U_t(X(s), T(s)) + |D_x U(X(s), T(s))| \right] \mathrm{d}s \\
&\quad + \int_0^\tau l(s) \left[U_t(X(s), T(s)) + D_x U(X(s), T(s)) \cdot \nu(X(s)) \right] \mathrm{d}s \\
&\le u_0(X(\tau)) + \tau.
\end{aligned}
$$

The above computation is easily justified by approximating u_0 by smooth functions (see Proposition A.7 and the remark after the proposition), and we conclude that

$$V(x,t) \ge u_0(x) - At \quad \text{for all} \quad (x,t) \in \bar{Q},$$

which yields

$$(28) \qquad V_*(x,0) \ge u_0(x) \quad \text{for all} \quad x \in \bar{\Omega},$$

where V_* denotes the lower semicontinuous envelope of the function V.

3. Next, we show that

$$(29) \qquad V^*(x,0) \le u_0(x) \quad \text{for all} \quad x \in \bar{\Omega},$$

where V^* denotes the upper semicontinuous envelope of the function V. To see this, we fix any $(x,t) \in \bar{Q}$, define the triple (X, l, τ) by

$$\tau := \varepsilon^{-1}t \quad \text{and} \quad X(s) := x, \quad l(s) := 0 \quad \text{for} \quad s \in [0,\tau],$$

and observe that $(X, l, \tau) \in \mathrm{SP}(1; x, t)$ and

$$V(x,t) \le \tau + u_0(x) = u_0(x) + \varepsilon^{-1}t.$$

This clearly shows that (29) holds.

4. We prove that V^* is a subsolution of (1). Let $\phi \in C^1(\bar{Q})$ and $(\hat{x}, \hat{t}) \in \bar{\Omega} \times (0, \infty)$, and assume that $V^* - \phi$ has a strict maximum at $(\hat{x}, \hat{t})$. We treat here only the case where

$\hat{x} \in \partial\Omega$. The other case can be handled similarly and more easily. We argue by contradiction and hence assume that

(30)
$$\varepsilon\phi_t(\hat{x},\hat{t}) + |D_x\phi(\hat{x},\hat{t})| > 1 \quad \text{and} \quad \phi_t(\hat{x},\hat{t}) + \nu(\hat{x}) \cdot D_x\phi(\hat{x},\hat{t}) > 0.$$

We choose a unit vector $e \in \mathbb{R}^n$ so that

$$|D_x\phi(\hat{x},\hat{t})| = -e \cdot D_x\phi(\hat{x},\hat{t}),$$

and then a constant $R \in (0,\hat{t})$ so that for all $(x,t) \in \bar{Q} \cap (\bar{B}_R(\hat{x}) \times [\hat{t}-R,\hat{t}+R])$,

$$\varepsilon\phi_t(x,t) - e \cdot D_x\phi(x,t) > 1 \quad \text{and} \quad \phi_t(x,t) + \nu(x) \cdot D_x\phi(x,t) > 0.$$

In view of (27), we fix an $r \in (0, R/2)$ so that

$$(\varepsilon^{-1}+1)r < R/2,$$

and choose a point $(\bar{x},\bar{t}) \in \bar{Q} \cap (B_{R/2}(\hat{x}) \times (\hat{t}-r/2, \hat{t}+r/2))$ so that

$$(V - \phi)(\bar{x},\bar{t}) > \max\{(V^* - \phi)(x, \hat{t}-r/2) \,:\, x \in \bar{\Omega}\}.$$

Such a choice is possible since

$$(V^* - \phi)(\hat{x},\hat{t}) > \max\{(V^* - \phi)(x, \hat{t}-r/2) \,:\, x \in \bar{\Omega}\}.$$

We set

$$\delta := \bar{t} - (\hat{t} - r/2),$$

and note that $0 < \bar{t} - (\hat{t}-r/2) < r$ and $(\varepsilon^{-1}+1)\delta < R/2$.

According to the existence results in [12, 13], there exists a solution $(X,l,\tau) \in \mathrm{SP}(1; \bar{x}, \delta)$ such that

$$\dot{X}(s) + l(s)\nu(X(s)) = e \quad \text{a.e.} \quad s \in [0,\tau].$$

We set

$$T(s) := \bar{t} - \varepsilon s - \int_0^s l(r)\,\mathrm{d}r \quad \text{for} \quad s \in [0,\tau].$$

By (26), we get

$$V(\bar{x},\bar{t}) \leq \tau + V(X(\tau),\bar{t}-\delta) = \tau + V(X(\tau), T(\tau)).$$

We may assume by adding a constant to the function ϕ that $(V-\phi)(\bar{x},\bar{t}) = 0$, which implies that

$$(V^* - \phi)(x, T(\tau)) = (V^* - \phi)(x, \hat{t}-r/2) < 0 \quad \text{for all } x \in \bar{\Omega}.$$

Hence, we get

(31)
$$
\begin{aligned}
0 &\leq \tau + V(X(\tau), T(\tau)) - V(\bar{x},\bar{t}) < \tau + \phi(X(\tau), T(\tau)) - \phi(\bar{x},\bar{t}) \\
&= \int_0^\tau \left[1 + \phi_t(X(s),T(s))\dot{T}(s) + D_x\phi(X(s),T(s)) \cdot \dot{X}(s)\right]\mathrm{d}s \\
&= \int_0^\tau \left[1 + e \cdot D_x\phi(X(s),T(s)) - \varepsilon\phi_t(X(s),T(s))\right]\mathrm{d}s \\
&\quad + \int_0^\tau l(s)\left[-\phi_t(X(s),T(s)) - \nu(X(s)) \cdot D_x\phi(X(s),T(s))\right]\mathrm{d}s.
\end{aligned}
$$

By estimate (27) and our choice of δ, we have

$$X(s) \in B_{R/2}(\bar{x}) \subset B_R(\hat{x}).$$

Now, using (30), we get

$$
0 < \int_0^\tau \left[1 + e \cdot D_x\phi(X(s), T(s)) - \varepsilon\phi_t(X(s), T(s))\right] \mathrm{d}s
$$
$$
+ \int_0^\tau l(s)\left[-\phi_t(X(s), T(s)) - \nu(X(s)) \cdot D_x\phi(X(s), T(s))\right] \mathrm{d}s < 0,
$$

which is a contradiction. Thus, V^* is a subsolution of (1).

5. We next prove that V_* is a supersolution of (1). The argument here is similar to that in the previous step. Let $\phi \in C^1(\bar{Q})$ and $(\hat{x}, \hat{t}) \in \bar{\Omega} \times (0, \infty)$, and assume that $V_* - \phi$ has a strict minimum at $(\hat{x}, \hat{t})$. Here again we treat only the case where $\hat{x} \in \partial\Omega$. We assume by contradiction that

$$
\varepsilon\phi_t(\hat{x}, \hat{t}) + |D_x\phi(\hat{x}, \hat{t})| < 1 \quad \text{and} \quad \phi_t(\hat{x}, \hat{t}) + \nu(\hat{x}) \cdot D_x\phi(\hat{x}, \hat{t}) < 0.
$$

We choose a constant $R \in (0, \hat{t})$ so that for all $(x, t) \in \bar{Q} \cap (\bar{B}_R(\hat{x}) \times [\hat{t} - R, \hat{t} + R])$,

$$
\varepsilon\phi_t(x, t) + |D_x\phi(x, t)| < 1 \quad \text{and} \quad \phi_t(x, t) + \nu(x) \cdot D_x\phi(x, t) < 0.
$$

As in the previous step, we may choose a point $(\bar{x}, \bar{t}) \in \bar{Q}$ and a constant $\delta > 0$ such that

$$
\begin{cases}
(V - \phi)(\bar{x}, \bar{t}) < \min\{(V_* - \phi)(x, \bar{t} - \delta) : x \in \bar{\Omega}\}, \\
\bar{B}_{R/2}(\bar{x}) \times [\bar{t} - \delta, \bar{t}] \subset \bar{B}_R(\hat{x}) \times [\hat{t} - R, \hat{t} + R], \\
(\varepsilon^{-1} + 1)\delta < R/2.
\end{cases}
$$

We may assume as before that $(V - \phi)(\bar{x}, \bar{t}) = 0$ and, hence, $(V_* - \phi)(x, \bar{t} - \delta) > 0$ for all $x \in \bar{\Omega}$. Setting

$$
\gamma := \min\{(V_* - \phi)(x, \bar{t} - \delta) : x \in \bar{\Omega}\},
$$

by (26), we may choose a triple $(X, l, \tau) \in \mathrm{SP}(1; \bar{x}, \bar{t})$ so that

$$
V(\bar{x}, \bar{t}) + \gamma > \tau + V(X(\tau), \bar{t} - \delta),
$$

which yields

$$
\phi(\bar{x}, \bar{t}) > \tau + \phi(X(\tau), \bar{t} - \delta).
$$

Noting that

$$
X(s) \in B_R(\hat{x}) \quad \text{for all} \quad s \in [0, \tau],
$$

and setting

$$
T(s) := \bar{t} - \varepsilon s - \int_0^s l(r)\, \mathrm{d}r \quad \text{and} \quad v(s) := \dot{X}(s) + l(s)\nu(X(s)) \quad \text{for} \quad s \in [0, \tau],
$$

we compute that

$$0 > \tau + \phi(X(\tau), T(\tau)) - \phi(\bar{x}, \bar{t})$$

$$= \int_0^\tau \left[1 + \phi_t(X(s), T(s))\dot{T}(s) + D_x\phi(X(s), T(s)) \cdot \dot{X}(s) \right] \mathrm{d}s$$

$$= \int_0^\tau \left[1 + v(s) \cdot D_x\phi(X(s), T(s)) - \varepsilon\phi_t(X(s), T(s)) \right] \mathrm{d}s$$

$$+ \int_0^\tau l(s) \left[-\phi_t(X(s), T(s)) - \nu(X(s)) \cdot D_x\phi(X(s), T(s)) \right] \mathrm{d}s$$

$$\geq \int_0^\tau \left[1 - |D_x\phi(X(s), T(s))| - \varepsilon\phi_t(X(s), T(s)) \right] \mathrm{d}s \geq 0,$$

which is a contradiction. We have thus shown that V_* is a supersolution of (1).

6. We now apply a comparison result (for instance, Theorem A.1), to conclude that

$$V^* \leq u^\varepsilon \leq V_* \quad \text{on} \ \ \bar{Q},$$

which obviously shows that $u^\varepsilon = V$ on $\bar{Q}$. $\qquad\qquad\square$

Replacing (l, t) by $(\varepsilon l, \varepsilon t)$ in (24) is a simple modification to get the right Skorokhod problem for (3). We thus denote by $\mathrm{SP}(3; x, t)$ the set of all triples (X, l, τ) of $\tau \in [0, \infty)$, $X \in \mathrm{Lip}\,([0, \tau], \mathbb{R}^n)$ and $l \in L^\infty([0, \tau], \mathbb{R})$ such that

(32)
$$\begin{cases} \dot{X}(s) + \varepsilon l(s)\nu(X(s)) \in \bar{B}_1(0) & \text{a.e.} \ \ s \in [0, \tau], \\ t = \int_0^\tau l(s)\,\mathrm{d}s + \tau, \\ X(0) = x, \quad X(s) \in \bar{\Omega} & \text{for all} \ \ s \in [0, \tau], \\ l(s) = 0 \ \text{if} \ X(s) \in \Omega, \quad l(s) \geq 0 & \text{a.e.} \ \ s \in [0, \tau]. \end{cases}$$

The following proposition is an immediate consequence of the previous theorem.

Corollary 5.2. *The solution $v^\varepsilon \in \mathrm{Lip}\,(\bar{Q})$ of (3) is represented as*

$$v^\varepsilon(x, t) = \inf\{\tau + u_0(X(\tau)) : (X, l, \tau) \in \mathrm{SP}(3; x, t)\} \quad \text{for all} \ \ (x, t) \in \bar{Q}.$$

The Skorokhod problem associated with problem (4) or (9) is given by the collection $\mathrm{SP}(4; x, t)$ of all triples (X, l, τ) of $\tau \in [0, \infty)$, $X \in \mathrm{Lip}\,([0, \tau], \mathbb{R}^n)$ and $l \in L^\infty([0, \tau], \mathbb{R})$ such that

(33)
$$\begin{cases} |\dot{X}(s)| \leq 1 & \text{a.e.} \ \ s \in [0, \tau], \\ t = \int_0^\tau l(s)\,\mathrm{d}s + \tau, \\ X(0) = x, \quad X(s) \in \bar{\Omega} & \text{for all} \ \ s \in [0, \tau], \\ l(s) = 0 \ \text{if} \ X(s) \in \Omega, \quad l(s) \geq 0 & \text{a.e.} \ \ s \in [0, \tau]. \end{cases}$$

The variational formula for the solution of (4) is stated as follows.

Theorem 5.3. *Let $v \in \mathrm{Lip}\,(\bar{Q})$ be the solution of (4). Then*

(34) $$v(x, t) = \inf\{\tau + u_0(X(\tau)) : (X, l, \tau) \in \mathrm{SP}(4; x, t)\} \quad \text{for all} \ \ (x, t) \in \bar{Q}.$$

Because of the lack of a "good" comparison theorem (see Lemma 3.2 and Example A.6), our strategy for the proof of the above theorem differs substantially from that of Theorem 5.1.

Proof. We write $V(x,t)$ for the right hand side of (34). The dynamic programming principle holds, which is stated as

$$(35) \quad V(x, t+h) = \inf\{\tau + V(X(\tau), t) \, : \, (X, l, \tau) \in \mathrm{SP}(4; x, h)\} \quad \text{for all} \ \ (x,t) \in \bar{Q}, \ h \geq 0.$$

1. Let $(x,t) \in \bar{Q}$ and $h > 0$. We set

$$X(t) = x \quad \text{and} \quad l(t) = 0 \quad \text{for} \ \ t \in [0,\, h],$$

and note that $(X, l, h) \in \mathrm{SP}(4; x, h)$. By (35), we have

$$V(x, t+h) \leq h + V(x,t).$$

That is, we have

$$(36) \qquad\qquad V(x, t+h) \leq V(x,t) + h \quad \text{for all} \ \ (x,t) \in \bar{Q}, \ h \geq 0.$$

2. Let $A > 0$ be a Lipschitz bound of the function u_0. Following Step 2 of the proof of Theorem 5.1, we obtain

$$V(x,t) \geq u_0(x) - At \quad \text{for all} \ \ (x,t) \in \bar{Q}.$$

Hence, using (35), we get

$$\begin{aligned}
V(x, t+h) &= \inf\{\tau + V(X(\tau), h) \, : \, (X, l, \tau) \in \mathrm{SP}(4; x, t)\} \\
(37) \qquad &\geq \inf\{\tau + u_0(X(\tau)) - Ah \, : \, (X, l, \tau) \in \mathrm{SP}(4; x, t)\} \\
&= V(x,t) - Ah \quad \text{for all} \ \ (x,t) \in \bar{Q}, \ h \geq 0.
\end{aligned}$$

Thus, combining (36) and (37), we see that the functions $t \mapsto V(x,t)$, with $x \in \bar{\Omega}$, are equi-Lipschitz continuous on $[0, \infty)$ with a Lipschitz bound $\max\{A,\, 1\}$.

3. Let $t \geq 0$, $B \subset \Omega$ be a ball and choose $x, y \in B$ so that $x \neq y$. Set

$$h = |x - y|, \quad l(s) = 0 \quad \text{and} \quad X(s) = x + sh^{-1}(y - x) \quad \text{for} \ \ s \in [0,\, h].$$

Noting that $(X, l, h) \in \mathrm{SP}(4; x, h)$, by (35), we get

$$V(x, t+h) \leq h + V(y,t).$$

Combining this with (37) yields

$$V(x,t) \leq V(y,t) + (A+1)|x - y|.$$

Hence, by the symmetry in x and y, we get

$$|V(x,t) - V(y,t)| \leq (A+1)|x - y|.$$

This shows that the functions $x \mapsto V(x,t)$, with $t \geq 0$, are equi-Lipschitz continuous on Ω with a Lipschitz bound $L_\Omega(A+1)$.

4. Let $x \in \partial\Omega$ and $t \geq 0$. Observe that if $\varepsilon > 0$ is small enough, then the line segment connecting the points x and $x - \varepsilon\nu(x)$ lies in $\bar{\Omega}$. Arguing as in Step 3, we deduce that

$$|V(x,t) - V(x - \varepsilon\nu(x), t)| \leq (A+1)\varepsilon.$$

This shows together with the observation in Step 3 that the functions $x \mapsto V(x,t)$, with $t \geq 0$, are continuous on $\bar{\Omega}$, which moreover implies that the functions $x \mapsto V(x,t)$, with

$t \geq 0$, are equi-Lipschitz continuous on $\bar{\Omega}$ with a Lipschitz bound $L_\Omega(A+1)$. This combined with the result in Step 2 assures that $V \in \mathrm{Lip}\,(\bar{Q})$.

5. Following the argument of the proof of Theorem 5.1, we see easily that V is a solution of (4). Lemma 3.2 guarantees that $v = V$ on $\bar{Q}$. □

For the problem (2) with initial condition, a natural choice is the Skorokhod problem $\mathrm{SP}(2; x, t)$, defined as the set of of all triples (X, l, τ) of $\tau \in [0, \infty)$, $X \in \mathrm{Lip}\,([0, \tau], \mathbb{R}^n)$ and $l \in L^\infty([0, \tau], \mathbb{R})$ such that

$$
(38) \quad
\begin{cases}
\dot{X}(s) + l(s)\nu(X(s)) \in \bar{B}_1(0) & \text{a.e.} \quad s \in [0, \tau], \\
t = \displaystyle\int_0^\tau l(s)\,\mathrm{d}s, \\
X(0) = x, \quad X(s) \in \bar{\Omega} & \text{for all} \quad s \in [0, \tau], \\
l(s) = 0 \ \text{if} \ X(s) \in \Omega, \quad l(s) \geq 0 & \text{a.e.} \quad s \in [0, \tau].
\end{cases}
$$

We remark that the first condition in (38) is equivalent that

$$
(39) \qquad\qquad |\dot{X}(s)|^2 + l(s)^2 \leq 1 \quad \text{a.e.}
$$

To see this, let $x, y \in \bar{\Omega}$ and $(X, l, \tau) \in \mathrm{SP}(x)$ be such that $X(\tau) = y$. Let ρ be a defining function of Ω and note that for any $t \in [0, \tau]$, if $X(t) \in \partial\Omega$, then the function $s \mapsto \rho(X(s))$ attains the maximum value 0 at t. Hence, if $t \in (0, \tau)$ is a point where $X(t) \in \partial\Omega$ and the function X is differentiable at t, then

$$
0 = \frac{\mathrm{d}}{\mathrm{d}s}\rho(X(s))\Big|_{s=t} = D\rho(X(t)) \cdot \dot{X}(t),
$$

that is, two vectors $\nu(X(t))$ and $\dot{X}(t)$ are perpendicular. Accordingly, we have

$$
\nu(X(s)) \cdot \dot{X}(s) = 0 \quad \text{if} \ X(s) \in \partial\Omega \qquad \text{a.e.}
$$

Thus, the first condition in (38) is equivalent to condition (39). It is now easily seen that $\lambda_\Omega(x, y) \geq 0$ for all $x, y \in \bar{\Omega}$.

Theorem 5.4. *Let $u \in \mathrm{Lip}\,(\bar{Q})$ be a (unique) solution of (2) satisfying the initial condition $u(\cdot, 0) = u_0$. Then*

$$
u(x, t) = \inf\{\tau + u_0(X(\tau)) : (X, l, \tau) \in \mathrm{SP}(2; x, t)\} \quad \textit{for all} \ (x, t) \in \bar{Q}.
$$

Before giving a proof of the above theorem, we make similar observations for v_0^-, v_∞ and u_∞. We introduce two "distance" functions d_Ω and λ_Ω on $\bar{\Omega} \times \bar{\Omega}$, where $d_\Omega(x, y)$ is defined as the infimum of all positive numbers τ for which there exists a function $X \in \mathrm{Lip}\,([0, \tau], \bar{\Omega})$ such that $X(t) \in \bar{\Omega}$ for all $t \in [0, \tau]$, $X(0) = x$ and $X(\tau) = y$, and $\lambda_\Omega(x, y)$ is defined by

$$
\lambda_\Omega(x, y) = \inf\left\{ \int_0^\tau (1 - l(s))\,\mathrm{d}s : (X, l, \tau) \in \mathrm{SP}(x), \ X(\tau) = y \right\},
$$

where $\mathrm{SP}(x)$ denotes the set of all triples (X, l, τ) of $\tau > 0$, $l \in L^\infty([0, \tau], \mathbb{R})$ and $X \in \mathrm{Lip}\,([0, \tau], \mathbb{R}^n)$ such that

$$
\begin{cases}
\dot{X}(s) + l(s)\nu(X(s)) \in \bar{B}_1(0) & \text{a.e.} \quad s \in [0, \tau], \\
X(0) = x, \quad X(s) \in \bar{\Omega} & \text{for all} \quad s \in [0, \tau], \\
l(s) = 0 \ \text{if} \ X(s) \in \Omega, \quad l(s) \geq 0 & \text{a.e.} \ \hat{s} \in [0, \tau].
\end{cases}
$$

Note that

$$\mathrm{SP}(x) = \bigcup_{t \geq 0} \mathrm{SP}(1; x, t) = \bigcup_{t \geq 0} \mathrm{SP}(2; x, t),$$

and that

$$d_\Omega(x, y) = \inf \left\{ \int_0^\tau (1 - l(s))\, \mathrm{d}s \, : \, (X, l, \tau) \in \mathrm{SP}(x), \ X(\tau) = y, \ l(s) \equiv 0 \right\} \geq \lambda_\Omega(x, y).$$

We note as well that $d_\Omega(x, x) = \lambda_\Omega(x, x) = 0$ for all $x \in \bar{\Omega}$.

Theorem 5.5. *The functions $v_0^- \in \mathrm{Lip}\,(\bar{\Omega})$, v_∞ and u_∞ are represented as*

$$(40) \qquad\qquad v_0^-(x) = \inf\{d_\Omega(x, y) + u_0(y) \, : \, y \in \bar{\Omega}\},$$

$$(41) \qquad\qquad v_\infty(x) = \inf\{d_\Omega(x, y) + v_0^-(y) \, : \, y \in \partial\Omega\},$$

$$(42) \qquad\qquad u_\infty(x) = \inf\{\lambda_\Omega(x, y) + v_0^-(y) \, : \, y \in \partial\Omega\}.$$

Lemma 5.6. *Let $y \in \bar{\Omega}$. (i) The function $x \mapsto d_\Omega(x, y)$ is a solution of (5) in $\Omega \setminus \{y\}$ and a subsolution of (5) in Ω. (ii) The function $x \mapsto \lambda_\Omega(x, y)$ is a solution of*

$$(43) \qquad \begin{cases} |Du(x)| = 1 & in \ \ \Omega \setminus \{y\}, \\ 1 + \nu(x) \cdot Du(x) = 0 & on \ \ \partial\Omega, \end{cases}$$

and a subsolution of (6).

Proof. 1. Note that the function $v(x) = |x - y|$ in $\mathbb{R}^n$, with $y \in \mathbb{R}^n$, is a solution of $|Dv(x)| = 1$ in $\mathbb{R}^n \setminus \{y\}$ and is a subsolution of $|Dv(x)| = 1$ in $\mathbb{R}^n$.

2. Let $y \in \bar{\Omega}$ and let $B \subset \Omega$ be an open ball such that $y \notin B$. According to the dynamic programming principle, we deduce that for any $x \in B$

$$d_\Omega(x, y) = \inf\{d_\Omega(x, z) + d_\Omega(z, y) \, : \, z \in \partial B\} = \inf\{|x - z| + d_\Omega(z, y) \, : \, z \in \partial B\},$$

and

$$\lambda_\Omega(x, y) = \inf\{|x - z| + \lambda_\Omega(z, y) \, : \, z \in \partial B\}.$$

By the observation in Step 1, using Proposition A.5, we see that both the functions $x \mapsto d_\Omega(x, y)$ and $x \mapsto \lambda_\Omega(x, y)$ are Lipschitz continuous in B and are solutions of $|Du(x)| = 1$ in B, which implies that they are both solutions of $|Du(x)| = 1$ in $\Omega \setminus \{y\}$.

Next let $y \in \Omega$ and let $B \subset \Omega$ be an open ball such that $y \in B$. For any $x \in B$, we have

$$d_\Omega(x, y) = \lambda_\Omega(x, y) = |x - y|,$$

from which we see that both the functions $x \mapsto d_\Omega(x, y)$ and $x \mapsto \lambda_\Omega(x, y)$ are subsolutions of $|Du(x)| = 1$ in B. This together with the previous observation, we conclude that the functions $x \mapsto d_\Omega(x, y)$ and $x \mapsto \lambda_\Omega(x, y)$ are subsolutions of $|Du(x)| = 1$ in Ω. This ensures that these functions are Lipschitz continuous in Ω with a Lipschitz bound L_Ω.

A consideration based on the dynamic programming principle similar to the above shows that if $\varepsilon > 0$ is sufficiently small, then

$$\max\{|d_\Omega(x - \varepsilon\nu(x), y) - d_\Omega(x, y)|, \ |\lambda_\Omega(x - \varepsilon\nu(x), y) - \lambda_\Omega(x, y)\} \leq \varepsilon \quad \text{for all } x \in \partial\Omega, \ y \in \bar{\Omega}.$$

This and the Lipschitz continuity of the functions $x \mapsto d_\Omega(x, y)$ and $x \mapsto \lambda_\Omega(x, y)$ in Ω guarantee that these functions are Lipschitz continuous on $\bar{\Omega}$.

3. By following the argument (Steps 4 and 5) of the proof of Theorem 5.1, it is now not hard to check that the function $x \mapsto \lambda_\Omega(x, y)$ on $\bar{\Omega}$ is a solution of (43). $\qquad\square$

Lemma 5.7. (i) *If $u \in \mathrm{Lip}\,(\bar{\Omega})$ is a subsolution of (5), then*

$$u(x) - u(y) \leq d_\Omega(x, y) \quad \text{for all } x, y \in \bar{\Omega}.$$

(ii) *If $u \in \mathrm{Lip}\,(\bar{\Omega})$ is a subsolution of (6), then*

$$u(x) - u(y) \leq \lambda_\Omega(x, y) \quad \text{for all } x, y \in \bar{\Omega}.$$

Proof. (i) Let $u \in \mathrm{Lip}\,(\bar{\Omega})$ be a subsolution of (5). We approximate u by a smooth function $u_\varepsilon \in C^1(\bar{\Omega})$ with $\varepsilon > 0$ (see the remark after Proposition A.7) such that $|Du_\varepsilon(x)| \leq 1 + \varepsilon$ in Ω, observe that for any $x, y \in \bar{\Omega}$,

$$u_\varepsilon(x) = u_\varepsilon(y) + \int_0^\tau Du_\varepsilon(X(s)) \cdot \dot{X}(s) \, \mathrm{d}s \leq u_\varepsilon(y) + (1 + \varepsilon)\tau,$$

where $(X, l, \tau) \in \mathrm{SP}(x)$ satisfies $X(\tau) = y$ and $l(s) \equiv 0$, and conclude that $u(x) \leq u(y) + d_\Omega(x, y)$ for all $x, y \in \bar{\Omega}$.

(ii) Let $u \in \mathrm{Lip}\,(\bar{\Omega})$ be a subsolution of (6). For each $\varepsilon > 0$, there is a function $u_\varepsilon \in C^1(\bar{\Omega})$ which satisfies

$$\begin{cases} |Du_\varepsilon(x)| \leq 1 + \varepsilon & \text{for all } x \in \Omega, \\ 1 + \nu(x) \cdot Du_\varepsilon(x) \leq 0 & \text{for all } x \in \partial\Omega, \\ \|u_\varepsilon - u\|_{\infty, \Omega} < \varepsilon. \end{cases}$$

(See [12, Theorem 4.2] for this.) Then, arguing as in the proof of (i) above, we easily conclude that $u(x) \leq u(y) + \lambda_\Omega(x, y)$ for all $x, y \in \bar{\Omega}$. $\qquad\square$

Lemma 5.8. (i) *If $u \in \mathrm{Lip}\,(\bar{\Omega})$ is a solution of (5), then*

$$u(x) = \min\{u(y) + d_\Omega(x, y) \,:\, y \in \partial\Omega\} \quad \text{for all } x \in \bar{\Omega}.$$

(ii) *If $u \in \mathrm{Lip}\,(\bar{\Omega})$ is a solution of (6), then*

$$u(x) = \min\{u(y) + \lambda_\Omega(x, y) \,:\, y \in \partial\Omega\} \quad \text{for all } x \in \bar{\Omega}.$$

Proof. (i) We set

$$V(x) = \min\{u(y) + d_\Omega(x, y) \,:\, y \in \partial\Omega\} \quad \text{for } x \in \bar{\Omega}.$$

By Lemma 5.7, we have

$$u(x) \leq V(x) \quad \text{for all } x \in \bar{\Omega}.$$

By the definition of V, we see that

$$V(x) \leq u(x) \quad \text{for all } x \in \partial\Omega.$$

Hence, we have $u(x) = V(x)$ for all $x \in \partial\Omega$. According to Proposition A.5, the function V is a solution of (5). Hence, by Lemma A.4, we conclude that $u = V$ on $\bar{\Omega}$. The proof of (ii) is similar to the above, and we skip it here. $\qquad\square$

Proof of Theorem 5.4. 1. We set

$$V(x,t) = \inf\{\tau + u_0(X(\tau)) : (X, l, \tau) \in \mathrm{SP}(2; x, t)\} \quad \text{for} \quad (x,t) \in \bar{Q}.$$

We show that

$$V^*(x, 0) \le u_0(x) \le V_*(x, 0) \quad \text{for all} \quad x \in \bar{\Omega}$$

as well as the locally boundedness of the function V. Once this is done, we just need to follow Steps 4, 5 and 6 of the proof of Theorem 5.1.

2. It is a standard observation that for each $t > 0$ the function $w : x \mapsto u(x,t)$ is a solution of the eikonal equation $|Dw(x)| = 1$ in Ω. By assumption, we have $u \in \mathrm{Lip}\,(\bar{Q})$. Hence, by the stability of the viscosity property, we see that u_0 is a solution of $|Dw(x)| = 1$ in Ω. As in Step 2 of the proof of Theorem 5.1, we easily find a constant $A > 0$ such that

$$V(x,t) \ge u_0(x) - At \quad \text{for all} \quad (x,t) \in \bar{Q},$$

which proves that V is locally bounded below in $\bar{Q}$ and that $V_*(x, 0) \ge u_0(x)$ for all $x \in \bar{\Omega}$.

3. Next, fix any $(x,t) \in \partial\Omega \times (0, \infty)$ and set

$$\tau = t, \quad l(s) = 1 \quad \text{and} \quad X(s) = x \quad \text{for} \quad s \in [0, \tau].$$

Observe that $(X, l, \tau) \in \mathrm{SP}(2; x, t)$ and that

$$(44) \qquad\qquad V(x,t) \le \tau + u_0(X(\tau)) = u_0(x) + t.$$

Now fix any $(x,t) \in \bar{\Omega} \times (0, \infty)$. By (i) of Lemma 5.8, there exists a point $y \in \partial\Omega$ such that

$$(45) \qquad\qquad u_0(x) = u_0(y) + d_\Omega(x, y).$$

By the dynamic programming principle, we have

$$V(x,t) \le d_\Omega(x, y) + V(y, t).$$

Combining this with (44) and using (45), we get

$$V(x,t) \le d_\Omega(x, y) + u_0(y) + t = u_0(x) + t,$$

which shows that V is locally bounded above on $\bar{Q}$ and that $V^*(x, 0) \le u_0(x)$ for all $x \in \bar{\Omega}$. $\qquad\square$

Proof of Theorem 5.5. 1. We write $V(x)$ for the right hand side of (40). Since v_0^- is a subsolution of (5), by Lemma 5.7 we have

$$v_0^-(x) \le V(x) \quad \text{for all} \quad x \in \bar{\Omega}.$$

On the other hand, in view of Proposition A.5 we see that V is a subsolution of (5). Also, we have

$$V(x) \le v_0^-(x) + d_\Omega(x, x) \le u_0(x) \quad \text{for all} \quad x \in \bar{\Omega}.$$

Now, the maximality of v_0^- ensures that $V \le v_0^-$ on $\bar{\Omega}$. Thus we conclude that $V = v_0^-$ on $\bar{\Omega}$.

2. Let $V(x)$ denote the right hand side of (41). By Lemma 5.7, we have

$$v_0^-(x) \le v_0^-(y) + d_\Omega(x, y) \quad \text{for all} \quad x, y \in \bar{\Omega},$$

and hence, $v_0^- \le V$ on $\bar{\Omega}$. In view of Proposition A.5, the function V is a solution of (5). By the minimality of v_∞, we see that $v_\infty \le V$ on $\bar{\Omega}$. Note that $v_\infty \ge v_0^-$ on $\bar{\Omega}$ and

$V(x) \leq v_0^-(x)$ for all $x \in \partial\Omega$. Hence, we have $v_\infty(x) = v_0^-(x) = V(x)$ for all $x \in \partial\Omega$. Now, by comparison (Lemma A.4), we get $v_\infty = V$ on $\bar{\Omega}$.

3. Let $V(x)$ denote the right hand side of (42). As noted before, the function v_∞ is a supersolution of (6). In Step 2 above, we have observed that $v_\infty = v_0^-$ on $\partial\Omega$. According to Lemma 5.6, the function V is a solution of (6). Since $V \leq v_\infty$ on $\partial\Omega$, by comparison, we get $V \leq v_\infty$ on $\bar{\Omega}$. Hence, by the maximality of u_∞, we see that $V \leq u_\infty$ on $\bar{\Omega}$. On the other hand, by (ii) of Lemma 5.8, we find that

$$V(x) = \min\{\lambda_\Omega(x,y) + v_\infty(y) \, : \, y \in \partial\Omega\}$$
$$\geq \min\{\lambda_\Omega(x,y) + u_\infty(y) \, : \, y \in \partial\Omega\} = u_\infty(x) \quad \text{for all } x \in \bar{\Omega}.$$

Thus, we have $u_\infty = V$ on $\bar{\Omega}$. $\qquad\square$

6. More on the function λ_Ω

By the assumption that Ω is a bounded, open connected subset of $\mathbb{R}^n$ and is of class C^1, we deduce that $\partial\Omega$ consists of a finite number of connected components Γ_i, with $i = 1, 2, ..., N$.

We have shown in the proof of Theorem 2.3 that the function $x \mapsto \text{dist}\,(x, \partial\Omega)$ on $\bar{\Omega}$ is a solution of (6). The same proof shows that for each $i = 1, ..., N$, the function $u(x) := \text{dist}\,(x, \Gamma_i)$ on $\bar{\Omega}$ is a solution of

$$(46) \qquad \begin{cases} |Du(x)| = 1 & \text{in } \Omega, \\ 1 + \nu(x) \cdot Du(x) = 0 & \text{on } \Gamma_i. \end{cases}$$

For $y \in \bar{\Omega}$ and $i, j = 1, ..., N$ we define

$$\gamma(y, i) := \text{dist}\,(y, \Gamma_i) = \min\{|y - z| \, : \, z \in \Gamma_i\},$$
$$\gamma(i, j) := \text{dist}\,(\Gamma_i, \Gamma_j) = \min\{|x - y| \, : \, x \in \Gamma_i, \ y \in \Gamma_j\}.$$

Let I denotes the set of all finite sequences $(i_1, ..., i_m)$ such that $i_j \in \{1, ..., N\}$ for all $j = 1, ..., m$ and $i_j \neq i_k$ if $j \neq k$. For $y \in \bar{\Omega}$ and $i = 1, ..., N$ we set

$$a_i(y) = \min\{\gamma(y, i_1) + \sum_{j=1}^{m-1} \gamma(i_j, i_{j+1}) \, : \, (i_1, ..., i_m) \in I, \ i_m = i\}.$$

Theorem 6.1. *We have*

$$(47) \quad \lambda_\Omega(x, y) = \min\{|x - y|, \min\{a_i(y) + \text{dist}\,(x, \Gamma_i) \, : \, i = 1, ..., N\}\} \quad \text{for all } x, y \in \bar{\Omega}.$$

Lemma 6.2. *For each $i = 1, ..., N$ we have*

$$\lambda_\Omega(x, y) = 0 \quad \text{for all } x, y \in \Gamma_i.$$

Before going into the proof of the above lemma, we remark that $\lambda_\Omega(x, y)$ is symmetric in x and y, that is,

$$\lambda_\Omega(x, y) = \lambda_\Omega(y, x) \quad \text{for all } x, y \in \bar{\Omega}.$$

To see this, let $x, y \in \bar{\Omega}$ and $(X, l, \tau) \in \text{SP}(x)$ be such that $X(\tau) = y$. We set

$$Y(s) = X(\tau - s) \quad \text{and} \quad m(s) = l(\tau - s) \quad \text{for } s \in [0, \tau],$$

then $(Y, m, \tau) \in \mathrm{SP}(y)$ and $Y(\tau) = x$. Moreover, we have

$$\int_0^\tau (1 - l(s)) \, \mathrm{d}s = \int_0^\tau (1 - m(s)) \, \mathrm{d}s,$$

and find that $\lambda_\Omega(x, y) = \lambda_\Omega(y, x)$ for all $x, y \in \bar\Omega$. From this symmetry, $\lambda_\Omega(x, y) = \lambda_\Omega(y, x)$, it follows that $\lambda_\Omega(x, y)$ is Lipschitz continuous on $\bar\Omega \times \bar\Omega$.

We remark also that the triangle inequality holds for λ_Ω:

$$\lambda_\Omega(x, y) \leq \lambda_\Omega(x, z) + \lambda_\Omega(z, y) \quad \text{for all } x, y, z \in \bar\Omega.$$

Indeed, for any $(X, l, \tau) \in \mathrm{SP}(x)$ and $(Z, m, \sigma) \in \mathrm{SP}(z)$ such that $X(\tau) = z$ and $Z(\sigma) = y$, we define $(\xi, p, \tau + \sigma) \in \mathrm{SP}(x)$ by concatenating (X, l) and (Z, m), i.e., by setting

$$(\xi(s), p(s)) = \begin{cases} (X(s), l(s)) & \text{for } s \in [0, \tau), \\ (Z(s), m(s)) & \text{for } s \in [\tau, \tau + \sigma] \end{cases}$$

and observe that $\xi(\tau + \sigma) = y$ and

$$\lambda_\Omega(x, y) \leq \int_0^{\tau + \sigma} (1 - p(s)) \, \mathrm{d}s = \int_0^\tau (1 - l(s)) \, \mathrm{d}s + \int_0^\sigma (1 - m(s)) \, \mathrm{d}s,$$

which implies that $\lambda_\Omega(x, y) \leq \lambda_\Omega(x, z) + \lambda_\Omega(z, y)$.

Proof of Lemma 6.2. Let $i = 1, ..., N$ and $x, y \in \Gamma_i$. By the connectedness and C^1 regularity of Γ_i, there exists a curve $X \in \mathrm{Lip}([0, \tau], \mathbb{R}^n)$ starting at x and ending at y such that $X(s) \in \Gamma_i$ for all $s \in [0, \tau]$. We may assume by an appropriate scaling if needed that $|\dot{X}(s)| \leq 1$ a.e. in $[0, \tau]$. Fix any $\varepsilon \in (0, 1)$ and set

$$X_\varepsilon(s) = X(\varepsilon s) \quad \text{and} \quad l_\varepsilon(s) = 1 - \varepsilon^2 \quad \text{for } s \in [0, \varepsilon^{-1}\tau].$$

Observe that

$$|\dot{X}_\varepsilon(s)|^2 + l_\varepsilon(s)^2 \leq \varepsilon^2 + (1 - \varepsilon^2)^2 < 1 \quad \text{a.e. in } [0, \varepsilon^{-1}\tau],$$

which assures that $(X_\varepsilon, l_\varepsilon, \varepsilon^{-1}\tau) \in \mathrm{SP}(x)$. Also , we have

$$\int_0^{\varepsilon^{-1}\tau} (1 - l_\varepsilon(s)) \, \mathrm{d}s = \varepsilon\tau.$$

Sending $\varepsilon \to 0$, we conclude that $\lambda_\Omega(x, y) = 0$. $\qquad\qquad\square$

We divide the proof of Theorem 6.1 into two parts.

Proof of Theorem 6.1, Part 1. We fix any $y \in \bar\Omega$ and write $v(x)$ for the right hand side of formula (47). Here we prove that

(48) $$v(x) \leq \lambda_\Omega(x, y) \quad \text{for all } x \in \bar\Omega.$$

1. We first prove that v is a solution of (6). Let $i = 1, ..., N$. By the definition of $a_i(y)$, for any sequence $(i_1, ..., i_m) \in I$ such that $i_m = i$, we have

$$a_i(y) \leq \gamma(y, i_1) + \sum_{k=1}^{m-1} \gamma(i_k, i_{k+1}).$$

In particular, if $m = 1$, then we get

$$a_i(y) \leq \gamma(y, i) \leq |y - x| \quad \text{for all} \quad x \in \Gamma_i.$$

Also, for any $j = 1, ..., N$, if we choose $(i_1, ..., i_m) \in I$, with $i_{m-1} = j$, optimally so that

$$a_j(y) = \gamma(y, i_1) + \sum_{k=1}^{m-2} \gamma(i_k, i_{k+1}),$$

then we get

$$a_i(y) \leq a_j(y) + \gamma(j, i) \leq a_j(y) + \text{dist}\,(x, \Gamma_j) \quad \text{for all} \quad x \in \Gamma_i.$$

Hence, by the definition of v, we see that $v(x) = a_i(y)$ for all $x \in \Gamma_i$. Note as well by the definition of v that $v(x) \leq a_i(y) + \text{dist}\,(x, \Gamma_i)$ for all $x \in \bar{\Omega}$.

Let $\varepsilon > 0$ and set

$$v_\varepsilon(x) = \min\{v(x), a_i(y) + \text{dist}\,(x, \Gamma_i) - \varepsilon\} \quad \text{for} \quad x \in \bar{\Omega}.$$

There exists an open neighborhood V_ε, relative to $\mathbb{R}^n$, of Γ_i such that

$$v_\varepsilon(x) = a_i(y) + \text{dist}\,(x, \Gamma_i) - \varepsilon \quad \text{for all} \quad x \in V_\varepsilon \cap \bar{\Omega}.$$

It is now a standard observation that v_ε is a solution of (46). It is clear that

$$\lim_{\varepsilon \to 0} v_\varepsilon(x) = v(x) \quad \text{uniformly on} \quad \bar{\Omega}.$$

Hence, by the stability of the viscosity property under uniform convergence, we see that v is a solution of (46). Since our choice of i is arbitrary, we may conclude that v is a solution of (6). Noting that $v(y) = 0$, by Lemma 5.7, we conclude that (48) holds. $\qquad\square$

Recalling the Jordan-Brouwer separation theorem (see for instance [11]), since the Γ_i are compact, connected C^1 hypersurfaces, we see that for each $i = 1, 2, ..., N$ the open subset $\mathbb{R}^n \setminus \Gamma_i$ of $\mathbb{R}^n$ has exactly two connected components O_i^+ and O_i^-. Since Ω is connected and does not intersect $\partial\Omega = \cup_i \Gamma_i$, for each i we have either $\Omega \subset O_i^+$ or $\Omega \subset O_i^-$. We choose our notation so that $\Omega \subset O_i^-$ for all $i = 1, ..., N$.

Lemma 6.3. *Let $i, j \in \{1, ..., N\}$. If $i \neq j$, then $\Gamma_j \subset O_i^-$.*

Proof. Since $\Gamma_j \subset \bar{\Omega} \subset \bar{O}_i^- = \Gamma_i \cup O_i^-$ and $\Gamma_i \cap \Gamma_j = \emptyset$, we have $\Gamma_j \subset O_i^-$. $\qquad\square$

Lemma 6.4. *We have $\Omega = \bigcap_{i=1}^{N} O_i^-$.*

Proof. 1. We first show that the set $\bigcap_{i=1}^{N} O_i^-$ is connected. To do this, fix $i = 1, ..., N$ and an open connected subset O of $\mathbb{R}^n$ such that $\Gamma_i \subset O$ and prove that $O \cap O_i^-$ is connected. Fix $x, y \in O \cap O_i^-$. Since O is arc-wise connected, there exists a curve $\xi \in C([0, 1], \mathbb{R}^n)$ such that $\xi(0) = x$, $\xi(1) = y$ and $\xi(t) \in O$ for all $t \in [0, 1]$. If $\xi(t) \in O_i^-$ for all $t \in [0, 1]$, then we are done. Otherwise, we may choose two numbers $0 < \sigma \leq \tau < 1$ so that $\xi(\sigma), \xi(\tau) \in \Gamma_i$ and $\xi(t) \in O_i^-$ for all $t \in [0, \sigma) \cup (\tau, 1]$. Now, since Γ_i is locally diffeomorphic to a hyperplane, it is not hard to find a small constant $\varepsilon > 0$ and a continuous curve $\eta \in C([\sigma - \varepsilon, \tau + \varepsilon], \mathbb{R}^n)$ such that $\eta(\sigma - \varepsilon) = \xi(\sigma - \varepsilon)$, $\eta(\tau + \varepsilon) = \xi(\tau + \varepsilon)$ and $\eta(t) \in O_i^-$ for all $t \in [\sigma - \varepsilon, \tau + \varepsilon]$. Here it is assumed that $0 < \sigma - \varepsilon < \tau + \varepsilon < 1$. Moreover, we may select the curve η so that the distance of the curve η to the hypersurface Γ_i,

$$\max_{t \in [\sigma - \varepsilon, \tau + \varepsilon]} \text{dist}\,(\eta(t), \Gamma_i),$$

is as small as required. Consequently, we may assume that $\eta(t) \in O$ for all $t \in [\sigma - \varepsilon, \tau + \varepsilon]$. Concatenating three curves $\xi|_{[0, \sigma-\varepsilon]}$ (the restriction of ξ to $[0, \sigma - \varepsilon]$), η and $\xi|_{[\tau+\varepsilon, 1]}$, we get a continuous curve in $O \cap O_i^-$ connecting x and y. Hence, $O \cap O_i^-$ is arc-wise connected, which shows that it is connected.

We assume that $N \geq 2$, note by Lemma 6.3 that $\Gamma_2 \subset O_1^-$ and apply the above observation to O_1^- and O_2^-, to see that $O_1^- \cap O_2^-$ is connected. If $N \geq 3$, then we note by Lemma 6.3 that $\Gamma_3 \subset O_1^- \cap O_2^-$ and use the above observation, to see that $O_1^- \cap O_2^- \cap O_3^-$ is connected. In general, by induction, we conclude that the set $\bigcap_{i=1}^N O_i^-$ is connected.

2. We know now that $\bigcap_{i=1}^N O_i^-$ is connected and includes the set Ω. To show the identity $\bigcap_{i=1}^N O_i^- = \Omega$, we suppose that there exists a point $x \in \bigcap_{i=1}^N O_i^- \setminus \Omega$ and will get a contradiction. Fix a point $x_0 \in \Omega$ and select a curve in $\bigcap_{i=1}^N O_i^-$ connecting x_0 and x. Since $x \notin \Omega$, the curve intersects $\partial\Omega$ at a point x_1. Since, for each i, Γ_i does not intersects O_i^-, the set $\partial\Omega = \bigcup_{i=1}^N \Gamma_i$ does not intersects $\bigcap_{i=1}^N O_i^-$. These together yield a contradiction:

$$x_1 \in \partial\Omega \cap \bigcap_{i=1}^N O_i^- = \emptyset. \qquad \square$$

Lemma 6.5. *For any $x, y \in \Omega$ we have*

(49) $$\lambda_\Omega(x, y) \leq |x - y| \quad \text{for all } x, y \in \bar{\Omega}.$$

Proof. By continuity, it is enough to show inequality (49) only for $x, y \in \Omega$. Fix any $x, y \in \Omega$ and consider the curve ϕ given by $\phi(t) := (1 - t)x + ty$ for $t \in [0, 1]$. Indeed, ϕ represents the line segment between x and y.

1. Assume first that $\phi(t) \in \Omega$ for all $t \in [0, 1]$. Fix any $\varepsilon > 0$, set $\tau_\varepsilon := |x - y| + \varepsilon$ and $\phi_\varepsilon(t) := \phi(\tau_\varepsilon^{-1}t)$ for $t \in [0, \tau_\varepsilon]$, and note that $(\phi_\varepsilon, \tau_\varepsilon, 0) \in \mathrm{SP}(x)$ (that is, $l(t) \equiv 0$ in the usual notation) and $\phi(\tau_\varepsilon) = y$. By the definition of λ_Ω, we get

$$\lambda_\Omega(x, y) \leq \tau_\varepsilon = |x - y| + \varepsilon,$$

which shows that (49) holds in this case.

2. Next assume that the curve ϕ intersects the complement of Ω. We show that there are sequences $\{s_k\}_{k=1}^m \subset (0, 1)$, $\{t_k\}_{k=1}^m \subset (0, 1)$ and $\{i_k\}_{k=1}^m \subset \{1, ..., N\}$ such that

(50)
$$\begin{cases} 0 < s_1 \leq t_1 < s_2 \leq t_2 < \cdots < s_m \leq t_m < 1, \\ i_k \neq i_j \quad \text{if } k \neq j, \\ \phi(s_k) \in \Gamma_{i_k}, \quad \phi(t_k) \in \Gamma_{i_k} \quad \text{for all } k = 1, ..., m, \\ \phi(t) \in \Omega \quad \text{for all } t \in [0, s_1) \cup \bigcup_{k=1}^{m-1}(t_k, s_{k+1}) \cup (t_m, 1]. \end{cases}$$

Here, since the i_k are mutually different, m is not more than N.

It is obvious that $\phi([0, 1]) \cap \partial\Omega \neq \emptyset$. We set

$$\begin{cases} s_1 = \min\{t \in [0, 1] : \phi(t) \in \partial\Omega\}, \\ t_1 = \max\{t \in [0, 1] : \phi(t) \in \Gamma_{i_1}\}, \end{cases}$$

where $i_1 \in \{1, ..., N\}$ is chosen so that $\phi(s_1) \in \Gamma_{i_1}$. Note that such an i_1 is uniquely determined.

Since both $\phi(0)$ and $\phi(1)$ are in Ω, it is clear that $0 < s_1 \leq t_1 < 1$ and also that $\phi(t) \in \Omega$ for all $t \in [0, s_1)$. Note that $\phi(1) \in \Omega \subset O_{i_1}^-$ and that $\phi((t_1, 1]) \cap \Gamma_{i_1} = \emptyset$. Hence, the connected set $\phi((t_1, 1])$ is included in $\mathbb{R}^n \setminus \Gamma_{i_1}$ and intersects $O_{i_1}^-$, which implies that

$$(51) \qquad\qquad\qquad\qquad \phi((t_1, 1]) \subset O_{i_1}^-.$$

By Lemma 6.3, we have $\phi(t_1) \in \Gamma_{i_1} \subset \bigcap_{i \neq i_1} O_i^-$, which implies that $\phi((t_1, \tau_1]) \subset \bigcap_{i \neq i_1} O_i^-$ for some $\tau_1 \in (t_1, 1]$. Combining this with (51) and using Lemma 6.4, we see that

$$\phi((t_1, \tau_1]) \subset \bigcap_{i=1}^N O_i^- = \Omega.$$

If $\phi((t_1, 1]) \subset \Omega$, then we set $m = 1$ and we are done. Otherwise, we repeat the previous argument, with the interval $[0, 1]$ replaced by $[\tau_1, 1]$. (Note that $t_1 < \tau_1 < 1$.) That is, we set

$$\begin{cases} s_2 = \min\{t \in [\tau_1, 1] : \phi(t) \in \partial\Omega\}, \\ t_2 = \max\{t \in [\tau_1, 1] : \phi(t) \in \Gamma_{i_2}\}, \end{cases}$$

where $i_2 \in \{1, ..., N\}$ is the integer such that $\phi(s_2) \in \Gamma_{i_2}$. By the choice of t_1, it is clear that $i_2 \neq i_1$. As in the first step of this iteration, we see that for some $\tau_2 \in (t_2, 1]$,

$$\phi((t_2, \tau_2]) \subset \Omega.$$

We repeat this procedure of finding (s_k, t_k, i_k) at most N times before arriving the situation that $\phi((t_k, 1]) \subset \Omega$, to conclude that there exist sequences $\{s_k\}_{k=1}^m \subset (0, 1)$, $\{t_k\}_{k=1}^m \subset (0, 1)$ and $\{i_k\}_{k=1}^m \subset \{1, ..., N\}$ such that all the conditions of (50) hold.

3. By the triangle inequality, we get

$$\lambda_\Omega(x, y) \leq \lambda_\Omega(\phi(0), \phi(s_1)) + \sum_{k=1}^m \lambda_\Omega(\phi(s_k), \phi(t_k))$$

$$+ \sum_{k=1}^{m-1} \lambda_\Omega(\phi(t_k), \phi(s_{k+1})) + \lambda_\Omega(\phi(t_m), \phi(1)).$$

According to Lemma 6.2, we have

$$\lambda_\Omega(\phi(s_k), \phi(t_k)) = 0 \quad \text{for all} \quad k = 1, ..., m.$$

Noting that

$$\phi(t) \in \Omega \quad \text{for all} \quad t \in [0, s_1) \cup \bigcup_{k=1}^{m-1} (t_k, s_{k+1}) \cup (t_m, 1]$$

and arguing as in Step 1, we get

$$\begin{cases} \lambda_\Omega(\phi(0), \phi(s_1)) \leq |\phi(0) - \phi(s_1)|, \\ \lambda_\Omega(\phi(t_k), \phi(s_{k+1})) \leq |\phi(t_k) - \phi(s_{k+1})| \quad \text{for all} \quad k = 1, ..., m-1, \\ \lambda_\Omega(\phi(t_m), \phi(1)) \leq |\phi(t_m) - \phi(1)|. \end{cases}$$

Adding these all together, we obtain

$$\lambda_\Omega(x,y) \le |\phi(0) - \phi(s_1)| + \sum_{k=1}^{m-1} |\phi(t_k) - \phi(s_{k+1})| + |\phi(t_m) - \phi(1)| \le |x - y|.$$

The proof is complete. □

Proof of Theorem 6.1, Part 2. As in Part 1, we fix any $y \in \bar{\Omega}$ and write $v(x)$ for the right hand side of formula (47). We show that

$$(52) \qquad\qquad \lambda_\Omega(x,y) \le v(x) \quad \text{for all} \ \ x \in \bar{\Omega},$$

which will complete the proof of the theorem.

Fix any $i \in \{1, ..., N\}$. There exists a sequence $(i_1, ..., i_m) \in I$ such that $i_m = i$ and

$$a_i(y) = \gamma(y, i_1) + \sum_{k=1}^{m-1} \gamma(i_k, i_{k+1}).$$

We may choose sequences $(x_1, ..., x_m) \in (\partial\Omega)^m$ and $(y_1, ..., y_{m-1}) \in (\partial\Omega)^{m-1}$ so that

$$\begin{cases} \gamma(y, i_1) = |y - x_1|, \quad x_1 \in \Gamma_{i_1}, \\ \gamma(i_k, i_{k+1}) = |y_k - x_{k+1}|, \quad y_k \in \Gamma_{i_k}, \ x_{k+1} \in \Gamma_{i_{k+1}} \quad \text{for all} \ \ k = 1, ..., m-1. \end{cases}$$

By the triangle inequality, we get

$$\lambda_\Omega(y, x_m) \le \lambda_\Omega(y, x_1) + \sum_{k=1}^{m} \lambda_\Omega(x_k, y_k) + \sum_{k=1}^{m-1} \lambda_\Omega(y_k, x_{k+1}).$$

Hence, using Lemmas 6.2 and 6.5, we obtain

$$\lambda_\Omega(y, x_m) \le |y - x_1| + \sum_{k=1}^{m-1} |y_k - x_{k+1}| = \gamma(y, i_1) + \sum_{k=1}^{m-1} \gamma(i_k, i_{k+1}) = a_i(y).$$

Thus we get

$$(53) \qquad \lambda_\Omega(y, x) \le \lambda_\Omega(y, x_m) + \lambda_\Omega(x_m, x) = \lambda_\Omega(y, x_m) \le a_i(y) \quad \text{for all} \ \ x \in \Gamma_i.$$

By the definition of $a_i(y)$, it is obvious that $v(x) = a_i(y)$ for all $x \in \Gamma_i$ and $i = 1, ..., N$. This together with (53) assures that $\lambda_\Omega(x,y) \le v(x)$ for all $x \in \partial\Omega$. By the standard comparison result, we conclude that $\lambda_\Omega(x,y) \le v(x)$ for all $x \in \bar{\Omega}$. □

Appendix A

We present a comparison theorem for the eikonal equation with the dynamical boundary condition. They are known in the literature (see for instance [5, 4, 12]), but for the reader's convenience we give here a proof.

We first consider the problem

$$(54) \qquad \begin{cases} a u_t(x,t) + |D_x u(x,t)| = f(x) & \text{in} \ \ Q_T := \Omega \times (0, T), \\ b u_t(x,t) + c\nu(x) \cdot D_x u(x,t) = g(x) & \text{on} \ \ \partial\Omega \times (0, T), \end{cases}$$

where $T > 0$, $a \ge 0$, $b \ge 0$ and $c > 0$ are constants and f and g are continuous functions on $\bar{\Omega}$ and $\partial\Omega$, respectively.

Theorem A.1. *Assume that $a+b > 0$ and that $\min_{\bar{\Omega}} f > 0$ if $a = 0$. Let $u \in \mathrm{USC}(\bar{\Omega} \times [0, T))$ and $v \in \mathrm{LSC}(\bar{\Omega} \times [0, T))$ be a subsolution and a supersolution of (54), respectively. Assume that $u(x, 0) \leq v(x, 0)$ for all $x \in \bar{\Omega}$ and that u and v are bounded on $\bar{\Omega} \times [0, T)$. Then $u \leq v$ on $\bar{\Omega} \times (0, T)$.*

In what follows we set

$$H(x, p) = |p| - f(x) \quad \text{for} \ \ (x, t) \in \bar{\Omega} \times \mathbb{R}^n,$$

so that our equation reads

$$au_t + H(x, D_x u) = 0 \quad \text{in } Q_T.$$

Lemma A.2. *Let $\gamma, e \in \mathbb{R}^{n+1}$ and assume that $\gamma \cdot e > 0$. Then there exists a function $\phi \in C^1(\mathbb{R}^{n+1})$ such that*

(55)
$$\begin{cases} \phi(t\xi) = t^2 \phi(\xi) & \text{for all} \ (t, \xi) \in \mathbb{R} \times \mathbb{R}^{n+1}, \\ \phi(\xi) > 0 & \text{if} \ \xi \neq 0, \\ \gamma \cdot D\phi(\xi) \geq 0 & \text{if} \ e \cdot \xi \geq 0, \\ \gamma \cdot D\phi(\xi) \leq 0 & \text{if} \ e \cdot \xi \leq 0. \end{cases}$$

See [15, 13, 12] for a proof of the above lemma. Indeed, the function

$$\phi(\xi) = \left| \xi - \frac{e \cdot \xi}{e \cdot \xi} \gamma \right|^2 + (e \cdot \xi)^2$$

has the required properties.

Lemma A.3. *Let $u \in \mathrm{USC}(\bar{\Omega} \times (0, T))$ be a subsolution of (54). Assume that the family $\{u(x, \cdot)\}_{\underline{x} \in \bar{\Omega}}$ of functions in $(0, T)$ is equi-Lipschitz continuous. Then u is Lipschitz continuous in $\bar{\Omega} \times (0, T)$. Moreover, if L is a Lipschitz bound of the family $\{u(x, \cdot)\}_{x \in \bar{\Omega}}$ in $(0, T)$, then the constant*

$$L_\Omega(aL + \|f\|_{\infty, \Omega})$$

is a Lipschitz bound of the function u in Q_T.

Notice that L_Ω indicates the Lipschitz constant introduced at the end of Section 1.

Proof. Let $L > 0$ be a Lipschitz bound for the family $\{u(x, \cdot)\}_{x \in \bar{\Omega}}$, i.e.,

$$|u(x, t) - u(x, s)| \leq L|t - s| \quad \text{for all} \ \ x \in \bar{\Omega} \ \ \text{and} \ \ t, s \in (0, T).$$

Let $(x, t) \in \bar{\Omega} \times (0, T)$ and $(p, q) \in D^+ u(x, t)$. Since u is a subsolution of (54), we have either

(56)
$$aq + H(x, p) \leq 0,$$

or

(57)
$$bq + c\nu(x) \cdot p \leq g(x) \quad \text{and} \quad x \in \partial\Omega.$$

Also, we have

$$|q| \leq L.$$

If $x \in \Omega$, then we get

$$|p| \leq aL + \|f\|_{\infty, \Omega},$$

which implies that for each $t \in (0, T)$, the function $u(\cdot, t)$ is a subsolution of

$$|D_x u(x,t)| \leq aL + \|f\|_{\infty,\Omega} \quad \text{in } \Omega.$$

This implies further that the family $\{u(\cdot,t)\}_{t\in(0,T)}$ of functions in Ω is equi-Lipschitz continuous with a Lipschitz bound given by $L_\Omega(aL + \|f\|_{\infty,\Omega})$, and we have

$$|u(x,t) - u(y,s)| \leq L_\Omega(aL + \|f\|_{\infty,\Omega})(|x-y|^2 + |t-s|^2)^{1/2} \quad \text{for all } (x,t),\, (y,s) \in Q_T.$$

Let $u|_{Q_T}$ denote the restriction of u to the domain Q_T. We may extend $u|_{Q_T}$ by continuity to the domain $\bar{\Omega} \times (0, T)$ and denote the resulting function as $\bar{u}$. Since u is upper semicontinuous on $\bar{\Omega} \times (0, T)$, $\bar{u} \in \text{Lip}\,(\bar{\Omega} \times (0, T))$ and $\bar{u} = u$ in Q_T, we have

$$\bar{u} \leq u \quad \text{on } \partial\Omega \times (0, T).$$

If $u = \bar{u}$ on $\bar{\Omega} \times (0, T)$, then we are done.

Thus we need only to show that

$$(58) \qquad\qquad u \leq \bar{u} \quad \text{on } \partial\Omega \times (0, T).$$

For this, we assume by contradiction that (58) were not the case, and will show a contradiction. By this assumption, we find a point $(x_0, t_0) \in \partial\Omega \times (0, T)$ such that

$$u(x_0, t_0) > \bar{u}(x_0, t_0).$$

We may choose an open interval $I \subset (0, T)$, a function $\psi \in C^1(\bar{I})$ and a constant $\delta > 0$ such that

$$\begin{cases} t_0 \in I \subset \bar{I} \subset (0, T), \\ \psi(t_0) = 0, \qquad \psi(t) \geq 0 \quad \text{for all } t \in \bar{I}, \\ u(x_0, t_0) - \bar{u}(x_0, t_0) > \delta, \\ u(x,t) - \bar{u}(x,t) - \psi(t) < 0 \quad \text{for all } (x,t) \in \bar{\Omega} \times \partial I. \end{cases}$$

By an approximation procedure, we find a function $\phi \in C^1(\bar{\Omega} \times \bar{I})$ such that

$$\|\bar{u} - \phi\|_{\infty, \bar{\Omega} \times \bar{I}} < \frac{\delta}{2}.$$

Note that if we set $\Phi(x,t) := u(x,t) - \phi(x,t) - \psi(t)$, then

$$(59) \qquad\qquad \max_{\bar{\Omega} \times \bar{I}} \Phi > \frac{\delta}{2} > \sup_{\Omega \times I \,\cup\, \bar{\Omega} \times \partial I} \Phi.$$

Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω. We may assume that $|D\rho(x)| \geq 1$ for all $x \in \partial\Omega$. Let $\zeta \in C^1(\mathbb{R})$ be a nondecreasing function such that

$$\zeta(0) = 0, \qquad 0 \leq \zeta'(r) \leq \zeta'(0) = 1, \qquad |\zeta(r)| \leq 1 \quad \text{for all } r \in \mathbb{R}.$$

Let $\varepsilon > 0$ be a small constant, and set

$$\Psi(x,t) = u(x,t) - \phi(x,t) - \psi(t) - \varepsilon\zeta(\varepsilon^{-2}\rho(x)) \quad \text{for } (x,t) \in \bar{\Omega} \times \bar{I}.$$

According to (59), if $\varepsilon > 0$ is sufficiently small, then the function Ψ attains its maximum at some point $(x_\varepsilon, t_\varepsilon) \in \partial\Omega \times I$. For such a small $\varepsilon > 0$ and point $(x_\varepsilon, t_\varepsilon)$, by the viscosity property of u (i.e., by (56) and (57)), we have either

$$(60) \qquad\qquad aq_\varepsilon + H(x_\varepsilon, p_\varepsilon + \varepsilon^{-1}D\rho(x_\varepsilon)) \leq 0,$$

or

(61)
$$bq_\varepsilon + c\nu(x_\varepsilon) \cdot (p_\varepsilon + \varepsilon^{-1} D\rho(x_\varepsilon)) \le g(x_\varepsilon),$$

where $p_\varepsilon := D_x\phi(x_\varepsilon, t_\varepsilon)$ and $q_\varepsilon := \phi_t(x_\varepsilon, t_\varepsilon) + \psi'(t_\varepsilon)$. Here we have used the fact that $\zeta'(\rho(x_\varepsilon)) = \zeta'(0) = 1$. Inequalities (60) and (61) yield

$$\varepsilon^{-1} \le \varepsilon^{-1}|D\rho(x_\varepsilon)| \le |p_\varepsilon| + |p_\varepsilon + \varepsilon^{-1} D\rho(x_\varepsilon)| \le |p_\varepsilon| + f(x_\varepsilon) - aq_\varepsilon,$$

or

$$c\varepsilon^{-1} \le c\varepsilon^{-1}|D\rho(x_\varepsilon)| = c\varepsilon^{-1}\nu(x_\varepsilon) \cdot D\rho(x_\varepsilon) \le g(x_\varepsilon) - bq_\varepsilon + c\nu(x_\varepsilon) \cdot p_\varepsilon$$

respectively. Since $|p_\varepsilon|$ and q_ε are bounded as $\varepsilon \to 0$, from these inequalities we get a contradiction. $\qquad\square$

Proof of Theorem A.1. We argue by contradiction: we assume that $\sup_{\bar\Omega \times [0, T)}(u - v) > 0$ and will get a contradiction. The following argument is divided into a few steps.

1. We may assume by replacing T by a number smaller than and close to T that $u \in \mathrm{USC}(\bar{Q}_T)$ and $v \in \mathrm{LSC}(\bar{Q}_T)$.

Let $\varepsilon \in (0, 1)$. If $a > 0$, then we set

$$U^\varepsilon(x, t) = u(x, t) - \frac{\varepsilon}{T - t + \varepsilon^2} \quad \text{for} \ (x, t) \in \bar{Q}_T.$$

Let $M > 0$ be a constant such that $\|g\|_{\infty,\Omega} \le M$. If $a = 0$, then we set

$$U^\varepsilon(x, t) = (1 - \varepsilon)u(x, t) - \frac{\varepsilon b^{-1} M (T + \varepsilon^2)^2}{T - t + \varepsilon^2} \quad \text{for} \ (x, t) \in \bar{Q}_T.$$

Observe that if ε is sufficiently small, then

$$\max_{\bar{Q}_T}(U^\varepsilon - v) > 0 > \max_{\bar\Omega \times \{0, T\}} (U^\varepsilon - v).$$

Observe also that if $a > 0$, then U^ε is a subsolution of

$$\begin{cases} aU_t^\varepsilon(x, t) + H(x, D_x U^\varepsilon(x, t)) \le -\dfrac{\varepsilon a}{(T + \varepsilon^2)^2} & \text{in } Q_T, \\ bU_t^\varepsilon(x, t) + c\nu(x) \cdot D_x U^\varepsilon(x, t) \le g(x) & \text{on } \partial\Omega \times (0, T). \end{cases}$$

Moreover, if $a = 0$, we compute informally that

$$aU_t^\varepsilon(x, t) + H(x, D_x U^\varepsilon(x, t)) = (1 - \varepsilon)|D_x u(x, t)| - f(x) \le -\varepsilon f(x),$$

and that for any $(x, t) \in \partial\Omega \times (0, T)$,

$$bU_t^\varepsilon(x, t) + c\nu(x) \cdot D_x U^\varepsilon(x, t) = (1 - \varepsilon)[bu_t(x, t) + c\nu(x) \cdot D_x u(x, t)] - \frac{\varepsilon M (T + \varepsilon^2)^2}{(T - t + \varepsilon^2)^2}$$

$$\le (1 - \varepsilon)g(x) - \varepsilon M \le g(x),$$

and deduce that U^ε is a subsolution of

$$\begin{cases} aU_t^\varepsilon(x, t) + H(x, D_x U^\varepsilon(x, t)) \le -\varepsilon f(x) & \text{in } Q_T, \\ bU_t^\varepsilon(x, t) + c\nu(x) \cdot D_x U^\varepsilon(x, t) \le g(x) & \text{on } \partial\Omega \times (0, T). \end{cases}$$

Thus, replacing u by U^ε, selecting $\varepsilon > 0$ sufficiently small and setting

$$\mu = \varepsilon \min\left\{ \frac{a}{(T + \varepsilon^2)^2}, \ \min_{\bar\Omega} f \right\},$$

we are in the situation that u is a subsolution of

(62)
$$\begin{cases} au_t(x,t) + H(x, D_x u(x,t)) \le -\mu & \text{in } Q_T, \\ bu_t(x,t) + c\nu(x) \cdot D_x u(x,t) \le g(x) & \text{on } \partial\Omega \times (0, T), \end{cases}$$

and

(63)
$$\max_{\bar{Q}_T}(u - v) > 0 > \max_{\bar{\Omega} \times \{0, T\}} (u - v).$$

2. Let $\varepsilon > 0$ be a small constant, and we consider the sup-convolution of $u(x,t)$ in t-variable. We set

(64)
$$U^\varepsilon(x,t) = \max_{s \in [0, T]} \left(u(x, s) - \frac{(t - s)^2}{\varepsilon} \right) \quad \text{for } (x, t) \in \bar{Q}_T.$$

This function $U^\varepsilon(x,t)$ is Lipschitz continuous in t-variable: indeed, we have

(65)
$$\begin{aligned} |U^\varepsilon(x, t_1) - U^\varepsilon(x, t_2)| &\le \varepsilon^{-1} \max_{s \in [0, T]} |(t_1 - s)^2 - (t_2 - s)^2| \\ &\le \varepsilon^{-1} 2T |t_1 - t_2| \quad \text{for all } t_1, t_2 \in [0, T], \ x \in \bar{\Omega}. \end{aligned}$$

It is easily seen that if $\mathcal{M}(x,t)$ denotes the set of all maximum points $s \in [0, T]$ in formula (64), then

$$|t - s| \le \left(2\varepsilon \|u\|_{\infty, \bar{Q}_T}\right)^{1/2} \quad \text{for all } s \in \mathcal{M}(x,t), \ (x, t) \in \bar{Q}_T.$$

Hence, setting

$$\delta(\varepsilon) = \left(2\varepsilon \|u\|_{\infty, \bar{Q}_T}\right)^{1/2},$$

we have

$$\mathcal{M}(x,t) \subset (0, T) \quad \text{for all } (x, t) \in \bar{\Omega} \times (\delta(\varepsilon), T - \delta(\varepsilon)).$$

It is now easily seen (and it is a standard observation) that U^ε is a subsolution of

(66)
$$\begin{cases} aU^\varepsilon_t(x,t) + H(x, D_x U^\varepsilon(x,t)) \le -\mu & \text{in } \Omega \times (\delta(\varepsilon), T - \delta(\varepsilon)), \\ bU_t(x,t) + c\nu(x) \cdot D_x U^\varepsilon(x,t) \le g(x) & \text{on } \partial\Omega \times (\delta(\varepsilon), T - \delta(\varepsilon)). \end{cases}$$

In view of Lemma A.3, by the Lipschitz property (65) and the viscosity property (66), we see that U^ε is Lipschitz continuous in $\bar{\Omega} \times (\delta(\varepsilon), T - \delta(\varepsilon))$. Noting that $u \in \text{USC}(\bar{Q}_T)$,

$$U^\varepsilon(x,t) \ge u(x,t) \quad \text{and} \quad \lim_{\varepsilon \to 0+} U^\varepsilon(x,t) = u(x,t) \quad \text{for all } (x, t) \in \bar{Q}_T,$$

we see that if $\varepsilon > 0$ is sufficiently small, then

$$\max_{\bar{\Omega} \times [2\delta(\varepsilon), T - 2\delta(\varepsilon)]} (U^\varepsilon - v) > 0 > \max_{\bar{\Omega} \times \{2\delta(\varepsilon), T - 2\delta(\varepsilon)\}} (U^\varepsilon - v).$$

Fixing $\varepsilon > 0$ small enough, replacing u by U^ε and writing $J = (2\delta(\varepsilon), T - 2\delta(\varepsilon))$, we get the situation that u is Lipschitz continuous on $\bar{\Omega} \times \bar{J}$, u is a subsolution of

(67)
$$\begin{cases} au_t(x,t) + H(x, D_x u(x,t)) \le -\mu & \text{in } \Omega \times J, \\ bu_t(x,t) + c\nu(x) \cdot D_x u(x,t) \le g(x) & \text{on } \partial\Omega \times J, \end{cases}$$

and

(68)
$$\max_{\bar{\Omega} \times \bar{J}}(u - v) > 0 > \max_{\bar{\Omega} \times \partial J}(u - v).$$

3. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω. We may assume that

$$1 \leq \min_{\partial\Omega} |D\rho| \leq \max_{\bar\Omega} |D\rho| \leq C \quad \text{for some constant } C.$$

Let $\varepsilon > 0$ and set

$$U^\varepsilon(x,t) = u(x,t) - \varepsilon\rho(x) \quad \text{for all} \ (x,t) \in \bar\Omega \times \bar{J}.$$

By assuming $\varepsilon > 0$ small enough, we may assume that

$$\max_{\bar\Omega \times \bar{J}}(U^\varepsilon - v) > 0 > \max_{\bar\Omega \times \partial J}(U^\varepsilon - v).$$

It is easily checked that U^ε is a subsolution of

$$\begin{cases} aU_t^\varepsilon(x,t) + H(x, D_xU^\varepsilon(x,t)) \leq -\mu + \varepsilon C & \text{in } \Omega \times J, \\ bU_t^\varepsilon(x,t) + c\nu(x) \cdot D_xU^\varepsilon(x,t) \leq g(x) - \varepsilon c & \text{on } \partial\Omega \times J. \end{cases}$$

Thus, selecting $\varepsilon > 0$ sufficiently small and replacing u by U^ε, we may assume that

$$u \in \mathrm{Lip}\,(\bar\Omega \times \bar{J}),$$
$$\max_{\bar\Omega \times \bar{J}}(u - v) > 0 > \max_{\bar\Omega \times \partial J}(u - v),$$

and for some constant $\lambda > 0$,

$$(69) \quad \begin{cases} au_t(x,t) + H(x, D_xu(x,t)) \leq -\lambda & \text{in } \Omega \times J, \\ bu_t(x,t) + c\nu(x) \cdot D_xu(x,t) \leq g(x) - \lambda & \text{on } \partial\Omega \times J. \end{cases}$$

4. Now, we choose a maximum point $(\hat{x}, \hat{t}) \in \bar\Omega \times J$ of $u - v$ over $\bar\Omega \times \bar{J}$. Replacing the function $u(x,t)$ by the function

$$u(x,t) - \varepsilon(|x - \hat{x}|^2 + (t - \hat{t})^2),$$

with $\varepsilon > 0$ sufficiently small and replacing λ by a smaller positive number, we may assume that $u - v$ has a strict maximum at $(\hat{x}, \hat{t})$.

If $\hat{x} \in \Omega$, then the standard argument leads to a contradiction, the detail of which is skipped here. We thus assume that $\hat{x} \in \partial\Omega$. According to Lemma A.2, there is a function $\phi \in C^1(\mathbb{R}^{n+1})$ having the properties (55) with $\gamma = (c\nu(\hat{x}), b)$ and $e = (\nu(\hat{x}), 0)$. For any $\alpha > 0$ we consider the function

$$\Phi(x,t,y,s) = u(x,t) - v(y,s) - \alpha\phi(x - y, t - s) - d\nu(\hat{x}) \cdot (x - y)$$

on $(\bar\Omega \times \bar{J})^2$, where d is the constant given by

$$cd = g(\hat{x}) - \frac{\lambda}{2}.$$

Let $(x_\alpha, t_\alpha, y_\alpha, s_\alpha)$ be a maximum point of the function Φ.

Let $L > 0$ be a Lipschitz bound of the function u in $\bar\Omega \times \bar{J}$. By the inequality $\Phi(x_\alpha, t_\alpha, y_\alpha, s_\alpha) \geq \Phi(y_\alpha, s_\alpha, y_\alpha, s_\alpha)$, we get

$$\alpha\phi(x_\alpha - y_\alpha, t_\alpha - s_\alpha) \leq L(|x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2)^{1/2} + |d||x_\alpha - y_\alpha|$$
$$\leq (L + |d|)(|x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2)^{1/2}.$$

We set

$$\theta = \min_{|\xi|=1} \phi(\xi) \ (>0),$$

and observe by the homogeneity of ϕ that

$$\phi(\xi) \geq \theta |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^{n+1}.$$

Combining these observations, we find that

(70) $$\alpha(|x_\alpha - y_\alpha|^2 + |t_\alpha - s_\alpha|^2)^{1/2} \leq \theta^{-1}(L + |d|).$$

Moreover, by the standard argument, we deduce that

$$\begin{cases} \lim_{\alpha \to \infty} (x_\alpha, t_\alpha) = (\hat{x}, \hat{t}), & \lim_{\alpha \to \infty} (y_\alpha, s_\alpha) = (\hat{x}, \hat{t}), \\ \lim_{\alpha \to \infty} \alpha\phi(x_\alpha, t_\alpha, y_\alpha, s_\alpha) = 0, \\ \lim_{\alpha \to \infty} u(x_\alpha, t_\alpha) = u(\hat{x}, \hat{t}), & \lim_{\alpha \to \infty} v(y_\alpha, s_\alpha) = v(\hat{x}, \hat{t}). \end{cases}$$

In particular, we may assume by choosing α large enough that $t_\alpha, s_\alpha \in J$.

Fix $\alpha > 0$ so that $t_\alpha, s_\alpha \in J$. By the viscosity property of u, writing

$$\alpha D\phi(x_\alpha - y_\alpha, t_\alpha - s_\alpha) =: (p_\alpha, q_\alpha) \in \mathbb{R}^n \times \mathbb{R},$$

we get either

(71) $$\begin{cases} aq_\alpha + H(x_\alpha, p_\alpha + d\nu(\hat{x})) \leq -\lambda & \text{or} \\ bq_\alpha + c\nu(x_\alpha) \cdot (p_\alpha + d\nu(\hat{x})) \leq g(x_\alpha) - \lambda & \text{and} \quad x_\alpha \in \partial\Omega. \end{cases}$$

Also, we get either

(72) $$\begin{cases} aq_\alpha + H(y_\alpha, p_\alpha + d\nu(\hat{x})) \geq 0 & \text{or} \\ bq_\alpha + c\nu(y_\alpha) \cdot (p_\alpha + d\nu(\hat{x})) \geq g(y_\alpha) & \text{and} \quad x_\alpha \in \partial\Omega. \end{cases}$$

Noting the homogeneity of the function $D\phi$, i.e., the property that

$$D\phi(t\xi) = tD\phi(\xi) \quad \text{for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^{n+1},$$

and setting $z_\alpha := \alpha(x_\alpha - y_\alpha)$ and $r_\alpha = \alpha(t_\alpha - s_\alpha)$, we have

$$(p_\alpha, q_\alpha) = D\phi(z_\alpha, r_\alpha).$$

By (70), the collection $\{(z_\alpha, r_\alpha)\}_{\alpha>0}$ is bounded in $\mathbb{R}^{n+1}$.

Consider the case where there is a sequence $\{\alpha_j\}$ diverging to infinity such that

(73) $$bq_\alpha + c\nu(x_\alpha) \cdot (p_\alpha + d\nu(\hat{x})) \leq g(x_\alpha) - \lambda \quad \text{and} \quad x_\alpha \in \partial\Omega \quad \text{for all } \alpha = \alpha_j.$$

We may assume by reselecting the subsequence $\{\alpha_j\}$ if needed that for some $(z, r) \in \mathbb{R}^{n+1}$,

$$\lim_{j \to \infty} (z_{\alpha_j}, r_{\alpha_j}) = (z, r).$$

For any defining function $\rho \in C^1(\mathbb{R}^n)$ of Ω, we have

$$0 \geq \alpha\rho(y_\alpha) = \alpha(\rho(y_\alpha) - \rho(x_\alpha)) = -D\rho(x_\alpha) \cdot z_\alpha + \alpha o(|y_\alpha - x_\alpha|) \quad \text{as } \alpha = \alpha_j, \ j \to \infty,$$

which yields

$$\nu(\hat{x}) \cdot z \geq 0.$$

Hence, we have $e \cdot (z, r) \geq 0$, where $e = (\nu(\hat{x}), 0)$. Now, since

$$\lim_{j \to \infty} (p_{\alpha_j}, q_{\alpha_j}) = \lim_{j \to \infty} D\phi(z_{\alpha_j}, r_{\alpha_j}) = D\phi(z, r),$$

by our choice of ϕ, we get

$$D\phi(z, r) \cdot (c\nu(\hat{x}), b) \geq 0.$$

Thus, sending $j \to \infty$ in (73) with $\alpha = \alpha_j$, we get

$$0 \geq \lambda - g(\hat{x}) + (c\nu(\hat{x}), b) \cdot D\phi(z, r) + cd \geq \frac{\lambda}{2} > 0,$$

which is a contradiction.

Next consider the case where there is a sequence $\{\alpha_j\}$ diverging to infinity such that

$$bq_\alpha + c\nu(y_\alpha) \cdot (p_\alpha + d\nu(\hat{x})) \geq g(y_\alpha) \quad \text{and} \quad y_\alpha \in \partial\Omega \quad \text{for all} \quad \alpha = \alpha_j.$$

An argument parallel to the above yields a contradiction.

What remains is the case where we have both

$$aq_\alpha + H(x_\alpha, p_\alpha + d\nu(\hat{x})) \leq -\lambda \quad \text{and} \quad aq_\alpha + H(y_\alpha, p_\alpha + d\nu(\hat{x})) \geq 0$$

if α is sufficiently large. Hence, sending $\alpha \to \infty$ along a sequence, we get

$$aq + H(\hat{x}, p + c\nu(\hat{x})) \leq -\lambda < 0 \leq aq + H(\hat{x}, p + c\nu(\hat{x}))$$

for some $(p, q) \in \mathbb{R}^n \times \mathbb{R}$, which is a contradiction. The proof is complete. $\square$

The stationary eikonal equation (5) is of a special importance in this article and the following is a well-known comparison result (see [14, 5, 2, 16] for instance) for (5).

Lemma A.4. *Let $v \in \mathrm{USC}(\bar{\Omega})$ and $w \in \mathrm{LSC}(\bar{\Omega})$ be a subsolution and a supersolution of (5), respectively. Assume that $v(x) \leq w(x)$ for all $x \in \partial\Omega$. Then $v \leq w$ on $\bar{\Omega}$.*

The following proposition is a well-known result for convex Hamilton-Jacobi equations (see for instance [8, 9, 12]).

Proposition A.5. *Let U be an open subset of $\mathbb{R}^n$ and $H \in C(U \times \mathbb{R}^n)$. Assume that for each $x \in U$, the function $p \mapsto H(x, p)$ is convex in $\mathbb{R}^n$. Let $\mathcal{F}$ be a nonempty collection of subsolutions of*

(74) $$H(x, Du(x)) = 0 \quad in \ \ U.$$

Assume that $\mathcal{F}$ is uniformly bounded and equi-Lipschitz in U. Set

$$u(x) = \inf\{v(x) : v \in \mathcal{F}\} \quad for \ \ x \in U.$$

Then the function u is a subsolution of (74).

As is well-known, if we replace "inf" by "sup" in the above definition of u, the same conclusion as above holds without the convexity of H.

Outline of proof. 1. Let $\phi \in \mathrm{Lip}\,(U)$. We show that ϕ is a (viscosity) subsolution of (74) if and only if $H(x, D\phi(x)) \leq 0$ a.e. in U. Note here by the Rademacher theorem that ϕ is differentiable almost everywhere (and the gradient $D\phi$ obtained in the pointwise sense is identical to that in the distributional sense). It is then obvious by the viscosity property that if ϕ is a subsolution of (74), we have $H(x, D\phi(x)) \leq 0$ a.e. Next, assume that $H(x, D\phi(x)) \leq 0$ a.e. in U. Fix any open ball B such that $\bar{B} \subset U$. Using the mollification

technique and Jensen's inequality, for each $\varepsilon > 0$, we can select a function $\phi_\varepsilon \in C^1(\bar{B})$ such that

$$
\begin{cases}
H(x, D\phi_\varepsilon(x)) \leq \varepsilon & \text{in } B, \\
\|\phi_\varepsilon - \phi\|_{\infty, B} < \varepsilon.
\end{cases}
$$

By sending $\varepsilon \to 0$, we see by the stability of the viscosity property under uniform convergence that ϕ is a subsolution of (74) in B. Since the choice of balls B is arbitrary, we conclude that ϕ is a subsolution of (74) in U.

2. Let $v \in \mathcal{F}$. By the above equivalence, we have $H(x, Dv(x)) \leq 0$ a.e. in U. Hence, setting $w := -v$, we have $H(x, -Dw(x)) \leq 0$ a.e. Note that for each $x \in U$, the function $p \mapsto H(x, -p)$ is convex in $\mathbb{R}^n$. Using the above equivalence again, we find that $w := -v$ is a subsolution of $H(x, -Dw(x)) \leq 0$ in U. Since

$$
-u(x) = \sup\{-v(x) : v \in \mathcal{F}\} \quad \text{for } x \in U,
$$

we see by the stability of the subsolution property under taking the pointwise supremum that $w := -u$ is a subsolution of $H(x, -Dw(x)) \leq 0$ in U, which implies thanks to the above equivalence that $H(x, Du(x)) \leq 0$ a.e. Hence, again by the equivalence, we conclude that u is a subsolution of (74). $\qquad\square$

Example A.6. Let $n = 1$ and $\Omega = (0, 1)$. Consider the problem

$$
(9) \qquad
\begin{cases}
u_t + |D_x u| = 1 & \text{in } \Omega \times (0, \infty), \\
u_t = 0 & \text{on } \partial\Omega \times (0, \infty),
\end{cases}
$$

together with the initial condition

$$
(75) \qquad\qquad u(x, 0) = 2x \quad \text{for } x \in \bar{\Omega}.
$$

As Lemma 3.2 states, the comparison principle holds for Lipschitz continuous subsolutions and supersolution of the above problem. In what follows we show that the comparison principle does not hold for semicontinuous subsolutions and supersolutions of (9).

We set

$$
\begin{cases}
v(x) = x, \\
w(x, t) = 2x - t, \\
u(x, t) = \max\{v(x), w(x, t)\}.
\end{cases}
$$

Note that v is a classical solution of (9) and that w is a classical solution of

$$
w_t + |D_x w| = 1 \quad \text{in } \Omega \times (0, \infty)
$$

and a classical subsolution of (9). Accordingly, u is a viscosity subsolution of (9).

Our claim here is that u is a solution of (9) satisfying the initial condition (75). It is clear that $u(x, 0) = 2x$ for all $x \in \bar{\Omega}$. We have already checked that u is a subsolution of (9). Note that

$$
u(x, t) = v(x) \quad \text{if } t > x,
$$

which shows that u is a classical solution of

$$
\begin{cases}
u_t + |D_x u| = 1 & \text{in } Q^+, \\
u_t = 0 & \text{on } \{0\} \times (0, \infty),
\end{cases}
$$

where $Q^+ := \{(x,t) : t > x\}$. Similarly, noting that

$$u = w \quad \text{if } t < x,$$

we see that u is a classical solution of

$$u_t(x,t) + |D_x u(x,t)| = 1 \quad \text{in } Q^-,$$

where $Q^- := \{(x,t) \in Q : t < x\}$.

Fix any $(\hat{x}, \hat{t}) \in \bar{\Omega} \times (0, \infty)$ such that $v(\hat{x}) = w(\hat{x}, \hat{t})$. Obviously, $0 < \hat{x} \le 1$ and $\hat{t} = \hat{x}$. Let $\phi \in C^1\bar{Q})$ and assume that $u - \phi$ has a minimum at $(\hat{x}, \hat{t}) = (\hat{x}, \hat{x})$. The function $r \mapsto (u - \phi)(r, r)$ on $(0, \hat{x}]$ has a minimum at $r = \hat{x}$ and we have

$$0 \ge \frac{\mathrm{d}}{\mathrm{d}r}(u - \phi)(r, r)\Big|_{r=\hat{x}} = 1 - \phi_x(\hat{x}, \hat{x}) - \phi_t(\hat{x}, \hat{x}),$$

which yields

$$\phi_t(\hat{x}, \hat{t}) + \phi_x(\hat{x}, \hat{t}) \ge 1.$$

Similarly, since the function $r \mapsto (u - \phi)(r, \hat{t})$ on $(0, \hat{x}]$ has a minimum at $r = \hat{x}$, we get

$$0 \ge \frac{\mathrm{d}}{\mathrm{d}r}(u - \phi)(r, \hat{t})\Big|_{r=\hat{x}} = 1 - \phi_x(\hat{x}, \hat{x}),$$

which shows that $\phi_x(\hat{x}, \hat{t}) \ge 1 > 0$. Hence, we get

$$\phi_t(\hat{x}, \hat{t}) + |\phi_x(\hat{x}, \hat{t})| \ge 1,$$

which assures that u is a supersolution of (9). Thus, we conclude that u is a viscosity solution of (9).

We set

$$U(x,t) = \begin{cases} u(x,t) & \text{for } (x,t) \in [0, 1) \times [0, \infty), \\ 2 & \text{for } (x,t) \in \{1\} \times [0, \infty). \end{cases}$$

The functions u and U differ only on the set $\{1\} \times (0, \infty)$ and the function U is upper semicontinuous on $\bar{Q}$. It is obvious to see that U is a viscosity subsolution of (9) and that $U(x, 0) = 2x = u(x, 0)$ for all $x \in [0, 1] = \bar{\Omega}$. Moreover, the inequality

$$U \le u \quad \text{on } \bar{Q}$$

does not hold. That is, in the framework of semicontinuous viscosity solutions, the comparison principle *does not hold*.

Proposition A.7. *For each $\delta > 0$ there exists a C^∞ diffeomorphism j_δ of $\mathbb{R}^n$ such that*

$$j_\delta(\bar{\Omega}) \subset \Omega \quad \text{and} \quad |Dj_\delta(x) - I| < \delta \quad \text{for all } x \in \mathbb{R}^n,$$

where $Dj_\delta(x)$ and I denote the Jacobian matrix of j_δ and the identity matrix, respectively, and, for $n \times n$ matrix A, $|A|$ denotes the operator norm of A, i.e., $|A| = \max_{|\xi| \le 1} |A\xi|$.

Let j_δ be as above. Since $\bar{\Omega}$ is compact, the set $j_\delta(\bar{\Omega})$ is a compact subset of Ω. Therefore, the set $U := j_\delta^{-1}(\Omega)$ is an open neighborhood of $\bar{\Omega}$ and $\bar{U} = j_\delta^{-1}(\bar{\Omega})$. These observations are useful to extend the domain of definition of functions on $\bar{\Omega}$ to a neighborhood of $\bar{\Omega}$. Let $u \in C(\bar{\Omega})$. The function $u \circ j_\delta$ is defined on $\bar{U}$, and $D(u \circ j_\delta)(x) = Dj_\delta(x)^* Du(j_\delta(x))$ if u is differentiable at $j_\delta(x)$, with $x \in U$, where $Dj_\delta(x)^*$ indicates the transposed matrix of $Dj_\delta(x)$. Moreover, this extension technique combined with the mollification can be used to approximate a function of $\bar{\Omega}$ by a smooth function on $\bar{\Omega}$.

Proof. 1. Extending the vector field ν on $\partial\Omega$ to the whole $\mathbb{R}^n$ as a bounded, continuous vector field and then mollifying the resulting vector field, we can find a bounded, C^∞ vector field μ on $\mathbb{R}^n$ such that $\nu(x) \cdot \mu(x) > 0$ for all $x \in \partial\Omega$. Since $\partial\Omega$ is bounded, we may assume that $\mu(x) = 0$ if $|x|$ is sufficiently large. We may assume by multiplying μ by a small positive number if needed that $|\mu(x)| + |D\mu(x)| < 1$ for all $x \in \mathbb{R}^n$. Let $\delta \in (0, 1)$ and define the mapping $j_\delta : \mathbb{R}^n \to \mathbb{R}^n$ by

$$j_\delta(x) = x - \delta\mu(x).$$

The mapping $x \mapsto \delta\mu(x)$ is a contraction on $\mathbb{R}^n$, which ensures that the mapping j_δ is invertible. Indeed, for every $x \in \mathbb{R}^n$, we have

$$|\delta D\mu(x)| < \delta$$

and, moreover,

$$|Dj_\delta(x) - I| < \delta.$$

2. Next we show that $j_\delta(\bar{\Omega}) \subset \Omega$ if $\delta \in (0, 1)$ is sufficiently small. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω. Observe that for any $x \in \mathbb{R}^n$,

$$\rho(j_\delta(x)) = \rho(x) - \int_0^\delta D\rho(x - t\mu(x)) \cdot \mu(x)\, \mathrm{d}t.$$

Note that if $x \in \partial\Omega$, then $\mu(x) \cdot D\rho(x) > 0$, which guarantees that there are a neighborhood V of $\partial\Omega$ and $\delta_0 \in (0, 1)$ such that

$$\int_0^\delta D\rho(x - t\mu(x)) \cdot \mu(x)\, \mathrm{d}t > 0 \quad \text{for all} \;\; x \in V, \; \delta \in (0, \delta_0).$$

On the other hand, we have

$$\sup_{\Omega \setminus V} \rho < 0,$$

and hence, replacing δ_0 by a smaller number if needed, we may assume that

$$\rho(j_\delta(x)) = \rho(x - \delta\mu(x)) < 0 \quad \text{for all} \;\; x \in \Omega \setminus V, \; \delta \in (0, \delta_0).$$

Thus, we have

$$\rho(j_\delta(x)) < 0 \quad \text{for all} \;\; x \in \bar{\Omega}, \; \delta \in (0, \delta_0),$$

and conclude that $j_\delta(x) \in \Omega$ for all $x \in \bar{\Omega}$ and $\delta \in (0, \delta_0)$. $\qquad\square$

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(Eman S. Al-Aidarous) Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.

(E. O. Alzahrani) Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.

(H. Ishii) Faculty of Education and Integrated Arts and Sciences, Waseda University, Nishi-Waseda, Shinjuku, Tokyo 169-8050, Japan/ Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.
E-mail address: `hitoshi.ishii@waseda.jp`

(Arshad. M. M. Younas) Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.