Asymptotic Solutions of Hamilton-Jacobi Equations for Large Time and Related Topics

Hitoshi ISHII*

Abstract. We discuss the recent developments related to the large-time asymptotic behavior of solutions of Hamilton-Jacobi equations.

Mathematics Subject Classification (2000). Primary 35B40; Secondary 35F25, 35B15.

Keywords. Hamilton-Jacobi equations, asymptotic solutions, large-time behavior, additive eigenvalue problem, weak KAM theory, Aubry sets.

1. Introduction

In this note we discuss recent developments related to the asymptotic behavior, as $t \to \infty$, of solutions $u = u(x,t)$ of the Cauchy problem

$$\text{(CP)} \qquad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \Omega \times (0,\infty), \\ u|_{t=0} = u_0, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is an open set, $H \in C(\Omega \times \mathbf{R}^n, \mathbf{R})$ is the Hamiltonian, $u \in C(\Omega \times [0,\infty), \mathbf{R})$ is the unknown, $u_t = \partial u/\partial t$, $Du = (\partial u/\partial x_1, ..., \partial u/\partial x_n)$, and $u_0 \in C(\Omega, \mathbf{R})$. In fact, we will be concerned in this note only with the cases where $\Omega = \mathbf{R}^n$ or Ω is the n-dimensional torus $\mathbf{T}^n := \mathbf{R}^n/\mathbf{Z}^n$.

Investigations on the asymptotic behavior of solutions $u(x,t)$ of (CP) as $t \to \infty$ go back to Kruzkov [30], Lions [31], and Barles [1]. An interesting feature of the recent developments concerning the asymptotic behavior of solutions is the interaction with weak KAM theory introduced by Fathi [15, 17] (see also [18, 19, 11, 14]). The large-time behavior of solution of (CP) is closely related to the "stationary" equation:

$$H(x, Dv) = c \quad \text{in } \Omega, \quad \text{where } c \text{ is a constant.}$$

Weak KAM theory is a useful tool to study the structure of solutions of this "stationary" equation.

*Supported in part by Grant-in-Aids for Scientific Research, No. 18204009, JSPS.

We will write $H[v]$ for $H(x, Dv(x))$ for notational simplicity and hence the above PDE can be written simply as $H[v] = c$.

Hamilton-Jacobi equations arise in calculus of variations (mechanics, geometric optics, geometry), optimal control, differential games, etc. They are called *Bellman* equations in optimal control and *Isaacs* equations in differential games, where they appear as dynamic programming equations. Basic references on these topics are books by Lions[31], Fleming-Soner[20] and Bardi-Capuzzo Dolcetta [2].

The right notion of weak solution for Hamilton-Jacob equations is that of *viscosity solution* introduced by Crandall-Lions [7]. This notion is based on the maximum principle while the notion of distribution by Schwartz is based on integration by parts. However, when we treat (CP) with greater generality, we will consider another notion (see (5) below) of solution based on the variational formula for solutions of (CP).

2. Additive eigenvalue problem

We begin with a formal expansion of the solution u of (CP). Consider the asymptotic expansion of the form

$$u(x,t) = a_0(x)t + a_1(x) + a_2(x)t^{-1} + \cdots \qquad \text{as } t \to \infty.$$

Plugging this expression into (CP), we get

$$a_0(x) + \frac{-a_1(x)}{t^2} + \cdots + H(x, Da_0(x)t + Da_1(x) + Da_2(x)t^{-1} + \cdots) = 0.$$

This suggests that

$$\begin{cases} a_0(x) \equiv a_0 \text{ for a constant } a_0, \\ a_0 + H(x, Da_1(x)) = 0, \end{cases}$$

and we are led to the *additive eigenvalue problem* for H. The problem is to find a pair $(c, v) \in \mathbf{R} \times C(\Omega)$ such that

$$H[v] = c \quad \text{ in } \Omega.$$

Given such a pair (c, v), we call c and v an (*additive*) *eigenvalue* and an (*additive*) *eigenfunction* for H, respectively.

Remark that if (c, v) is a solution of the additive eigenvalue problem for H, then the function $u(x,t) := v(x) - ct$ is a solution of $u_t + H[u] = 0$, and conversely, if a solution u of (CP) has the form $u(x,t) = v(x) - ct$, with $(c, v) \in \mathbf{R} \times C(\Omega)$, then (c, v) is a solution of the additive eigenvalue problem for H. We call the function $v(x) - ct$ an *asymptotic solution* for $u_t + H[u] = 0$ if (c, v) is a solution of the additive eigenvalue problem for H.

Additive eigenvalue problems arise in *ergodic control problems*, where one seeks to minimize the *long-time average* of cost

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t), \alpha(t))\, \mathrm{d}\, t,$$

where $\alpha : [0, \infty) \to A$ is a control, A is the control region, $X : [0, \infty) \to \mathbf{R}^n$ describes the state of the system under considerations which is the solution of the state equation

$$\dot{X}(t) = g(X(t), \alpha(t)), \quad X(0) = x \in \mathbf{R}^n,$$

with a given function $g : \mathbf{R}^n \times A \to \mathbf{R}^n$, and $f : \mathbf{R}^n \times A \to \mathbf{R}$ represents the running cost of the system. Such an ergodic control problem is closely related to the problem of finding the limit

$$\lim_{t\to\infty} \frac{1}{t} u(x, t)$$

for the solution of

$$u_t + H[u] = 0 \ \text{ in } \Omega \times (0, \infty), \quad u|_{t=0} = 0,$$

where

$$H(x, p) = \sup_{a\in A}(-g(x, a) \cdot p - f(x, a)).$$

Additive eigenvalue problems play an important role in homogenization for Hamilton-Jacobi equations, where they are referred to as *cell problems.* In this theory one is concerned with the *macroscopic effects* of small scale oscillating phenomena.

As an example, consider the Hamilton-Jacobi equation

$$\lambda u^\varepsilon(x) + H(x, x/\varepsilon, Du^\varepsilon(x)) = 0 \quad \text{ in } \Omega,$$

where $\lambda > 0$ is a given constant and $\varepsilon > 0$ is a small parameter to be sent to zero. Here the Hamiltonian $H(x, y, p)$ is $\mathbf{Z}^n$-periodic in the variable y.

The basic scheme in periodic homogenization is (i) to solve the additive eigenvalue problem for $G(y, q) := H(x, y, p + q)$ with fixed (x, p), i.e., to find a $(c, v) \in \mathbf{R} \times C(\mathbf{T}^n)$ such that

$$H(x, y, p + D_y v(y)) = c \quad \text{ for } y \in \mathbf{T}^n,$$

(ii) to define the so-called *effective Hamiltonian* $\bar{H} \in C(\mathbf{R}^n \times \mathbf{R}^n)$ by setting $\bar{H}(x, p) = c$, and (iii) to solve the Hamilton-Jacobi equation

$$\lambda\bar{u} + \bar{H}(x, D\bar{u}(x)) = 0 \quad \text{in } \Omega,$$

in order to find the limit function $\bar{u}(x) := \lim_{\varepsilon\to 0+} u^\varepsilon(x)$.

In this article we often identify any $\mathbf{Z}^n$-periodic function f on $\mathbf{R}^n$ with the function $\bar{f}$ on $\mathbf{T}^n$ defined by $\bar{f}(x + \mathbf{Z}^n) = f(x)$ for $x \in \mathbf{R}^n$.

Some of basic references on homogenization of Hamilton-Jacobi equations are: Lions-Papanicolaou-Varadhan [32], Evans [12, 13] (perturbed test functions method), Ishii [26], Lions-Souganidis [33] (almost periodic homogenization), Souganidis [41], Rezakhanlou-Tarver [39], Lions-Souganidis [34] (random homogenization).

3. Convex Hamilton-Jacobi equations

We assume throughout this paper that

(A1) H is continuous and convex,

That is, $H \in C(\Omega \times \mathbf{R}^n)$ and $H(x,p)$ is convex in $p \in \mathbf{R}^n$ for every $x \in \Omega$. If H is convex in this sense, we call PDE $H[u] = 0$ or $u_t + H[u] = 0$ a convex Hamilton-Jacobi equation. We are here concerned with viscosity solutions (resp., viscosity subsolutions, viscosity supersolutions) of Hamilton-Jacobi equations and call them simply solutions (resp., subsolutions, supersolutions). We use the following notation:

$$\begin{aligned} \mathcal{S}_H^- \equiv& \mathcal{S}_H^-(\Omega) := \{u \text{ subsolution of } H[u] = 0 \text{ in } \Omega\}, \\ \mathcal{S}_H^+ \equiv& \mathcal{S}_H^+(\Omega) := \{u \text{ supersolution of } H[u] = 0 \text{ in } \Omega\}, \\ \mathcal{S}_H \equiv& \mathcal{S}_H(\Omega) := \mathcal{S}_H^- \cap \mathcal{S}_H^+. \end{aligned}$$

According to the theory of semicontinuous viscosity solutions due to Barron–Jensen [6] (see also [27]) we know that under the assumption that $H(x,p)$ is convex in $p \in \mathbf{R}^n$, if $S \subset \mathcal{S}_H^-$ and $u \in C(\Omega)$ is given by $u(x) := \inf\{v(x) \mid v \in S\}$, then $u \in \mathcal{S}_H^-$. Classical observations similar to this are the following. For any $H \in C(\Omega \times \mathbf{R}^n)$, which may not be convex in p, we have

$$\begin{aligned} S \subset \mathcal{S}_H^-,\ u(x) = \sup\{v(x) \mid v \in S\} \text{ for all } x \in \Omega,\ u \in C(\Omega) \quad &\Longrightarrow \quad u \in \mathcal{S}_H^-, \\ S \subset \mathcal{S}_H^+,\ u(x) = \inf\{v(x) \mid v \in S\} \text{ for all } x \in \Omega,\ u \in C(\Omega) \quad &\Longrightarrow \quad u \in \mathcal{S}_H^+. \end{aligned}$$

Hence, if $H(x,p)$ is convex in p, then we have

$$S \subset \mathcal{S}_H,\ u(x) = \sup\{v(x) \mid v \in S\} \text{ for all } x \in \Omega,\ u \in C(\Omega) \quad \Longrightarrow \quad u \in \mathcal{S}_H.$$

That is, the viscosity property is closed under the operation of taking pointwise infimum.

The above general observation can be applied to showing the Hopf-Lax-Oleinik formula for the solution of (CP) as in the next example.

Example 1 (Hopf-Lax-Oleinik). Let $H \in C(\mathbf{R}^n)$ be a convex function. Let L denote the convex conjugate of H. That is, $L(\xi) = \sup_{p \in \mathbf{R}^n}(\xi \cdot p - H(p))$. As is well-known, L is a proper, lower semicontinuous, convex in $\mathbf{R}^n$ and satisfies $\lim_{|\xi| \to \infty} L(\xi)/|\xi| = \infty$. We assume that $L \in C^1(\mathbf{R}^n)$, for simplicity, and consider the function $v(x,t) = tL((x-y)/t)$ on $\mathbf{R}^n \times (0, \infty)$, where $y \in \mathbf{R}^n$. Compute that

$$v_t(x,t) = L\left(\frac{x-y}{t}\right) - \frac{x-y}{t} \cdot DL\left(\frac{x-y}{t}\right), \quad Dv(x,t) = DL\left(\frac{x-y}{t}\right).$$

Observe by the convex duality that $H(p) = p \cdot \xi - L(\xi)$ if and only if $p = DL(\xi)$, and hence that $H(DL(\xi)) = \xi \cdot DL(\xi) - L(\xi)$ for all $\xi \in \mathbf{R}^n$. Therefore we have

$$H(Dv(x,t)) = H(DL(\frac{x-y}{t})) = \frac{x-y}{t} \cdot DL(\frac{x-y}{t}) - L\left(\frac{x-y}{t}\right) = -v_t(x,t).$$

That is, the function v is a classical solution of $v_t(x,t) + H(D_x v(x,t)) = 0$. Fix any $u_0 \in \mathrm{BUC}(\mathbf{R}^n)$, where $\mathrm{BUC}(\Omega)$ denotes the space of all bounded, uniformly continuous functions on Ω. Thus we see that the formula

$$u(x,t) = \inf_{y\in\mathbf{R}^n} \left(u_0(y) + tL\left(\frac{x-y}{t}\right)\right)$$

gives a solution of $u_t + H(Du) = 0$ in $\mathbf{R}^n \times (0,\infty)$.

4. A result in $\mathbf{T}^n$

Since the works of Namah-Roquejoffre [38] and Fathi [16], there has been much interest in the large-time asymptotic behavior of the solution u of (CP). See for this also [3, 4, 22, 10, 23, 24, 25, 28, 36, 37, 38, 40] and references therein. A typical result obtained in this development is stated as in Theorem 1 below. In addition to (A1), we need these hypotheses:

(A2) H is locally coercive, i.e., for any compact $K \subset \Omega$,

$$\lim_{r\to\infty} \inf\{H(x,p) \mid (x,p) \in K \times \mathbf{R}^n,\ |p| \geq r\} = \infty.$$

(A3) $H(x,p)$ is strictly convex in p.

Theorem 1. *Let* $\Omega = \mathbf{T}^n$ *and* $u_0 \in C(\mathbf{T}^n)$. *Assume that* (A1) *and* (A2) *hold.* (i) *The additive eigenvalue problem for* H *has a solution* $(c,v) \in \mathbf{R} \times C(\mathbf{T}^n)$. *Moreover the constant* c *is uniquely determined.* (ii) *The Cauchy problem* (CP) *has a unique solution* $u \in C(\mathbf{T}^n \times [0,\infty))$. (iii) *Assume in addition that* (A3) *holds. Then there exists an additive eigenfunction* $u_\infty \in C(\mathbf{T}^n)$ *such that*

$$\lim_{t\to\infty} \max_{x\in\mathbf{T}^n} |u(x,t) - u_\infty(x) + ct| = 0.$$

We remark that assertion (i) of the theorem above is a classical result due to Lions-Papanicolaou-Varadhan [32], assertion (ii) is a more classical result due to Crandall-Lions [7], Crandall-Evans-Lions [8] and others, and assertion (iii) can be found in Barles-Souganidis [4] and Davini-Siconolfi [10].

The following example shows that the convexity and coercivity of H are not enough to assure that the solution $u(x,t)$ of (CP) "converges" to an asymptotic solution as $t \to \infty$.

Example 2 (Barles-Souganidis [4]). Consider the Cauchy problem

$$u_t + |Du + 1| = 1 \quad \text{in } \mathbf{R}\times(0,\infty) \quad \text{and} \quad u(x,0) = \sin x.$$

Then $u(x,t) := \sin(x-t)$ is a classical solution and as $t \to \infty$,

$$u(x,t) \not\to v(x) - ct$$

for any $(c, v) \in \mathbf{R}\times C(\mathbf{R})$. Note that $H(x, p) = |p+1|-1$ is convex and coercive, but not strictly convex. Finally, since $\sin x$ and $H(x, p) = |p+1| - 1$ are 2π-periodic as functions of x, the spatial domain Ω in the Cauchy problem above can be regarded as one-dimensional torus $\mathbf{R}/2\pi\mathbf{Z}$ as in Theorem 1, as far as 2π-periodic solutions are concerned.

This example justifies somehow assumption (A3) in Theorem 1 (iii) although it is far from necessary for convergence of the solution of (CP) to an asymptotic solution as a general result in [4] indicates.

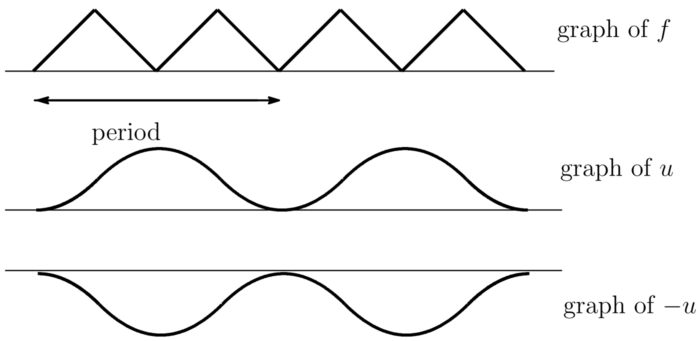
5. Weak KAM theory in terms of PDE

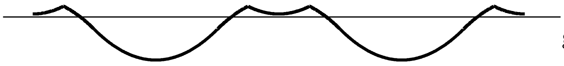
Analysis on the asymptotic behavior such as Theorem 1 (iii) is rather difficult because of the complex structure of eigenfunctions for H. A first remark in this regard is that if v is an eigenfunction for H, then so is $v+a$, with $a \in \mathbf{R}$. Actually, the complexity is far beyond this as the following example shows.

Example 3. Consider the eikonal equation $|Du| = f(x)$ in $\mathbf{R}$, where $f \in C(\mathbf{R}/2\mathbf{Z})$ is the function defined by $f(x) = \min\{x, 1-x\}$ for $x \in [0, 1]$ and $= \min\{x-1, 2-x\}$ for $x \in [1, 2]$. Let $u \in C(\mathbf{R}/2\mathbf{Z})$ be the function given by

$$u(x) = \begin{cases} \displaystyle\int_0^x f(t)\,\mathrm{d}\,t & \text{for } x \in [0,\, 1], \\ \displaystyle\int_0^1 f(t)\,\mathrm{d}\,t - \int_1^x f(t)\,\mathrm{d}\,t \equiv \frac{1}{4} - \int_1^x f(t)\,\mathrm{d}\,t & \text{for } x \in [1,\, 2]. \end{cases}$$

Then both u and $-u$ are classical solutions of $|Du| = f$ in $\mathbf{R}$. Moreover, for any $c \in [0,\, 1/2]$, the function $v_c(x) = \min\{u(x),\, -u(x)+c\}$ is a solution of $|Du| = f$ in $\mathbf{R}$. The difference $v_{c_1} - v_{c_2}$ for any $c_1, c_2 \in [0,\, 1/2]$, with $c_1 \neq c_2$, is not a constant function.





Weak KAM theory is a useful tool to study the structure of the space of additive eigenfunctions. Here we give a quick review of Aubry sets from weak KAM theory in terms of PDE, which we follow Fathi-Siconolfi [18, 19].

In what follows (except in Theorem 3 and Proposition 4), we assume that H satisfies (A1) and (A2) and that $\mathcal{S}_H^- \neq \emptyset$. If, instead, $\mathcal{S}_{H-c}^- \neq \emptyset$ for some $c > 0$, we may reduce to the case where $c = 0$ by replacing H by $H - c$. Define the function d_H on $\Omega \times \Omega$ by

$$d_H(x, y) = \sup\{w(x) - w(y) \mid w \in \mathcal{S}_H^-(\Omega)\}.$$

By the assumption that $c = 0$, we have $\mathcal{S}_H^-(\Omega) \neq \emptyset$, which implies that $d_H(x, y) > -\infty$ for all $x, y \in \Omega$. The coercivity assumption guarantees that the function d_H is a locally Lipschitz function. By definition, $d_H(\cdot, y)$ is the maximum subsolution of $H[u] = 0$ in Ω among those satisfying $u(y) = 0$. Here are some basic properties of d_H:

$$\begin{aligned}
&d_H(y, y) = 0 \quad \text{for all } y \in \Omega,\\
&d_H(\cdot, y) \in \mathcal{S}_H^-(\Omega) \quad \text{for all } y \in \Omega,\\
&d_H(\cdot, y) \in \mathcal{S}_H(\Omega \setminus \{y\}) \quad \text{for all } y \in \Omega,\\
&d_H(x, y) \le d_H(x, z) + d_H(z, y) \quad \text{for all } x, y, z \in \Omega.
\end{aligned}$$

We have the formula for d_H:

$$d_H(x, y) = \inf\Big\{\int_0^t L[\eta]\,\mathrm{d}\,s \;\Big|\; t > 0,\ \eta \in \mathrm{AC}([0, t], \Omega),\ \eta(t) = x,\ \eta(0) = y\Big\},$$

where L denotes the Lagrangian of H, i.e., $L(x, \xi) = \sup_{\xi \in \mathbf{R}^n}(\xi \cdot p - H(x, p))$, $\mathrm{AC}([0, t], \Omega)$ denotes the space of all absolutely continuous curves $\eta : [0, t] \to \Omega$, and $L[\eta]$ is an abbreviated notation for $L(\eta(s), \dot\eta(s))$.

The (projected) *Aubry set* $\mathcal{A}_H \subset \Omega$ is defined as

$$\mathcal{A}_H := \{y \in \Omega \mid d_H(\cdot, y) \in \mathcal{S}_H(\Omega)\}.$$

A characterization of the Aubry set is given by the following: any point $y \in \Omega$ is in $\mathcal{A}_H$ if and only if

$$\inf\Big\{\int_0^t L[\eta]\,\mathrm{d}\,s \;\Big|\; t \ge \varepsilon,\ \eta \in \mathrm{AC}([0, t]),\ \eta(0) = \eta(t) = y\Big\} = 0, \tag{1}$$

where ε is an arbitrary fixed positive constant.

Let $\Omega = \mathbf{T}^n$. One of main observations related to Aubry sets is this.

Theorem 2 (representation of solutions). *If u is a solution of $H[u] = 0$ in $\mathbf{T}^n$, then*

$$u(x) = \inf\{u(y) + d_H(x, y) \mid y \in \mathcal{A}_H\} \quad \textit{for all } x \in \mathbf{T}^n.$$

This theorem says that any solution u of $H[u]=0$ in $\mathbf{T}^n$ is determined by its restriction $u|_{\mathcal{A}_H}$ to $\mathcal{A}_H$.

In the following two propositions we do not necessarily assume that $\mathcal{A}_H \neq \emptyset$.

Theorem 3. *Under the hypotheses and notation of Theorem* 1 (iii)*, we have for any* $x \in \Omega$*,*

$$u_\infty(x) = \inf\{u_0(y) + d_{H-c}(x,y) + d_{H-c}(y,z) \mid y \in \Omega,\ z \in \mathcal{A}_{H-c}\}. \tag{2}$$

The above result is due to Davini-Siconolfi [10] (see also [21]). The formula in the above theorem is interpreted as follows (see [23]).

Proposition 4. *Under the hypotheses of the above theorem, we have*

$$u_\infty(x) = \inf\{v(x) \mid v \in \mathcal{S}_{H-c},\ v \geq u_0^- \ \textit{in}\ \Omega\} \quad \textit{for}\ x \in \Omega, \tag{3}$$

where u_0^- *is the maximum subsolution of* $H[u]=c$ *"below"* u_0*, i.e.,*

$$u_0^-(x) := \sup\{v(x) \mid v \in \mathcal{S}_{H-c}^-,\ v \geq u_0 \ \textit{in}\ \Omega\} \quad \textit{for}\ x \in \Omega. \tag{4}$$

Indeed, it is not hard to see that the function: $x \mapsto \inf\{u_0(y) + d_{H-c}(x,y) \mid y \in \Omega\}$ is the maximum subsolution of $H[u]=c$ below u_0 and also from Theorem 2 that the function: $x \mapsto \inf\{u_0^-(z) + d_{H-c}(x,z) \mid z \in \mathcal{A}_{H-c}\}$ is the minimum solution of $H[u]=c$ in Ω above u_0^-. Consequently, from (2) we get

$$\begin{aligned} u_\infty(x) &= \inf\{\inf\{u_0(y) + d_{H-c}(x,y) \mid y \in \Omega\} + d_{H-c}(y,z) \mid z \in \mathcal{A}_{H-c}\} \\ &= \inf\{u_0^-(y) + d_{H-c}(y,z) \mid z \in \mathcal{A}_{H-c}\} \\ &= \inf\{v(x) \mid v \in \mathcal{S}_{H-c}, v \geq u_0^- \ \text{in}\ \Omega\}. \end{aligned}$$

We note (see [25]) that the formula (3) is valid for $\Omega = \mathbf{R}^n$ as well under the assumptions (A1), (A2), (A4), and (A5). (See below for (A4) and (A5).)

Now, we discuss the case where $\Omega = \mathbf{R}^n$. The following representation theorem of solutions of $H=0$ is taken from Ishii-Mitake [29] (see also [25, Section 3]).

Theorem 5. *Let* $u \in \mathcal{S}_H(\mathbf{R}^n)$*. Then*

$$u(x) = \min\{w_{\mathcal{A}}(x), w_\infty(x)\},$$

where

$$\begin{aligned} w_{\mathcal{A}}(x) &:= \inf\{u(y) + d_H(x,y) \mid y \in \mathcal{A}_H\}, \\ w_\infty(x) &:= \inf\{d(x) + c \mid d \in D_\infty,\ c \in C(u,d)\}, \\ D_\infty &:= \{\phi \in \mathcal{S}_H(\mathbf{R}^n) \mid \exists y_j \in \mathbf{R}^n \ \textit{such that}\ |y_j| \to \infty, \\ &\qquad \phi(y_j) + d_H(\cdot, y_j) \to \phi \ \ \textit{in}\ C(\mathbf{R}^n)\}, \\ C(u,d) &:= \{c \in (-\infty, \infty] \mid u \leq d + c \ \ \textit{in}\ \mathbf{R}^n\}. \end{aligned}$$

This theorem asserts that any $u \in \mathcal{S}_H(\mathbf{R}^n)$ is "factorized" as the minimum of two functions $w_{\mathcal{A}}$ and w_∞, and the function $w_{\mathcal{A}}$ depends only on the restriction $u|_{\mathcal{A}_H}$ to $\mathcal{A}_H$ and the function w_∞ depends only on the behavior of u as $|x| \to \infty$.

Contrary to the situation of Theorem 1 (i), uniqueness of additive eigenvalues in unbounded domains Ω does not hold: indeed, if we set

$$c_H = \inf\{a \in \mathbf{R} \mid \in S^-_{H-a} \neq \emptyset\},$$

then for any $b \geq c_H$ there exists a solution v of $H[v] = b$ in Ω. See, for instance, Barles-Roquejoffre [3].

When $\Omega = \mathbf{R}$, we have

$$d_H(x,y) = \max\{d_+(x) - d_+(y),\, d_-(x) - d_-(y)\},$$

where

$$d_+(x) := \lim_{y\to\infty}(d_H(x,y) - d_H(0,y)),$$
$$d_-(x) := \lim_{y\to-\infty}(d_H(x,y) - d_H(0,y)).$$

Thus, any solution of $H[u] = 0$ in $\mathbf{R}$ has a representation

$$u(x) = \min\{d_-(x) + c_-,\, d_+(x) + c_+\},$$

where $c_\pm$ are constants possibly being $+\infty$.

6. One-dimensional case

We wish to find sufficient conditions for (H, u_0) and $\Omega = \mathbf{R}^n$ so that the solution $u(x,t)$ of (CP) "converges" to an asymptotic solution $v(x) - ct$. For simplicity of presentation, we consider the case where $c = 0$, which can be attained by replacing H by $H - c$ if necessary. We introduce a new condition on H.

(A4) For $\phi \in \mathcal{S}_H$ there exist a constant $C > 0$ and a function $\psi \in \mathcal{S}_{H-C}$ such that $\lim_{|x|\to\infty}(\phi - \psi)(x) = \infty$.

Let $\Omega = \mathbf{R}^n$ and assume that (A1), (A2), and (A4) hold. These hypotheses do not guarantee the unique solvability of (CP) in the sense of viscosity solutions. We thus regard the function

$$u(x,t) = \inf\Big\{\int_0^t L[\eta]\,\mathrm{d}\,s + u_0(\eta(0)) \;\Big|\; \eta \in \mathrm{AC}([0,\,t],\Omega),\ \eta(t) = x\Big\} \tag{5}$$

on $\mathbf{R}^n \times [0,\,\infty)$ as the unique *solution* of (CP). An important remark here is that this function u may take the value $-\infty$. That is, $u(x,t) \in [-\infty,\,\infty)$ in general.

It is easily seen that if $H(x,p) = H_0(p) - f(x)$, with $H_0,\ f \in C(\mathbf{R}^n)$, and H_0 is uniformly continuous in $\mathbf{R}^n$, then H satisfies (A4). Also, if $H \in \mathrm{BUC}(\mathbf{R}^n \times B(0,R))$ for any $R > 0$ and H is globally coercive, i.e.,

$$\lim_{r\to\infty} \inf\{H(x,p) \mid x \in \mathbf{R}^n,\ |p| \geq r\} = \infty, \tag{6}$$

then H satisfies (A4). Here and henceforth we denote by $B(a,r)$ the closed ball of $\mathbf{R}^n$ with center at a and radius $r \geq 0$.

We now follow arguments in [24, 25]. As in (4), we set

$$u_0^-(x) = \sup\{v(x) \mid v \in \mathcal{S}_H^-,\ v \leq u_0 \ \text{ in } \mathbf{R}^n\} \ \text{ for } x \in \mathbf{R}^n.$$

Clearly we have $u_0^- \leq u_0$ in $\mathbf{R}^n$. As before, we have

$$u_0^-(x) = \inf\{u_0(y) + d_H(x,y) \mid y \in \mathbf{R}^n\} \ \text{ for } x \in \mathbf{R}^n. \tag{7}$$

Proposition 6. *If $u_0^-(x_0) = -\infty$ for some $x_0 \in \mathbf{R}^n$, then $u_0^-(x) \equiv -\infty$ and*

$$\liminf_{t\to\infty} u(x,t) = -\infty \quad \textit{for all } x \in \mathbf{R}^n,$$

where u is the solution of (CP).

To continue, we need to assume that

$$u_0^-(x) > -\infty \quad \text{for } x \in \mathbf{R}^n, \tag{8}$$

and set

$$u_\infty(x) = \inf\{w(x) \mid w \in \mathcal{S}_H,\ w \geq u_0^- \ \text{ in } \mathbf{R}^n\} \ \text{ for } x \in \mathbf{R}^n.$$

It is clear that $u_\infty \geq u_0^-$ in $\mathbf{R}^n$.

Proposition 7. *If $u_\infty(x_0) = \infty$ for some $x_0 \in \mathbf{R}^n$, then $u_\infty(x) \equiv \infty$ and*

$$\lim_{t\to\infty} u(x,t) = \infty \quad \textit{for } x \in \mathbf{R}^n.$$

Thus, in order to get an asymptotic solution $v \in \mathcal{S}_H$ for the solution of (CP), we have to assume that

(A5) $-\infty < u_0^-(x) \leq u_\infty(x) < \infty$ for all $x \in \mathbf{R}^n$.

This condition can be stated equivalently as

$$\{\phi \in \mathcal{S}_H^- \mid \phi \leq u_0 \ \text{ in } \mathbf{R}^n\} \neq \emptyset \quad \text{and} \quad \{\phi \in \mathcal{S}_H \mid \phi \geq u_0^- \ \text{ in } \mathbf{R}^n\} \neq \emptyset.$$

Any curve $\gamma \in C((-\infty,\, 0])$ is said to be *extremal* for $\phi \in \mathcal{S}_H$ if it satisfies the conditions: $\gamma \in \mathrm{AC}([-t,\, 0])$ for all $t > 0$ and

$$\int_a^b L[\gamma]\,\mathrm{d}\,s = \phi(\gamma(b)) - \phi(\gamma(a)) \quad \text{ for } a, b \leq 0. \tag{9}$$

The main role of assumption (A4) is to guarantee the existence of extremal curves. Indeed, we have (see [25]):

Proposition 8. *Assume that* (A1)*,* (A2)*, and* (A4) *hold. For any* $\phi \in \mathcal{S}_H$ *and* $x \in \mathbf{R}^n$*, there is an extremal curve* γ *such that* $\gamma(0) = x$*.*

We denote by $\mathcal{E}_x(\phi)$ the set of such extremal curves γ and set $\mathcal{E}(\phi) = \bigcup_{x\in\mathbf{R}^n} \mathcal{E}(\phi)$. We recall (see e.g. [28]) that for any $\psi \in \mathcal{S}_H^-$ and $\eta \in \mathrm{AC}([a,\, b])$, with $a < b$,

$$\psi(\gamma(b)) - \psi(\gamma(a)) \le \int_a^b L[\gamma]\,\mathrm{d}\,s. \tag{10}$$

Thus, for any $\gamma \in \mathcal{E}(\phi)$, with $\phi \in \mathcal{S}_H$, and any $a < b \le 0$, we have

$$\int_a^b L[\gamma]\,\mathrm{d}\,s = \inf\Big\{\int_a^b L[\eta]\,\mathrm{d}\,s \;\Big|\; \eta \in \mathrm{AC}(a,\, b]),\ \eta(t) = \gamma(t) \ \text{ at } t = a,\, b\Big\}.$$

Every curve $\gamma \in \mathcal{E}(\phi)$, with $\phi \in \mathcal{S}_H$, is “extremal” in this sense.

Let $\phi \in \mathcal{S}_H$, $\psi \in \mathcal{S}_H^-$, and $\gamma \in \mathcal{E}(\phi)$. Combining (9) and (10) yield

$$(\phi - \psi)(\gamma(a)) \le (\phi - \psi)(\gamma(b)) \quad \text{ for all } a \le b \le 0. \tag{11}$$

That is, the fucntion: $t \mapsto (\phi - \psi)(\gamma(-t))$ is non-increasing on $[0,\, \infty)$.

We now assume that $n = 1$ and explain the main result in Ichihara-Ishii [23]. Fix any $y \in \mathbf{R}$ and choose an *extremal curve* $\gamma \in \mathcal{E}_y(u_\infty)$.

Theorem 9. *Assume in addition to the hypotheses* (A1)*,* (A2) *and* (A4) *that* (A5) *holds. Then we have*

$$u(y,t) \to u_\infty(y) \qquad \text{as } t \to \infty$$

provided one of the following five conditions is satisfied:

$$\gamma((-\infty,\, 0]) \ \text{ is bounded.} \tag{12}$$

$$\begin{cases} \sup \gamma((-\infty,\, 0]) = \infty, \\ \lim_{x\to\infty}(u_0(x) - u_0^-(x)) = 0. \end{cases} \tag{13}$$

$$\begin{cases} \sup \gamma((-\infty,\, 0]) = \infty, \\ \liminf_{x\to\infty}(u_0(x) - u_0^-(x)) > 0. \end{cases} \tag{14}$$

$$\begin{cases} \inf \gamma((-\infty,\, 0]) = -\infty, \\ \lim_{x\to-\infty}(u_0(x) - u_0^-(x)) = 0. \end{cases} \tag{15}$$

$$\begin{cases} \inf \gamma((-\infty,\, 0]) = -\infty, \\ \liminf_{x\to-\infty}(u_0(x) - u_0^-(x)) > 0. \end{cases} \tag{16}$$

In other words, the convergence of $u(y,t)$ to $u_\infty(y)$, as $t \to \infty$, holds except in either of the following two cases:

$$\begin{cases} \sup \gamma((-\infty,\, 0]) = \infty, \\ \liminf_{x\to\infty}(u_0(x) - u_0^-(x)) = 0, \\ \limsup_{x\to\infty}(u_0(x) - u_0^-(x)) > 0, \end{cases} \tag{17}$$

$$\begin{cases} \inf \gamma((-\infty,\, 0]) = -\infty, \\ \liminf_{x\to-\infty}(u_0(x) - u_0^-(x)) = 0, \\ \limsup_{x\to-\infty}(u_0(x) - u_0^-(x)) > 0. \end{cases} \tag{18}$$

Let us recall some examples from the literature.

First of all we go back to Example 2, where $H(x,p) \equiv H(p) = |p+1| - 1$ and $u_0(x) = \sin x$. The corresponding Lagrangian L is given by $L(\xi) = \delta_{[-1,\,1]}(\xi) - \xi + 1$, where $\delta_{[-1,\,1]}$ denotes the indicator function of the interval $[-1,\,1]$, i.e., $\delta_{[-1,\,1]}(\xi) = 0$ if $\xi \in [-1,\,1]$ and $= \infty$ otherwise. The minimum of L is attained at $\xi = 1$, which implies that if γ is an extremal curve on $(-\infty,\,0]$, then $\dot\gamma(s) = 1$ a.e. $s \in (-\infty,\,0]$. Therefore, we have $\inf \gamma((-\infty,\,0]) = -\infty$. Since $\{p \mid H(p) \le 0\} = [-2,\,0]$, we may check easily that

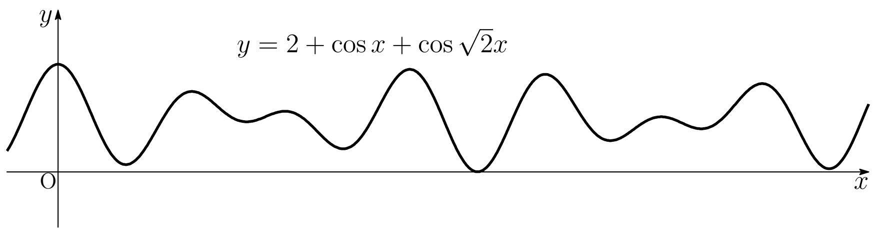
$$d_H(x,y) = \begin{cases} 0 & \text{for } x \ge y, \\ -2(x-y) & \text{for } x \le y. \end{cases}$$
$$d_-(x) \equiv 0, \qquad d_+(x) = -2x \ \text{ for } x \in \mathbf{R}.$$

It is now easy to see that

$$u_0^-(x) = \inf\{u_0(y) + d_H(x,y) \mid y \in \mathbf{R}\} \equiv -1, \qquad u_\infty(x) \equiv -1.$$

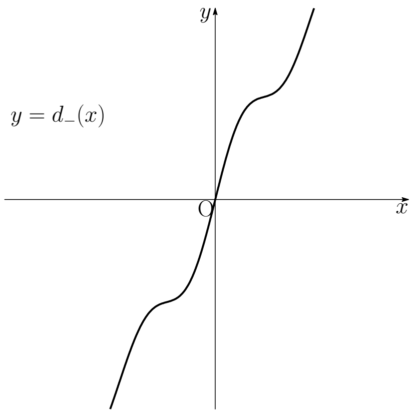
Hence condition (18) is valid in this example.

Example 4 (Lions-Souganidis [35]). Let $f(x) = 2 + \cos x + \cos\sqrt{2}x$ and $H(x,p) = |p|^2 - f(x)^2$. Note that f is quasi-periodic, $\inf_{\mathbf{R}} f = 0$ and $f(x) > 0$ for all $x \in \mathbf{R}$.



The Cauchy problem (CP) with the current H and with $u_0(x) \equiv 0$ has a unique solution $u \in C(\mathbf{R} \times [0,\infty))$ satisfying $u \ge 0$ in $\mathbf{R} \times [0,\,\infty)$. It is easy to see that

$$d_H(x,y) = \Big|\int_y^x f(t)\,\mathrm{d}\,t\Big|,$$
$$d_-(x) = \int_0^x f(t)\,\mathrm{d}\,t = 2x + \sin x + \frac{1}{\sqrt{2}}\sin\sqrt{2}x,$$
$$d_+(x) = -\,d_-(x).$$



It is not hard to check that $u_0^-(x) \equiv 0$ and $u_\infty(x) \equiv \infty$. By Proposition 7, we conclude that $\lim_{t\to\infty} u(x,t) = \infty$ for all $x \in \mathbf{R}$. On the other hand, since $\inf_{\mathbf{R}} f = 0$, we see easily that $\mathcal{S}_{H-c} = \emptyset$ for any $c < 0$, which assures that there is no asymptotic solution $v(x) - ct$, with $c < 0$. Thus, the solution u does not converge to any asymptotic solution.

By Theorem 1 (iii), we know that if H satisfies (A1)-(A3) with $\Omega = \mathbf{R}$ and the functions $H(\cdot, p)$ and $u_0 \in C(\mathbf{R})$ are $\mathbf{Z}$-periodic, then the convergence of the solution u of (CP) to an asymptotic solution holds. However, the above example shows that, in this assertion, the periodicity of $H(\cdot, p)$ cannot be replaced by the quasi-periodicity of $H(\cdot, p)$.

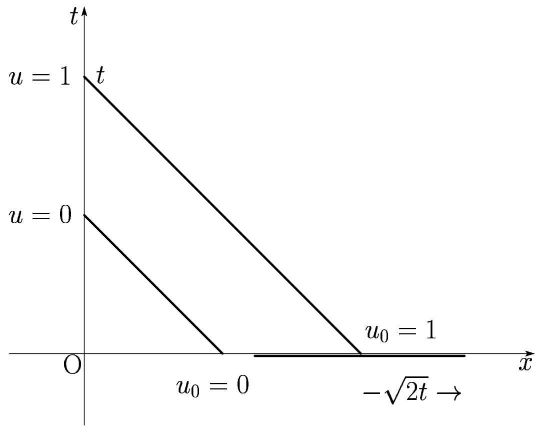
Example 5 (Barles-Souganidis [4]). Consider the Cauchy problem (CP), with $H(p) = |p|^2/2 - p$. The Lagrangian of H is given by $L(\xi) = (\xi + 1)^2/2$. Since L attains its minimum value zero at $\xi = -1$, we see that any extremal curve $\gamma \in C((-\infty, 0])$ satisfies $\dot\gamma(s) = -1$ a.e. $s \in (-\infty, 0)$. Hence we have $\gamma((-\infty, 0]) = \infty$ for any extremal curve $\gamma \in C((-\infty, 0])$. If the initial data $u_0 \in C(\mathbf{R})$ is periodic, then Theorem 1 (iii) guarantees that the solution u of (CP) converges to the asymptotic solution u_∞ as $t \to \infty$. We will see in Theorem 11 below that the same convergence assertion is valid if u_0 is almost periodic. Here we examine the asymptotic behavior of the solution u of (CP) with initial data u_0 which "oscillates slowly" at $+\infty$. We first recall the Hopf-Lax-Oleinik formula

$$u(x,t) = \inf_{y\in\mathbf{R}} \left(u_0(y) + tL\Big(\frac{x-y}{t}\Big)\right) = \inf_{y\in\mathbf{R}} \left(u_0(y) + \frac{1}{2t}|x - y + t|^2 \right).$$

In particular, we have

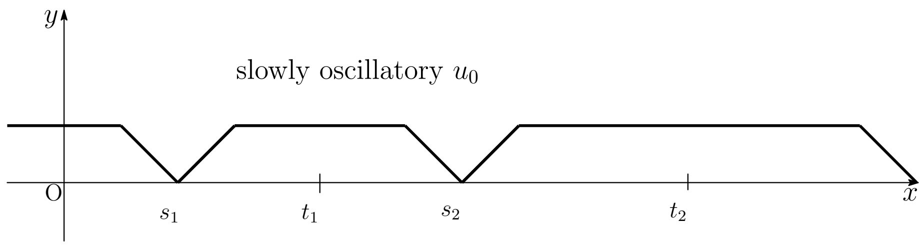
$$u(0,t) = \inf_{y\in\mathbf{R}} \left(u_0(y) + \frac{1}{2t}|y - t|^2 \right).$$

We assume that $0 \le u_0(x) \le 1$ for all $x \in \mathbf{R}$, and observe that $0 \le u(x,t) \le 1$ for all (x,t), that if $u_0(t) = 0$ for some $t > 0$, then $u(0,t) = 0$, and that if $u_0(x) = 0$ for all $x \in [t - \sqrt{2t},\ t + \sqrt{2t}]$ and for some $t > 0$, then $u(0,t) = 1$.



We choose two increasing sequences $\{s_k\}$, $\{t_k\} \subset (0, \infty)$ so that $s_k + 1 \leq t_k - \sqrt{2t_k}$ and $t_k + \sqrt{2t_k} \leq s_{k+1} - 1$ for all $k \in \mathbf{N}$. For instance, the sequences $t_k = (2k)^2$ and $s_k = t_k - 1 - \sqrt{2t_k}$, $k \in \mathbf{N}$, have the required properties. We define u_0 by

$$u_0(x) = \min\{1,\ \mathrm{dist}(x, \{s_k \mid k \in \mathbf{N}\})\}. \tag{19}$$



For this initial data u_0 the solution u of (CP) has the oscillatory property: $u(0, s_k) = 0$ and $u(0, t_k) = 1$ for all $k \in \mathbf{N}$. In particular, u does not converge any asymptotic solution. Thus, roughly speaking, if the data u_0 oscillates slowly at $+\infty$, the solution of (CP) may reflect the oscillatory behavior of u_0 and may not converge to any asymptotic solution. Considerations similar to the above show that the solution of (CP) satisfying the initial condition $u|_{t=0} = -u_0$ converges to -1 uniformly on $\mathbf{R}$ as $t \to \infty$. Finally, noting that $H(p) \leq 0$ if and only if $0 \leq p \leq 2$, we observe that d_H is given by

$$d_H(x, y) = \begin{cases} 0 & \text{if } x \leq y, \\ 2(x - y) & \text{if } x > y, \end{cases}$$

that $u_0^-(x) \equiv 0$ by (7), and that $u_\infty(x) \equiv 0$, from which we see that (17) holds.

7. Some results in $\mathbf{R}^n$

In this section we discuss some results on the asymptotic behavior of solutions of (CP) in the case where $\Omega = \mathbf{R}^n$.

We begin with a result obtained in [28] and introduce the following assumption.

(A6) There exist functions $\phi_i \in C(\mathbf{R}^n)$ and $\sigma_i \in C(\mathbf{R}^n)$, with $i = 0, 1$, such that

$$H[\phi_i] \le -\sigma_i \quad \text{in } \mathbf{R}^n,$$
$$\lim_{|x|\to\infty} \sigma_i(x) = \infty,$$
$$\lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty.$$

We next introduce the function spaces Φ_0, Ψ_0 as

$$\Phi_0 = \{f \in C(\mathbf{R}^n) \mid \inf_{\mathbf{R}^n}(f - \phi_0) > -\infty\},$$
$$\Psi_0 = \big\{g \in C(\mathbf{R}^n \times [0, \infty)) \;\big|\; \inf_{\mathbf{R}^n \times [0,T]} (g - \phi_0) > -\infty \;\text{ for all } T > 0\big\}.$$

Theorem 10. *Assume that* (A1)*,* (A2) *and* (A6) *hold. Let $u_0 \in \Phi_0$.* (i) *The additive eigenvalue problem for H has a solution $(c, v) \in \mathbf{R} \times \Phi_0$. The additive eigenvalue c is uniquely determined.* (ii) *There exists a unique solution $u \in \Psi_0$ of the Cauchy problem* (CP)*.* (iii) *Assume in addition that* (A3) *holds. Then there exists a function $u_\infty \in \Phi_0 \cap \mathcal{S}_{H-c}$ for which*

$$u(\cdot, t) + ct - u_\infty \to 0 \quad \textit{in } C(\mathbf{R}^n) \;\; \textit{as} \;\; t \to \infty.$$

Some remarks on this result are in order: the additive eigenvalue c is unique since additive eigenfunctions are sought in Φ_0 and it is given by

$$c = \inf\{a \in \mathbf{R} \mid \mathcal{S}^-_{H-a} \ne \emptyset\}.$$

In other words, for the constant c defined by the formula above, we have

$$\mathcal{S}_{H-a} \cap \Phi_0 \ne \emptyset \;\text{ if and only if }\; a = c$$

and

$$\mathcal{S}_{H-a} \ne \emptyset \;\text{ if and only if }\; a \ge c.$$

In what follows we assume that $\phi_0, \phi_1 \in \mathcal{S}^-_{H-c}$, which can be realized by modifying ϕ_i, $i = 0, 1$, appropriately (see [28] for the details). The functions ϕ_0, ϕ_1 have a kind of role of Lyapunov functions for underlying dynamical systems. Indeed, for any $v \in \mathcal{S}_{H-c} \cap \Phi_0$ and extremal curve $\gamma \in C((-\infty, 0])$, which satisfies by definition

$$\int_{-t}^0 (L[\gamma] - c)\,\mathrm{d}\,s = v(\gamma(0)) - v(\gamma(-t)) \quad \text{for all } t > 0,$$

we have the monotonicity (11), with v and ϕ_1 in place of ϕ and ψ, respectively, from which we may deduce that $\{\gamma(-t) \mid t \ge 0\}$ is bounded in $\mathbf{R}^n$ and furthermore that the Aubry set $\mathcal{A}_{H-c}$ is a nonempty compact set. Theorem 10 thus gives a sufficient condition in higher dimensions that any extremal curves γ for $v \in \mathcal{S}_{H-c} \cap \Phi_0$ satisfy (12) in Theorem 9.

For any $v \in \mathcal{S}_{H-c} \cap \Phi_0$, we have

$$v(x) = \inf\{v(y) + d_{H-c}(x,y) \mid y \in \mathcal{A}_{H-c}\} \quad \text{for all } x \in \mathbf{R}^n.$$

This representation assertion differs from Theorem 5 in that the "factor" w_∞ is missing in the formula above. In fact, the restriction $v \in \Phi_0$ suppresses the influence of v from infinity points. To be more precise, let $v \in \mathcal{S}_{H-c} \cap \Phi_0$ and let D_∞ and w_∞ denote respectively the subset of $\mathcal{S}_{H-c}$ and the function on $\mathbf{R}^n$ defined as in Theorem 5, with d_H replaced by d_{H-c}. Then $w_\infty(x) = \infty$ for all $x \in \mathbf{R}^n$. That is, we have $\sup_{\mathbf{R}^n}(v - d) = \infty$ for all $d \in D_\infty$. To check this, we argue by contradiction. Fix $d \in D_\infty$ and suppose that $\sup_{\mathbf{R}^n}(v - d) < \infty$. We choose a constant $a \in \mathbf{R}$ so that $v \le d + a$ in $\mathbf{R}^n$. Also, we choose a sequence $\{y_k\} \subset \mathbf{R}^n$ such that

$$|y_k| \to \infty \quad \text{and} \quad d(y_k) + d_{H-c}(\cdot, y_k) \to d \quad \text{in } C(\mathbf{R}^n) \quad \text{as } k \to \infty.$$

We may moreover assume that $|d(y_k) + d_{H-c}(0, y_k) - d(0)| \le 1$ for all $k \in \mathbf{N}$. Since $v \in \Phi_0$, there is a constant $C_0 > 0$ such that $v \ge \phi_0 - C_0$ in $\mathbf{R}^n$. Combining these, we observe that

$$\begin{aligned} d(0) &\ge d(y_k) + d_{H-c}(0, y_k) - 1 \ge v(y_k) - a + \phi_1(0) - \phi_1(y_k) - 1 \\ &\ge \phi_0(y_k) - \phi_1(y_k) - C_0 - a + \phi_1(0) - 1 \to \infty \quad \text{as} \quad k \to \infty. \end{aligned}$$

This is a contradiction, which proves that $w_\infty(x) = \infty$ for all $x \in \mathbf{R}^n$.

Prior to [28], Fujita-Ishii-Loreti [21] studied a similar situation for the Hamilton-Jacobi equation $u_t + \alpha x \cdot Du + H(Du) = f(x)$, where $\alpha > 0$ and H has the superlinear growth, $\lim_{|p|\to\infty} H(p)/|p| = \infty$. A result similar to the results above has been obtained by Barles-Roquejoffre [3, Theorem 4.1] which generalizes to the unbounded case the previous result due to Namah-Roquejoffre [38, Theorem 1].

A very simple example to which Theorem 10 applies is the equation $u_t + |Du|^2 = |x|$ in $\mathbf{R}^n$. Then a possible choice of (ϕ_i, σ_i) is the following.

$$\phi_1(x) = -|x|, \quad \sigma_1(x) = |x| - 1, \quad \phi_0(x) = -\frac{1}{2}|x|, \quad \sigma_0(x) = |x| - \frac{1}{4}.$$

We now discuss the generalization of Theorem 1 (iii) obtained in Ichihara-Ishii [23]. We consider the Cauchy problem (CP) for $\Omega = \mathbf{R}^n$. Assume that H satisfies all the assumptions in Theorem 1 (iii). That is, H satisfies (A1)-(A3) and $H(x,p)$ is $\mathbf{Z}^n$-periodic in x. Let c be the additive eigenvalue given by Theorem 1 (i). In other words, c is the unique constant such that there exists a function $v \in \mathcal{S}_{H-c}(\mathbf{R}^n)$ which is $\mathbf{Z}^n$-periodic. Under these hypotheses we have:

Theorem 11. *Assume that u_0 is almost periodic in $\mathbf{R}^n$.* (i) *There exists a unique solution u of* (CP) *such that $u \in \mathrm{BUC}(\mathbf{R}^n \times [0,T])$ for all $T > 0$.* (ii) *There exists an almost periodic solution u_∞ of $H[u_\infty] = c$ in $\mathbf{R}^n$ for which $u(\cdot,t) - u_\infty + ct \to 0$ in $C(\mathbf{R}^n)$ as $t \to \infty$.*

Theorem 1 deals with Hamilton-Jacobi equations on the n-dimensional torus Ω which is compact and hence can be regarded as an n-dimensional generalization of Theorem 9 in case (12). On the other hand, if we view Theorem 1 in the periodic setting with $\Omega = \mathbf{R}^n$, then it (and also Theorem 11) may deal with the n-dimensional situations of Theorem 9 where cases (12) and/or (13) holds. To see this, as in Example 5 let us consider the Hamiltonian $H(p) := |p|^2/2 - p$ and the domain $\Omega := \mathbf{R}$. Arguments similar to Example 5, if $u_0 \in C(\mathbf{R})$ is almost periodic and is not constant, then we see that

$$u_0^-(x) \equiv \inf_{\mathbf{R}} u_0 \quad \text{and} \quad u_\infty(x) \equiv \inf_{\mathbf{R}} u_0$$

and that (17) holds.

Example 5 tells us that for the function u_0 defined by (19), the solution u of (CP), with u_0 as its initial data, does not converge to any asymptotic solution, while the solution v of (CP), with $-u_0$ as its initial data, does converge to the asymptotic solution $v_\infty(x) \equiv -1$.

Motivated by this observation, we introduce the notion of semi-almost periodicities in what follows. We begin by recalling the definition of almost periodicity. A function $f \in C(\mathbf{R}^n)$ is called almost periodic if for any sequence $\{y_j\} \subset \mathbf{R}^n$ there exist a subsequence $\{z_j\}$ of $\{y_j\}$ and a function $g \in C(\mathbf{R}^n)$ such that $f(x + z_j) \to g(x)$ uniformly on $\mathbf{R}^n$ as $j \to \infty$. A function $f \in C(\mathbf{R}^n)$ is called lower (resp. upper) semi-almost periodic if for any sequence $\{y_j\} \subset \mathbf{R}^n$ and any $\varepsilon > 0$, there exist a subsequence $\{z_j\}$ of $\{y_j\}$ and a function $g \in C(\mathbf{R}^n)$ such that $f(\cdot + z_j) \to g$ in $C(\mathbf{R}^n)$ as $j \to \infty$ and $f(x + z_j) + \varepsilon > g(x)$ (resp. $f(x + z_j) - \varepsilon < g(x)$) for all $(x, j) \in \mathbf{R}^n \times \mathbf{N}$. Remark that if u_0 is the function defined by (19), then the function $-u_0$ is lower semi-almost periodic.

Theorem 12. *Assume in addition to* (A1)–(A3) *that u_0 is lower semi-almost periodic. Let c be the unique constant given by Theorem 1. Then we have:* (i) *There exists a unique solution u of* (CP) *such that $u \in \mathrm{BUC}(\mathbf{R}^n \times [0, T])$ for all $T > 0$.* (ii) *There exists a solution u_∞ of $H[u_\infty] = c$ in $\mathbf{R}^n$ for which $u(\cdot, t) - u_\infty + ct \to 0$ in $C(\mathbf{R}^n)$ as $t \to \infty$.*

See [23] for the proof of the theorem above and further generalizations.

8. General criteria.

Throughout this section we let $\Omega = \mathbf{R}^n$ and assume that H satisfies (A1), (A2), (A4) and (A5). Let $u_0 \in C(\mathbf{R}^n)$ and u be the solution of (CP) defined as in Section 6. Let u_∞ and u_0^- be the functions on $\mathbf{R}^n$ defined by (2) and (3), respectively. Following [25], we discuss three general criteria for the pointwise convergence of

$$u(x, t) \to u_\infty(x) \text{ as } t \to \infty. \tag{20}$$

As stated in Proposition 4, we have

$$\liminf_{t \to \infty} u(x, t) = u_\infty(x) \quad \text{for all } x \in \mathbf{R}^n. \tag{21}$$

We remark also that the pointwise convergence (20) for all $x \in \mathbf{R}^n$ implies the locally uniform convergence

$$u(\cdot, t) \to u_\infty \quad \text{in } C(\mathbf{R}^n) \;\text{ as } t \to \infty. \tag{22}$$

We fix any $z \in \mathbf{R}^n$ and consider the pointwise convergence

$$u(z, t) \to u_\infty(z) \quad \text{ as } t \to \infty. \tag{23}$$

We fix any $\gamma \in \mathcal{E}_z(u_\infty)$ and introduce the first criterion

(C1) $\lim_{t\to\infty} (u_0 - u_\infty)(\gamma(-t)) = 0.$

An interesting obervation in [25] is that $\lim_{t\to\infty}(u_\infty - u_0^-)(\gamma(-t)) = 0$. Hence condition (C1) is equivalent to the condition

$$\lim_{t\to\infty} (u_0 - u_0^-)(\gamma(-t)) = 0.$$

Theorem 13. *Assume that* (C1) *holds. Then the convergence* (23) *holds.*

Proof. By the definition of extremal curves, we see that

$$\begin{aligned} u(z, t) &\le \int_{-t}^{0} L[\gamma] \,\mathrm{d}\, s + u_0(\gamma(-t)) \\ &= u_\infty(z) - u_\infty(\gamma(-t)) + u_0(\gamma(-t)) \qquad \text{for all } \; t > 0. \end{aligned}$$

This together with (C1) and (21) yields

$$\limsup_{t\to\infty} u(z,t) \le u_\infty(z) + \lim_{t\to\infty}(u_0 - u_\infty)(\gamma(-t)) = u_\infty(z) = \liminf_{t\to\infty} u(z,t),$$

which shows (23). □

The theorem above can be applied to the cases (13) and (15) of Theorem 9. Indeed, in the case when (13) is satisfied, we have

$$\lim_{t\to\infty} \gamma(-t) = \infty,$$

and hence $\lim_{t\to\infty}(u_0 - u_0^-)(\gamma(-t)) = 0$. That is, (C1) holds. Similarly, in the case when (15) is satisfied, we see that (C1) holds.

We consider the following situation similar to that of [3, Theorem 4.2]. That is, we assume that there are a constant $\delta > 0$ and a function $\psi \in C(\mathbf{R}^n)$ such that $H[\psi] \le -\delta$ in $\mathbf{R}^n$. Moreover let $\phi_0 \in \mathcal{S}_H$ and assume that

$$\lim_{|x|\to\infty} (u_0 - \phi_0)(x) = 0. \tag{24}$$

The main conclusion of [3, Theorem 4.2] is then that (22) holds and $u_\infty = \phi_0$. This conclusion is valid in the current situation.

We show just that (C1) holds for all $z \in \mathbf{R}^n$ and hence (22) holds by Theorem 13. Note that the existence of $\psi \in \mathcal{S}_{H+\delta}$, with $\delta > 0$, implies that $\mathcal{A}_H = \emptyset$. Therefore, $\lim_{t\to\infty} |\gamma(-t)| = \infty$ for any $\gamma \in \mathcal{E}(\phi)$ and any $\phi \in \mathcal{S}_H$. There is a constant $C_0 > 0$ such that $\sup_{\mathbf{R}^n} |u_0 - \phi_0| \le C_0$. From this we see that $\phi_0 + C_0 \ge u_0 \ge u_0^- \ge \phi_0 - C_0$ in $\mathbf{R}^n$. Hence, $u_0^- \le u_\infty \le \phi_0 + C_0$ in $\mathbf{R}^n$. Consequently, we get $\sup_{\mathbf{R}^n} |u_0^- - \phi_0| \le C_0$ and $\sup_{\mathbf{R}^n} |u_\infty - \phi_0| \le C_0$.

Now let $\gamma \in \mathcal{E}(u_\infty)$. By (10) and (9), we obtain

$$\begin{aligned}\phi(\gamma(0)) - \psi(\gamma(-t)) \le & \int_{-t}^{0} (L[\gamma] - \delta)\, \mathrm{d}\, s \\ & = u_\infty(\gamma(0)) - u_\infty(\gamma(-t)) - \delta t \quad \text{for all } t \ge 0.\end{aligned}$$

This shows that $\lim_{t\to\infty}(\psi - u_\infty)(\gamma(-t)) = \infty$. Moreover, since $\sup_{\mathbf{R}^n} |\phi_0 - u_\infty| < \infty$, we see that $\lim_{t\to\infty}(\psi - \phi_0)(\gamma(-t)) = \infty$.

Next we fix any $\varepsilon > 0$ and choose $A_\varepsilon > 0$ so large that

$$\psi_\varepsilon(x) := \min\{\psi(x) - A_\varepsilon,\, \phi_0(x) - \varepsilon\} \le u_0(x) \quad \text{for all } x \in \mathbf{R}^n.$$

This is possible because $\lim_{|x|\to\infty}(u_0 - \phi_0)(x) = 0$. Observe that $\psi_\varepsilon \in \mathcal{S}_H^-$ and hence that $\psi_\varepsilon \le u_0^-$ in $\mathbf{R}^n$. Then, recalling that $\lim_{t\to\infty}(\psi - \phi_0)(\gamma(-t)) = \infty$, we observe that $\psi_\varepsilon(\gamma(-t)) = \phi_0(\gamma(-t)) - \varepsilon$ if t is sufficiently large and that

$$\begin{aligned}&\limsup_{t\to\infty}(u_0 - u_0^-)(\gamma(-t)) \le \limsup_{t\to\infty}(u_0 - \psi_\varepsilon)(\gamma(-t)) \\ &= \lim_{t\to\infty}(u_0 - \phi_0)(\gamma(-t)) + \varepsilon = \varepsilon.\end{aligned}$$

Since $u_0 \ge u_0^-$ in $\mathbf{R}^n$ and $\varepsilon > 0$ is arbitrary, we now conclude that $\lim_{t\to\infty}(u_0 - u_0^-)(\gamma(-t)) = 0$, that is, (C1) holds.

Next we introduce the second criterion.

(C2) For each $\varepsilon > 0$ there exists a $\tau > 0$ such that for any $t > 0$ and for some $\eta \in \mathrm{AC}([-t,\, 0])$,

$$\eta(-t) = \eta(0) = \gamma(-\tau) \quad \text{and} \quad \int_{-t}^{0} L[\eta]\, \mathrm{d}\, s < \varepsilon.$$

Theorem 14. *Under the assumption* (C2)*, the convergence* (23) *holds.*

Proof. Fix any $\varepsilon > 0$ and let $\tau > 0$ be the constant from assumption (C2). Set $y = \gamma(-\tau)$ and choose a $\sigma > 0$ in view of (21) so that $u(y, \sigma) < u_\infty(y) + \varepsilon$. Fix any $t > 0$. By (C2), we may choose an $\eta \in \mathrm{AC}([-t,\, 0])$ such that $\eta(-t) = \eta(0) = y$ and

$$\int_{-t}^{0} L[\eta]\, \mathrm{d}\, s < \varepsilon.$$

Now, using the dynamic programming principle, we compute that

$$\begin{aligned} u(z,\,\tau+\sigma+t) &\leq \int_{-\tau}^{0} L[\gamma]\,\mathrm{d}\,s + u(\gamma(-\tau),\,t+\sigma) \\ &\leq u_\infty(z) - u_\infty(y) + \int_{-t}^{0} L[\eta]\,\mathrm{d}\,s + u(\eta(-t),\,\sigma) \\ &< u_\infty(z) - u_\infty(y) + \varepsilon + u(y,\,\sigma) \\ &< u_\infty(z) - u_\infty(y) + u_\infty(y) + 2\varepsilon = u_\infty(z) + 2\varepsilon. \end{aligned}$$

Consequently we obtain

$$\limsup_{t\to\infty} u(z,\,t) \leq u_\infty(z) = \liminf_{t\to\infty} u(z,\,t),$$

which concludes the proof. □

Motivated by the main results in [38, 22], we formulate a proposition as follows.

Theorem 15. *Assume in addition to* (A1) (A2) *and* (A4) *that there are two functions* $\phi_0,\,\phi_1 \in \mathcal{S}_H^-$ *such that*

$$\lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty \quad \textit{and} \quad \inf_{\mathbf{R}^n} (u_0 - \phi_0) > -\infty. \tag{25}$$

Assume moreover that

$$\mathcal{A}_H \neq \emptyset \quad \textit{and} \quad L(x,0) = 0 \;\; \textit{for all } x \in \mathcal{A}_H. \tag{26}$$

Then the convergence (22) *holds.*

In the above theorem we do not need to assume (A5). Indeed, (A5) holds as a consequence of the hypotheses of Theorem 15. We remark that any point $x \in \mathcal{A}_H$ which satisfies the second condition of (26) is said to be an *equilibrium.*

Proof. We may assume by adding a constant to ϕ_0 that $u_0 \geq \phi_0$ in $\mathbf{R}^n$. We then have $\phi_0 \leq u_0^- \leq u_0$ in $\mathbf{R}^n$. Fix a $y \in \mathcal{A}_H$ and observe that $u_\infty \leq u_0^-(y) + d_H(\cdot, y)$ in $\mathbf{R}^n$. Hence, (A5) is valid.

Fix any $\gamma \in \mathcal{E}_z(u_\infty)$, with $z \in \mathbf{R}^n$, and recall the monotonicity (11), with ϕ and ψ replaced by u_∞ and ϕ_1, resepectively. Since $u_\infty \geq \phi_0$ in $\mathbf{R}^n$, this monotonicity and (25) enssure that $\gamma(-t) \in B(0,R)$ for all $t \geq 0$ and some $R > 0$. This together with (1) implies that $\mathcal{A}_H \neq \emptyset$ and $\mathrm{dist}(\gamma(-t), \mathcal{A}_H) \to 0$ as $t \to \infty$. Fix any $t > 0$ and choose a point $y \in \mathcal{A}_H$ so that $|\gamma(-t) - y| = \mathrm{dist}(\gamma(-t), \mathcal{A}_H)$. (Recall that $\mathcal{A}_H$ is a closed subset of $\mathbf{R}^n$.) There are constants $\delta_R > 0$ and $C_R > 0$ (see e.g. [28]) so that $L(x,\xi) \leq C_R$ for all $(x,\xi) \in B(0,R) \times B(0,\delta_R)$. Let $r > 0$, set $\rho = \mathrm{dist}(\gamma(-t), \mathcal{A}_H)$ and $\xi = \delta_R(y - \gamma(-t))/\rho$, and define the curve $\eta \in \mathrm{AC}([-r,\,0])$ by

$$\eta(s) = \begin{cases} \gamma(-t) - s\xi & \text{for } s \in [-\rho/\delta_R,\,0], \\ y & \text{for } s \in [-r+\rho/\delta_R,\,-\rho/\delta_R], \\ \gamma(-t) + (s+r)\xi & \text{for } s \in [-r,\,-r+\rho/\delta_R] \end{cases}$$

if $\delta_R r > 2\rho$ and $\eta(s) = \gamma(-t)$ if $\delta_R r \le 2\rho$. It is easy to see that

$$\int_{-r}^{0} L[\eta]\,\mathrm{d}\,s \le \frac{2C_R}{\delta_R}\rho = \frac{2C_R}{\delta_R}\,\mathrm{dist}(\gamma(-t), \mathcal{A}_H).$$

It is now obvious that (C2) holds for all $\gamma \in \mathcal{E}(u_\infty)$. Thus, applying Theorem 14, we conclude that the convergence (22) holds. □

Under the hypotheses of the theorem above, we have

$$u_\infty(x) = \inf\{u_0^-(y) + d_H(x,y) \mid y \in \mathcal{A}_H\} \quad \text{for all } x \in \mathbf{R}^n.$$

Here the term w_∞ of Theorem 5 is missing, which is due to the assumption that $\inf_{\mathbf{R}^n}(u_0 - \phi_0) > -\infty$.

Condition (C2) covers another situation, where "nearly optimal" curves in the formula (5) for the solution of (CP) exhibit a "switch-back" motion for large t. We discuss just a simple example and refer to [25] for further generalities.

Let $n = 1$ and consider the case where the Hamiltonian H is given by $H(x,p) := |p| - e^{-|x|}$ and u_0 is given by $u_0(x) = \min\{|x| - 2, 0\}$. It is clear that (A1), (A2), and (A4) are satisfied. It is easy to see that $d_H(x,y) = \left|\int_y^x e^{-s}\,\mathrm{d}\,s\right|$ for all $x, y \in \mathbf{R}$. By the formula

$$u_0^-(x) = \inf\{u_0(y) + d_H(x,y) \mid y \in \mathbf{R}\},$$

we see that $u_0^-(x) = -e^{-|x|} - 1$ for $x \in \mathbf{R}$. We define the functions $d_\pm \in \mathcal{S}_H$ as before by $d_\pm(x) = \lim_{y\to\pm\infty}(d_H(x,y) - d_H(0,y))$, and observe that $d_\pm(x) = e^{\mp x} - 1$ for $x \in \mathbf{R}$ and by Theorem 5 that $u_\infty(x) = e^{-|x|} - 1$ for $x \in \mathbf{R}$. We know now that (A5) holds. Note that the Lagrangian L is given by $L(x,\xi) = \delta_{[-1,1]}(\xi) + e^{-|x|}$.

Given $z \in \mathbf{R}$, we define the curve $\gamma \in C((-\infty,0])$ by $\gamma(s) = z - \mathrm{sgn}\,(z)\,s$, where $\mathrm{sgn}\,(z) = 1$ for $z \ge 0$ and $= -1$ for $z < 0$. Then, it is easy to see that $\gamma \in \mathcal{E}_z(u_\infty)$ and $|\gamma(-t)| \to \infty$ as $t \to \infty$. Fix any $\varepsilon > 0$ and choose a $\tau > 0$ so that

$$2\int_{|\gamma(-\tau)|}^{\infty} e^{-s}\,\mathrm{d}\,s < \varepsilon.$$

We define $\eta \in \mathrm{AC}([-t,\,0])$ for any fixed $t > 0$ by

$$\eta(s) := \begin{cases} \gamma(-\tau) - \mathrm{sgn}\,(z)\,s & \text{for } -\dfrac{t}{2} \le s \le 0, \\ \gamma(-\tau) + \mathrm{sgn}\,(z)\,(s+t) & \text{for } -t \le s \le -\dfrac{t}{2}, \end{cases}$$

and observe that $\eta(0) = \eta(-t) = \gamma(-\tau)$ and

$$\int_{-t}^{0} L[\eta]\,\mathrm{d}\,s < 2\int_{|\gamma(-\tau)|}^{\infty} e^{-s}\,\mathrm{d}\,s < \varepsilon,$$

so that condition (C2) is valid for the given γ. Now, Theorem 14 guarantees that the convergence (22) holds.

We remark that the curve $\eta \in \mathrm{AC}([-t,\,0])$ built here has a switch-back motion in which the point $\eta(-s)$, with $s \in [0,\,t]$, moves toward ∞ or $-\infty$ with a unit speed up to the time $t/2$ and then moves back to the starting point. It is also worth mentioning that condition (C1) does not hold in this case. Indeed, we have $\lim_{t\to\infty}(u_0 - u_\infty)(\gamma(-t)) = 1 > 0$.

Similar switch-back motions appear in "nearly optimal" curves in (5) in the cases (14) and (16) of Theorem 9.

The third criterion is the following.

(C3) For any $\varepsilon > 0$, there exists a $\tau > 0$ and for each $t \ge \tau$, a $\sigma(t) \in [0,\,\tau]$ such that

$$u_\infty(\gamma(-t)) + \varepsilon > u(\gamma(-t), \sigma(t)).$$

Note that the above inequality is equivalent to the condition that there is an $\eta \in \mathrm{AC}([-\sigma(t),\,0])$ such that $\eta(0) = \gamma(-t)$ and

$$u_\infty(\gamma(-t)) + \varepsilon > \int_{-\sigma(t)}^{0} L[\eta]\,\mathrm{d}\,s + u_0(\eta(-\sigma(t))).$$

In our next theorem, condition (C3) is used together with one of the conditions $(\mathrm{A7})_\pm$ on H, which are certain strict convexity requirements on H. We set $Q := \{(x,p) \in \mathbf{R}^{2n} \mid H(x,p) = 0\}$ and

$$S := \{(x,\xi) \in \mathbf{R}^{2n} \mid (x,p) \in Q, \quad \xi \in D_2^- H(x,p) \ \text{ for some } p \in \mathbf{R}^n\},$$

where $D_2^- H(x,p)$ stands for the subdifferential of H with respect to the p variable.

$(\mathbf{A7})_+$ There exists a modulus ω satisfying $\omega(r) > 0$ for $r > 0$ such that for all $(x,p) \in Q$, $\xi \in D_2^- H(x,p)$ and $q \in \mathbf{R}^n$.

$$H(x,p+q) \ge \xi \cdot q + \omega((\xi \cdot q)_+).$$

$(\mathbf{A7})_-$ There exists a modulus ω satisfying $\omega(r) > 0$ for $r > 0$ such that for all $(x,p) \in Q$, $\xi \in D_2^- H(x,p)$ and $q \in \mathbf{R}^n$,

$$H(x,p+q) \ge \xi \cdot q + \omega((\xi \cdot q)_-).$$

Here $r_\pm := \max\{\pm r, 0\}$ for $r \in \mathbf{R}$.

Roughly speaking, $(\mathrm{A7})_+$ (resp., $(\mathrm{A7})_-$) means that $H(x,\,\cdot\,)$ is strictly convex "upward" (resp., "downward") at the zero-level set of H uniformly in $x \in \mathbf{R}^n$. We note that condition $(\mathrm{A7})_+$ has already been used in [4] to replace the strict convexity of $H(x,\cdot)$ in order to get the convergence (22). Condition $(\mathrm{A7})_-$ has been introduced in [24, 25].

Theorem 16. *Assume that* (C3) *and either* $(\mathrm{A7})_+$ *or* $(\mathrm{A7})_-$ *are satisfied. Then the convergence* (23) *holds.*

We refer to [25] for a proof of the theorem above. A variant of Theorem 15 is given by the next proposition, which can be also regarded as a version of Theorem 10 (iii) and where (A5) is not assumed to hold.

Theorem 17. *Assume that* (A1)*,* (A2)*,* (A4) *and either of* $(A7)_+$ *or* $(A7)_-$ *hold and that there are two functions* ϕ_0, $\phi_1 \in \mathcal{S}_H^-$ *such that*

$$\lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty \quad \textit{and} \quad \inf_{\mathbf{R}^n} (u_0 - \phi_0) > -\infty. \tag{27}$$

Assume moreover that $\mathcal{A}_H \neq \emptyset$*. Then the convergence* (22) *holds.*

Proof. As in the proof of Theorem 15, we see that (A5) holds. It remains to show that (C3) holds for any $\gamma \in \mathcal{E}(u_\infty)$. Fix $\gamma \in \mathcal{E}_z$, with $z \in \mathbf{R}^n$, and observe as in Theorem 15 that there is a constant $R > 0$ such that $\gamma(s) \in B(0, R)$ for all $s \leq 0$. Then we fix any $\varepsilon > 0$ and choose, in view of (2), a $\tau_y > 0$ for each $y \in B(0, R)$ so that $u_\infty(y) + \varepsilon > u(y, \tau_y)$. Next, using the compactness of $B(0, R)$ and the continuity of u_∞ and u, we deduce that there exists a $\tau > 0$ such that $u_\infty(x) + \varepsilon > u(x, \tau_x)$ for any $x \in B(0, R)$ and some $\tau_x \in [0, \tau]$. That is, (C3) is valid for any $\gamma \in \mathcal{E}(u_\infty)$. □

The assertion of Theorems 1, with (A3) replaced either by $(A7)_+$ or $(A7)_-$, is valid, which can be proved similarly to the above proof by applying Theorem 16. This remark applies to Theorems 11 and 12 as well.

References

[1] Barles, G, *Solutions de viscosité des équations de Hamilton-Jacobi*, Mathématiques & Applications (Berlin), 17, Springer-Verlag, Paris, 1994.

[2] Bardi, M., Capuzzo-Dolcetta, I., *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia,* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.

[3] Barles, G., Roquejoffre, J.-M., Ergodic type problems and large time behaviour of unbounded solutions of Hamilton-Jacobi equations. *Comm. Partial Differential Equations* **31** (2006), no. 8, 1209–1225.

[4] Barles, G., Souganidis, P. E., On the large time behavior of solutions of Hamilton-Jacobi equations. *SIAM J. Math. Anal.* **31** (2000), no. 4, 925–939.

[5] Barles, G., Souganidis, P. E., Space-time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations. *SIAM J. Math. Anal.* **32** (2001), no. 6, 1311–1323.

[6] Barron, E. N., Jensen, R., Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians. *Comm. Partial Differential Equations* **15** (1990), no. 12, 1713–1742.

[7] Crandall, M. G., Lions, P.-L., Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.* **277** (1983), no. 1, 1–42.

[8] Crandall, M. G., Evans, L. C., Lions, P.-L. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.* **282** (1984), no. 2, 487–502.

[9] Crandall, M. G., Ishii, H., Lions, P.-L., User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67.

[10] Davini, A., Siconolfi, A., A generalized dynamical approach to the large time behavior of solutions of Hamilton-Jacobi equations. *SIAM J. Math. Anal.* **38** (2006) no. 2, 478–502.

[11] E, Weinan, Aubry-Mather theory and periodic solutions of the forced Burgers equation. *Comm. Pure Appl. Math.* **52** (1999) no. 7, 811–828.

[12] Evans, L. C., The perturbed test function method for viscosity solutions of nonlinear PDE. *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989), no. 3-4, 359–375.

[13] Evans, L. C., Periodic homogenisation of certain fully nonlinear partial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* **120** (1992), no. 3-4, 245–265.

[14] Evans, L. C., A survey of partial differential equations methods in weak KAM theory. *Comm. Pure Appl. Math.* 57 (2004) no. 4, 445–480.

[15] Fathi, A., Théorème KAM faible et théorie de Mather pour les systèmes lagrangiens. *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), no. 9, 1043–1046.

[16] Fathi, A., Sur la convergence du semi-groupe de Lax-Oleinik. *C. R. Acad. Sci. Paris Sér. I Math.* **327** (1998), no. 3, 267–270.

[17] Fathi, A., *Weak KAM theorem in Lagrangian dynamics*, to appear.

[18] Fathi, A., Siconolfi, A., Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation. *Invent. Math.* **155** (2004), no. 2, 363–388.

[19] Fathi, A., Siconolfi, A., PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians. *Calc. Var. Partial Differential Equations* **22** (2005), no. 2, 185–228.

[20] Fleming, W. H., Soner, H. M., *Controlled Markov Processes and Viscosity Solutions. Second edition.* Stochastic Modelling and Applied Probability, 25. Springer, New York, 2006.

[21] Fujita, Y., Ishii, H., Loreti, P., Asymptotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator. *Comm. Partial Differential Equations* **31** (2006), no. 6, 827–848.

[22] Fujita, Y., Ishii, H., Loreti, P., Asymptotic solutions of Hamilton-Jacobi equations in Euclidean n space. *Indiana Univ. Math. J.* **55** (2006), no. 5, 1671–1700.

[23] Ichihara, N., Ishii, H., Asymptotic solutions of Hamilton-Jacobi equations with semi-periodic Hamiltonians, to appear in *Comm. in Partial Differential Equations.*

[24] Ichihara, N., Ishii, H., The large-time behavior of solutions of Hamilton-Jacobi equations on the real line, to appear in *Methods Appl. Anal.*

[25] Ichihara, N., Ishii, H., Long-time behavior of solutions of Hamilton-Jacobi equations with convex and coercive Hamiltonians, preprint.

[26] Ishii, H., Almost periodic homogenization of Hamilton-Jacobi equations. *International Conference on Differential Equations*, Vol. 1, 2 (Berlin, 1999), pp. 600–605, World Sci. Publ., River Edge, NJ, 2000.

[27] Ishii, H., A generalization of a theorem of Barron and Jensen and a comparison theorem for lower semicontinuous viscosity solutions. *Proc. Roy. Soc. Edinburgh Sect. A* **131** (2001), no. 1, 137–154.

[28] Ishii, H., Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008) no. 2, 231–266.

[29] Ishii, H., Mitake, H., Representation formulas for solutions of Hamilton-Jacobi equations with convex Hamiltonians. *Indiana Univ. Math. J.* **56** (2007), no. 5, 2159–2184.

[30] Kružkov, S. N., Generalized solutions of nonlinear equations of the first order with several independent variables. II *Math. USSR Sbornik* **1** (1967) 93–116.

[31] Lions, P.-L., *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Mathematics, Vol. 69, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[32] Lions, P.-L., Papanicolaou, G., Varadhan, S. R. S., Homogenization of Hamilton-Jacobi equations, unpublished preprint.

[33] Lions, P.-L., Souganidis, P. E. Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), no. 5, 667–677.

[34] Lions, P.-L., Souganidis, P. E., Homogenization of "viscous" Hamilton-Jacobi equations in stationary ergodic media. *Comm. Partial Differential Equations* **30** (2005), no. 1-3, 335–375.

[35] Lions, P.-L., Souganidis, P. E., Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting. *Comm. Pure Appl. Math.* **56** (2003), no. 10, 1501–1524.

[36] Mitake, H., Asymptotic solutions of Hamilton-Jacobi equations with state constraints. to appear in *Appl. Math. Optim.*

[37] Mitake, H., The large-time behavior of solutions of the Cauchy-Dirichlet problem of Hamilton-Jacobi equations. to appear in *NoDEA Nonlinear Differential Equations Appl.*

[38] Namah, G., Roquejoffre, J.-M, Remarks on the long time behaviour of the solutions of Hamilton-Jacobi equations. *Commun. Partial Differential Equations* **24** (1999), no. 5–6, 883–893.

[39] Rezakhanlou, F., Tarver, J. E., Homogenization for stochastic Hamilton-Jacobi equations. *Arch. Ration. Mech. Anal.* **151** (2000), no. 4, 277–309.

[40] Roquejoffre, J.-M., Convergence to steady states or periodic solutions in a class of Hamilton-Jacobi equations, *J. Math. Pures Appl.* (9) **80** (2001), no. 1, 85–104.

[41] Souganidis, P. E., Stochastic homogenization of Hamilton-Jacobi equations and some applications. *Asymptot. Anal.* **20** (1999), no. 1, 1–11.

Department of Mathematics, Faculty of Education and Integrated Arts and Sciences,
Waseda University, Shinjuku, Tokyo, 169-8050 Japan
E-mail: hitoshi.ishii@waseda.jp