A CLASS OF INTEGRAL EQUATIONS AND APPROXIMATION OF p-LAPLACE EQUATIONS

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $1 < p < \infty$ and $\sigma < p$. We study the nonlinear singular integral equation

$$M[u](x) = f_0(x) \quad \text{in } \Omega$$

with the boundary condition $u = g_0$ on $\partial\Omega$, where $f_0 \in C(\overline{\Omega})$ and $g_0 \in C(\partial\Omega)$ are given functions and M is the singular integral operator given by

$$M[u](x) = \text{p.v.} \int_{B(0,\rho(x))} \frac{p-\sigma}{|z|^{n+\sigma}} |u(x+z) - u(x)|^{p-2}(u(x+z) - u(x))\,\mathrm{d}z,$$

with some choice of $\rho \in C(\overline{\Omega})$ having the property, $0 < \rho(x) \leq \operatorname{dist}(x, \partial\Omega)$. We establish the solvability (well-posedness) of this Dirichlet problem and the convergence uniform on $\overline{\Omega}$, as $\sigma \to p$, of the solution u_σ of the Dirichlet problem to the solution u of the Dirichlet problem for the p-Laplace equation $\nu\Delta_p u = f_0$ in Ω with the Dirichlet condition $u = g_0$ on $\partial\Omega$, where the factor ν is a positive constant (see (7.2)).

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ and $\rho \in C(\overline{\Omega})$ a given function satisfying

$$\lambda_0 \operatorname{dist}(x, \partial\Omega) \leq \rho(x) \leq \operatorname{dist}(x, \partial\Omega),$$

where $0 < \lambda_0 \leq 1$ is a fixed constant.

Let $p > 1$ and $\sigma < p$. We introduce the nonlinear singular integral operator $M = M_\sigma$ given formally by

$$M[\phi](x) = \text{p.v.} \int_{B(0,\rho(x))} G(\phi(x+z) - \phi(x))K(z)\,\mathrm{d}z$$

for bounded measurable functions ϕ on Ω, where G is the function on $\mathbb{R}$ given by $G(x) = |x|^{p-2}x$ and the kernel $K = K_\sigma$ is given by

$$K(z) = \frac{\mu}{|z|^{n+\sigma}}, \qquad \text{with } \mu = \mu_\sigma := p - \sigma.$$

The symbol "p.v." stands for the principal value of the integral. That is,

$$M[\phi](x) = \lim_{r \to 0+} \int_{r<|z|\leq\rho(x)} G(\phi(x+z) - \phi(x))K(z)\,\mathrm{d}z \quad \text{if the limit exists.}$$

The constant σ will be often regarded as a parameter to be sent to p.

We deal with the integral equation

$$M[u](x) = f_0(x) \quad \text{in } \Omega, \tag{1.1}$$

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where f_0 is a given continuous, real-valued function on Ω and u represents the unknown function on Ω. Associated with (1.1) is the boundary condition

$$u(x) = g_0(x) \quad \text{for } x \in \partial\Omega, \tag{1.2}$$

where g_0 is a given continuous function on $\partial\Omega$.

Our primary purpose is to investigate the solvability of the Dirichlet problem (1.1) and (1.2), and the secondary interest here is to study the asymptotic behavior of solutions u_σ of (1.1)–(1.2) as $\sigma \to p$.

In the next section, we establish some basic estimates of the singular integral operator M. In view of application to the asymptotic analysis as $\sigma \to p$, it is important to obtain estimates of the operators $M = M_\sigma$ which are independent of σ in a range close to p.

The notion of solution of (1.1) in this paper is an adaptation of viscosity solutions of differential equations and it is defined as follows. We begin by introducing the spaces $\mathcal{T}_p(\Omega)$ of test functions. We set $\mathcal{T}_p(\Omega) = C^2(\Omega)$ for $p \ge 2$. For $1 < p < 2$ let $\mathcal{T}_p(\Omega)$ denote the space of functions $\phi \in C^2(\Omega)$ having the property: for each compact $R \subset \Omega$ there exist a neighborhood $V \subset \Omega$ of R and constants $\beta > 1/(p-1)$ and $A > 0$ such that for any $y \in R$, if $D\phi$ vanishes at y, then

$$|\phi(x) - \phi(y)| \le A|x-y|^{\beta+1} \quad \text{for all } x \in V.$$

We call any bounded function u in Ω a (viscosity) *subsolution* of (1.1) if we have

$$M^+[u^*](x) \ge f_0(x)$$

whenever $(x, \phi) \in \Omega \times \mathcal{T}_p(\Omega)$ and $u^* - \phi$ has a maximum at x. Here the operator M^+ is defined by

$$M^+[v](x) = \limsup_{\delta \to 0+} \int_{\delta < |z| < \rho(x)} G(v(x+z) - v(x))K(z)\,\mathrm{d}z$$

and u^* denotes the upper semicontinuous envelope of u. Similarly, we call any bounded function u a (viscosity) *supersolution* of (1.1) if we have

$$M^-[u_*](x) \le f_0(x)$$

whenever $(x, \phi) \in \Omega \times \mathcal{T}_p(\Omega)$ and $u_* - \phi$ has a minimum at x, where the operator M^- is defined by

$$M^-[v](x) = \liminf_{\delta \to 0+} \int_{\delta < |z| < \rho(x)} G(v(x+z) - v(x))K(z)\,\mathrm{d}z$$

and u_* denotes the lower semicontinuous envelope of u. Finally, we call any bounded function u in Ω a (viscosity) *solution* of (1.1) if it is both a subsolution and a supersolution of (1.1).

In Section 3 we prove the stability of solutions of (1.1) under certain limiting processes and under taking the pointwise supremum or infimum. Also, in Section 3 the Perron method is established for the integral equation (1.1). In Section 4 we establish a comparison theorem between sub and supersolutions of (1.1). In Section 5, we build sub and supersolutions which attain the boundary condition (1.2) and prove the existence of a continuous solution of (1.1)–(1.2).

In Section 6, we recall basic results concerning weak solutions in $W^{1,p}_{\mathrm{loc}}(\Omega)$ of the inhomogeneous p-Laplace equation

$$\Delta_p u(x) = f_0(x) \quad \text{in } \Omega, \tag{1.3}$$

and formulate comparison results for (1.3), where we mostly follow the argument of [12].

In Section 7 we are concerned with the asymptotic behavior of solutions u_σ of (1.1)–(1.2), and we show that under appropriate hypotheses, u_σ converges uniformly to the solution u of the Dirichlet problem

$$\nu\Delta_p u(x) = f_0(x) \quad \text{in } \Omega,$$

where ν is an appropriate positive constant (see (7.2) for the precise value of ν), with the Dirichlet condition (1.2).

In section 8, we give a few comments on possible generalizations or variants of the results presented in the preceding sections.

Recently, while this paper was in preparation, Andreu-Mazón-Rossi-Toledo [1, 2] have studied problems similar to ours. In [1] they study the evolution equation

$$u_t(x,t) = M_D[u(\cdot,t)](x) \quad \text{in } \Omega \times (0, T). \tag{1.4}$$

Here the unknown function u is defined on $\Omega \times (0, T)$, $0 < T < \infty$, u_t denotes the derivative of u with respect to the time variable t and the operator M_D is given by

$$\begin{aligned} M_D[\phi](x) = &\int_\Omega G(\phi(y) - \phi(x))J(x-y)\,\mathrm{d}y \\ &+ \int_{\Omega_J\setminus\Omega} G(g_0(y) - \phi(x))J(x-y)\,\mathrm{d}y, \end{aligned} \tag{1.5}$$

where the function J is a nonnegative continuous radial function on $\mathbb{R}^n$ with compact support, $\Omega_J := \Omega + \operatorname{supp} J$ and g_0 is a given function on $\mathbb{R}^n$ belonging to $L^p(\mathbb{R}^n)$. In [1] they have established, among others, the solvability in the space

$$C([0, T], L^1(\Omega)) \cap W^{1,1}((0, T), L^1(\Omega)),$$

of the Cauchy problem for (1.4) with initial data $u_0 \in L^p(\Omega)$ and, under some additional assumptions on J and g_0, the convergence in the space $C([0, T], L^p(\Omega))$, as $\varepsilon \to 0+$, of the solution u_ε of the Cauchy problem for (1.4), with the kernel function $J(x)$ replaced by $J_{p,\varepsilon}(x) := C_p J(x/\varepsilon)/\varepsilon^{n+p}$ with $C_p := (1/2)\int J(x)|x_n|^p\,\mathrm{d}x$, to the solution u of the initial-boundary value problem for

$$u_t(x,t) = \Delta_p u(x,t) \quad \text{for } (x,t) \in \Omega \times (0,\infty) \tag{1.6}$$

with the Dirichlet boundary condition $u = g_0$ on $\partial\Omega \times (0, T)$ and the initial data $u(\cdot, 0) = u_0$. In [2], they have treated the evolution equation similar to (1.4), but with M_D replaced by the operator M_N defined by

$$M_N[\phi](x) = \int_\Omega G(\phi(y) - \phi(x))J(x-y)\,\mathrm{d}y,$$

and have obtained solvability and convergence results similar to the above, where the limit problem is the initial-boundary problem for (1.6) with the Neumann boundary condition $\partial u/\partial n = 0$, with n denoting the outer unit normal vectors at points on $\partial\Omega$.

In [1] they treat the evolution problem while we study here the stationary problem, and the operator M_D in [1] is different from our M. Beyond these apparent differences, there are two important differences between [1] and ours. One is of the qualitative property between the operators M and M_D: the kernel K_σ of M is singular at the origin while the kernel J of M_D is continuous. Indeed, it is not clear if the Cauchy problem for (1.4), with singular kernel J is solvable or not, while it seems difficult to solve the Dirichlet problem for (1.1) with a continuous kernel K. The second is that the results [1, 2] are formulated in the L^p framework while the viscosity solutions approach is taken here.

We refer the reader to [1, 2] and the references therein for some applications of nonlocal diffusion equations like (1.1), (1.4), or (1.4) with M_N in place of M_D. For the viscosity solutions approach to integro-differential equations with singular kernels, we refer to the article [4]. We refer to [3, 6] for regularity results for integro-differential equations. We refer to [9, 10] and the references therein for analysis of nonlocal Hamilton-Jacobi equations describing dislocation dynamics.

Before closing the introduction we introduce a few of notation used below: $a\wedge b := \min\{a, b\}$, $a\vee b := \max\{a, b\}$, $a_+ := a\vee 0$ for $a, b \in \mathbb{R}$ and $\|u\|_{\infty,\Omega} := \sup_{x\in\Omega} |u(x)|$ for real-valued function u on Ω. We write int$B$ for the interior of the set B in a topological space.

2. Estimates of operators $M^\pm$

We note that for any bounded measurable function ϕ on Ω and for any $x \in \Omega$, if $0 < \delta \le \rho(x)$, then

$$M^+[\phi](x) = M_\delta^+[\phi](x) + \int_{\delta<|z|\le\rho(x)} G(\phi(x+z) - \phi(x))K(z)\,\mathrm{d}z,$$

where

$$M_\delta^+[\phi](x) = \limsup_{\varepsilon\to 0+} \int_{\varepsilon<|z|<\delta} G(\phi(x+z) - \phi(x))K(z)\,\mathrm{d}z.$$

In this section, we fix $x \in \mathbb{R}^n$, $\delta > 0$ and u a bounded measurable function on the ball $B(x, \delta)$, and establish some upper bounds of $M_\delta^+[u](x)$.

We note that the function G has the properties: (i) $G(a) < G(b)$ if $a < b$ and (ii) $G(ab) = G(a)G(b)$ for all $a, b \in \mathbb{R}$.

The following lemma (see, e.g., [8, Exercise 6.65]) will be useful when carrying out our computations and can be checked easily.

Lemma 2.1. *Let $p_i > 0$ for $i = 1, ..., n$ and let $f : (0, 1] \to [0, \infty)$ be a continuous function which satisfies the integrability condition at the origin:*

$$\int_0^1 f(t)t^{p_1+p_2+\cdots+p_n-1}\,\mathrm{d}t < \infty.$$

Set $\Theta = \{x = (x_1, ..., x_n) \in B(0, 1) \mid x_i \ge 0$ for all $i\}$. Then

$$\begin{aligned}\int_\Theta & f(x_1^2 + x_2^2 + \cdots + x_n^2)x_1^{2p_1-1}x_2^{2p_2-1}\cdots x_n^{2p_n-1}\,\mathrm{d}x \\ &= \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{2^n\Gamma(p_1+p_2+\cdots+p_n)}\int_0^1 f(t)t^{p_1+p_2+\cdots+p_n-1}\,\mathrm{d}t,\end{aligned}$$

where Γ denotes the gamma function, i.e., $\Gamma(t) = \int_0^\infty \mathrm{e}^{-x}x^{t-1}\,\mathrm{d}x$.

Theorem 2.2. *Assume that $p \ge 2$ and that there are a vector $q \in \mathbb{R}^n$ and a constant $C > 0$ such that*

$$u(x+z) - u(x) \le q\cdot z + C|z|^2 \quad \text{for all } z \in B(0, \delta). \tag{2.1}$$

Then there is a constant $C_1 > 0$, depending only on n, such that

$$M_\delta^+[u](x) \le C_1C(|q| + \delta C)^{p-2}\delta^{p-\sigma}.$$

A warning here is that $M_\delta^+[u](x)$ can be $-\infty$ in the above theorem. Also, we remark that if we replace (2.1) by the inequality

$$u(x+z)-u(x) \geq q\cdot z - C|z|^2 \quad \text{for all } z \in B(0,\delta)$$

in the above theorem, we have the following conclusion:

$$M_\delta^-[u](x) \geq -C_1C(|q|+\delta C)^{p-2}\delta^{p-\sigma},$$

where

$$M_\delta^-[u](x) := \liminf_{\varepsilon\to 0-}\int_{\varepsilon<|z|<\delta} G(u(x+z)-u(z))K(z)\,\mathrm{d}z.$$

This result follows from the above theorem applied to $v := -u$. Indeed, we have

$$v(x+z)-v(x) \leq -q\cdot z + C|z|^2$$

for all $z \in B(0,\delta)$. Hence, as a consequence of Theorem 2.2, we obtain

$$M_\delta^+[v](x) \leq C_1C(|q|+\delta C)^{p-2}\delta^{p-\sigma},$$

while we obviously have

$$M_\delta^-[u](x) = -M_\delta^+[v](x).$$

Combining these yields the desired conclusion.

Another important remark is that Theorem 2.2 readily shows that under the assumptions of Theorem 2.2 we have $M_\delta^+[u](x) = M_\delta^-[u](x)$. Indeed, under the assumptions of Theorem 2.2, we see that

$$M_\varepsilon^+[u](x) \leq C_1C(|q|+\varepsilon C)^{p-2}\varepsilon^{p-\sigma} \quad \text{for any } 0<\varepsilon<\delta,$$

from which one deduces easily that $M_\delta^+[u](x) \leq M_\delta^-[u](x)$. That is, under the assumptions of Theorem 2.2, the following identity holds:

$$M[u](x) = M^+[u](x) = M^-[u](x). \tag{2.2}$$

In what follows we denote by σ_n the surface area of $(n-1)$-dimensional unit sphere, i.e.,

$$\sigma_n := \frac{2\Gamma(1/2)^n}{\Gamma(n/2)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Proof. It is enough to show that the assertion of Theorem 2.2 is valid for $x=0$ and $\delta=1$. Indeed, if we define the function u_δ on $B(0,1)$ by $u_\delta(z)=u(x+\delta z)$, then we have

$$u_\delta(z)-u_\delta(0) \leq \delta q\cdot z + \delta^2 C|z|^2 \quad \text{for all } z\in B(0,1).$$

If we assume in addition that the assertion of Theorem 2.2 holds true for $x=0$ and $\delta=1$, then we get

$$M_1^+[u_\delta](0) \leq C_1\delta^2C(\delta|q|+\delta^2C)^{p-2} = C_1C(|q|+\delta C)^{p-2}\delta^p. \tag{2.3}$$

On the other hand, one observes that

$$\begin{aligned} M_1^+[u_\delta](0) &= \limsup_{\varepsilon\to 0+}\int_{\varepsilon<|z|<1} G(u(x+\delta z)-u(x))K(z)\,\mathrm{d}z \\ &= \limsup_{\varepsilon\to 0+}\int_{\varepsilon<|z|<\delta} G(u(x+y)-u(x))K(y/\delta)\delta^{-n}\,\mathrm{d}y = \delta^\sigma M_\delta^+[u](x). \end{aligned}$$

Combining this with (2.3) ensures that

$$M_\delta^+[u](x) \leq C_1(|q|+\delta C)^{p-2}\delta^{p-\sigma}.$$

We may thus assume that $x = 0$ and $\delta = 1$. Fix any $0 < \varepsilon < 1$. Let $z \in \mathbb{R}^n$ be such that $\varepsilon < |z| \le 1$. Observe that

$$G(u(z) - u(0)) \le G(q \cdot z + C|z|^2) \le G(q \cdot z) + G'(q \cdot z + \theta C|z|^2)C|z|^2$$

for some $\theta = \theta(z) \in (0,\, 1)$, where $G'(t) := \,\mathrm{d}G(t)/\,\mathrm{d}t$, and

$$G'(q \cdot z + \theta C|z|^2) \le (p-1)\,(|q||z| + C|z|^2)^{p-2} \le (p-1)\,(|q| + C)^{p-2}|z|^{p-2}.$$

By symmetry, we have

$$\int_{\varepsilon<|z|<1} G(q \cdot z)K(z)\,\mathrm{d}z = 0.$$

Hence, we get

$$\begin{aligned}
&\int_{\varepsilon<|z|<1} G(u(z) - u(0))K(z)\,\mathrm{d}z \\
&\le \int_{\varepsilon<|z|<1} (G(q \cdot z) + C(p-1)(|q| + C)^{p-2}|z|^p)K(z)\,\mathrm{d}z \\
&= \mu C(|q| + C)^{p-2} \int_{\varepsilon<|z|<1} |z|^{p-n-\sigma}\,\mathrm{d}z \\
&= \mu C(|q| + C)^{p-2}\sigma_n \int_{\varepsilon}^{1} r^{p-1-\sigma}\,\mathrm{d}r < \sigma_n C(|q| + C)^{p-2},
\end{aligned}$$

which completes the proof. □

Theorem 2.3. *Assume that $1 < p < 2$ and there are a vector $q \in \mathbb{R}^n \setminus \{0\}$ and a constant $C > 0$ such that $u(x + z) - u(x) \le q \cdot z + C|z|^2$ for all $z \in B(0,\, \delta)$. Then there is a constant $C_1 > 0$, depending only on p and n, such that*

$$M_\delta^+[u](x) \le C_1 C|q|^{p-2}\delta^{p-\sigma}.$$

For the proof of the above theorem, we need the following lemma.

Lemma 2.4. *Suppose that $n \ge 2$. Let $0 < a < 1$ and $e \in \mathbb{R}^n$ be a unit vector. Set*

$$S(a) = \{x \in \mathbb{R}^n \mid |x| = 1,\, |e \cdot x| \le a\}.$$

Let $|S(a)|$ denote the $(n-1)$-dimensional surface measure of $S(a)$. Then we have $|S(a)| \le \pi\sigma_{n-1}a$.

Proof. We begin with the formula from Advanced Calculus

$$|S(a)| = 2\sigma_{n-1} \int_0^{\sin^{-1} a} \cos^{n-2} t\,\mathrm{d}t.$$

Since $\sin^{-1} a \le \pi a/2$, we immediately get

$$|S(a)| \le 2\sigma_{n-1} \sin^{-1}(a) \le \pi\sigma_{n-1}a.$$

□

Proof of Theorem 2.3. We first prove that the conclusion of Theorem 2.3 is valid under the additional assumption that

$$|q| \ge 4\delta C. \tag{2.4}$$

As in the proof of the previous theorem, we may assume that $x = 0$ and $\delta = 1$.

In the case where $n \geq 2$, we make an orthogonal transformation if needed and assume that $q = |q|e_n$, where $e_n \in \mathbb{R}^n$ denotes the unit vector $e_n = (0, ..., 0, 1)$. We write $z = (z', z_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for generic $z \in \mathbb{R}^n$ in what follows.

Fix any $0 < \varepsilon < 1$. Set $a := C/|q| \in (0, 1/4]$, $\Theta = \{z \in \mathbb{R}^n \mid \varepsilon < |z| < 1\}$, $\Theta^+ = \{z = (z', z_n) \in \Theta \mid |z_n| > 2a|z|^2\}$ and $\Theta^- = \{z = (z', z_n) \in \Theta \mid |z_n| \leq 2a|z|^2\}$. Setting

$$\begin{aligned} I &:= \int_{\Theta} G(u(z) - u(0))K(z)\,\mathrm{d}z, \\ I^+ &:= \int_{\Theta^+} G(u(z) - u(0))K(z)\,\mathrm{d}z, \\ I^- &:= \int_{\Theta^-} G(u(z) - u(0))K(z)\,\mathrm{d}z, \end{aligned}$$

we observe that $I = I^+ + I^-$ and

$$\begin{aligned} I^+ := \int_{\Theta^+} G(u(z) - u(0))K(z)\,\mathrm{d}z &\leq \int_{\Theta^+} G(|q|z_n)G(1 + a|z|^2/z_n)K(z)\,\mathrm{d}z \\ &= |q|^{p-1} \int_{\Theta^+} |z_n|^{p-2}(z_n + (p-1)|1 + \lambda(z)|^{p-2}a|z|^2)K(z)\,\mathrm{d}z, \end{aligned}$$

where $\lambda(z)$ is a real-valued function on Θ^+ satisfying $|\lambda(z)| < 1/2$. Here we have used that $a|z|^2/|z_n| \leq 1/2$ for $z \in \Theta^+$. Hence we get

$$I^+ \leq 2^{2-p}(p-1)|q|^{p-1}a\mu \int_{\Theta^+} |z_n|^{p-2}|z|^{2-n-\sigma}\,\mathrm{d}z.$$

Applying Lemma 2.1, we obtain

$$I^+ < C_2|q|^{p-1}a\mu \int_0^1 t^{\frac{p-\sigma}{2}-1}\,\mathrm{d}t = 2C_2|q|^{p-1}a = 2C_2C|q|^{p-2},$$

where

$$C_2 = \frac{2^{2-p}(p-1)\Gamma(1/2)^{n-1}\Gamma((p-1)/2)}{\Gamma((p+n-2)/2)}.$$

Now, we compute

$$\begin{aligned} I^- &\leq |q|^{p-1} \int_{\Theta^-} G(|z_n| + a|z|^2)K(z)\,\mathrm{d}z \leq |q|^{p-1} \int_{\Theta^-} G(3a|z|^2)K(z)\,\mathrm{d}z \\ &\leq |q|^{p-1}\mu \int_{\Theta^-} |z|^{2p-2-\sigma-n}\,\mathrm{d}z \leq |q|^{p-1}\mu \int_{\Theta^-} |z|^{p-1-n-\sigma}\,\mathrm{d}z. \end{aligned} \tag{2.5}$$

For $z = (z', z_n) \in \Theta^-$, since $a \leq 1/4$, we have $|z_n| \leq 2a|z|^2 \leq 2a|z'|^2 + \frac{|z_n|}{2}$, and $|z_n| \leq 4a|z'|^2$. We now assume that $p - \sigma < 2$. Since $p - 1 - n - \sigma < 0$, we get

$$\int_{\Theta^-} |z|^{p-1-n-\sigma}\,\mathrm{d}z \leq \int_{\Theta^-} |z'|^{p-1-n-\sigma}\,\mathrm{d}z$$

and

$$\begin{aligned} \mu \int_{\Theta^-} |z|^{p-1-n-\sigma}\,\mathrm{d}z &\leq \mu \int_{|z'|<1} \mathrm{d}z' \int_0^{4a|z'|^2} |z'|^{p-1-n-\sigma}\,\mathrm{d}z_n \\ &\leq 4a\mu \int_{|z'|<1} |z'|^{p+1-\sigma-n}\,\mathrm{d}z' = 4a\sigma_{n-1}. \end{aligned}$$

We next treat the other case, i.e., the case where $p - \sigma \geq 2$. Let $S(t)$ denote the portion of the unit sphere defined by Lemma 2.4, with $e = e_n$, for $t \in (0, 1)$. Since

$|z_n| \le 2a|z|^2$ for $z \in \Theta^-$, we see that $\Theta^- \subset \{ty \mid y \in S(2a),\, 0 \le t \le 1\}$. Thus, using Lemma 2.4, we find that

$$\mu \int_{\Theta^-} |z|^{p-1-n-\sigma}\,\mathrm{d}z \le \mu|S(2a)| \int_0^1 t^{p-2-\sigma}\,\mathrm{d}t \le 2\pi\sigma_{n-1}\frac{p-\sigma}{p-1-\sigma}a \le 4\pi\sigma_{n-1}a.$$

Thus we get $I^- \le 4\pi\sigma_{n-1}|q|^{p-2}$ in view of (2.5) and

$$I \le C_3 C|q|^{p-2}, \tag{2.6}$$

where $C_3 = 2C_2 + 4\pi\sigma_{n-1}$.

Next we consider the case where $n = 1$. We follow the above argument for higher dimensions. Noting that $C|z|/|q| < 1/2$ for all $z \in (-1,\, 1)$, we compute that for any $0 < \varepsilon < 1$ and for some function $\lambda(z) \in (-1/2,\, 1/2)$,

$$\begin{aligned} I &\le \int_{\varepsilon<|z|<1} G(qz)\Big(1 + (p-1)|1+\lambda(z)|^{p-2}\frac{Cz}{q}\Big)K(z)\,\mathrm{d}z \\ &\le 2^{3-p}(p-1)\,C|q|^{p-2}\mu\int_\varepsilon^1 |z|^{p-1-\sigma}\,\mathrm{d}z < 2^{3-p}(p-1)\,C|q|^{p-2}. \end{aligned}$$

This together with (2.6) guarantees that the conclusion of the theorem holds under condition (2.4).

Now, we turn to the general case. We may assume that $x = 0$ and $\delta = 1$. If $|q| \ge 4C$, then we are done. Thus, we may assume that $|q| < 4C$.

We set $r := |q|/(4C) \in (0,\, 1)$ and observe that condition (2.4), with r in place of δ, is satisfied. We apply what we have proved above, to see that

$$M_r^+[u](0) \le C_3C|q|^{p-2}r^{p-\sigma} < C_3C|q|^{p-2}.$$

Also, we have

$$\int_{r<|z|<1} G\big(u(z)-u(0)\big)K(z)\,\mathrm{d}z \le \int_{r<|z|<1} \big(G(|q||z|) + G(C|z|^2)\big)\,K(z)\,\mathrm{d}z,$$

$$\begin{aligned} \int_{r<|z|<1} G(|q||z|)K(z)\,\mathrm{d}z &\le |q|^{p-1}r^{-1}\mu\int_{r<|z|<1}|z|^{p-n-\sigma}\,\mathrm{d}z \\ &\le \sigma_n|q|^{p-1}r^{-1} = 4\sigma_nC|q|^{p-2}, \end{aligned}$$

and

$$\begin{aligned} \int_{r<|z|<1} G(C|z|^2)K(z)\,\mathrm{d}z &\le C^{p-1}r^{p-2}\mu\int_{r<|z|<1}|z|^{p-n-\sigma}\,\mathrm{d}z \\ &\le \sigma_nC(Cr)^{p-2} \le 4\sigma_nC|q|^{p-2}. \end{aligned}$$

Combining these, we get

$$I \le \big(C_3 + 8\sigma_n\big)C|q|^{p-2},$$

which completes the proof. □

Now let $1 < p < 2$ and $\beta > 1/(p-1)$. Let $\phi \in C^2(\mathbb{R}^n)$ be the function given by $\phi(x) = |x|^{\beta+1}$. We note that for all $x, y \in \mathbb{R}^n$,

$$D\phi(x) = (\beta+1)|x|^{\beta-1}x \quad \text{and} \quad |D^2\phi(x)y\cdot y| \le \beta(\beta+1)|x|^{\beta-1}|y|^2.$$

Lemma 2.5. *We have*

$$M_\delta^+[\phi](0) \le \sigma_n\,\delta^{(\beta+1)(p-1)-\sigma}.$$

We remark that $(\beta+1)(p-1) - \sigma > p - \sigma > 0$.

Proof. Observe that for any $z \in \mathbb{R}^n$,

$$G(\phi(z) - \phi(0))K(z) = G(|z|^{\beta+1})K(z) = \mu|z|^{(\beta+1)(p-1)-n-\sigma}.$$

Hence, we get for any $0 < \varepsilon < \delta$,

$$\begin{aligned}\int_{\varepsilon<|z|<\delta} G(\phi(z) - \phi(0))K(z)\,\mathrm{d}z =& \sigma_n\mu \int_{\varepsilon}^{\delta} r^{(\beta+1)(p-1)-\sigma-1}\,\mathrm{d}z \\ <& \frac{\sigma_n\mu}{(\beta+1)(p-1)-\sigma}\delta^{(\beta+1)(p-1)-\sigma}.\end{aligned}$$

Thus

$$M_\delta^+[\phi](0) \le \sigma_n \delta^{(\beta+1)(p-1)-\sigma}.$$

□

Theorem 2.6. *There is a constant $C_1 > 0$ depending only on β, p and n such that for any $x \in B(0, \delta)$,*

$$M_\delta^+[\phi](x) \le C_1 \delta^{(\beta+1)(p-1)-\sigma}.$$

Proof. Fix any $x \in B(0, \delta)$. In view of Lemma 2.5, if $x = 0$, then we have nothing to prove, and hence we may assume that $x \ne 0$. Observe that for any $z \in B(0, |x|)$ and for some $\theta = \theta(z) \in (0, 1)$,

$$\begin{aligned}\phi(x+z) - \phi(x) \le& (\beta+1)|x|^{\beta-1}x \cdot z + \frac{\beta(\beta+1)}{2}|x+\theta z|^{\beta-1}|z|^2 \\ \le& (\beta+1)|x|^{\beta-1}x \cdot z + \beta(\beta+1)2^{\beta-2}|x|^{\beta-1}|z|^2.\end{aligned}$$

Using Theorem 2.3, we get

$$M_{|x|}[\phi](x) \le C_2 2^{\beta-2}\beta(\beta+1)^{p-1}|x|^{(\beta+1)(p-1)-\sigma},$$

where C_2 is a constant depending only on p and n.

Next, setting

$$I = \int_{|x|<|z|<\delta} G(\phi(x+z) - \phi(x))K(z)\,\mathrm{d}z,$$

we have

$$M_\delta^+[\phi](x) \le C_2 2^{\beta-2}\beta(\beta+1)^{p-1}\delta^{(\beta+1)(p-1)-\sigma} + I. \tag{2.7}$$

Observe that $G(\phi(x+z) - \phi(x)) \le G(\phi(x+z)) \le G(\phi(2z))$ for $z \in \mathbb{R}^n \setminus B(0, |x|)$ and

$$I \le 2^{(\beta+1)(p-1)}\mu \int_{|x|<|z|<\delta} |z|^{(\beta+1)(p-1)-n-\sigma}\,\mathrm{d}z \le 2^{(\beta+1)(p-1)}\sigma_n\delta^{(\beta+1)(p-1)-\sigma}.$$

This combined with (2.7) completes the proof. □

We close this section with the following remark. Theorems 2.2, 2.3 and 2.6 guarantee that identity (2.2) holds true for every $x \in \Omega$ and $u \in \mathcal{T}_p$.

3. Stability properties and the Perron method

In this section we establish some stability properties of subsolutions of (1.1) as well as the Perron method. Analogous stability properties are valid for supersolutions of (1.1), but we leave the details to the reader.

Lemma 3.1. *Let $\delta > 0$, $\{x_k\} \subset \Omega$ and $x_0 \in \Omega$. Let $\{u_k\}$ be a sequence of bounded measurable functions on Ω and u a bounded measurable function on Ω. Assume that $\{u_k\}$ is uniformly bounded on Ω and $(x_k,\, u_k(x_k)) \to (x_0,\, u(x_0))$ as $k \to \infty$. Moreover assume that*

$$\lim_{j\to\infty} \sup\{u_k(y) \mid y \in B(z, j^{-1}) \cap \Omega,\ k \geq j\} \leq u(z) \quad \text{for all } z \in \Omega. \tag{3.1}$$

Then

$$\limsup_{k\to\infty} \int_{B(0,\,\rho(x_k))\setminus B(0,\,\delta)} G(u_k(x_k+z) - u_k(x_k))K(z)\,\mathrm{d}z \\ \leq \int_{B(0,\,\rho(x_0))\setminus B(0,\,\delta)} G(u(x_0+z) - u(x_0))K(z)\,\mathrm{d}z.$$

Proof. Set

$$f_k(z) = \begin{cases} G(u_k(x_k+z) - u_k(x_k)) & \text{for } z \in B(0,\,\rho(x_0)) \cap B(0,\,\rho(x_k)), \\ 0 & \text{for } z \in B(0,\,\rho(x_0)) \setminus B(0,\,\rho(x_k)), \end{cases}$$

$$I_k = \int_{B(0,\,\rho(x_0))\setminus B(0,\,\delta)} f_k(z)K(z)\,\mathrm{d}z.$$

Choose a constant $C > 0$ so that $|u_k(z)| \leq C$ for all $(z, k) \in \Omega \times \mathbb{N}$, and note that $|f_k(z)|K(z) \leq G(2C)K(z)$ for all $z \in B(0,\,\rho(x_0))$ and all $k \in \mathbb{N}$. By the continuity of ρ, we find that

$$\limsup_{k\to\infty} \int_{B(0,\,\rho(x_k))\setminus B(0,\,\delta)} G(u_k(x_k+z) - u_k(x_k))K(z)\,\mathrm{d}z = \limsup_{k\to\infty} I_k.$$

By the Fatou lemma, we have

$$\limsup_{k\to\infty} I_k \leq \int_{B(0,\,\rho(x_0))\setminus B(0,\delta)} \limsup_{k\to\infty} f_k(z)\,K(z)\,\mathrm{d}z.$$

Since G is continuous and nondecreasing in $\mathbb{R}$, using (3.1), we see that for any $z \in \mathrm{int}B(0,\,\rho(x_0))$,

$$\limsup_{k\to\infty} f_k(z) \leq G(u(x_0+z) - u(x_0)).$$

Thus we obtain

$$\limsup_{k\to\infty} I_k \leq \int_{B(0,\,\rho(x_0))\setminus B(0,\,\delta)} G(u(x_0+z) - u(x_0))K(z)\,\mathrm{d}z,$$

which completes the proof. □

Theorem 3.2. *Let $\{u_k\}$ be a sequence of bounded measurable functions on Ω and u a bounded measurable function on Ω. Let $\phi \in \mathcal{T}_p$ and let $\{x_k\} \subset \Omega$ be a sequence converging to a point $x_0 \in \Omega$. Assume that for each $k \in \mathbb{N}$ the function $u_k - \phi$ attains a maximum at x_k, the sequence $\{u_k\}$ is uniformly bounded on Ω, $u_k(x_k) \to u(x_0)$ as $k \to \infty$ and*

$$\lim_{j\to\infty} \sup\{u_k(y) \mid y \in B(x, j^{-1}) \cap \Omega,\ k \geq j\} \leq u(x) \quad \text{for all } x \in \Omega.$$

Then

$$\limsup_{k\to\infty} M^+[u_k](x_k) \le M^+[u](x_0).$$

A useful remark concerning the above theorem is that the global maximum assumption can be replaced by the following "uniform" local maximum condition: there exists a constant $r > 0$, independent of k, such that $u_k - \phi$ attains a maximum over $B(x_0, r) \cap \Omega$.

Proof. Fix an $r \in (0, \rho(x_0)/2)$. By selecting a subsequence if necessary, we may assume that $x_k \in B(x_0, r)$ for all $k \in \mathbb{N}$. Noting that $B(x_k, r) \subset B(x_0, 2r) \subset \Omega$, we choose a constant $C > 0$ so that

$$\phi(x_k + z) - \phi(x_k) \le D\phi(x_k) \cdot z + C|z|^2 \quad \text{for all } z \in B(0, r).$$

Then we have

$$u_k(x_k + z) - u_k(x_k) \le D\phi(x_k) \cdot z + C|z|^2 \quad \text{for all } z \in B(0, r).$$

We first treat the case where $p \ge 2$. By Theorem 2.2, there is a constant $C_1 > 0$, independent of k, such that for any $0 < \delta < r$ and any $k \in \mathbb{N}$,

$$M_\delta^+[u_k](x_k) \le C_1 C(|D\phi(x_k)| + \delta C)^{p-2}\delta^{p-\sigma}. \tag{3.2}$$

Thus, we have

$$\begin{aligned} M^+[u_k](x_k) \le & C_1 C(|D\phi(x_k)| + \delta C)^{p-2}\delta^{p-\sigma} \\ & + \int_{B(0,\, \rho(x_k))\setminus B(0,\, \delta)} G(u_k(x_k + z) - u_k(x_k))K(z)\,\mathrm{d}z. \end{aligned}$$

We now apply Lemma 3.1 to the second term on the right hand side of the above inequality, to get

$$\begin{aligned} \limsup_{k\to\infty} M^+[u_k](x_k) \le & C_1 C(|D\phi(x_0)| + \delta C)^{p-2}\delta^{p-\sigma} \\ & + \int_{B(0,\, \rho(x_0))\setminus B(0,\, \delta)} G(u(x_0 + z) - u(x_0))K(z)\,\mathrm{d}z, \end{aligned}$$

from which we conclude that

$$\limsup_{k\to\infty} M^+[u_k](x_k) \le M^+[u](x_0).$$

Next, we consider the case where $1 < p < 2$. We follow the above argument with some modifications. In the case where $D\phi(x_0) \ne 0$, we may assume by selecting a subsequence if needed that $\inf_{k\in\mathbb{N}} |D\phi(x_k)| > 0$, and instead of (3.2), by applying Theorem 2.3, we get

$$M_\delta^+[u_k](x_k) \le C_1|D\phi(x_k)|^{p-2}\delta^{p-\sigma}.$$

In the case where $D\phi(x_0) = 0$, we may replace the test function ϕ by the function

$$\phi(x) = A|x - x_0|^{\beta+1},$$

where A is a sufficiently large constant, and using Theorem 2.6, we get

$$M_\delta^+[u_k](x_k) \le M_\delta^+[\phi](x_k) \le AC_1\delta^{(\beta+1)(p-1)-\sigma} \quad \text{if } |x_k - x_0| \le \delta$$

in place of (3.2), where C_1 is a constant depending only on p, β and n. Then the rest of argument is the same as the previous case. □

Theorem 3.3. *Let $\mathcal{S}_0$ be a nonempty set of subsolutions of* (1.1)*. Assume that the family $\mathcal{S}_0$ is uniformly bounded on Ω. Define the bounded function u on Ω by $u(x) = \sup\{v(x) \mid v \in \mathcal{S}_0\}$. Then u is a subsolution of* (1.1).

Proof. Let $x_0 \in \Omega$ and $\phi \in \mathcal{T}_p(\Omega)$, and assume that $u^* - \phi$ attains a strict maximum at x_0. By the definition of u^*, there are sequences $\{x_k\} \subset B(x_0, r)$, where $r > 0$ is chosen so that $B(x_0, r) \subset \Omega$, and $\{v_k\} \subset \mathcal{S}_0$ such that $v_k(x_k) \to u^*(x_0)$ and $x_k \to x_0$ as $k \to \infty$. By the definition of u, we have $v_k^* \le u^*$ in Ω.

For each $k \in \mathbb{N}$ let $y_k \in B(x_0, r)$ be a maximum point, over $B(x_0, r)$, of the function $v_k^* - \phi$. Observe as usual that

$$\begin{aligned}(u^* - \phi)(x_0) &= \lim_{k\to\infty}(v_k - \phi)(x_k) \le \liminf_{k\to\infty}(v_k^* - \phi)(y_k)\\ &\le \limsup_{k\to\infty}(v_k^* - \phi)(y_k) \le \limsup_{k\to\infty}(u^* - \phi)(y_k) \le (u^* - \phi)(x_0).\end{aligned}$$

This shows that $v_k^*(y_k) \to u^*(x_0)$ and $(u^* - \phi)(y_k) \to (u^* - \phi)(x_0)$ as $k \to \infty$. It is now easy to deduce that $y_k \to x_0$ as $k \to \infty$. Passing to a subsequence if necessary, we may assume that $y_k \in \mathrm{int}B(x_0, r)$ for all k. Since v_k is a subsolution of (1.1), we have $M^+[v_k^*](y_k) \ge f(y_k)$ for all $k \in \mathbb{N}$. Since $v_k^* \le u^*$, we see that for all $x \in \Omega$,

$$\lim_{j\to\infty} \sup\{v_k^*(y) \mid k \ge j,\ y \in B(x, j^{-1}) \cap \Omega\} \le u^*(x).$$

We may now invoke Theorem 3.2, to conclude that $M^+[u^*](x_0) \ge f_0(x_0)$, which completes the proof. □

Theorem 3.4. *Let $\{u_k\}$ be a sequence of subsolutions of* (1.1)*. Assume that the collection $\{u_k\}$ is uniformly bounded on Ω. Define the bounded function u on Ω by*

$$u(x) = \lim_{j\to\infty} \sup\{u_k(y) \mid y \in B(x, j^{-1}) \cap \Omega,\ k \ge j\}.$$

Then u is a subsolution of (1.1).

Proof. First of all, we remark that $u \in \mathrm{USC}(\Omega)$. Next, let $x_0 \in \Omega$ and $\phi \in \mathcal{T}_p(\Omega)$. Assume that $u - \phi$ attains a strict maximum at x_0. By the definition of u, there are sequences $\{k_j\} \subset \mathbb{N}$ diverging to infinity and $\{x_j\} \subset \Omega$ such that $u_{k_j}(x_j) \to u(x_0)$ and $x_j \to x_0$ as $j \to \infty$. Here we also assume by passing to a subsequence if necessary that $\{x_j\} \subset B(x_0, r)$, where $r > 0$ is chosen so that $B(x_0, r) \subset \Omega$.

Set $v_j = u_{k_j}$ for $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ let $y_j \in B(x_0, r)$ be a maximum point, over $B(x_0, r)$, of $v_j^* - \phi$. We observe that

$$(u - \phi)(x_0) = \lim_{j\to\infty}(v_j - \phi)(x_j) \le \liminf_{j\to\infty}(v_j^* - \phi)(y_j). \tag{3.3}$$

By selecting a subsequence if necessary, we may assume that $y_j \to y$ as $j \to \infty$ for some $y \in B(x_0, r)$. By the definition of u, we see that

$$\limsup_{j\to\infty}(v_j^* - \phi)(y_j) = \limsup_{k\to\infty} v_j^*(y_j) - \phi(y) \le u(y) - \phi(y).$$

This together with (3.3) guarantees that $y = x_0$. That is, the sequence $\{y_j\}$ converges to x_0. Also, it follows that $v_j^*(y_j) \to u(x_0)$ as $j \to \infty$.

For sufficiently large j, we have $y_j \in \mathrm{int}B(x_0, r)$ and $M^+[v_j^*](y_j) \ge f_0(y_j)$. Applying Theorem 3.2, we find that $M^+[u](x_0) \ge f_0(x_0)$. This finishes the proof. □

To formulate a basic existence result (Perron method) for (1.1), we let $g^- \in \mathrm{LSC}(\Omega)$ and $g^+ \in \mathrm{USC}(\Omega)$ be a subsolution and a supersolution of (1.1), respectively. Assume furthermore that $g^\pm$ are bounded in Ω and $g^- \le g^+$ in Ω. Set

$$u(x) = \sup\{v(x) \mid v \text{ is a subsolution of (1.1)}, \ g^- \le v \le g^+ \ \text{ in } \ \Omega\}. \tag{3.4}$$

Note that u is bounded in Ω.

Theorem 3.5. *The function u given by* (3.4) *is a solution of* (1.1).

Proof. We note by Theorem 3.4 that u^* is a subsolution of (1.1). We thus need to show that u_* is a supersolution of (1.1).

Let $x_0 \in \Omega$ and $\phi \in \mathcal{T}_p(\Omega)$. Assume that $u_* - \phi$ attains a strict minimum at x_0, with minimum value zero. We intend to show that the inequality

$$M^-[u_*](x_0) \le f_0(x_0) \tag{3.5}$$

holds.

It is clear by the definition of u that $g^- \le u \le g^+$ in Ω. Consequently we have $g^- \le u_* \le g^+_*$ in Ω. Consider first the case where $u_*(x_0) = g^+_*(x_0)$. Then, since $u_* \le g^+_*$ in Ω, it follows that $g^+_* - \phi$ attains a minimum at x_0. As g^+ is a supersolution of (1.1), we have

$$M^-[g^+_*](x_0) \le f_0(x_0). \tag{3.6}$$

But, since $u_* \le g^+_*$ in Ω and $g^+_*(x_0) = u_*(x_0)$, we see that

$$M^-[g^+_*](x_0) \ge M^-[u_*](x_0),$$

from which together with (3.6) we conclude that (3.5) holds.

Next we assume that $u_*(x_0) < g^+_*(x_0)$. We deduce by the semicontinuity of g^+_* that $g^+_* > \phi + \varepsilon$ in a neighborhood of x_0 for some constant $\varepsilon \in (0, 1)$. Furthermore, we may assume, by modifying ϕ on a set away from the point x_0 if necessary, that $g^+_*(x) > \phi(x) + \varepsilon$ for all $x \in \Omega$.

Define

$$u_k = u \vee \left(\phi + \tfrac{1}{k}\right) \quad \text{in } \Omega.$$

Note that $(u_k)_*(x_0) = \phi(x_0) + 1/k > u_*(x_0)$ and therefore $u_k \not\le u$. Since $\phi + \varepsilon < g^+$ in Ω, we see that $g^- \le u_k \le g^+$ for sufficiently large k, say, $k \ge j$, for some $j \in \mathbb{N}$.

In what follows we are concerned only with u_k having $k \ge j$. Since $u_k \not\le u$ and $g^- \le u_k \le g^+$ on Ω, by the definition of u, we find that u_k is not a subsolution of (1.1). Thus, for each k there are a point $x_k \in \Omega$ and a function $\psi_k \in \mathcal{T}_p(\Omega)$ such that x_k is a maximum point of $u_k^* - \psi_k$ and the inequality

$$M^+[u_k^*](x_k) < f_0(x_k) \tag{3.7}$$

holds.

Set $\phi_k(x) = \phi(x) + \frac{1}{k}$ for $x \in \Omega$ and $V_k = \{x \in \Omega \mid \phi_k(x) > u^*(x)\}$. Note that V_k is an open subset of Ω and $u_k = \phi_k$ on V_k.

We claim that $x_k \in V_k$. Indeed, if this were not the case, then we would have $\phi_k(x_k) \le u^*(x_k)$ and therefore $u_k^*(x_k) = u^*(x_k) \vee \phi_k(x_k) = u^*(x_k)$.

Now, since $u_k^* \ge u^*$ in Ω, we see that x_k is a maximum point of $u^* - \psi_k$. Hence we have $M^+[u^*](x_k) \ge f_0(x_k)$. Since $u_k^*(x_k) = u^*(x_k)$ and $u_k^* \ge u^*$ in Ω, we have $M^+[u^*](x_k) \le M^+[u_k^*](x_k)$. From these we obtain $M^+[u_k^*](x_k) \ge f_0(x_k)$, which contradicts (3.7). Thus we have $x_k \in V_k$.

As noted above, V_k is an open subset of Ω and $u_k = \phi_k$ on V_k. Therefore, $(u_k)_*(x_k) = \phi_k(x_k)$. By the definition of u_k, we have $u_k \ge \phi_k$ on Ω and hence

$(u_k)_* \geq \phi_k$ on Ω. Thus, $(u_k)_* - \phi$ takes a minimum at x_k. Also, since $u_* \leq (u_k)_* \leq u_* + 1/k$ in Ω, we find that, as $k \to \infty$, $(u_k)_* \to u_*$ uniformly on Ω and $x_k \to x_0$. Hence, applying Theorem 3.2, we obtain

$$\liminf_{k\to\infty} M^-[(u_k)_*](x_k) \geq M^-[u_*](x_0).$$

Combining this with (3.7) yields $f_0(x_0) \geq M^-[u_*](x_0)$, which finishes the proof. □

4. Comparison theorem

In this section we prove the following comparison theorem.

Theorem 4.1. *Let* $\lambda_0 = 1$. *Let* $u \in \mathrm{USC}(\overline{\Omega})$ *and* $v \in \mathrm{LSC}(\overline{\Omega})$ *be a subsolution and a supersolution of* (1.1), *respectively. Assume that* $u \leq v$ *on* $\partial\Omega$ *and* u *and* v *are bounded on* $\overline{\Omega}$. *Then* $u \leq v$ *in* Ω.

Proof. We argue by contradiction, and thus suppose that $m := \sup_\Omega (u-v) > 0$ and will show a contradiction. We fix a constant $C > 0$ so that $\|u\|_{\infty,\overline{\Omega}} \vee \|v\|_{\infty,\overline{\Omega}} \leq C$. Since G is strictly increasing, we can choose a nondecreasing positive function γ on $(0, m)$ so that

$$G(t+s) \geq G(t) + \gamma(s) \quad \text{for all } |t| \leq 2C,\ 0 < s < m.$$

For $\alpha > 0$ we consider the function Φ_α on $\overline{\Omega} \times \overline{\Omega}$ defined by

$$\Phi_\alpha(x,y) = u(x) - v(y) - \alpha|x-y|^{\beta+1},$$

where $\beta > \max\{1,\ 1/(p-1)\}$. For each $\alpha > 0$, let $(x_\alpha, y_\alpha) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of Φ_α. As usual in viscosity solutions theory, we observe that there are a sequence $\{\alpha_k\}$, diverging to infinity, and a point $x_0 \in \overline{\Omega}$ for which $x_{\alpha_k} \to x_0$, $y_{\alpha_k} \to x_0$, $u(x_{\alpha_k}) \to u(x_0)$ and $v(y_{\alpha_k}) \to v(x_0)$ as $j \to \infty$. Also, it is easy to see that $(u-v)(x_0) = m$. Since $\max_{\partial\Omega}(u-v) \leq 0$ by assumption, we have $x_0 \in \Omega$.

For notational simplicity, we write x_k and y_k for x_{α_k} and y_{α_k}, respectively. Passing to a subsequence if necessary, we may assume that $x_k, y_k \in \Omega$ for all $k \in \mathbb{N}$. Hence, by the definition of sub and supersolutions of (1.1), we have $M^+[u](x_k) \geq f_0(x_k)$ and $f_0(y_k) \geq M^-[v](y_k)$ for all $k \in \mathbb{N}$. As a remark after Theorem 2.2, we see from Theorems 2.2, 2.3 and 2.6 that $M^+[u](x_k) = M^-[u](x_k)$ for all $\in \mathbb{N}$.

Since $\rho(x_0) = \operatorname{dist}(x_0, \partial\Omega)$ and $m > 0$, by the upper semicontinuity of $u - v$, we can choose a point $\xi \in \operatorname{int} B(x_0, \rho(x_0))$ so that $(u-v)(\xi) < m/2$. Then, in view of the semicontinuity of u and v, we can choose an $0 < r < \operatorname{dist}(\xi, \partial B(x_0, \rho(x_0)))$ so that $u(x) - v(y) < m/2$ for all $x, y \in B(\xi,\ r)$. Setting $\rho_k = \rho(x_k) \wedge \rho(y_k)$ and passing to a subsequence if necessary, we may assume that

$$B(\xi, r) \subset B(x_k, \rho_k) \cap B(y_k, \rho_k) \quad \text{for all } k \in \mathbb{N},$$

which can be stated as

$$B(\xi - x_k, r) \cup B(\xi - y_k, r) \subset B(0, \rho_k) \quad \text{for } k \in \mathbb{N}.$$

Again, passing to a subsequence if needed, we may assume that

$$B(\xi - x_0,\ r/2) \subset B(\xi - x_k,\ r) \cap B(\xi - y_k,\ r) \quad \text{for } k \in \mathbb{N}.$$

Note that for $z \in B(\xi - x_0,\ r/2)$,

$$x_k + z,\ y_k + z \in B(\xi,\ r)$$

and

$$u(x_k + z) - v(y_k + z) < \frac{m}{2}.$$

Since $u(x_k) - v(y_k) \geq m$, we have

$$u(x_k + z) - u(x_k) < v(y_k + z) - v(y_k) - \frac{m}{2} \quad \text{for } z \in B(\xi - x_0,\, r/2).$$

Note also that $B(\xi - x_0) \subset B(0, \rho_k)$ for $k \in \mathbb{N}$.

We have

$$\Phi(x_k, y_k) \geq \Phi(x_k + z, y_k + z) \quad \text{for all } z \in B(0,\, \rho_k),\ k \in \mathbb{N},$$

and hence

$$u(x_k) - v(y_k) \geq u(x_k + z) - v(y_k + z) \quad \text{for all } z \in B(0,\, \rho_k),\ k \in \mathbb{N}.$$

We set $\eta = \xi - x_0$. Using the above observations, we compute that

$$\begin{aligned} f_0(x_k) \leq & M^-[u](x_k) \\ \leq & \liminf_{\varepsilon\to 0+} \int_{B(0,\,\rho_k)\setminus(B(\eta,\,r/2)\cup B(0,\,\varepsilon))} G(v(y_k + z) - v(y_k))K(z)\,\mathrm{d}z \\ & + \int_{B(\eta,\,r/2)} G(u(x_k + z) - u(x_k))K(z)\,\mathrm{d}z \\ & + \int_{\rho_k<|z|<\rho(x_k)} G(2C)K(z)\,\mathrm{d}z \\ \leq & \liminf_{\varepsilon\to 0+} \int_{B(0,\,\rho_k)\setminus(B(\eta,\,r/2)\cup B(0,\,\varepsilon))} G(v(y_k + z) - v(y_k))K(z)\,\mathrm{d}z \\ & + \int_{B(\eta,\,r/2)} G(v(y_k + z) - v(y_k) - m/2)K(z)\,\mathrm{d}z \\ & + \int_{\rho_k<|z|<\rho(x_k)} G(2C)K(z)\,\mathrm{d}z \\ \leq & \liminf_{\varepsilon\to 0+} \int_{B(0,\,\rho_k)\setminus B(0,\,\varepsilon)} G(v(y_k + z) - v(y_k))K(z)\,\mathrm{d}z \\ & - \gamma(m/2)\int_{B(\eta,\,r/2)} K(z)\,\mathrm{d}z + \int_{\rho_k<|z|<\rho(x_k)} G(2C)K(z)\,\mathrm{d}z \\ \leq & M^-[v](y_k) - \gamma(m/2)\int_{B(\eta,\,r/2)} K(z)\,\mathrm{d}z \\ & + 2\int_{\rho_k<|z|<\rho(x_k)\vee\rho(y_k)} G(2C)K(z)\,\mathrm{d}z \\ \leq & f_0(y_k) - \gamma(m/2)\int_{B(\eta,\,r/2)} K(z)\,\mathrm{d}z \\ & + 2\int_{\rho_k<|z|<\rho(x_k)\vee\rho(y_k)} G(2C)K(z)\,\mathrm{d}z. \end{aligned} \tag{4.1}$$

Sending $k \to \infty$ yields

$$\gamma(m/2)\int_{B(\eta,\,r/2)} K(z)\,\mathrm{d}z < 0,$$

which is a contradiction. □

Remark 4.1. In the (linear) case where $p = 2$, the same conclusion as the above theorem is valid without assuming $\lambda_0 = 1$.

Proof of Remark 4.1. Let $p = 2$ and $0 < \lambda_0 < 1$. As in the proof of the previous theorem, we suppose that $m := \max_{\overline{\Omega}}(u - v) > 0$ and will show a contradiction. We set $\Gamma = \{x \in \overline{\Omega} \mid (u - v)(x) = m\}$. Obviously, the set Γ is a nonempty closed subset of Ω and there are a point $x_0 \in \Gamma$ and a ball $B(\xi, r)$, with $r > 0$, such that

$$B(\xi, r) \subset \mathrm{int}B(x_0, \rho(x_0)) \setminus \Gamma.$$

Here we may assume by choosing $r > 0$ small enough that $u(x) - v(y) \le m_0$ for all $x, y \in B(\xi,\, r)$ and some constant $m_0 < m$.

Let $\varepsilon > 0$, and note that the function $u(x) - v(x) - \varepsilon|x - x_0|^2$ has a strict maximum at $x = x_0$, and, introducing the function

$$\Psi_\alpha(x, y) = u(x) - v(y) - \varepsilon|x - x_0|^2 - \alpha|x - y|^2$$

on $\overline{\Omega} \times \overline{\Omega}$, we find that there are a sequence $\{\alpha_k\}$ diverging to infinity and sequences $\{x_k\}$ and $\{y_k\}$ converging to x_0 such that Ψ_{α_k} attains a maximum at (x_k, y_k).

Selecting a subsequence if necessary, we may assume that $x_k, y_k \notin B(\xi, r)$ and $B(\xi, r) \subset B(x_k, \rho(x_k)) \cap B(y_k, \rho(y_k))$ for all $k \in \mathbb{N}$. Setting $\eta = \xi - x_0$, we may assume that for all $k \in \mathbb{N}$,

$$\big(x_k + B(\eta, r/2)\big) \cup \big(y_k + B(\eta, r/2)\big) \subset B(\xi, r).$$

As u and v are sub and supersolutions of (1.1), respectively, we get

$$M^+[u](x_k) = M^-[u](x_k) \ge f_0(x_k) \quad \text{and} \quad M^-[v](y_k) = M^+[v](y_k) \le f_0(y_k).$$

Since

$$\Psi_{\alpha_k}(x_k, y_k) \ge \Psi_{\alpha_k}(x_k + z, y_k + z) \quad \text{for all } z \in B(0, \rho(x_k) \wedge \rho(y_k)),\ k \in \mathbb{N},$$

we have

$$\begin{aligned} u(x_k + z) - u(x_k) \le& v(y_k + z) - v(y_k) + \varepsilon\left(2(x_k - x_0)\cdot z + |z|^2\right) \\ &\text{for all } z \in B(0,\, \rho(x_k) \wedge \rho(y_k)),\ k \in \mathbb{N}. \end{aligned}$$

Hence, computing similarly to (4.1), we get

$$\begin{aligned} f_0(x_k) \le& f_0(y_k) - \gamma(m - m_0)\int_{B(\eta,\, r/2)} K(z)\,\mathrm{d}z + 2\int_{N_k} G(2C)K(z)\,\mathrm{d}z \\ &+ \varepsilon\int_{B(0,\, \rho(x_k))} |z|^2 K(z)\,\mathrm{d}z, \end{aligned}$$

where

$$N_k = \big(B(x_k,\, \rho(x_k)) \cup B(y_k,\, \rho(y_k))\big) \setminus \big(B(x_k,\, \rho(x_k) \cap B(y_k,\, \rho(y_k))\big),$$

from which we obtain a contradiction in the limit as $k \to \infty$ if $\varepsilon > 0$ is sufficiently small. □

5. Existence of continuous solutions

In this section we establish an existence result for the Dirichlet problem for (1.1)–(1.2). We need the following additional hypotheses on Ω and f_0.

(H1) The set Ω satisfies the *uniform exterior sphere condition.* That is, there is an $R > 0$ and, for each $x \in \partial\Omega$, a point $y \in \mathbb{R}^n$ such that

$$B(y,\, R) \cap \overline{\Omega} = \{x\}.$$

(H2) There exist constants $\varepsilon_0 \in (0, 1)$ and $C_0 > 0$ such that

$$|f_0(x)| \le C_0 \left(\lambda_0 \operatorname{dist}(x, \partial\Omega)\right)^{\varepsilon_0(p-1)-\sigma} \quad \text{for all } x \in \Omega.$$

Remark 5.1. (i) Although we are mainly concerned with bounded f_0, but assumption (H2), with $\varepsilon_0(p-1) - \sigma < 0$, allows f_0 to blow up at points of the boundary $\partial\Omega$. (ii) Fix a bounded function f_0 on Ω and constants $p > 1$ and $0 < \sigma_0 < p$. We may choose constants $\varepsilon_0 \in (0, 1)$ and $C_0 > 0$ so that $\varepsilon_0(p-1) - \sigma_0 \le 0$ and $|f_0(x)| \le C_0(\lambda_0 \operatorname{dist}(x, \partial\Omega))^{\varepsilon_0(p-1)-\sigma_0}$ for $x \in \Omega$. Then, for any $\sigma_0 \le \sigma < p$, we have $|f_0(x)| \le C_1 \left(\lambda_0 \operatorname{dist}(x, \partial\Omega)\right)^{\varepsilon_0(p-1)-\sigma}$ for all $x \in \Omega$ and for some constant $C_1 > 0$ independent of σ. This remark is important when we discuss the asymptotic behavior of of the solution u_σ of (1.1)–(1.2) as $\sigma \to p$.

Henceforth in this section, we assume that the above hypotheses are valid, and we fix $R > 0$, $\varepsilon_0 \in (0, 1)$ and $C_0 > 0$ taken from (H1)–(H2).

The main result in this section is stated as follows.

Theorem 5.1. *Assume that* $\lambda_0 = 1$ *if* $p \ne 2$. *Then there exists a unique solution* $u \in C(\overline{\Omega})$ *of* (1.1)–(1.2).

Proof. In view of the Perron method (Theorem 3.5) and the comparison theorem (Theorem 4.1 and Remark 4.1), we need only to show that there exist a subsolution $\psi^- \in \mathrm{LSC}(\overline{\Omega})$ and a supersolution $\psi^+ \in \mathrm{USC}(\overline{\Omega})$ of (1.1) such that $\psi^- \le \psi^+$ in $\overline{\Omega}$ and $\psi^- = \psi^+$ on $\partial\Omega$, which is exactly what the next theorem guarantees. □

Theorem 5.2. *There exist functions* $\psi^+ \in \mathrm{USC}(\overline{\Omega})$ *and* $\psi^- \in \mathrm{LSC}(\overline{\Omega})$ *such that* ψ^+ *(resp.,* ψ^-*) is a supersolution (resp., subsolution) of* (1.1), $\psi^- \le \psi^+$ *on* $\overline{\Omega}$ *and* $\psi = g_0$ *on* $\partial\Omega$. *Moreover, the functions* $\psi^\pm$ *can be chosen independently of* σ.

Remark 5.2. The hypotheses of Theorem 4.1 exclude the case where $0 < \lambda_0 < 1$ and $p \ne 2$. But, even in this case, the proof of Theorem 5.1 ensures the existence of a solution u of (1.1)-(1.2) which is continuous at points of the boundary $\partial\Omega$, that is,

$$\lim_{\Omega \ni x \to y} u(x) = g_0(y) \quad \text{for } y \in \partial\Omega,$$

and may not be continuous in Ω.

The above theorem is an easy consequence of the following lemma.

Lemma 5.3. *Let* $g \in C^2(\overline{\Omega})$. *Then there is a function* $\psi \in C(\overline{\Omega})$ *such that* ψ *is a supersolution of* (1.1), $g \le \psi$ *on* $\overline{\Omega}$ *and* $\psi = g$ *on* $\partial\Omega$. *The choice of* ψ *does not depends on* σ.

Assuming the above lemma as true for the moment, we prove Theorem 5.2 as follows.

Proof of Theorem 5.2. We extend the domain of definition of g_0 to $\overline{\Omega}$ so that the resulting function, denoted again by g_0, is continuous on $\overline{\Omega}$. For any $0 < \varepsilon < 1$ we choose a function $g_\varepsilon \in C^2(\overline{\Omega})$ so that $\|g_\varepsilon - g_0\|_{\infty,\overline{\Omega}} < \varepsilon$. We apply Lemma 5.3 with $g = \varepsilon + g_\varepsilon$, to find a supersolution $\psi_\varepsilon^+ \in C(\overline{\Omega})$ of (1.1) such that $\psi_\varepsilon^+ \ge \varepsilon + g_\varepsilon$ on $\overline{\Omega}$ and $\psi_\varepsilon^+ = \varepsilon + g_\varepsilon$ on $\partial\Omega$. Here the choice of ψ_ε^+ is independent of σ. Now, we set

$$\psi^+(x) = \inf\{\psi_\varepsilon^+(x) \mid 0 < \varepsilon < 1\} \quad \text{for } x \in \overline{\Omega}.$$

This function ψ^+ is upper semicontinuous on $\overline{\Omega}$, is a supersolution of (1.1) due to Theorem 3.3 and satisfies the conditions that $g_0 \le \psi^+$ on $\overline{\Omega}$ and $g_0 = \psi^+$ on $\partial\Omega$.

Next we apply Lemma 5.3 with $-f_0$ and $\varepsilon - g_\varepsilon$ in place of f_0 and g, respectively, to obtain a supersolution ϕ_ε of

$$M[u](x) = -f_0 \quad \text{in } \Omega.$$

Setting $\psi_\varepsilon^- = -\phi_\varepsilon$, we observe that ψ_ε^- is a subsolution of (1.1) and satisfies the conditions that $\psi_\varepsilon^- \geq -\varepsilon + g_\varepsilon$ on $\overline{\Omega}$ and $\psi_\varepsilon^- = -\varepsilon + g_\varepsilon$ on $\partial\Omega$. As before, setting

$$\psi^-(x) = \sup\{\psi_\varepsilon^-(x) \mid 0 < \varepsilon < 1\} \quad \text{for } x \in \overline{\Omega},$$

we get a subsolution $\psi^- \in \mathrm{LSC}(\overline{\Omega})$ of (1.1), the choice of which is independent of σ, having the properties: $\psi^- \leq g_0$ on $\overline{\Omega}$ and $\psi^- = g_0$ on $\partial\Omega$. Noting that $\psi^- \leq g_0 \leq \psi^+$ on $\overline{\Omega}$, we conclude the proof. □

In this section we put $d(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \overline{\Omega}$ and

$$\Omega_\delta = \{x \in \Omega \mid d(x) > \delta\} \quad \text{for } \delta > 0.$$

To prove Lemma 5.3, we need the following lemma.

Lemma 5.4. *Let $\varepsilon \in (0, 1)$. Define the function $\phi_\varepsilon \in C(\overline{\Omega})$ by $\phi_\varepsilon(x) = d(x)^\varepsilon$. Then there are constants $\delta = \delta_{\varepsilon,R}$, $C = C_R > 0$, $\gamma = \gamma_{\varepsilon,R} > 0$ and, for each $x \in \Omega \setminus \Omega_\delta$, a unit vector $e = e_x \in \mathbb{R}^n$ such that for any $z \in B(0, d(x))$,*

$$\phi_\varepsilon(x+z) - \phi_\varepsilon(x) \leq \begin{cases} \varepsilon d(x)^{\varepsilon-1}\left(e\cdot z + C|z|^2\right), \\ \varepsilon d(x)^{\varepsilon-1}\left(e\cdot z - \gamma d(x)^{-1}|z|^2\right) & \text{if } |e\cdot z| \geq |z|/2. \end{cases} \tag{5.1}$$

Now, assuming Lemma 5.4 as true, we give the proof of Lemma 5.3.

Proof of Lemma 5.3. In this proof we write ε for ε_0 for notational simplicity. Let ϕ_ε, C_R, $\gamma = \gamma_{\varepsilon,R}$ and $\delta = \delta_{\varepsilon,R}$ be as in Lemma 5.4. Fix a constant $C \geq C_R \vee 1$ so that

$$C_0 \vee \|g\|_{\infty,\Omega} \vee \|Dg\|_{\infty,\Omega} \vee \|D^2 g\|_{\infty,\Omega} \leq C.$$

Here, to be sure, we write $\|D^2 g\|_{\infty,\Omega} := \sup\{|D^2 g(x)\xi\cdot\xi| \mid x \in \Omega,\ \xi \in B(0, 1)\}$.

It is easy to see that there is a quadratic function $\psi_0 \in C^2(\mathbb{R}^n)$ such that

$$\psi_0(x+z) - \psi_0(x) \leq D\psi_0(x)\cdot z - |z|^2 \quad \text{for all } x, z \in \mathbb{R}^n$$

and

$$\operatorname{diam}(\Omega) + 1 \leq |D\psi_0(x)| \leq 3\operatorname{diam}(\Omega) + 1 \quad \text{for all } x \in \overline{\Omega}.$$

We may moreover assume that $\psi_0 \geq 0$ on $\overline{\Omega}$. We fix such a function ψ_0.

Now, we fix $x \in \Omega$ and set $q_0 = D\psi_0(x)$ and

$$\Sigma_0 = \{z \in B(0, \rho(x)) \mid |q_0\cdot z| \geq |q_0||z|/2\}.$$

Note that Σ_0 is symmetric, i.e., $-\Sigma_0 = \Sigma_0$ and the volume of Σ_0 is comparable to that of $B(0, \rho(x))$, i.e., $|\Sigma_0| = \tau_n|B(0, \rho(x))|$ for some constant $\tau_n \in (0, 1)$. We observe that for some $\theta \in (0, 1)$,

$$\begin{aligned} &G(\psi_0(x+z) - \psi_0(x)) = G(q_0\cdot z) - G'(q_0\cdot z - \theta|z|^2)|z|^2 \\ &\leq \begin{cases} G(q_0\cdot z) & \text{for all } z \in B(0, \rho(x)) \\ G(q_0\cdot z) - (p-1)|q_0\cdot z - \theta|z|^2|^{p-2}|z|^p & \text{for all } z \in \Sigma_0. \end{cases} \end{aligned}$$

Let $z \in \Sigma_0$ and $\theta \in (0, 1)$, and observe that if $p \geq 2$, then

$$|q_0\cdot z - \theta|z|^2|^{p-2} \geq 2^{2-p}|z|^{p-2}\big||q_0| - 2|z|\big|^{p-2} \geq 2^{2-p}|z|^{p-2}$$

and if $p<2$, then

$$|q_0\cdot z-\theta|z|^2|^{p-2}\geq |z|^{p-2}||q_0|+|z||^{p-2}\geq (4\,\mathrm{diam}\,(\Omega)+1)^{p-2}|z|^{p-2}.$$

Here we have used the condition that $\mathrm{diam}\,(\Omega)+1\leq |q_0|\leq 3\,\mathrm{diam}\,(\Omega)+1$. Setting

$$b_p=\begin{cases} (p-1)\,2^{2-p} & \text{if } p\geq 2,\\ (p-1)(4\,\mathrm{diam}\,(\Omega)+1)^{p-2} & \text{if } p<2,\end{cases}$$

we have for $z\in\Sigma_0$,

$$G(\psi_0(x+z)-\psi_0(x))\leq G(q_0\cdot z)-b_p|z|^2,$$

and obtain

$$\begin{aligned} M[\psi_0](x)\leq &\int_{B(0,\,\rho(x))\setminus\Sigma_0} G(q_0\cdot z)K(z)\,\mathrm{d}z\\ &+\int_{\Sigma_0}\big(G(q_0\cdot z)-b_p|z|^p\big)K(z)\,\mathrm{d}z\\ =&-b_p\mu\int_{\Sigma_0}|z|^{p-n-\sigma}\,\mathrm{d}z=-b_p\tau_n\sigma_n\rho(x)^{p-\sigma}.\end{aligned}$$

Thus, noting that $p=\varepsilon(p-1)+(1-\varepsilon)p+\varepsilon$ and setting $b_0=b_p\tau_n\sigma_n(\lambda_0\delta/2)^{(1-\varepsilon)p+\varepsilon}$, we get

$$M[\psi_0](x)\leq -b_0\rho(x)^{\varepsilon(p-1)-\sigma}\quad \text{for all } x\in\Omega_{\delta/2}. \tag{5.2}$$

Let $A\geq 1$ be a constant to be fixed later on, and set

$$v(x)=g(x)+A\phi_\varepsilon(x)\quad \text{for } x\in\Omega.$$

Fix $x\in\Omega\setminus\Omega_\delta$ and let $e\in\mathbb{R}^n$ be a unit vector which satisfies (5.1). We set $\Sigma=\{z\in B(0,\,\rho(x))\mid |e\cdot z|\geq |z|/2\}$, $q_1=Dg(x)+\varepsilon d(x)^{\varepsilon-1}Ae$ and $\gamma_1=\gamma\varepsilon d(x)^{\varepsilon-2}A/2$. We may assume by replacing γ and δ by smaller positive numbers if needed that $\delta\leq 4\gamma\leq 1$. We now assume that $4C\leq\varepsilon\delta^{\varepsilon-1}A$. Then we have $C\leq\gamma\varepsilon\delta^{\varepsilon-2}A$ and

$$\frac{C}{2}-\gamma\varepsilon d(x)^{\varepsilon-2}A\leq -\frac{\gamma\varepsilon d(x)^{\varepsilon-2}A}{2}=\gamma_1.$$

Hence, by (5.1), we have for any $z\in\Sigma$,

$$v(x+z)-v(x)\leq q_1\cdot z-\gamma_1|z|^2.$$

Observe also that for any $z\in\Sigma$,

$$\begin{aligned} &\gamma_1|z|^2\leq\frac{\gamma\varepsilon d(x)^{\varepsilon-1}A|z|}{2}\leq\frac{\varepsilon d(x)^{\varepsilon-1}A|z|}{8},\\ &|q_1\cdot z|\geq\varepsilon d(x)^{\varepsilon-1}A|e\cdot z|-C|z|\geq\frac{\varepsilon d(x)^{\varepsilon-1}|z|A}{4},\\ &|q_1\cdot z|\leq\varepsilon d(x)^{\varepsilon-1}A|z|+C|z|\leq 2\varepsilon d(x)^{\varepsilon-1}A|z|.\end{aligned}$$

Hence, for any $z\in\Sigma$ and $\theta\in(0,\,1)$, if $p\geq 2$, then

$$G'(q_1\cdot z-\theta\gamma_1|z|^2)=(p-1)\big|q_1\cdot z-\theta\gamma_1|z|^2\big|^{p-2}\geq (p-1)\Big(\frac{\varepsilon d(x)^{\varepsilon-1}A|z|}{8}\Big)^{p-2},$$

and if $1<p<2$, then

$$G'(q_1\cdot z-\theta\gamma_1|z|^2)\geq (p-1)\,\big(2\varepsilon d(x)^{\varepsilon-1}A|z|\big)^{p-2}.$$

Thus, setting

$$c_p = \begin{cases} (p-1)\, 8^{2-p} & \text{if } p \ge 2, \\ (p-1)\, 2^{p-2} & \text{if } p < 2, \end{cases}$$

we get

$$G(v(x+z) - v(x)) \le G(q_1 \cdot z) - c_p \left(\varepsilon d(x)^{\varepsilon-1} A\right)^{p-2} \gamma_1 |z|^p \quad \text{for } z \in \Sigma,$$

and consequently

$$\begin{aligned} \int_\Sigma G(v(x+z) - v(x)) K(z)\, \mathrm{d}z &\le -c_p \left(\varepsilon d(x)^{\varepsilon-1} A\right)^{p-2} \gamma_1 \mu \int_\Sigma |z|^{p-n-\sigma}\, \mathrm{d}z \\ &= -\frac{1}{2} c_p \gamma (\varepsilon A)^{p-1} d(x)^{-(1-\varepsilon)p-\varepsilon} \tau_n \sigma_n \rho(x)^{p-\sigma} \\ &\le -\frac{1}{2} c_p \tau_n \sigma_n \gamma (\varepsilon A)^{p-1} \lambda_0^{(1-\varepsilon)p+\varepsilon} \rho(x)^{\varepsilon(p-1)-\sigma}. \end{aligned} \tag{5.3}$$

Next, we give an estimate of the integral

$$I := \int_{B(0,\, \rho(x)) \setminus \Sigma} G(v(x+z) - v(x)) K(z)\, \mathrm{d}z.$$

We have $v(x+z) - v(x) \le q_1 \cdot z + C(1 + \varepsilon d(x)^{\varepsilon-1} A)|z|^2$ for $z \in B(0,\, \rho(x))$. Noting that

$$|q_1| \vee C(1 + \varepsilon d(x)^{\varepsilon-1} A) \le Q := 2\varepsilon d(x)^{\varepsilon-1} AC$$

and arguing as in the proofs of Theorems 2.2 and 2.3, we find a constant $C_1 > 0$, depending only on p and n, such that if $p \ge 2$, then

$$\begin{aligned} I \le& C_1 Q(Q + \rho(x)Q)^{p-2} \rho(x)^{p-\sigma} = C_1 Q^{p-1} (1 + \rho(x))^{p-2} \rho(x)^{p-\sigma} \\ &\le 2^{p-2} C_1 \left(2\varepsilon AC\right)^{p-1} d(x)^{(\varepsilon-1)(p-1)} \rho(x)^{p-\sigma} \le C_1 (4\varepsilon AC)^{p-1} \rho(x)^{(\varepsilon-1)(p-1)+p-\sigma} \\ &= C_1 (4\varepsilon AC)^{p-1} \rho(x)^{\varepsilon(p-1)-\sigma+1}, \end{aligned}$$

and if $p < 2$, then

$$I \le C_1 Q^{p-1} \rho(x)^{p-\sigma} \le C_1 (2\varepsilon AC)^{p-1} d(x)^{(\varepsilon-1)(p-1)} \rho(x)^{p-2} \le C_1 (2\varepsilon AC)^{p-1} \rho(x)^{\varepsilon(p-1)-\sigma+1}.$$

Here we have used that $\rho(x) \le \delta \le 1$. From these and (5.3), we get

$$M^+[v](x) \le (\varepsilon A)^{p-1} \Big((4C)^{p-1} C_1 \delta - \frac{1}{2} c_p \tau_n \sigma_n \gamma \lambda_0^{(1-\varepsilon)p+\varepsilon} \Big) \rho(x)^{\varepsilon(p-1)-\sigma}.$$

Set $c_0 = c_p \tau_n \sigma_n \gamma \lambda_0^{(1-\varepsilon)p+\varepsilon}/4$. Replacing $\delta > 0$ by a smaller number if needed, we may assume that $(4C)^{p-1} C_1 \delta \le c_0$. Then we have

$$M^+[v](x) \le -c_0 (\varepsilon A)^{p-1} \rho(x)^{\varepsilon(p-1)-\sigma} \quad \text{for all } x \in \Omega \setminus \Omega_\delta.$$

We now assume that $c_0 (\varepsilon A)^{p-1} \ge C$, and then we get

$$M^+[v](x) \le -C \rho(x)^{\varepsilon(p-1)-\sigma} \quad \text{for all } x \in \Omega \setminus \Omega_\delta. \tag{5.4}$$

At this stage, our requirement on A is that $A \ge A_1$, where

$$A_1 := \max\Big\{1,\, \frac{4C}{\varepsilon \delta^{\varepsilon-1}},\, \frac{1}{\varepsilon}\Big(\frac{C}{c_0}\Big)^{\frac{1}{p-1}}\Big\}.$$

By (5.2), for any constant $B > 0$, we have

$$M[B\psi_0](x) \le -B^{p-1} b_0 \rho(x)^{\varepsilon(p-1)-\sigma} \quad \text{for } x \in \Omega_{\delta/2}.$$

We fix $B > 0$ so that $B^{p-1}b_0 \geq C$, and we have

$$M[B\psi_0](x) \leq -C\rho(x)^{\varepsilon(p-1)-\sigma} \quad \text{for all } x \in \Omega_{\delta/2}. \tag{5.5}$$

We set

$$L := B \max_{\overline{\Omega}} \psi_0 \in (0, \infty) \quad \text{and} \quad j_\varepsilon(t) = t^\varepsilon \quad \text{for} \quad t \geq 0,$$

and observe that

$$\begin{aligned} \sup_{\Omega \setminus \Omega_{\delta/2}} v \leq & C + A j_\varepsilon(\delta/2), \\ \inf_{\Omega_\delta} v \geq & - C + A j_\varepsilon(\delta). \end{aligned}$$

Since $j_\varepsilon(\delta) > j_\varepsilon(\delta/2)$, we may choose a constant $A_2 > 0$ so that

$$A_2 \left(j_\varepsilon(\delta) - j_\varepsilon(\delta/2) \right) \geq L + 2C.$$

We finally fix $A = A_1 \vee A_2$, and define the functions $w, \psi \in C(\overline{\Omega})$ by

$$\begin{aligned} w(x) =& C + A j_\varepsilon(\delta/2) + B\psi_0(x), \\ \psi(x) =& \begin{cases} v(x) & \text{if } x \in \overline{\Omega} \setminus \Omega_{\delta/2}, \\ v(x) \wedge w(x) & \text{if } x \in \Omega_{\delta/2} \setminus \Omega_\delta, \\ w(x) & \text{if } x \in \Omega_\delta. \end{cases} \end{aligned}$$

It is easily checked that $\psi \geq g$ on Ω and $\psi = g$ on $\partial\Omega$ and also that $\psi(x) = v(x) \wedge w(x)$ on Ω.

It remains to check that ψ is a supersolution of (1.1). Let $\phi \in \mathcal{T}_p(\Omega)$ and $y \in \Omega$, and assume that $\psi - \phi$ attains a minimum at y. We may assume that $(\psi - \phi)(y) = 0$, so that $\psi \geq \phi$ in Ω. We divide our consideration into three cases. First, we consider the case where $y \in \Omega_{\delta/2}$ and $\psi(y) = w(y)$. Since $\phi \leq \psi = v \wedge w$ in Ω, we see from (5.5) that

$$M[\phi](y) \leq M[w](y) \leq f_0(y).$$

Next, consider the case where $y \in \Omega_{\delta/2}$ and $\psi(y) \neq w(y)$. Then we have $y \in \Omega_{\delta/2} \setminus \Omega_\delta$ and $\psi(y) = v(y)$. Hence, from (5.4), we get

$$M[\phi](y) \leq M^+[v](y) \leq f_0(y).$$

The last case is where $y \in \Omega \setminus \Omega_{\delta/2}$. But then we have $\phi(y) = \psi(y) = v(y)$ and, as in the previous case, we get

$$M[\phi](y) \leq M^+[v](y) \leq f_0(y),$$

which completes the proof. □

We need the following lemma for the proof of Lemma 5.4.

Lemma 5.5. *Let $r > 0$, $0 < \varepsilon < 1$, and $e \in \mathbb{R}^n$ be a unit vector. Set $x = (R + r)e$. Then there are positive constants $c_{\varepsilon,R}$ and $\delta_{\varepsilon,R}$, depending only on ε and R, such that for any $z \in B(0, r)$, if $r \leq \delta_{\varepsilon,R}$, then*

$$(|x + z| - R)^\varepsilon - (|x| - R)^\varepsilon \leq \begin{cases} \varepsilon r^{\varepsilon-1} \Big(e \cdot z + \dfrac{|z|^2}{2R} \Big), \\ \varepsilon r^{\varepsilon-1} \Big(e \cdot z - c_{\varepsilon,R} r^{-1} |z|^2 \Big) & \text{if} \ \ |e \cdot z| \geq \frac{|z|}{2}. \end{cases}$$

Proof. We fix any $z \in B(0, r)$ and observe that for some $\theta \in (0,\, 1)$,

$$(|x+z|-R)^{\varepsilon}-(|x|-R)^{\varepsilon} \leq \varepsilon(|x|-R)^{\varepsilon-1}(|x+z|-|x|) - \frac{\varepsilon(1-\varepsilon)}{2}(|x+\theta z|-R)^{\varepsilon-2}(|x+z|-|x|)^2.$$

We set $f(y)=|x+y|$ for $y \in \mathbb{R}^n$ and compute that if $x+y \neq 0$, then

$$Df(y)=\frac{x+y}{|x+y|} \quad \text{and} \quad D^2f(y)=\frac{1}{|x+y|}\Big(I-\frac{(x+y)\otimes(x+y)}{|x+y|^2}\Big),$$

where I denotes the identity matrix of order n and $v \otimes v := (v_iv_j)_{1\leq i,j\leq n}$ for $v=(v_1,v_2,...,v_n)$. Hence, we have

$$|x+z|-|x| \leq e\cdot z+\frac{|z|^2}{2|x+\theta z|}$$

for some $\theta \in (0,\, 1)$. Thus, noting that $R \leq |x+\theta z| \leq R+2r$ for $\theta \in (0,\, 1)$, we get

$$(|x+z|-R)^{\varepsilon}-(|x|-R)^{\varepsilon} \leq \varepsilon r^{\varepsilon-1}\Big(e\cdot z+\frac{|z|^2}{2R}\Big) - \frac{\varepsilon(1-\varepsilon)}{2}(2r)^{\varepsilon-2}(|x+z|-|x|)^2. \tag{5.6}$$

In particular, we have

$$(|x+z|-R)^{\varepsilon}-(|x|-R)^{\varepsilon} \leq \varepsilon r^{\varepsilon-1}\Big(e\cdot z+\frac{|z|^2}{2R}\Big).$$

We assume henceforth that $|e\cdot z| \geq |z|/2$. Note that

$$2|x\cdot z|-|z|^2 \geq (R+r)|z|-r|z|=R|z|,$$

and

$$(|x+z|-|z|)^2=\frac{(|x+z|^2-|x|^2)^2}{(|x+z|+|x|)2} \geq \frac{(R|z|)^2}{(2R+3r)^2}. \tag{5.7}$$

We choose $\delta_{\varepsilon,R}>0$ so that

$$\frac{1}{R} \leq (1-\varepsilon)2^{\varepsilon-3}\frac{R^2}{(2R+3\delta)^2},$$

and set

$$c_{\varepsilon,R} := (1-\varepsilon)2^{\varepsilon-3}\frac{R^2}{(2R+3\delta)^2}.$$

From (5.6) and (5.7), if $r \leq \delta_{\varepsilon,R}$, we get

$$(|x+z|-R)^{\varepsilon}-(|x|-R)^{\varepsilon} \leq \varepsilon r^{\varepsilon-1}\left(e\cdot z-c_{\varepsilon,R}r^{-1}|z|^2\right),$$

which completes the proof. □

Proof of Lemma 5.4. Let $c=c_{\varepsilon,R}$ and $\delta=\delta_{\varepsilon,R}$ be positive constants from Lemma 5.5. Fix any $x \in \Omega\setminus\Omega_\delta$. Set $r := d(x) \in (0,\, \delta]$ and select a point $\xi \in \partial\Omega$ so that $r=|\xi-x|$. By the uniform exterior sphere condition (H1), there is a point $\eta \in \mathbb{R}^n$ such that $B(\eta,\, R)\cap\overline{\Omega}=\{\xi\}$. By translation, we may assume that $\eta=0$. Setting $e=x/|x|$, we have $x=(R+r)e$ and $\xi=Re$. Note also that $d(x)^{\varepsilon}=r^{\varepsilon}=(|x|-R)^{\varepsilon}$. Let $z \in B(0,r)$. Setting $\bar{e}=(x+z)/|x+z|$, we observe that $R\bar{e} \notin \Omega$,

$$d(x+z) \leq |x+z-R\bar{e}|=|x+z|-R,$$

and

$$d(x+z)^\varepsilon - d(x)^\varepsilon \le (|x+z|-R)^\varepsilon - (|x|-R)^\varepsilon.$$

Now, by virtue of Lemma 5.5, we see that

$$\phi_\varepsilon(x+z) - \phi_\varepsilon(x) \le \begin{cases} \varepsilon r^{\varepsilon-1}\Big(e\cdot z + \dfrac{|z|^2}{2R}\Big), \\ \varepsilon r^{\varepsilon-1}\left(e\cdot z - cr^{-1}|z|^2\right) \quad \text{if } |e\cdot z| \ge \frac{|z|}{2}. \end{cases}$$

This completes the proof. □

6. Comparison results for the p-Laplace equation

In this section we recall some of basic results on the inhomogeneous p-Laplace equation

$$\Delta_p u = f_0(x) \quad \text{in } \Omega \tag{6.1}$$

and formulate comparison results for (6.1). The results in this section are more or less well-known (see [12]), and thus we give only a brief sketch of their proofs. We refer to [12] for results and proofs similar to those in this section.

We are concerned with the Dirichlet problem for (6.1) with the Dirichlet condition (1.2), i.e., the condition $u = g_0$ on $\partial\Omega$. We may assume that g_0 is a continuous function on $\overline{\Omega}$ and moreover $g_0 \in C^2(\Omega)$.

We call any function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ a *weak solution* of (6.1) if

$$-\int_\Omega |Du(x)|^{p-2}Du(x)\cdot D\psi(x)\,\mathrm{d}x = \int_\Omega f_0(x)\psi(x)\,\mathrm{d}x \quad \text{for all } \psi \in C_0^\infty(\Omega).$$

Also we call any function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ a *weak subsolution* (resp., *supersolution*) of (1.1) if

$$-\int_\Omega |Du(x)|^{p-2}Du(x)\cdot D\psi(x)\,\mathrm{d}x \ge \int_\Omega f_0(x)\psi(x)\,\mathrm{d}x,$$

$$\text{(resp.,}\quad -\int_\Omega |Du(x)|^{p-2}Du(x)\cdot D\psi(x)\,\mathrm{d}x \le \int_\Omega f_0(x)\psi(x)\,\mathrm{d}x\ )$$

for all $\psi \in C_0^\infty(\Omega)$ with $\psi \ge 0$.

In this paper, the Dirichlet condition (1.2) for weak solutions u of (6.1) is understood in the pointwise sense, i.e.,

$$\lim_{x\to\partial\Omega}(u-g_0)(x) = 0.$$

Next, following [11, 14], we recall the definition of viscosity solutions of (6.1). We call any bounded function u in Ω a *viscosity subsolution* (resp., *supersolution*) of (6.1) provided that for any $(x,\phi) \in \Omega \times \mathcal{T}_p(\Omega)$ for which $u^* - \phi$ (resp., $u_* - \phi$) attains a local maximum (resp., minimum) at x, we have

$$\Delta_p\phi(x) \ge f_0(x) \qquad (\text{resp., } \Delta_p\phi(x) \le f_0(x)\,) \quad \text{if } D\phi(x) \ne 0,$$

and

$$0 \ge f_0(x) \qquad (\text{resp., } 0 \le f_0(x)\,) \quad \text{if } D\phi(x) = 0.$$

We call any bounded function u on Ω a *viscosity solution* of (6.1) if it is both a viscosity sub and supersolution of (6.1).

We assume throughout this section that the uniform exterior sphere condition (H1) holds and that $f_0 \in C(\Omega)$ is bounded on Ω.

Theorem 6.1. *Let $u, v \in W^{1,p}_{\rm loc}(\Omega)$ be weak sub and supersolutions of* (6.1)*, respectively. Assume that*

$$\limsup_{x\to\partial\Omega}(u-v)(x)\le 0.$$

Then $u \le v$ a.e. in Ω.

Proof. Fix any $\varepsilon > 0$ and replace u by $u - \varepsilon$. Then $w := (u-v)_+ \in W^{1,p}_0(\Omega)$, and we get

$$-\int_{w>0}\left(|Du|^{p-2}Du-|Dv|^{p-2}Dv\right)\cdot(Du-Dv)\,\mathrm{d}x\ge 0,$$

which implies that

$$\int_{w>0}\left(|Du|^{p-2}Du-|Dv|^{p-2}Dv\right)\cdot(Du-Dv)\,\mathrm{d}x= 0.$$

Observe (see [15, Lemma 1]) that there is a constant $\gamma_p > 0$ such that for all $a, b \in \mathbb{R}^n$,

$$(|a|^{p-2}-|b|^{p-2}b)\cdot(a-b)\ge\begin{cases}\gamma_p|a-b|^p & \text{if } p\ge 2,\\ \dfrac{\gamma_p|a-b|^2}{(|a|+|b|)^{2-p}} & \text{if } p<2.\end{cases}$$

Accordingly, if $p \ge 2$, then we have

$$\begin{aligned}&\int_{w>0}|D(u-v)|^p\,\mathrm{d}x\\ &\le\gamma_p^{-1}\int_{w>0}\left(|Du|^{p-2}Du-|Dv|^{p-2}Dv\right)\cdot(Du-Dv)=0,\end{aligned}$$

and, if $1 < p < 2$, then we have

$$\begin{aligned}&\int_{w>0}|Du-Dv|^p\,\mathrm{d}x\\ &\le\left(\int_{w>0}\frac{|Du-Dv|^2}{(|Du|+|Dv|)^{2-p}}\,\mathrm{d}x\right)^{p/2}\left(\int_{w>0}(|Du|+|Dv|)^p\,\mathrm{d}x\right)^{(2-p)/2}\\ &\le\left(\gamma_p^{-1}\int_{w>0}\left(|Du|^{p-2}Du-|Dv|^{p-2}Dv\right)\cdot(Du-Dv)\,\mathrm{d}x\right)^{p/2}\\ &\quad\times\left(\int_{w>0}(|Du|+|Dv|)^p\,\mathrm{d}x\right)^{(2-p)/2}=0.\end{aligned}$$

Thus we find that $w = 0$ and hence $u \le v + \varepsilon$ a.e. in Ω, which shows that $u \le v$ a.e. in Ω. □

Lemma 6.2. *For each $x \in \partial\Omega$ and $\varepsilon > 0$ there exist a weak supersolution $\psi^+_{x,\varepsilon} \in C^\infty(\overline{\Omega})$ and a weak subsolution $\psi^-_{x,\varepsilon} \in C^\infty(\overline{\Omega})$ of* (6.1) *such that $\psi^-_{x,\varepsilon} \le g_0 \le \psi^+_{x,\varepsilon}$ in $\overline{\Omega}$ and $\psi^+_{x,\varepsilon}(x) - \varepsilon \le g_0(x) \le \psi^-_{x,\varepsilon}(x) + \varepsilon$.*

Proof. Fix any $x \in \partial\Omega$ and $\varepsilon > 0$. Let $y \in \mathbb{R}^n$ and $R > 0$ be those from condition (H1). Let $C > 0$ and $\alpha > 0$ be constants to be selected later. We define the function $f \in C^\infty(\mathbb{R}^n)$ by

$$f(z)=C(\mathrm{e}^{-\alpha R^2}-\mathrm{e}^{-\alpha|z-y|^2}).$$

By a simple computation, we get

$$\Delta_p f(z)=(2\alpha C)^{p-1}\mathrm{e}^{-\alpha(p-1)|z-y|^2}|z-y|^{p-2}\left(n+p-2-2\alpha(p-1)|z-y|^2\right).$$

We choose $\alpha > 0$ so that $2\alpha(p-1)R^2 > n+p-2$ and then $C > 0$ so that

$$\Delta_p f(z) \le f_0(z) \quad \text{and} \quad \varepsilon + f(z) \ge g_0(z) \quad \text{for all } z \in \overline{\Omega}.$$

The function $f(z)+\varepsilon$ has the properties required of the function $\psi^+_{x,\varepsilon}$ in the lemma. The function $\psi^-_{x,\varepsilon}$ can be constructed in a similar way. □

We need the following well-known Hölder gradient estimate for the solutions of (6.1). We refer to [7, 13, 15] for this estimate.

Lemma 6.3. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of* (6.1). *There is a constant $\alpha \in (0, 1)$, depending only on p and n, and for each ball $B := B(x_0, 2r) \subset \Omega$ a constant $C > 0$, depending only on p, n, r, $\|u\|_{\infty,B}$ and $\|f_0\|_{\infty,B}$, such that*

$$|Du(x) - Du(x')| \le C|x-x'|^\alpha \quad \textit{for all } x, x' \in B(x_0, r).$$

The constant C can be chosen so that it is nondecreasing in $\|u\|_{\infty,B}$ and $\|f_0\|_{\infty,B}$.

Theorem 6.4. *There is a unique weak solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ of* (6.1) *and* (1.2).

Proof. We choose a sequence $\{g_k\} \subset C^1(\overline{\Omega})$ such that, as $k \to \infty$, $g_k \to g_0$ uniformly on $\overline{\Omega}$ and $Dg_k \to Dg_0$ locally uniformly in Ω. For each $k \in \mathbb{N}$ we consider the convex minimization problem

$$\inf\{I(v) \mid v \in g_k + W^{1,p}_0(\Omega)\}, \tag{6.2}$$

where $k \in \mathbb{N}$ and

$$I(v) = \int_\Omega \Big(\frac{1}{p}|Dv|^p + f_0 v\Big)\,\mathrm{d}x.$$

It is a standard observation that for each $k \in \mathbb{N}$, the minimization problem (6.2) has a unique solution $u_k \in g_k + W^{1,p}_0(\Omega)$ and it is a weak solution of (6.1).

According to Lemma 6.2, there are functions $\psi^\pm \in C^\infty(\overline{\Omega})$ such that ψ^+ (resp., ψ^-) is a weak supersolution (resp., subsolution) of (6.1) and $\psi^- \le g_k \le \psi^+$ on $\overline{\Omega}$ for all $k \in \mathbb{N}$. By an argument similar to the proof of Theorem 6.1, we see that $\psi^- \le u_k \le \psi^+$ a.e. in Ω for all $k \in \mathbb{N}$. By Lemma 6.3, we may assume that $u_k \in C^{1,\alpha}(\Omega)$ for all n and for some $\alpha \in (0, 1)$ and that the sequence $\{u_k\}$ is precompact in $C^1(\Omega)$. Thus, the sequence u_k has a subsequence $\{u_{k_j}\}$ such that $(u_{k_j}, Du_{k_j}) \to (u, Du)$ locally uniformly in Ω for some function $u \in C^{1,\alpha}(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ as $j \to \infty$. It is easily seen that u is a weak solution of (6.1). We extend the domain of definition of u up to $\partial\Omega$ by setting $u(x) = g_0(x)$ for all $x \in \partial\Omega$.

We now show that $u \in C(\overline{\Omega})$. Fix any $x \in \partial\Omega$ and $\varepsilon > 0$. Let $\psi^\pm_{x,\varepsilon} \in C^\infty(\overline{\Omega})$ be two functions from Lemma 6.2. If $k \in \mathbb{N}$ is sufficiently large, then we have

$$\psi^-_{x,\varepsilon}(z) - 2\varepsilon \le g_k(z) \le \psi_{x,\varepsilon}(z) + 2\varepsilon \quad \text{for all } z \in \overline{\Omega}.$$

By comparison, we see that if k is sufficiently large, then

$$\psi_{x,\varepsilon}(z) - 2\varepsilon \le u_k(z) \le \psi^+_{x,\varepsilon}(z) + 2\varepsilon \quad \text{for all } z \in \overline{\Omega},$$

which obviously implies that u is continuous at x. Thus u is a continuous function on $\overline{\Omega}$.

The uniqueness of weak solutions of (6.1) and (1.2) follows from Theorem 6.1. □

Theorem 6.5. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{USC}(\Omega)$* (*resp., $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{LSC}(\Omega)$*) *be a weak subsolution* (*resp., supersolution*) *of* (6.1). *Then it is a viscosity subsolution* (*resp., supersolution*) *of* (6.1).

Proof. Note that $w \in W^{1,p}_{\text{loc}}(\Omega) \cap \text{LSC}(\Omega)$ is a weak (resp., viscosity) supersolution of (6.1) if and only if $-w \in W^{1,p}_{\text{loc}}(\Omega) \cap \text{USC}(\Omega)$ is a weak (resp., viscosity) subsolution of (6.1) with $-f_0$ in place of f_0. Hence, we need only to prove the subsolution part of the assertion.

Let $U \subset \Omega$ be an open ball such that $\overline{U} \subset \Omega$. Suppose that u is not a viscosity subsolution of (6.1) in U. Then there is a function $\phi \in \mathcal{T}_p(U) \cap C(\overline{U})$ such that $u - \phi$ attains a strict maximum over $\overline{U}$ at some point $x_0 \in U$ and

$$\begin{cases} \Delta_p \phi(x_0) < f_0(x_0) & \text{if } D\phi(x_0) \neq 0, \\ 0 < f_0(x_0) & \text{if } D\phi(x_0) = 0. \end{cases}$$

By replacing the function $\phi(x)$ by the function $C|x - x_0|^{\beta+1}$ with a sufficiently large $C > 0$ and a $\beta > 1/(p-1)$ if $1 < p < 2$ and $D\phi(x_0) = 0$, we may assume that $|D\phi|^{p-2} D\phi \in C^1(U)$, and then it is easily checked that ϕ is a weak solution of (6.1) in a neighborhood $V \subset U$ of x_0. Adding a constant to u, we may assume that $(u-\phi)(x_0) > 0$ and $\max_{\partial V}(u-\phi) < 0$. By the comparison theorem (Theorem 6.1), we find that $u \leq \phi$ in V, which is a contradiction. This guarantees that u is a viscosity subsolution of (6.1). □

Proposition 6.6. *Let $f_1, f_2 \in C(\overline{\Omega})$ satisfy $f_1 > f_2$ on $\overline{\Omega}$. Let $u \in \text{USC}(\overline{\Omega})$ (resp., $v \in \text{LSC}(\overline{\Omega})$) be a viscosity subsolution (resp., supersolution) of (6.1) with f_1 (resp., f_2) in place of f_0. Assume that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

Proof. We argue by contradiction, and thus assume that $\max_{\overline{\Omega}}(u - v) > 0$. Fix a $\beta \geq 1$ so that $\beta > 1/(p-1)$, and set $\phi(x) = |x|^{\beta+1}$ for $x \in \mathbb{R}^n$. For any $\alpha > 1$ we consider the function

$$u(x) - v(y) - \alpha\phi(x - y) \quad \text{on } \overline{\Omega} \times \overline{\Omega}$$

and choose a maximum point (x_α, y_α) of it. Restricting our attention to sufficiently large α, we may assume that $x_\alpha, y_\alpha \in \Omega$. Setting

$$q_\alpha := \alpha D\phi(x_\alpha - y_\alpha) = \alpha(\beta+1)|x_\alpha - y_\alpha|^{\beta-1}(x_\alpha - y_\alpha),$$

noting that

$$0 \leq D^2\phi(x) \leq (\beta+1)\beta|x|^{\beta-1} I \quad \text{for all } x \in \mathbb{R}^n,$$

and using, for instance, [5, Theorem 3.2], we find an $n \times n$ real matrix X_α such that

$$(q_\alpha, X_\alpha) \in \overline{J}^{2,+} u(x_\alpha) \quad \text{and} \quad (q_\alpha, X_\alpha) \in \overline{J}^{2,-} v(y_\alpha).$$

Here we refer the reader to [5] for the definition of semijets $\overline{J}^{2,\pm}$. Note that for every $\psi \in \mathcal{T}_p(\Omega)$, if $D\psi(x) \neq 0$, then

$$\Delta_p \psi(x) = |D\psi(x)|^{p-4} \operatorname{tr} \left(|D\psi(x)|^2 D^2\psi(x) + (p-2)(D\psi(x) \otimes D\psi(x)) D^2\psi(x) \right).$$

Now, by the viscosity property of u and v, we get

$$|q_\alpha|^{p-4} \operatorname{tr} \left(|q_\alpha|^2 X_\alpha + (p-2)(q_\alpha \otimes q_\alpha) X_\alpha \right) \geq f_1(x_\alpha),$$
$$|q_\alpha|^{p-4} \operatorname{tr} \left(|q_\alpha|^2 X_\alpha + (p-2)(q_\alpha \otimes q_\alpha) X_\alpha \right) \leq f_2(y_\alpha)$$

if either $p \geq 2$ or $q_\alpha \neq 0$, and

$$0 \geq f_1(x_\alpha) \quad \text{and} \quad 0 \leq f_2(y_\alpha)$$

otherwise. From these, we see that $f_1(x_\alpha) \leq f_2(y_\alpha)$. Sending $\alpha \to 0$, we conclude that $f_1(x_0) \leq f_2(x_0)$ for some $x_0 \in \overline{\Omega}$, but this contradicts our assumption that $f_1 > f_2$ on $\overline{\Omega}$. □

The following Theorem improves the previous proposition.

Theorem 6.7. *Let* $u \in \mathrm{USC}(\overline{\Omega})$ *and* $v \in \mathrm{LSC}(\overline{\Omega})$ *be, respectively, viscosity sub and supersolutions of* (6.1). *Assume that* $u \le v$ *on* $\partial\Omega$. *Then* $u \le v$ *in* Ω.

Proof. According to Theorem 6.4, there is a unique weak solution $w \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ of (6.1) and (1.2).

Now, we prove that $u \le w$ in Ω. Fix any $\gamma \in (0,\,1)$, and let $w_\gamma \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be the unique weak solution of (6.1), with $f_0 - \gamma$ in place of f_0, and (1.2).

Since w_γ is a viscosity solution of (6.1) with $f_0 - \gamma$ in place of f_0, applying Proposition 6.6, we see that $u \le w_\gamma$ on $\overline{\Omega}$.

Using Lemma 6.3, we deduce that there is a sequence $\gamma_j \to 0$ such that as $j \to \infty$, $(w_{\gamma_j}, Dw_{\gamma_j}) \to (w_0, Dw_0)$ locally uniformly in Ω for some weak solution w_0 of (6.1).

Let $\psi^+_{x,\varepsilon} \in C^\infty(\overline{\Omega})$, with $x \in \partial\Omega$ and $\varepsilon \in (0,\,1)$, be those functions given by Lemma 6.2 with $f_0 - 1$ in place of f_0. By Theorem 6.1, we have

$$w_\gamma(z) \le \psi^+(z) := \inf\{\varepsilon + \psi_{x,\varepsilon}(z) \mid x \in \partial\Omega,\ \varepsilon \in (0,\,1)\} \quad \text{for all } z \in \overline{\Omega}.$$

Since $\psi^+ = g_0$ on $\partial\Omega$ and $\psi^+ \in \mathrm{USC}(\overline{\Omega})$, we see that if we set $w_0(x) = g_0(x)$ for $x \in \partial\Omega$, then $w_0 \in C(\overline{\Omega})$. Hence, by the uniqueness of weak solutions of (6.1) and (1.2), we find that $w_0 = w$. This shows that $u \le w$ on $\overline{\Omega}$.

An argument similar to the above yields the inequality $w \le v$ on $\overline{\Omega}$. The proof is now complete. □

7. p-Laplace equation in the limit as $\sigma \to p$

Throughout this section we assume that the uniform exterior sphere condition (H1) is satisfied, $f_0 \in C(\Omega)$ is bounded on Ω and $1/2 \le \sigma < p$. The last two assumptions assure, in particular, that there are constants $\varepsilon_0 \in (0,\,1)$ and $C_0 > 0$, independent of σ, such that

$$|f_0(x)| \le C_0 \left(\lambda_0 \operatorname{dist}(x, \partial\Omega)\right)^{\varepsilon_0(p-1)-\sigma} \quad \text{for } x \in \Omega.$$

That is, condition (H2) is satisfied. Hence, according to Lemma 5.3, there are functions $\psi^\pm \in C(\overline{\Omega})$, independent of σ, such that $\psi^\pm = g_0$ on $\partial\Omega$, $\psi^- \le \psi^+$ in Ω and ψ^+ (resp., ψ^-) is a supersolution (resp., subsolution) of (1.1). By virtue of Theorem 3.5, there is a solution u of (1.1) (see also Theorem 5.1 and Remark 5.2) such that $\psi^- \le u \le \psi^+$ in Ω. We fix such a solution and denote it by u_σ. According to Theorem 5.1, under the additional assumption that $\lambda_0 = 1$ if $p \ne 2$, u_σ is a unique solution of the problem (1.1)–(1.2) and it is continuous on $\overline{\Omega}$.

As the limit equation for (1.1), we introduce the p-Laplace equation

$$\nu \Delta_p u(x) = f_0(x) \quad \text{for } x \in \Omega. \tag{7.1}$$

with the factor $\nu = \nu_{n,p}$ given by

$$\nu = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{n+p}{2}\right)}. \tag{7.2}$$

By Theorem 6.4, the Dirichlet problem (7.1) and (1.2) has a unique weak solution in $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ which is also a unique viscosity solution of (7.1) and (1.2), by Theorems 6.5 and 6.7.

Theorem 7.1. *Let $v \in W^{1,p}_{\rm loc} \cap C(\overline{\Omega})$ be the unique weak solution of* (7.1) *and* (1.2). *Then*

$$\lim_{\sigma \to p-} u_\sigma(x) = v(x) \quad \textit{uniformly on } \overline{\Omega}.$$

Proof. As usual in viscosity solutions theory, we introduce the half relaxed limits $u^\pm$ of u_σ by

$$u^+(x) = \lim_{r \to 0+} \sup\{u_\sigma(y) \mid y \in B(x,r) \cap \overline{\Omega},\, p - r < \sigma < p\} \ \text{ for } x \in \overline{\Omega},$$
$$u^-(x) = \lim_{r \to 0+} \inf\{u_\sigma(y) \mid y \in B(x,r) \cap \overline{\Omega},\, p - r < \sigma < p\} \ \text{ for } x \in \overline{\Omega}.$$

Observe that $u^+ \in \mathrm{USC}(\overline{\Omega})$, $u^- \in \mathrm{LSC}(\overline{\Omega})$ and $\psi^- \le u^- \le u^+ \le \psi^+$ on $\overline{\Omega}$. We intend to show that u^+ (resp., u^-) is a viscosity subsolution (resp., supersolution) of (7.1). Once this was done, Theorem 6.7 guarantees that $u^- = u^+$ on $\overline{\Omega}$ and, as $\sigma \to p-$, u_σ converges uniformly on $\overline{\Omega}$ to the unique viscosity solution of (7.1) and (1.2) which is equal to v, thanks to Theorem 6.5. In fact, we prove here only that u^+ is a viscosity subsolution of (7.1), and leave it to the reader to check that u^- is a viscosity supersolution of (7.1).

Let $\phi \in \mathcal{T}_p(\Omega)$, and assume that $u^+ - \phi$ attains a strict maximum at $x_0 \in \Omega$. By translation, we may assume that $x_0 = 0$, and then set $q = D\phi(0)$ and $A = D^2\phi(0)$. We choose a constant $\delta_0 \in (0,\, 1/2)$ so that $B(0, 2\delta_0) \subset \Omega$. Fix a constant $C_1 > 0$ so that $(1/2)|D^2\phi(x)\xi \cdot \xi| \le C_1|\xi|^2$ for all $x \in B(0,\, 2\delta_0)$ and $\xi \in \mathbb{R}^n$. It is easy to find a sequence $\{\sigma_k\} \subset \big(1/2,\, p\big)$ converging to p such that for each $k \in \mathbb{N}$, $u^*_{\sigma_k} - \phi$ attains a maximum over $B(0, 2\delta_0)$ at some point $x_k \in B(0, \delta_0)$, where x_k converges to the origin. Note that $M_{\sigma_k}[u^*_{\sigma_k}](x_k) \ge f_0(x_k)$ for all $k \in \mathbb{N}$. We may assume that $\delta_0 < \rho(x)$ for all $x \in B(0, 2\delta_0)$.

We first consider the case where $q = 0$ and $p \ne 2$. Note that $\Delta_p\phi(0) = 0$ if $p > 2$. Thus we need to show that $f_0(0) \le 0$. If $1 < p < 2$, we may replace the test function ϕ by a function $C|x|^{\beta+1}$, with some constants $C > 0$ and $\beta > 1/(p-1)$. Applying Theorem 2.2 or Theorem 2.6, we see that there is a constant $C_2 > 0$, independent of σ, such that for any $0 < \delta < \delta_0$ and $x \in B(0, \delta)$, if $p \ge 2$, then

$$M_\sigma[u^*_{\sigma_k}](x_k) \le C_2(|D\phi(x_k)| + \delta)^{p-2}\delta^{p-\sigma} + \int_{\delta < |z| < \rho(x_k)} G(C_3)\frac{p-\sigma}{|z|^{n+\sigma}}\,\mathrm{d}z,$$

and if $1 < p < 2$, then

$$M_\sigma[u^*_{\sigma_k}](x_k) \le C_2\delta^{(\beta+1)(p-1)-\sigma} + \int_{\delta < |z| < \rho(x_k)} G(C_3)\frac{p-\sigma}{|z|^{n+\sigma}}\,\mathrm{d}z,$$

where $C_3 := \|\psi^+\|_{\infty,\Omega} + \|\psi^-\|_{\infty,\Omega}$. From this observation, since $M_{\sigma_k}[u^*_{\sigma_k}](x_k) \ge f_0(x_k)$, we find that $f_0(0) \le 0$, which was to be shown.

Next, we consider the case where $q \ne 0$ and will show that $f_0(0) \le \nu\Delta_p\phi(0)$. Fix any $\varepsilon \in (0,\, 1)$. We may assume by reselecting δ_0 if needed that

$$|(A - D^2\phi(x))\xi \cdot \xi| \le \varepsilon|\xi|^2 \quad \text{ for all } x \in B(0, 2\delta_0) \text{ and } \xi \in \mathbb{R}^n.$$

We may also assume that $|q|/2 \le |D\phi(x)| \le 2|q|$ for all $x \in B(0, \delta_0)$.

Fix any $x \in B(0, \delta_0)$. For each $z \in B(0,\, \delta_0)$ we can choose a constant $\theta(z) \in (0,\, 1)$ so that

$$\phi(x+z) - \phi(x) = q_x \cdot z + \frac{1}{2}D^2\phi(x + \theta(z)z)z \cdot z,$$

where $q_x := D\phi(x)$, and note that

$$G(\phi(x+z)-\phi(x)) \le G\Big(q_x\cdot z + \frac{1}{2}A_\varepsilon z\cdot z\Big),$$

where $A_\varepsilon := A+\varepsilon I$. Let $\delta \in (0,\,\delta_0)$. We set $C_4 = C_1+1$ and

$$W_\delta(x) = \{z\in B(0,\,\delta) \mid C_4|z|^2 < \varepsilon|q_x\cdot z|\}.$$

Let $z\in W_\delta(x)$ and compute that

$$\begin{aligned} G(\phi(x+z)-\phi(x)) \le& G(q_x\cdot z)G\Big(1+\frac{A_\varepsilon z\cdot z}{2q_x\cdot z}\Big)\\ =& G(q_x\cdot z)\Big(1+G'(1+\lambda(z))\frac{A_\varepsilon z\cdot z}{2q_x\cdot z}\Big)\\ =& G(q_x\cdot z)+(p-1)|q_x\cdot z|^{p-2}|1+\lambda(z)|^{p-2}\frac{A_\varepsilon z\cdot z}{2} \end{aligned}$$

for some $\lambda(z)\in\mathbb{R}$ satisfying

$$|\lambda(z)| \le \left|\frac{A_\varepsilon z\cdot z}{2q_x\cdot z}\right| \le \frac{C_4|z|^2}{2|q_x\cdot z|} < \varepsilon.$$

Noting that if $1<p<2$, then

$$(1+\varepsilon)^{p-2} \le |1+\lambda(z)|^{p-2} \le (1-\varepsilon)^{p-2}$$

and if $p\ge 2$, then

$$(1-\varepsilon)^{p-2} \le |1+\lambda(z)|^{p-2} \le (1+\varepsilon)^{p-2},$$

we find that

$$\big|\big(|1+\lambda(z)|^{p-2}-1\big)A_\varepsilon z\cdot z\big| \le \big|(1+\varepsilon)^{p-2}-(1-\varepsilon)^{p-2}\big|C_4|z|^2.$$

Setting $\gamma_\varepsilon = \varepsilon + \big|(1-\varepsilon)^{p-2}-(1+\varepsilon)^{p-2}\big|$ and $B_\varepsilon = A+\gamma_\varepsilon I$, we observe that

$$\begin{aligned} |1+\lambda(z)|^{p-2}A_\varepsilon z\cdot z \le& B_\varepsilon z\cdot z,\\ G(\phi(x+z)-\phi(x)) \le& G(q_x\cdot z)+\frac{(p-1)|q_x\cdot z|^{p-2}B_\varepsilon z\cdot z}{2}, \end{aligned}$$

and $\lim_{\varepsilon\to 0}\gamma_\varepsilon = 0$.

Now, we write $\bar q_x = q_x/|q_x|$ and reselect δ_0, if needed, so small that $C_4\delta_0 \le \varepsilon|q_x|/2$. Observe that if $z\in B(0,\,\delta)\setminus W_\delta(x)$, then

$$\begin{aligned} \varepsilon|q_x\cdot z| \le C_4|z|^2 =& C_4\big(|z-(\bar q_x\cdot z)\bar q|^2+(\bar q_x\cdot z)^2\big)\\ \le& C_4\big(|z-(\bar q_x\cdot z)\bar q_x|^2+\delta|\bar q_x\cdot z|\big)\\ \le& C_4|z-(\bar q_x\cdot z)\bar q|^2+\frac{\varepsilon}{2}|q_x\cdot z|. \end{aligned}$$

That is, for any $z\in B(0,\,\delta)\setminus W_\delta(x)$, we have $\varepsilon|q_x\cdot z|\le 2C_4|z-(\bar q_x\cdot z)\bar q_x|^2$. Hence, setting

$$V_\delta(x) = \big\{z\in B(0,\,\delta) \bigm| \varepsilon|q_x\cdot z| \le 2C_4|z-(\bar q_x\cdot z)\bar q_x|^2\big\},$$

we get $B(0,\,\delta)\subset W_\delta(x)\cup V_\delta(x)$.

Next we observe that

$$\begin{aligned} I_1(x) :=& \int_{W_\delta(x)} G(\phi(x+z)-\phi(x))K(z)\,\mathrm{d}z \\ \le& \int_{W_\delta(x)} \big(G(q_x\cdot z)+\frac{p-1}{2}|q_x\cdot z|^{p-2}B_\varepsilon z\cdot z\big)K(z)\,\mathrm{d}z \\ =&\frac{p-1}{2}\int_{W_\delta(x)} |q_x\cdot z|^{p-2}(B_\varepsilon z\cdot z)K(z)\,\mathrm{d}z. \end{aligned}$$

We make an orthogonal change of variables in the above integral. Indeed, for each $x\in B(0,\delta)$, we introduce an orthogonal matrix U_x of order n for which $U_xe_n=\bar q_x$ and compute as follows:

$$\begin{aligned} I_1(x) \le& \frac{p-1}{2}\int_{W_\delta^n} |q_x\cdot U_xy|^{p-2}(B_\varepsilon U_xy\cdot U_xy)K(y)\ dy \\ =&\frac{p-1}{2}|q_x|^{p-2}\sum_{j=1}^n\int_{W_\delta^n}|y_n|^{p-2}b_{jj}(x)y_j^2K(y)\,\mathrm{d}y \\ \le&\frac{p-1}{2}|q_x|^{p-2}\sum_{j=1}^n\Big(\int_{|y|<\delta}|y_n|^{p-2}b_{jj}(x)y_j^2K(y)\,\mathrm{d}y \\ &+\int_{V_\delta^n}|b_{jj}(x)||y_n|^{p-2}y_j^2K(y)\,\mathrm{d}y\Big), \end{aligned}$$

where $b_{ij}(x)$ denotes the (i,j)-entry of the matrix $U_x^{-1}B_\varepsilon U_x$ and

$$\begin{aligned} W_\delta^n :=&\{y=(y',y_n)\in B(0,\,\delta)\mid C_4|y|^2<\varepsilon|q_x||y_n|\}, \\ V_\delta^n :=&\{y=(y',y_n)\in B(0,\delta)\mid \varepsilon|q_x||y_n|\le 2C_4|y'|^2\}. \end{aligned}$$

For $1\le j\le n$ we compute

$$\begin{aligned} J_{1,j}(x) :=& \int_{|y|<\delta}|y_n|^{p-2}y_j^2K(y)\,\mathrm{d}y \\ =&\mu\delta^{p-\sigma}\int_{|y|<1}|y_n|^{p-2}y_j^2|y|^{-n-\sigma}\,\mathrm{d}y. \end{aligned}$$

We use Lemma 2.1, to find that if $j<n$, then

$$J_{1,j}(x)=\frac{\mu\delta^{p-\sigma}\Gamma\big(\frac32\big)\Gamma\big(\frac12\big)^{n-2}\Gamma\big(\frac{p-1}{2}\big)}{\Gamma\big(\frac{n+p}{2}\big)}\int_0^1 t^{\frac{p-\sigma}{2}-1}\,\mathrm{d}t=\frac{2\nu\delta^{p-\sigma}}{p-1},$$

and

$$J_{1,n}(x)=\frac{\mu\delta^{p-\sigma}\Gamma\big(\frac12\big)^{n-1}\Gamma\big(\frac{p+1}{2}\big)}{\Gamma\big(\frac{n+p}{2}\big)}\int_0^1 t^{\frac{p-\sigma}{2}-1}\,\mathrm{d}t=2\nu\delta^{p-\sigma}.$$

Next, we set $C_5 = 4C_4/(|q|\varepsilon)$, so that $|y_n| \le C_5|y'|^2$ for $y = (y', y_n) \in V_\delta^n$. We compute that for $1 \le j < n$,

$$\begin{aligned} J_{2,j}(x) :=& \int_{V_\delta^n} |y_n|^{p-2} y_j^2 K(y)\,\mathrm{d}y \\ \le& 2\mu \int_{|y'|<\delta} |y'|^{2-n-\sigma}\,\mathrm{d}y' \int_0^{C_5|y'|^2} y_n^{p-2}\,\mathrm{d}y_n \\ =& \frac{2C_5^{p-1}\mu}{p-1} \int_{|y'|<1} |y'|^{2p-n-\sigma}\,\mathrm{d}y' = \frac{2C_5^{p-1}\sigma_{n-1}\mu}{(p-1)(2p-1-\sigma)}. \end{aligned}$$

Similarly we get

$$\begin{aligned} J_{2,n}(x) :=& \int_{V_\delta^n} |y_n|^p K(y)\,\mathrm{d}y \\ \le& 2\mu \int_{|y'|<\delta} |y'|^{-n-\sigma}\,\mathrm{d}y' \int_0^{C_5|y'|^2} y_n^p\,\mathrm{d}y_n \\ =& \frac{2C_5^{p+1}\mu}{p+1} \int_{|y'|<1} |y'|^{2p+2-n-\sigma}\,\mathrm{d}y' \\ =& \frac{2C_5^{p+1}\sigma_{n-1}\mu}{(p+1)(2p+1-\sigma)} < \frac{2C_5^{p+1}\sigma_{n-1}\mu}{(p+1)(2p-1-\sigma)}. \end{aligned}$$

Furthermore, noting that

$$\phi(x+z) - \phi(x) \le q_x \cdot z + C_4|z|^2 \quad \text{for } z \in B(0,\delta)$$

and

$$|q_x| \cdot |y_n| + C_4|y|^2 \le (2|q| + C_4)|y_n| + C_4|y'|^2 \le C_6|y'|^2 \quad \text{for } y \in V_\delta^n,$$

where $C_6 := (2|q| + C_4)C_5 + C_4$, we compute that

$$\begin{aligned} I_2(x) :=& \int_{V_\delta(x)} G(\phi(x+z) - \phi(x))K(z)\,\mathrm{d}z \\ \le& \int_{V_\delta^n} G(|q_x||y_n| + C_4|y|^2)K(y)\,\mathrm{d}y \\ \le& 2C_6^{p-1}\mu \int_{|y'|<1} |y'|^{2p-2-n-\sigma}\,\mathrm{d}y' \int_0^{C_5|y'|^2} \mathrm{d}y_n \\ \le& 2C_5C_6^{p-1}\mu \int_{|y'|<1} |y'|^{2p-n-\sigma}\,\mathrm{d}y' \\ \le& \frac{2C_5C_6^{p-1}\sigma_{n-1}\mu}{2p-1-\sigma}. \end{aligned}$$

We combine the above observations, to obtain

$$\begin{aligned} &\limsup_{r\to 0+} \int_{r<|z|<\delta} G(\phi(x+z) - \phi(x))K(z)\,\mathrm{d}z \\ &\le |q_x|^{p-2}\nu \left(\sum_{j=1}^n b_{jj}(x) + (p-2)b_{nn}(x) \right) + \frac{C_7\mu}{2p-1-\sigma}, \end{aligned} \tag{7.3}$$

where C_7 is a positive constant depending only on C_1, p, $|q|$, ε and n. Since $f_0(x_k) \le M_{\sigma_k}[u^*_{\sigma_k}](x_k)$, we have

$$f_0(x_k) \le \limsup_{r\to 0+} \int_{r<|z|<\delta} G(\phi(x_k+z)-\phi(x_k))K_{\sigma_k}(z)\,\mathrm{d}z + \int_{\delta<|z|<\rho(x_k)} G(C_3)K_{\sigma_k}(z)\,\mathrm{d}z.$$

Here, as before, we have

$$\lim_{\sigma\to p-} \int_{\delta<|z|<\rho(x_k)} G(C_3)K_{\sigma_k}(z)\,\mathrm{d}z = 0.$$

Observe that

$$\sum_{j=1}^n b_{jj}(x) = \operatorname{tr}\left(U_x^{-1}B_\varepsilon U_x\right) = \operatorname{tr} B_\varepsilon,$$
$$b_{nn}(x) = U_x^{-1}B_\varepsilon U_x e_n\cdot e_n = B_\varepsilon \bar q_x\cdot \bar q_x.$$

Now, from(7.3), we get

$$f_0(0) \le \nu\big(|q|^{p-2}\operatorname{tr} B_\varepsilon + (p-2)|q|^{p-4}B_\varepsilon q\cdot q\big),$$

and, because of the arbitrariness of $\varepsilon > 0$,

$$f_0(0) \le \nu\left(|q|^{p-2}\Delta\phi(0) + (p-2)|q|^{p-4}D^2\phi(0)q\cdot q\right) = \nu\Delta_p\phi(0),$$

which is the desired inequality.

It remains to check the case where $p = 2$ and $q = 0$. For each $\varepsilon > 0$, selecting $\delta_0 > 0$ as in the previous case and setting $A_\varepsilon = (a_{ij}) := A + \varepsilon I$, we have for any $0 < r < \delta < \delta_0$ and any $x \in B(0, \delta_0)$,

$$\int_{r<|z|<\delta} G(\phi(x+z)-\phi(x))K(z)\,\mathrm{d}z \le \int_{r<|z|<\delta}\Big(q_x\cdot z + \frac{1}{2}A_\varepsilon z\cdot z\Big)K(z)\,\mathrm{d}z = \frac{1}{2}\sum_{j=1}^n \int_{r<|z|<\delta} a_{jj}z_j^2 K(z)\,\mathrm{d}z.$$

By applying Lemma 2.1, we find that for any $1 \le j \le n$,

$$\int_{|z|<\delta} z_j^2 K(z)\,\mathrm{d}z = 2\nu\delta^{2-\sigma}.$$

Hence we have

$$\limsup_{r\to 0+}\int_{r<|z|<\delta} G(\phi(x+z)-\phi(x))K(z)\,\mathrm{d}z \le \nu\delta^{2-\sigma}\operatorname{tr} A_\varepsilon.$$

Using this and arguing as in the previous case, we see easily that $f_0(0) \le \nu\Delta\phi(0)$. This completes the proof. □

8. Final remarks

In this section we discuss a few possible extensions and variants of the formulations and results presented in the previous sections.

Let $c \in C(\overline{\Omega})$ be a given function satisfying $\inf_\Omega c > 0$. We consider the integral equation

$$M_\sigma[u](x) = c(x)u(x) + f_0(x) \quad \text{in } \Omega, \tag{8.1}$$

together with the Dirichlet condition (1.2). The p-Laplace equation corresponding to (8.1) is

$$\nu\Delta_p u(x) = c(x)u(x) + f_0(x) \quad \text{in } \Omega, \tag{8.2}$$

where $\nu = \nu_{n,p}$ is the constant given by (7.2). Because of the new term "cu", two equations (8.1) and (8.2) are tractable. Indeed, for the Dirichlet problem for (8.1)-(1.2), without the restriction that $\lambda_0 = 1$ if $p \neq 2$, a comparison assertion similar to Theorem 4.1 and consequently the existence of a unique continuous solution as in Theorem 5.1 hold true. Also, for the Dirichlet problem (8.2)-(1.2), a comparison theorem for viscosity sub and supersolutions similar to Proposition 6.7, but with $f_1 = f_2$, is valid. The same assertion as Theorem 7.1, with (8.1) and (8.2) in place of (1.1) and (7.1) respectively, is valid.

A remark similar to the above applies to the evolution problem. The equations are now

$$M_\sigma[u(\cdot,t)](x) = u_t(x,t) + f_0(x,t) \quad \text{in } Q_T, \tag{8.3}$$

and

$$\nu\Delta_p u(x,t) = u_t(x,t) + f_0(x,t) \quad \text{in } Q_T, \tag{8.4}$$

where $0 < T < \infty$ is a fixed constant, $Q_T := \Omega \times (0,T)$, $u_t := \partial u/\partial t$ and $f_0 \in C(\overline{Q}_T)$ is a given function. The initial-boundary condition for (8.3) or (8.4) is of the form

$$u = g_0 \quad \text{on the parabolic boundary, } \partial_p Q_T = \overline{\Omega} \times \{0\} \cup \partial\Omega \times (0,T), \tag{8.5}$$

where $g_0 \in C(\partial_p Q_T)$. With an obvious modification (see for instance [11]) of the definition of spaces of test functions, we have well-posedness and convergence results similar to those for (8.1) and (8.2). That is, the Cauchy-Dirichlet problems for (8.3) and for (8.4) are well-posed in the space $C(\overline{Q}_T)$ and the solution u_σ of the problem (8.3) and (8.5) converges uniformly on $\overline{Q}_T$ as $\sigma \to p-$ to the solution of the problem (8.4) and (8.5).

It would be interesting to treat the Neumann boundary problem for (1.1) as in [2], and we hope to come back to this issue in a future publication.

Another interesting question would be to seek for the possibility of replacing the operator M_σ, in the well-posedness problem of Sections 3-5 or in the convergence problem of Section 6 for (1.1), by the operator

$$\widetilde{M}_\sigma[\phi](x) := \text{p.v.}\int_{B(x)} G(\phi(x+z) - \phi(x))K_\sigma(z)\,\mathrm{d}z,$$

where $B(x)$, with $x \in \Omega$, are given measurable subsets of $\mathbb{R}^n$ satisfying the condition that $x + B(x) \subset \overline{\Omega}$ for all $x \in \Omega$.

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