HAMILTON-JACOBI EQUATIONS WITH THEIR HAMILTONIANS DEPENDING LIPSCHITZ CONTINUOUSLY ON THE UNKNOWN

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ABSTRACT. We study the Hamilton-Jacobi equations $H(x, Du, u) = 0$ in M and $\partial u/\partial t + H(x, D_x u, u) = 0$ in $M \times (0, \infty)$, where the Hamiltonian $H = H(x, p, u)$ depends Lipschitz continuously on the variable u. In the framework of the semicontinuous viscosity solutions due to Barron-Jensen, we establish the comparison principle, existence theorem, and representation formula as value functions for extended real-valued, lower semicontinuous solutions for the Cauchy problem. We also establish some results on the long-time behavior of solutions for the Cauchy problem and classification of solutions for the stationary problem.

CONTENTS

1. INTRODUCTION

We study the Hamilton-Jacobi equation

$$(1.1) \qquad u_t + H(x, D_x u, u) = 0 \quad \text{in } M \times (0, T),$$

where M is a connected, closed, smooth Riemannian manifold of dimension n, T is either a positive number or $+\infty$, H is the Hamiltonian on $T^*M \times \mathbb{R}$, u is the unknown function on $M \times [0, T)$, u_t denotes the partial derivative $\partial u / \partial t = \partial_t u$, $D = D_x$ denotes the differential map, so that $(x, Du) = (x, Du(x))$ denotes an element of the cotangent space T_x^*M. The Riemannian structure on M induces a norm $|\cdot| = |\cdot|_x$ on the tangent space $T_x M$. The canonical pairing between T_x^*M and $T_x M$ is denoted by $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_x$, which defines naturally a norm $|\cdot| = |\cdot|_x$ on T_x^*M.

The following list collects our main assumptions on the Hamiltonian H.

(H1) The function $(x, p, u) \mapsto H(x, p, u)$ is continuous on $T^*M \times \mathbb{R}$.

(H2) For any $R > 0$ there exists $K > 0$ such that

$$H(x, p, u) > R \quad \text{if } |u| \le R \text{ and } |p| \ge K.$$

(H3) For any $(x, u) \in M \times \mathbb{R}$, the function $p \mapsto H(x, p, u)$ is convex on T_x^*M.

(H4) The functions $u \mapsto H(x, p, u)$ are equi-Lipschitz continuous on $\mathbb{R}$, with $(x, p) \in T^*M$.

When (H4) is assumed, the symbol Λ denotes a positive Lipschitz bound:

$$|H(x, p, u) - H(x, p, v)| \le \Lambda |u - v| \quad \text{for all } (x, p) \in T^*M, \ u, v \in \mathbb{R},$$

and it is fixed throughout this paper. Remark also that under the assumption (H4), condition (H2) is equivalent to say that for any $R > 0$, there exists $K > 0$ such that $H(x, p, 0) > R$ if $|p| \ge K$.

In recent work K. Wang-L. Wang-J. Yan [18], the authors have studied the Cauchy problem for (1.1) with the initial condition of the form, with given $y \in M$ and $c \in \mathbb{R}$,

$$(1.2) \qquad u(x, 0) = \begin{cases} c & \text{if } x = y \\ \infty & \text{otherwise.} \end{cases}$$

To assure the existence of a solution belonging in $C(M \times (0, T), \mathbb{R})$, they assume a coercivity assumption stronger than (H2) above. The purpose of this paper is to adopt the notion of semicontinuous viscosity solutions due to Barron-Jensen [2] and to give an existence and uniqueness result for the Cauchy problem for (1.1) with general lower semicontinuous data. Consider the Hamilton-Jacobi equation

$$(1.3) \qquad u_t + |D_x u| - u = 0 \quad \text{in } M \times (0, \infty),$$

with the initial condition (1.2). This equation has the property of a finite speed propagation, by which any solution has necessarily discontinuities for positive times as far as the initial data is discontinuous. Indeed, the main concern of the paper [2] is to take into this kind of singular behavior of the solutions into the notion of solution, and the solution u of (1.3) and (1.2) is given by the formula (see the proof of Proposition 5.3 below for a related discussion)

$$u(x,t) = \begin{cases} c\,e^t & \text{if } d(x,y) \leq t, \\ +\infty & \text{otherwise.} \end{cases}$$

New features of our adaptation of the semicontinuous viscosity solutions to (1.1) are the use of structure conditions on (1.1), that is, the hypotheses (H1)–(H4) different from those employed in [1–3, 10], and the extension of the notion of solution which applies to extended real-value functions.

We recall the definition of lower semicontinuous viscosity sub and supersolutions of (1.1) following [2, 10, 13]: let $u : M \times (0, T) \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function, that is, $u \in \mathrm{LSC}(M \times (0, T), \mathbb{R} \cup \{\infty\})$. The function u is called a lower semicontinuous viscosity subsolution (resp., supersolution) of (1.1) if, whenever $\phi \in C^1(M \times (0, T), \mathbb{R})$ and $\min(u - \phi) = (u - \phi)(x, t)$ for some $(x, t) \in M \times (0, T)$, we have $\phi_t(x, t) + H(x, D\phi(x, t), u(x, t)) \leq 0$ (resp., ≥ 0). If u is both lower semicontinuous sub and supersolutions of (1.1), then we call it a lower semicontinuous solution of (1.1). Henceforth, for simplicity of notation, we write simply "BJ" for "lower semicontinuous viscosity". For instance, we say a BJ subsolution instead of a lower semicontinuous viscosity subsolution.

We remark that a function $u \in \mathrm{LSC}(M \times (0, T), \mathbb{R} \cup \{\infty\})$ is a BJ subsolution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, T)$ if and only if it is a viscosity supersolution, in the sense of Crandall-Lions ([5, 6]), of $-u_t - H(x, D_x u, u) = 0$ in $M \times (0, T)$.

Let $u \in \mathrm{LSC}(M \times (0, T), \mathbb{R} \cup \{\infty\})$ and $(x, t) \in M \times (0, T)$ be such that $u(x, t) < \infty$. Let $D^- u(x, t)$ denote the subdifferential of u at (x, t) defined as the set of all $(p, q) \in T_x^* M \times \mathbb{R}$ such that in local coordinates, as $|y| + |s| \to 0$,

$$u(x + y, t + s) \geq u(x, t) + \langle p, y \rangle + qs + o(|y| + |s|).$$

It is straightforward to generalize the notion of BJ solution to a general Hamilton-Jacobi equation $F(x, Du, u) = 0$ in U. We henceforth write $\mathcal{S}_{\mathrm{BJ}}(F) = \mathcal{S}_{\mathrm{BJ}}(F, U)$ (resp., $\mathcal{S}_{\mathrm{BJ}}^+(F) = \mathcal{S}_{\mathrm{BJ}}^+(F, U)$ and $\mathcal{S}_{\mathrm{BJ}}^-(F) = \mathcal{S}_{\mathrm{BJ}}^-(F, U)$) for the set of all BJ solutions (resp., BJ supersolutions and BJ subsolutions) $u \in \mathrm{LSC}(U, \mathbb{R} \cup \{\infty\})$ of $F(x, Du, u) = 0$ in U. For instance, $\mathcal{S}_{\mathrm{BJ}}(\partial_t + H) = \mathcal{S}_{\mathrm{BJ}}(\partial_t + H, M \times (0, T))$ denotes the set of all BJ solutions of (1.1). Similarly, we write $\mathcal{S}(F) = \mathcal{S}(F, U)$ (resp., $\mathcal{S}^+(F) = \mathcal{S}^+(F, U)$ and

$\mathcal{S}^-(F) = \mathcal{S}^-(F, U))$ for the set of all viscosity solutions $u \in \mathrm{LSC}(U, \mathbb{R})$ (resp., viscosity supersolutions $u \in \mathrm{LSC}(U, \mathbb{R} \cup \{\infty\})$ and viscosity subsolutions $u \in \mathrm{USC}(U, \mathbb{R} \cup \{-\infty\})$) of $F(x, Du, u) = 0$ in U in the Crandall-Lions sense.

The remark after the introduction of BJ solutions above, can be stated as $\mathcal{S}^-_{\mathrm{BJ}}(F) = \mathcal{S}^+(-F)$.

Another simple remark here is that for any continuous function u, we have $u \in \mathcal{S}^-(F)$ if and only if $-u \in \mathcal{S}^-_{\mathrm{BJ}}(F^\ominus)$, where $F^\ominus$ is given by $F^\ominus(x, p, u) = F(x, -p, -u)$.

After this introduction, we establish a comparison principle for BJ sub and super-solutions of (1.1) in Section 2, an existence result for the Cauchy problem for (1.1) is proved in Section 3, and a representation formula for BJ solutions of (1.1), based on the idea of value functions of optimal control associated with (1.1) is presented in Section 4. In Section 5, we introduce the notion of fundamental solutions to (1.1) as well as their basic properties. In Section 6, we investigate the long-time behavior of solutions of (1.1), which is applied in Section 7 to classification of solutions to the corresponding stationary problem together with several suggestive examples. The appendix presents a proof of a classical existence result of Lipschitz continuous solutions to the Cauchy problem for (1.1).

2. COMPARISON PRINCIPLE

In this work, it is of major importance to establish the comparison principle for BJ solutions of (1.1).

Theorem 2.1. *Assume* (H1)–(H4). *Let* $v, w \in \mathrm{LSC}(M \times [0, T), \mathbb{R} \cup \{\infty\})$ *be, respectively, BJ sub and supersolutions of*

$$(2.1) \qquad u_t + H(x, D_x u, u) = 0 \quad in \ \ M \times (0, T).$$

Assume that

$$(2.2) \qquad w(x, 0) \geq \liminf_{t \to 0+} v(x, t) \quad for \ all \ \ x \in M.$$

Then, $v \leq w$ *on* $M \times [0, T)$.

A stronger inequality than (2.2) implies a stronger conclusion in the above theorem, as stated in the following corollary.

Corollary 2.2. *Under the hypotheses of Theorem 2.1, but with* (2.2) *replaced by the condition that for some constant* $C > 0$,

$$w(x, 0) \geq C + \liminf_{t \to 0+} v(x, t) \quad for \ all \ x \in M,$$

we have $v(x,t) + Ce^{-\Lambda t} \leq w(x,t)$ *for all* $(x,t) \in M \times [0,T)$. *Similarly, if* (2.2) *is replaced by the condition that for some constant* $C > 0$,

$$w(x,0) + C \geq \liminf_{t \to 0+} v(x,t) \quad \textit{for all } x \in M,$$

then we have $v(x,t) \leq w(x,t) + Ce^{\Lambda t}$ *for all* $(x,t) \in M \times [0,T)$.

Proof. Set $z(x,t) = v(x,t) + Ce^{-\Lambda t}$ and compute in a slightly informal fashion that

$$z_t + H(x, D_x z, z) \leq v_t - \Lambda Ce^{-\Lambda t} + H(x, D_x v, v) + \Lambda Ce^{-\Lambda t} \leq 0,$$

to find that z is a BJ subsolution of (2.1). It is then easily seen by Theorem 2.1 that $z \leq w$ in $M \times (0,T)$. Instead, if we set $z(x,t) = w(x,t) + Ce^{\Lambda t}$, then we find that z is a BJ supersolution of (2.1) and conclude by Theorem 2.1 that $v \leq w$ in $M \times [0,T)$. $\square$

The next two lemmas constitute the primary part of the proof of Theorem 2.1.

Lemma 2.3. *In addition to the hypotheses of Theorem 2.1, assume that v is Lipschitz continuous on $M \times [0,T)$. Then $v \leq w$ on $M \times [0,T)$.*

It is a classical observation in the literature that the Lipschitz property of a viscosity subsolution or supersolution simplifies the formulation and proof of the comparison theorem.

We remark that, under the hypotheses of Lemma 2.3, condition (2.2) is equivalent to the inequality $v(x,0) \leq w(x,0)$ for all $x \in M$.

Proof. Thanks to the Lipschitz regularity of v, the function v is a viscosity subsolution of (2.1) in the Crandall-Lions sense, which is a classical observation due to [2] (see also [13, Theorem 2.3]). By the standard change of the unknown functions (i.e., by considering the new unknowns $v(x,t)e^{-(\Lambda+1)t}$ and $w(x,t)e^{-(\Lambda+1)t}$), we may assume that $u \mapsto H(x,p,u) - u$ is nondecreasing.

To the contrary to the conclusion, we suppose that $\sup_{M \times [0,T)}(v-w) > 0$, and we will obtain a contradiction. We can choose $S \in (0,T]$ so that $\sup_{M \times [0,S)}(v-w) > 0$. Let $\varepsilon > 0$ and consider the function

$$u(x,t) = v(x,t) - \frac{\varepsilon}{S + \varepsilon^2 - t} \quad \text{on } M \times [0,S].$$

Note that, since $-w$ is bounded from above on $M \times [0,S]$, if $\varepsilon > 0$ is sufficiently small, then

$$\max_{x \in M}(u(x,S) - w(x,S)) = \max_{x \in M}(v(x,S) - w(x,S)) - \frac{1}{\varepsilon} < 0.$$

By assumption, we have

$$\max_{x \in M}(u(x,0) - w(x,0)) \leq -\frac{1}{S + \varepsilon^2} < 0.$$

Choosing a point $(x_0, t_0) \in M \times [0, S)$ such that

$$(v - w)(x_0, t_0) > 0,$$

we observe that for sufficiently small $\varepsilon > 0$.

$$(u - w)(x_0, t_0) = (v - w)(x_0, t_0) - \frac{\varepsilon}{S + \varepsilon^2 - t_0} > 0.$$

Hence, fixing $\varepsilon > 0$ small enough, we find that

$$\max_{M \times [0,S]} (u - w) > 0 \quad \text{and} \quad \max_{\partial(M \times [0,S])} (u - w) < 0.$$

Note also that

$$u_t + H(x, D_x u, u) = v_t - \frac{\varepsilon}{(S + \varepsilon^2 - t)^2} + H\left(x, D_x v, v - \frac{\varepsilon}{S + \varepsilon^2 - t}\right)$$
$$\leq v_t + H(x, D_x v, v) \leq 0 \quad \text{in } M \times (0, S).$$

Fix a maximum point $(\hat{x}, \hat{t}) \in M \times (0, S)$ of the function $u - w$ on $M \times [0, S]$. We choose a chart (U, ϕ) such that $\hat{x} \in U$. We identify U with $\phi(U)$, so that $\hat{x} \in U \subset \mathbb{R}^n$. Consider the function

$$\Phi_\alpha : (x, t, y, s) \mapsto u(x, t) - w(y, s) - \alpha|x - y|^2 - \alpha(t - s)^2 - |x - \hat{x}|^2 - (t - \hat{t})^2$$

on $U \times [0, S] \times U \times [0, S]$. We fix a compact neighborhood $B \subset U \times [0, S]$ of $(\hat{x}, \hat{t})$. Let $(x_\alpha, t_\alpha, y_\alpha, s_\alpha) \in B \times B$ be a maximum point of Φ_α on the set $B \times B$. It is a standard observation that as $\alpha \to \infty$,

$$(x_\alpha, t_\alpha, y_\alpha, s_\alpha) \to (\hat{x}, \hat{t}, \hat{x}, \hat{t}) \quad \text{and} \quad w(y_\alpha, s_\alpha) \to w(\hat{x}, \hat{t})$$

Assuming α large enough, we may assume that x_α, y_α are in the interior of B. By the viscosity properties of u and w, we find that

$$2\alpha(t_\alpha - s_\alpha) + 2(t_\alpha - \hat{t}) + H(x_\alpha, 2\alpha(x_\alpha - y_\alpha) + 2(x_\alpha - \hat{x}), u(x_\alpha, t_\alpha)) \leq 0,$$
$$2\alpha(t_\alpha - s_\alpha) + H(y_\alpha, 2\alpha(x_\alpha - y_\alpha), w(y_\alpha, s_\alpha)) \geq 0$$

The Lipschitz continuity of u implies that the collections $\{\alpha(x_\alpha - y_\alpha)\}$ and $\{\alpha(t_\alpha - s_\alpha)\}$ are bounded in $\mathbb{R}^n$ and $\mathbb{R}$, respectively. We may choose a sequence $\{\alpha_j\} \subset (0, \infty)$ so that, as $j \to \infty$,

$$2\alpha_j(x_{\alpha_j} - y_{\alpha_j}) \to \hat{p} \quad \text{and} \quad 2\alpha_j(t_{\alpha_j} - s_{\alpha_j}) \to \hat{q}.$$

Sending $j \to \infty$, we get

$$\hat{q} + H(\hat{x}, \hat{p}, u(\hat{x}, \hat{t})) \leq 0,$$
$$\hat{q} + H(\hat{x}, \hat{p}, w(\hat{x}, \hat{t})) \geq 0,$$

and, subtracting one from the other and recalling that $u \mapsto H(x, p, u) - u$ is nondecreasing,

$$0 \geq H(\hat{x}, \hat{p}, u(\hat{x}, \hat{t})) - H(\hat{x}, \hat{p}, w(\hat{x}, \hat{t})) \geq (u - w)(\hat{x}, \hat{t}) > 0.$$

This is a contradiction, which proves that $v \leq w$ on $M \times [0, T)$. $\qquad \square$

Lemma 2.4. *Assume* (H1)–(H4). *Let* $v \in \mathrm{LSC}(M \times [0, T), \mathbb{R} \cup \{\infty\})$ *be a BJ subsolution of* $u_t + H(x, D_x u, u) = 0$ *in* $M \times (0, T)$, *and assume that* v *is bounded from below on* $M \times [0, T)$. *Let* $C_0, C_1 > 0$ *be constants such that*

$$H(x, p, 0) \geq -C_0 \quad \text{for all } (x, p) \in T^*M \quad \text{and} \quad v \geq -C_1 \quad \text{on } M \times [0, T).$$

Fix $y \in M$. *If* $\liminf_{t \to 0+} v(y, t) < \infty$, *then*

(2.3) $\qquad v(y, t) \leq e^{\Lambda t} \liminf_{t \to 0+} v(y, t) + (C_0 \Lambda^{-1} + 2C_1)(e^{\Lambda t} - 1) \quad \text{for all } t \in (0, T),$

Furthermore, for any $s \in (0, T)$, *if* $v(y, s) < \infty$, *then*

(2.4) $\qquad v(y, t) \leq v(y, s) e^{\Lambda(t-s)} + (C_0 \Lambda^{-1} + 2C_1)(e^{\Lambda(t-s)} - 1) \quad \text{for all } t \in (s, T).$

We remark that if $u \in \mathrm{LSC}(M \times [0, T), \mathbb{R} \cup \{\infty\})$, then u is bounded from below on $M \times [0, S]$ for any $0 < S < T$.

Proof. We first show that (2.3) is valid. We set $C_2 = C_0 + 2\Lambda C_1$ and compute in an informal way that

$$0 \geq v_t + H(x, D_x v, v) \geq v_t + H(x, p, 0) - \Lambda |v| \geq v_t - C_0 - \Lambda(v + 2C_1),$$

and deduce that v is a subsolution of

(2.5) $\qquad\qquad\qquad\qquad v_t - \Lambda v - C_2 = 0 \quad \text{in } M \times (0, T).$

Set $c = \liminf_{t \to 0+} v(y, t)$. Suppose, to the contrary to (2.3), that there is $\tau \in (0, T)$ such that

$$v(y, \tau) > c e^{\Lambda \tau} + C_2 \Lambda^{-1}(e^{\Lambda \tau} - 1).$$

Note that if we define the function v_0 on $[0, T)$ by

$$v_0(t) = \begin{cases} c & \text{if } t = 0, \\ v(y, t) & \text{otherwise} \end{cases}$$

then $v_0 \in \mathrm{LSC}([0, T), \mathbb{R} \cup \{\infty\})$. Choose $\varepsilon > \delta > 0$ so small that

$$v_0(\tau) > \varepsilon + v_0(0) e^{\Lambda \tau} + (C_2 + \varepsilon)\Lambda^{-1}(e^{\Lambda \tau} - 1) \quad \text{and} \quad \Lambda \delta < \varepsilon.$$

For any $a \in \mathbb{R}$, set

$$\psi_a(t) = a e^{\Lambda t} + (C_2 + \varepsilon)\Lambda^{-1}(e^{\Lambda t} - 1),$$

and note that

$$\psi_a'(t) - \Lambda \psi_a(t) - C_2 = \varepsilon \quad \text{for all } t \geq 0.$$

Fix a function $\chi \in C^1([0, \infty), \mathbb{R})$ such that

$$\chi(0) = 0, \quad \chi'(t) \geq 0 \quad \text{for all } t \geq 0, \quad \text{and} \quad \chi(t) = \delta \quad \text{for all } t \geq \tau/2,$$

and for any $k \in \mathbb{N}$, set

$$\chi_k(t) = \chi(kt) \quad \text{and} \quad f_{a,k}(t) = \psi_a(t) + \chi_k(t) \quad \text{for } t \geq 0.$$

Observe that for all $t \geq 0$,

(2.6) $\qquad f'_{a,k}(t) - \Lambda f_{a,k}(t) - C_2 = \varepsilon + \chi'_k(t) - \Lambda \chi_k(t) \geq \varepsilon - \Lambda \delta > 0.$

Consider the function $f_k := \psi_{a,k}$ with $a := v_0(0)$. Since $v_0(0) = c = \liminf_{t \to 0+} v_0(t)$, we may choose $s \in (0, \tau)$ such that $v_0(s) < v_0(0)e^{\Lambda s} + \delta$. If $k \in \mathbb{N}$ is so large that $ks > \tau/2$, then $\chi_k(s) = \delta$, and hence,

$$f_k(s) = v_0(0)e^{\Lambda s} + (C_2 + \varepsilon)(e^{\Lambda s} - 1) + \delta > v_0(s).$$

Fix such $k \in \mathbb{N}$ and note that

$$\min_{t \in [0,\tau]} (v_0 - f_k)(t) \leq (v_0 - f_k)(s) < 0,$$

and, by the choice of τ, ε, δ,

$$(v_0 - f_k)(0) = 0, \qquad (v_0 - f_k)(\tau) > 0.$$

It is clear that the function

$$\Phi : a \mapsto \min_{t \in [0,\tau]} (v_0 - f_{a,k})(t)$$

is monotone decreasing, continuous, and satisfies

$$\lim_{a \to \infty} \Phi(a) = -\infty \quad \text{and} \quad \lim_{a \to -\infty} \Phi(a) = \infty.$$

These observations show that there is a unique $a < v_0(0)$ such that

$$\min_{t \in [0,\tau]} (v_0 - f_{a,k})(t) = 0,$$

while

$$\min_{t=0,\tau} (v_0 - f_{a,k})(t) > \min_{t=0,\tau} (v_0 - f_k)(t) = 0.$$

Thus, there is $t_0 \in (0, \tau)$ such that $t \mapsto v_0(t) - f_{a,k}(t)$ takes a minimum 0 at $t = t_0$. Since $f_{a,k}(t) = v_0(t) = v(y,t)$ at $t = t_0$, by (2.6) we have

(2.7) $\qquad f'_{a,k}(t_0) - \Lambda v(y, t_0) - C_2 = f'_{a,k}(t_0) - \Lambda f_{a,k}(t_0) - C_2 > 0.$

Let a, k be as above and set $f = f_{a,k}$ for simplicity of notation. Fix $\sigma > 0$ so that $\sigma < t_0 < \tau$, and consider the function

$$\Psi_\alpha(x, t) := v(x, t) - f(t) + (t - t_0)^2 + \alpha d(x, y)^2$$

on the set $M \times [\sigma, \tau]$, where $\alpha > 0$ is a constant to be sent to ∞. Let (x_α, t_α) be a minimum point of Ψ_α. It is easily seen that $(x_\alpha, t_\alpha) \to (y, t_0)$ and $v(x_\alpha, t_\alpha) \to v(y, t_0)$

as $\alpha \to \infty$. Thus, if α is large enough, (x_α, t_α) is an interior point of $M \times [\sigma, \tau]$ and hence, by (2.5),

$$f'(t_\alpha) - 2(t_\alpha - t_0) - \Lambda v(x_\alpha, t_\alpha) - C_0 \leq 0.$$

Sending $\alpha \to \infty$ yields

$$f'_{a,k}(t_0) - \Lambda v(y, t_0) - C_0 \leq 0,$$

which contradicts (2.7) and completes the proof of (2.3).

Now, we prove (2.4). Fix any $s \in (0, T)$ such that $v(y, s) \in \mathbb{R}$, and set

$$\eta(t) = v(y, s)e^{\Lambda(t-s)} + C_2 \Lambda^{-1}(e^{\Lambda(t-s)} - 1) \quad \text{for } t \in [0, T).$$

If $\liminf_{t \to 0+} v(y, t) \leq \eta(0)$, then we find by (2.3) that for all $t \in [0, T)$,

$$v(y, t) \leq \eta(0)e^{\Lambda t} + C_2 \Lambda^{-1}(e^{\Lambda t} - 1) = v(y, s)e^{\Lambda(t-s)} + C_2 \Lambda^{-1}(e^{\Lambda(t-s)} - 1),$$

which shows that (2.4) holds.

Next, consider the case $\liminf_{t \to 0+} v(y, t) > \eta(0)$. As before, define $v_0 \in \mathrm{LSC}([0, T), \mathbb{R} \cup \{\infty\})$ by

$$v_0(t) = \begin{cases} \displaystyle\liminf_{t \to 0+} v(y, t) & \text{if } t = 0, \\ v(y, t) & \text{if } t > 0. \end{cases}$$

Notice that the value $v_0(0)$ can be $+\infty$.

To the contrary to (2.4), we suppose that for some $\tau \in (s, T)$,

$$v_0(\tau) > \eta(\tau),$$

and will get a contradiction. Note that

$$v_0(0) > \eta(0), \quad v_0(\tau) > \eta(\tau), \quad \text{and} \quad v_0(s) = \eta(s).$$

We choose $\varepsilon > 0$ so that

$$v_0(\tau) > \eta(\tau) + \varepsilon C_2 \Lambda^{-1}(e^{\Lambda(\tau-s)} - 1),$$

and, for $a \in \mathbb{R}$, define the smooth function g_a on $[0, T)$ by

$$g_a(t) = ae^{\Lambda(t-s)} + (C_2 + \varepsilon)\Lambda^{-1}(e^{\Lambda(t-s)} - 1).$$

Observe that if $a = v_0(s)$, then $g_a(t) = \eta(t) + \varepsilon C_2 \Lambda^{-1}(e^{\Lambda(t-s)} - 1)$ for all $t \in [0, T)$ and

(2.8) $$v_0(0) > g_a(0), \quad v_0(\tau) > g_a(\tau), \quad \text{and} \quad v_0(s) = g_a(s),$$

which implies

$$\min_{[0,\tau]}(v_0 - g_a) \leq 0.$$

Note also that, as $a \to -\infty$, $\min_{[0,\tau]}(v_0 - g_a)$ goes monotonically and continuously to $+\infty$. Consequently, there is $a \leq v_0(s)$ such that

$$\min_{[0,\tau]}(v_0 - g_a) = 0,$$

while, by (2.8), $(v_0 - g_a)(t) > 0$ for $t = 0, \tau$. Hence, there is $t_0 \in (0, \tau)$ such that $t \mapsto v(y, t) - g_a(t)$ takes a local minimum at $t = t_0$, with value 0. As in the proof of (2.3) above, (2.5) and the above observation yield

$$g_a'(t_0) - \Lambda g_a(t_0) - C_2 = g_a'(t_0) - \Lambda v(y, t_0) - C_2 \leq 0,$$

while the function g_a satisfies

$$g_a'(t) - \Lambda g_a(t) - C_2 = \varepsilon \quad \text{for all } t > 0.$$

This contradiction completes the proof. □

The following lemma is a standard observation about BJ subsolutions.

Lemma 2.5. *Let $u, v \in \mathcal{S}_{\mathrm{BJ}}^-(F, U)$, where (F, U) is either (H, M) or $(\partial_t + H, M \times (0, T))$. Then, $\min\{u, v\} \in \mathcal{S}_{\mathrm{BJ}}^-(F, U)$. If $v, w \in \mathcal{S}_{\mathrm{BJ}}(F, U)$ instead, then $\min\{v, w\} \in \mathcal{S}_{\mathrm{BJ}}(F, U)$.*

Proof. Let $\psi \in C^1(U, \mathbb{R})$ and $z \in U$ be a minimum point of the function $\min\{u, v\} - \psi$. Observe that if $\min\{u, v\} = u$ at z, then $u - \psi$ takes a minimum at z, and otherwise $v - \psi$ takes a minimum at z. If $u, v \in \mathcal{S}_{\mathrm{BJ}}^-(F, U)$ (resp., $u, v \in \mathcal{S}_{\mathrm{BJ}}(F, U)$), we find that $F(z, D\psi(z), \min\{u, v\}(z)) \leq 0$ (resp., $F(z, D\psi(z), \min\{u, v\}(z)) = 0$), which completes the proof. □

The following proof of Theorem 2.1 has a strong similarity to that of [14, Theorem 4.1] (see also [12, Theorem 1]).

Proof of Theorem 2.1. We need only to prove that $v \leq w$ on $M \times (0, S)$ for all $0 < S < T$. Thus, we may assume in this proof that $T < \infty$ and v is bounded from below on $M \times [0, T]$.

We first observe that we may assume that v is a bounded function on $M \times [0, T]$. Choosing $C > 0$ so large that

$$H(x, 0, 0) \leq C\Lambda e^{-2\Lambda T},$$

and setting $z(x, t) = Ce^{-2\Lambda t}$ for $(x, t) \in M \times [0, T]$, we compute

$$z_t + H(x, D_x z, z) \leq -2\Lambda z + H(x, 0, 0) + \Lambda z \leq 0 \quad \text{on } M \times [0, T],$$

to find that z is a subsolution of (2.1). Thanks to Lemma 2.5, the function $\min\{v, z\}$ is a BJ subsolution of (2.1), which is bounded on $M \times [0, T]$. It is obvious that (2.2)

is valid with $\min\{v, z\}$ in place of v. To conclude that $v \le w$ on $M \times [0, T)$, we only need to show that $\min\{v, z\} \le w$ on $M \times [0, T)$ for all C large enough. Thus, we may assume by replacing v by $\min\{v, z\}$, with C sufficiently large, if necessary that v is bounded on $M \times [0, T)$.

Next, we regularize v by the inf-convolution in the variable t. We define the function v_0 on $M \times [0, T)$ by

$$v_0(x, t) = \begin{cases} \liminf_{t \to 0+} v(x, t) & \text{for } t = 0, \\ v(x, t) & \text{for } t > 0, \end{cases}$$

let $\varepsilon > 0$, and set

$$v_\varepsilon(x, t) = \inf_{s \in [0, T)} \left(v_0(x, s) + \frac{e^{-\Lambda t}}{2\varepsilon}(t - s)^2 \right) \quad \text{for } (x, t) \in M \times [0, T).$$

Fix a constant $C_0 > 0$ such that $H(x, p, 0) \ge -C_0$ for all $(x, p) \in T^*M$ and $|v(x, t)| \le C_0$ for all $(x, t) \in M \times [0, T)$. Using (2.3) in Lemma 2.4, we obtain

(2.9) $$v_0(x, t) \le v_0(x, 0) + C_0(1 + \Lambda)(e^{\Lambda t} - 1) \le v_0(x, 0) + C_1 t,$$

where $C_1 := C_0 e^{\Lambda T} \Lambda(1 + \Lambda)$.

We set

$$T_1(\varepsilon) = 2\varepsilon C_1 e^{\Lambda T} \quad \text{and} \quad T_2(\varepsilon) = T - 2\sqrt{\varepsilon(C_0 + 1)e^{\Lambda T}},$$

assume that $\varepsilon > 0$ is sufficiently small so that $T_1(\varepsilon) < T_2(\varepsilon)$, and prove that

(2.10) $$v_\varepsilon(x, t) = \min_{s \in (0, T)} \left(v_0(x, s) + \frac{e^{-\Lambda t}}{2\varepsilon}(t - s)^2 \right) \quad \text{for all } (x, t) \in M \times [T_1(\varepsilon), T_2(\varepsilon)],$$

that is, the minimum above is attained at some $s \in [0, T)$.

We first prove that

(2.11) $$v_\varepsilon(x, t) = \min_{s \in [0, T)} \left(v_0(x, s) + \frac{e^{-\Lambda t}}{2\varepsilon}(t - s)^2 \right) \quad \text{for all } (x, t) \in M \times [0, T_2(\varepsilon)],$$

To see this, we fix any $(x, t) \in M \times [0, T_2(\varepsilon)]$. If $s \in (0, T)$ is chosen so that $v_\varepsilon(x, t) + 1 > v_0(x, s) + e^{-\Lambda t}(t - s)^2/(2\varepsilon)$, then

$$C_0 + 1 \ge v_\varepsilon(x, t) + 1 \ge -C_0 + \frac{e^{-\Lambda t}}{2\varepsilon}(t - s)^2,$$

and, consequently,

$$|t - s| < 2\sqrt{\varepsilon(C_0 + \tfrac{1}{2})e^{\Lambda T}} \quad \text{and} \quad s < T_2(\varepsilon) + 2\sqrt{\varepsilon(C_0 + \tfrac{1}{2})e^{\Lambda T}} < T.$$

Applying this estimate to any minimizing sequence $\{s_k\}_{k \in \mathbb{N}} \subset (0, T)$ of the minimization problem in the definition of $v_\varepsilon(x, t)$, we find that (2.11) holds. Also, applying the

estimates above to the minimizer s in (2.11), we find that for all $(x,t) \in M \times [0, T_2(\varepsilon)]$,

$$(2.12) \qquad v_\varepsilon(x,t) \geq \inf \left\{ v_0(x,s) \,:\, 0 \leq s < T, \ |s - t| < 2\sqrt{\varepsilon(C_0 + 1)e^{\Lambda T}} \right\}.$$

To complete the proof of (2.10), we fix $(x,t) \in M \times [T_1(\varepsilon), T_2(\varepsilon)]$ and note by (2.9) and (2.11) that

$$(2.13) \qquad v_\varepsilon(x,t) \leq v_0(x,t) \leq v_0(x,0) + C_1 t.$$

Hence, if $s = 0$ were a minimizer of (2.11), then, by the choice of $T_1(\varepsilon)$, we would have

$$v_0(x,0) + C_1 t \geq v_\varepsilon(x,t) = v_0(x,0) + \frac{e^{-\Lambda t}}{2\varepsilon} t^2$$
$$> v_0(x,0) + \frac{T_1(\varepsilon)e^{-\Lambda T}}{2\varepsilon} t = v_0(x,0) + C_1 t,$$

which is a contradiction. This together with (2.11) assures that (2.10) is valid.

By (2.13) and (2.2), we have for all $x \in M$,

$$(2.14) \qquad v_\varepsilon(x, T_1(\varepsilon)) \leq v_0(x,0) + C_1 T_1(\varepsilon) \leq w(x,0) + C_1 T_1(\varepsilon).$$

By the definition of v_ε, the family of the functions $t \mapsto v_\varepsilon(x,t)$, with $x \in M$, is equi-Lipschitz continuous on $[0, T)$.

Now, we claim that v_ε satisfies, in the BJ sense,

$$v_{\varepsilon,t} + H(x, D_x v_\varepsilon, v_\varepsilon) \leq 0 \quad \text{in } M \times (T_1(\varepsilon), T_2(\varepsilon)).$$

To check this, let $\phi \in C^1(M \times (T_1(\varepsilon), T_2(\varepsilon)), \mathbb{R})$ and assume that $v_\varepsilon - \phi$ takes a minimum at $(\hat{x}, \hat{t}) \in M \times (T_1(\varepsilon), T_2(\varepsilon))$ and for some $\hat{s} \in (0, T)$,

$$v_\varepsilon(\hat{x}, \hat{t}) = v_0(\hat{x}, \hat{s}) + \frac{e^{-\Lambda \hat{t}}}{2\varepsilon}(\hat{t} - \hat{s})^2.$$

Then, the function

$$(x, t, s) \mapsto v_0(x,s) + \frac{e^{-\Lambda t}}{2\varepsilon}(t - s)^2 - \phi(x,t)$$

takes a (local) minimum at $(\hat{x}, \hat{t}, \hat{s})$. This implies that

$$\left(D_x \phi(\hat{x}, \hat{t}), \frac{e^{-\Lambda \hat{t}}}{\varepsilon}(\hat{t} - \hat{s}) \right) \in D^- v_0(\hat{x}, \hat{s}),$$
$$\phi_t(\hat{x}, \hat{t}) = \frac{e^{-\Lambda \hat{t}}}{\varepsilon}(\hat{t} - \hat{s}) - \frac{\Lambda e^{-\Lambda \hat{t}}}{2\varepsilon}(\hat{t} - \hat{s})^2.$$

Since v_0 is a BJ subsolution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, T)$, we have

$$0 \geq \frac{e^{-\Lambda \hat{t}}}{\varepsilon}(\hat{t} - \hat{s}) + H(\hat{x}, D_x \phi(\hat{x}, \hat{t}), v_0(\hat{x}, \hat{s}))$$

$$= \phi_t(\hat{x}, \hat{t}) + \Lambda \frac{e^{-\Lambda \hat{t}}}{2\varepsilon}(\hat{t} - \hat{s})^2 + H\left(\hat{x}, D_x \phi(\hat{x}, \hat{t}), v_\varepsilon(\hat{x}, \hat{t}) - \frac{e^{-\Lambda \hat{t}}}{2\varepsilon}(\hat{t} - \hat{s})^2\right)$$

$$\geq \phi_t(\hat{x}, \hat{t}) + \Lambda \frac{e^{-\Lambda \hat{t}}}{2\varepsilon}(\hat{t} - \hat{s})^2 + H(\hat{x}, D_x \phi(\hat{x}, \hat{t}), v_\varepsilon(\hat{x}, \hat{t})) - \frac{\Lambda e^{-\Lambda \hat{t}}}{2\varepsilon}(\hat{t} - \hat{s})^2$$

$$= \phi_t(\hat{x}, \hat{t}) + H(\hat{x}, D_x \phi(\hat{x}, \hat{t}), v_\varepsilon(\hat{x}, \hat{t})),$$

which assures that v_ε is a BJ subsolution of $u_t + H(x, D_x u, u) = 0$ in $M \times (T_1(\varepsilon), T_2(\varepsilon))$.

Recalling that the functions $t \mapsto v_\varepsilon(x, t)$, with $x \in M$, are equi-Lipschitz continuous on $[T_1(\varepsilon), T_2(\varepsilon)]$, we choose a constant $C(\varepsilon) > 0$ as a Lipschitz bound of the functions above, we find that v_ε is a BJ subsolution of

$$-C(\varepsilon) + H(x, D_x u, u) \leq 0 \quad \text{in } M \times (T_1(\varepsilon), T_2(\varepsilon)).$$

Thus, since v_ε is a bounded function, we deduce that $v_\varepsilon \in \mathrm{Lip}(M \times [T_1(\varepsilon), T_2(\varepsilon)], \mathbb{R})$.

Noting that the functions $(x, t) \mapsto v_\varepsilon(x, t + T_1(\varepsilon)) - C_1 T_1(\varepsilon) e^{\Lambda t}$ is a subsolution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, T_2(\varepsilon) - T_1(\varepsilon))$ and recalling (2.14), we invoke Lemma 2.3, to obtain

$$w(x, t) \geq v_\varepsilon(x, t + T_1(\varepsilon)) - C_1 T_1(\varepsilon) e^{\Lambda t} \quad \text{for all } (x, t) \in M \times [0, T_2(\varepsilon) - T_1(\varepsilon)).$$

Combine this with (2.12), to get for all $(x, t) \in M \times [0, T_2(\varepsilon) - T_1(\varepsilon))$,

$$w(x, t) + C_1 T_1(\varepsilon) e^{\Lambda t} \geq v_\varepsilon(x, t + T_1(\varepsilon))$$

$$\geq \inf \left\{ v_0(x, s) \, : \, s \in [0, T), \, |s - t| \leq T_1(\varepsilon) + 2\sqrt{\varepsilon(C_0 + 1)e^{\Lambda T}} \right\},$$

which yields, in view of the lower semicontinuity of $t \mapsto v_0(x, t)$, that for all $(x, t) \in M \times [0, T)$,

$$v(x, t) \leq v_0(x, t) \leq \liminf_{[0,T) \ni s \to t} v_0(x, s) \leq \liminf_{\varepsilon \to 0+} v_\varepsilon(x, t + T_1(\varepsilon)) \leq w(x, t).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

3. EXISTENCE OF BJ SOLUTIONS

We consider the Cauchy problem

(3.1)
$$\begin{cases} u_t + H(x, D_x u, u) = 0 & \text{in } M \times [0, \infty), \\ u(x, 0) = \phi(x) & \text{for } x \in M, \end{cases}$$

where

(3.2)
$$\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\}).$$

In view of Theorem 2.1, we understand the initial condition in (3.1) as

$$\phi(x) = \liminf_{t \to 0+} u(x, t) \quad \text{for all } x \in M.$$

We call a function $u \in \mathrm{LSC}(M \times [0, \infty), \mathbb{R} \cup \{\infty\})$ a BJ solution of (3.1) if u is a BJ solution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, \infty)$ and

$$(3.3) \qquad \phi(x) = u(x, 0) = \liminf_{t \to \infty} u(x, t) \quad \text{for all } x \in M.$$

Theorem 3.1. *Assume* (H1)–(H4) *and* (3.2). *Then there exists a BJ solution of* (3.1).

This theorem and Theorem 2.1 assure that for each $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$, there exists a unique BJ solution u of (3.1). In what follows we write $S_t = S_t^H$ for the map $\phi \mapsto u(\cdot, t)$ for all $t \geq 0$. It follows from Theorem 2.1 and Lemma 2.4 that for every $s, t \geq 0$, $S_t \circ S_s = S_{t+s}$.

Proof. We fix a constant $C_0 > 0$ so that

$$H(x, p, 0) \geq -C_0 \quad \text{for all } (x, p) \in T^*M \quad \text{and} \quad \phi \geq -C_0 \quad \text{on } M.$$

Select a sequence $\{\phi_k\}_{k \in \mathbb{N}} \subset C^1(M, \mathbb{R})$ such that for all $x \in M$,

$$-C_0 \leq \phi_k(x) \leq \phi_{k+1}(x) \quad \text{and} \quad \lim_{k \to \infty} \phi_k(x) = \phi(x).$$

It is well-known (see Theorem A.1 in the appendix) that for each $k \in \mathbb{N}$ there is a Crandall-Lions viscosity solution $u_k \in \mathrm{Lip}(M \times [0, \infty), \mathbb{R})$ of (3.1), with ϕ_k in place of ϕ, which is aslo a BJ solultion of

$$(3.4) \qquad u_t + H(x, D_x u, u) = 0 \quad \text{in } M \times (0, \infty).$$

Note that if we set $v(x, t) = -C_0 e^{\Lambda t}$ for $(x, t) \in M \times [0, \infty)$, then v is a subsolution of (3.4).

By the classical comparison theorem (or Theorem 2.1 above), we find that for all $k \in \mathbb{N}$,

$$-C_0 e^{\Lambda t} \leq u_k(x, t) \leq u_{k+1}(x, t) \quad \text{for all } (x, t) \in M \times [0, \infty).$$

We define a function u_∞ on $M \times [0, \infty)$ by

$$u_\infty(x, t) = \lim_{k \to \infty} u_k(x, t).$$

It is obvious that $u_\infty \in \mathrm{LSC}(M \times [0, \infty), \mathbb{R} \cup \{\infty\})$.

We claim that u_∞ is a BJ solution of (3.1). By the standard stability property of viscosity solutions, we find that u_∞ is a BJ solution of $u_t + H(x, D_x u, u) = 0$ in

$M \times (0, \infty)$. Indeed, as we see below, u_∞ is the so-called lower relaxed limit of $\{u_k\}$. It is clear that for any $(x, t) \in M \times (0, T)$,

$$(3.5) \qquad u_\infty(x, t) \geq \sup_{j \in \mathbb{N}} \inf\{u_k(y, s) \,:\, d(y, x) + |s - t| < j^{-1}, \, k \geq j\}.$$

On the other hand, for any $a < u_\infty(x, t)$, by the continuity of the u_k, we may choose $l, m \in \mathbb{N}$ such that

$$a < u_l(y, s) \quad \text{if} \quad d(y, x) + |s - t| < m^{-1},$$

which implies that for $j := \max\{l, m\}$,

$$a < u_k(y, s) \quad \text{if} \quad k \geq j, \, d(y, x) + |s - t| < j^{-1}.$$

This combined with (3.5) shows that for all $(x, t) \in M \times (0, T)$,

$$(3.6) \qquad u_\infty(x, t) = \sup_{j \in \mathbb{N}} \inf\{u_k(y, s) \,:\, d(y, x) + |s - t| < j^{-1}, \, k \geq j\},$$

where the formula on the right hand side is the lower relaxed limit of $\{u_k\}$. The identity above can be stated as

$$(3.7) \qquad -u_\infty(x, t) = \inf_{j \in \mathbb{N}} \sup\{-u_k(y, s) \,:\, d(y, x) + |s - t| < j^{-1}, \, k \geq j\},$$

where the formula on the right hand side is the upper relaxed limit of $\{-u_k\}$. Noting that $-u_k \in \mathcal{S}(-(\partial_t + H)^\ominus, M \times (0, T)) \subset \mathcal{S}^-((\partial_t + H)^\ominus, M \times (0, T))$, we find by [7, Lemma 6.1], for instance, that $-u_\infty \in \mathcal{S}^-((\partial_t + H)^\ominus, M \times (0, T))$, which implies that $u_\infty \in \mathcal{S}_{\mathrm{BJ}}^-(\partial_t + H, M \times (0, T))$. It is straightforward from (3.6) and [7, Remark 6.2] to find that $u_\infty \in \mathcal{S}^+(\partial_t + H, M \times (0, T))$. These inclusions together prove that u_∞ is a BJ solution of (3.4).

It remains to check (3.3). Fix any $x \in M$. Note first that

$$u_\infty(x, 0) = \lim_{k \to \infty} u_k(x, 0) = \lim_{k \to \infty} \phi_k(x) = \phi(x).$$

For any $k \in \mathbb{N}$, we have

$$\liminf_{t \to 0+} u_\infty(x, t) \geq \lim_{t \to 0+} u_k(x, t) = \phi_k(x),$$

and hence,

$$\phi(x) \leq \liminf_{t \to 0+} u_\infty(x, t).$$

Now, (2.3) assures that for all $0 < T < \infty$ and $(x, t) \in M \times [0, T)$,

$$u_k(x, t) \leq u_k(x, 0)e^{\Lambda t} + C_0(\Lambda^{-1} + 2)e^{\Lambda T}(e^{\Lambda t} - 1) \quad \text{for all } k \in \mathbb{N},$$

which implies

$$u_\infty(x, t) \leq \phi(x)e^{\Lambda t} + C_0(\Lambda^{-1} + 2)e^{\Lambda T}(e^{\Lambda t} - 1),$$

and moreover,

$$\liminf_{t \to 0+} u_\infty(x,t) \le \phi(x) = u_\infty(x,0) \quad \text{for all } x \in M.$$

Thus, we find that (3.3) holds and u_∞ is a BJ solution of (3.1). $\qquad \square$

Corollary 3.2. *Under the hypotheses of Theorem 3.1, there exists a unique BJ solution of* (3.1).

Proof. The existence and uniqueness assertions follow from Theorems 3.1 and 2.1, respectively. $\qquad \square$

4. Value function representation

We give a value function representation or the Hopf-Lax-Oleink formula for the solution of (3.1).

Let L denote the Lagrangian associated with H, that is,

$$L(x,\xi,u) = \sup_{p \in T_x^* M} (\langle p, \xi \rangle - H(x,p,u)) \quad \text{for } (x,\xi,u) \in TM \times \mathbb{R}.$$

Note that L is lower semicontinuous on $TM \times \mathbb{R}$, and that (x,ξ,u) is coercive and, furthermore, has a superlinear growth in ξ. To see the superlinear growth, let $A > 0$ be any constant and observe that

$$(4.1) \qquad L(x,\xi,u) \ge \max_{p \in T_x^* M, |p|=A} (\langle p, \xi \rangle - C(A,x,u)) \ge A|\xi| - C(A,x,u),$$

where $C(A,x,u) := \max_{p \in T_x^* M, |p|=A} H(x,p,u)$.

Let $u \in \mathrm{LSC}(M \times [0,\infty), \mathbb{R} \cup \{+\infty\})$. Note that for each $T > 0$, u is bounded from below on $M \times [0,T]$ by a constant. For $(x,t) \in M \times (0,\infty)$, let $\mathcal{C}(x,t,u)$ denote the set of all $\gamma \in \mathrm{AC}([0,t], M)$ such that $\gamma(t) = x$ and

$$(4.2) \qquad \int_0^t (|L(\gamma(s), \dot{\gamma}(s), 0)| + |u(\gamma(s), s)|) ds < \infty.$$

Theorem 4.1. *Assume* (H1)–(H4). *Let* $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$ *and let* u *be the BJ solution of* (3.1). *Fix* $(x,t) \in M \times (0,\infty)$ *so that* $u(x,t) < \infty$. *Then*

$$(4.3) \qquad u(x,t) = \min_{\gamma \in \mathcal{C}(x,t,u)} \int_0^t L(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s)) ds + \phi(\gamma(0)),$$

and the minimum above is attained.

It is convenient to convert our Hamilton-Jacobi equation to the one whose Lagrangian $\widetilde{L}(x,\xi,t,u)$ is increasing in u. Let u be as in Theorem 4.1 and for $\lambda \in \mathbb{R}$, set

$$v(x,t) = e^{\lambda t} u(x,t),$$

and calculate in a slightly informal way that

$$v_t = e^{\lambda t}(u_t + \lambda u) = -e^{\lambda t}H(x, D_x u, u) + \lambda v = -e^{\lambda t}H(x, e^{-\lambda t}D_x v, e^{-\lambda t}v) + \lambda v,$$

to find that in the viscosity sense,

$$v_t + \widetilde{H}(x, D_x v, t, v) = 0 \quad \text{in } M \times (0, \infty),$$

where $\widetilde{H}$ is given by

$$\widetilde{H}(x, p, t, u) = e^{\lambda t}H(x, e^{-\lambda t}p, e^{-\lambda t}u) - \lambda u \quad \text{for } (x, p, t, u) \in T^*M \times \mathbb{R}^2.$$

The Lagrangian $\widetilde{L}(x, \xi, t, u)$ corresponding to $\widetilde{H}$ is defined as

$$\widetilde{L}(x, \xi, t, u) := \sup_{p \in T_x^*M} \langle p, \xi \rangle - \widetilde{H}(x, p, t, u)$$

$$= e^{\lambda t}L(x, \xi, e^{-\lambda t}u) + \lambda u \quad \text{for } (x, \xi, t, u) \in TM \times \mathbb{R}^2.$$

Note that if $\lambda \geq \Lambda$, then $u \mapsto \widetilde{L}(x, \xi, t, u)$ is nondecreasing on $\mathbb{R}$. Henceforth, we fix

$$\lambda = \Lambda + 1.$$

As in the proof of Theorem 3.1, we select a sequence $\{\phi_k\} \subset C^1(M, \mathbb{R})$ such that for all $x \in M$,

$$\phi_k(x) \leq \phi_{k+1}(x) \quad \text{and} \quad \lim_{k \to \infty} \phi_k(x) = \phi(x).$$

Let u_k be the solution of (3.1), with ϕ_k in place of ϕ. Note that $u_k \in \text{Lip}(M \times [0, \infty), \mathbb{R})$, $u_k \leq u_{k+1}$ on M for all $k \in \mathbb{N}$, and

$$u(x, t) = \lim_{k \to \infty} u_k(x, t) \quad \text{for all } (x, t) \in M \times [0, \infty).$$

Set $v_k(x, t) = e^{\lambda t}u_k(x, t)$ for $(x, t) \in M \times [0, \infty)$. We recall that for every $k \in \mathbb{N}$,

$$(4.4) \qquad v_k(x, t) = \min_{\gamma \in \text{AC}([0,t],M), \, \gamma(t)=x} \int_0^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v_k(\gamma(s), s))ds + \phi_k(\gamma(0)),$$

and the minimum above is achieved at some $\gamma \in \text{AC}([0, t], M)$ satisfying $\gamma(t) = x$. Note that, in the above formula,

$$(4.5) \qquad \int_0^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v_k(\gamma(s), s))ds$$

$$= \int_0^t e^{\lambda s}[L(\gamma(s), \dot{\gamma}(s), u_k(\gamma(s), s)) + \lambda u_k(\gamma(s), s)]ds.$$

Lemma 4.2. *Assume (H1)–(H4). Let $(x, t) \in M \times (0, \infty)$ and $\gamma \in \text{AC}([0, t], M)$ be such that $\gamma(t) = x$. Let $u \in \text{LSC}(M \times [0, t], \mathbb{R} \cup \{+\infty\})$. Then, $\gamma \in \mathcal{C}(x, t, u)$ if and only if*

$$(4.6) \qquad \int_0^t e^{\lambda s}[L(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s)) + \lambda u(\gamma(s), s)]ds < \infty.$$

Notice that if $u \geq -C_0$ for some $C_0 > 0$, then

$$L(y, \xi, u) \geq L(y, \xi, -C_0) - \lambda(u + C_0),$$

and

$$e^{\lambda s}[L(y, \xi, u) + \lambda u] \geq e^{\lambda s}[L(x, \xi, -C_0) + \lambda(-C_0)] \geq -C$$

for all $(y, \xi) \in TM$ and some constant $C > 0$. Hence, the condition (4.6) makes sense.

Proof. We fix a constant $C_0 > 0$ so that for all $(y, s) \in M \times [0, t]$ and $\xi \in T_y M$,

$$u(y, s) \geq -C_0 \quad \text{and} \quad L(y, \xi, 0) \geq -C_0,$$

which, in particular, yield

$$|u(y, s)| \leq 2C_0 + u(y, s) \quad \text{and} \quad |L(y, \xi, 0)| \leq 2C_0 + L(y, \xi, 0).$$

Assume first that $\gamma \in \mathcal{C}(x, t, u)$. Note that for any $(y, \xi, u) \in TM \times \mathbb{R}$,

$$L(y, \xi, u) + \lambda u \leq L(y, \xi, 0) + \Lambda|u| + \lambda u \leq |L(y, \xi, 0)| + 2\lambda|u|,$$

and hence,

$$\int_0^T e^{\lambda s}[L(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s)) + \lambda u(\gamma(s), s)]ds$$

$$\leq \int_0^T e^{\lambda s}(|L(\gamma(s), \dot{\gamma}(s), 0)| + 2\lambda|u(\gamma(s), s)|)ds$$

$$\leq e^{\lambda T} \int_0^T (|L(\gamma(s), \dot{\gamma}(s), 0)| + 2\lambda|u(\gamma(s), s)|)ds.$$

This shows that (4.6) holds.

Next, assume that (4.6) is satisfied. Note that for any $(y, \xi, u) \in TM \times \mathbb{R}$, if $u \geq -C_0$, then

$$|L(y, \xi, 0)| + |u| \leq 2C_0 + L(y, \xi, 0) + |u| \leq 2C_0 + L(y, \xi, u) + \Lambda|u| + |u|$$

$$\leq 2C_0(1 + \lambda) + L(y, \xi, u) + \lambda u.$$

Hence, we have

$$\int_0^t (|L(\gamma(s), \dot{\gamma}(s), 0)| + |u(\gamma(s), s)|)ds \leq \int_0^t e^{\lambda s}[|L(\gamma(s), \dot{\gamma}(s), 0)| + |u(\gamma(s), s)|]ds$$

$$\leq \int_0^t e^{\lambda s}[L(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s)) + \lambda u(\gamma(s), s)]ds + 2C_0(1 + \lambda)\lambda^{-1}(e^{\lambda t} - 1),$$

which shows that $\gamma \in \mathcal{C}(x, t, u)$. The proof is complete.   □

Proof of Theorem 4.1. By formula (4.4), we have

$$v(x,t) \geq v_k(x,t) = \min_{\gamma(0)=x} \int_0^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v_k(\gamma(s), s))ds + \phi_k(\gamma(0)).$$

We select a minimizer $\gamma_k \in \mathrm{AC}([0,t], M)$, with $\gamma_k(t) = x$, for each $k \in \mathbb{N}$ in the above, to obtain

$$v(x,t) \geq \int_0^t \widetilde{L}(\gamma_k(s), \dot{\gamma}_k(s), s, v_k(\gamma_k(s), s)))ds + \phi_k(\gamma_k(0))$$

$$\geq \int_0^t \widetilde{L}(\gamma_k(s), \dot{\gamma}_k(s), s, v_j(\gamma_k(s), s)))ds + \phi_j(\gamma_k(0)) \quad \text{if } k \geq j,$$

where we have used the fact that $u \mapsto \widetilde{L}(y, \xi, s, u)$ is nondecreasing. Since $\xi \mapsto \widetilde{L}(y, \xi, s, u)$ has a superlinear growth (see (4.1)), we may select a subsequence of $\{\gamma_k\}$, which will be denoted again by the same symbol, such that for some $\gamma \in \mathrm{AC}([0,t], M)$, as $k \to \infty$,

$$\gamma_k \to \gamma \quad \text{in } C([0,t], M) \quad \text{and} \quad \dot{\gamma}_k \to \dot{\gamma} \quad \text{weakly in } L^1([0,t], TM).$$

It follows that for any $j \in \mathbb{N}$,

$$v(x,t) \geq \int_0^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v_j(\gamma(s), s))ds + \phi_j(\gamma(0)).$$

Furthermore, by the monotone convergence theorem,

$$(4.7) \qquad v(x,t) \geq \int_0^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds + \phi(\gamma(0)).$$

As noted in (4.5), we have

$$\int_0^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds = \int_0^t e^{\lambda s}[L(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s)) + \lambda u(\gamma(s), s)]ds,$$

and hence, (4.7) assures together with Lemma 4.2 that $\gamma \in \mathcal{C}(x, t, u)$.

On the other hand, from (4.4), we have for any $\eta \in \mathcal{C}(x, t, u)$,

$$v_k(x,t) \leq \int_0^t \widetilde{L}(\eta(s), \dot{\eta}(s), s, v_k(\eta(s), s))ds + \phi_k(\eta(0))$$

$$\leq \int_0^t \widetilde{L}(\eta(s), \dot{\eta}(s), s, v(\eta(s), s))ds + \phi(\eta(0)),$$

and moreover,

$$v(x,t) \leq \int_0^t \widetilde{L}(\eta(s), \dot{\eta}(s), s, v(\eta(s), s))ds + \phi(\eta(0)).$$

Thus, we have

$$(4.8) \qquad v(x,t) = \min_{\eta \in \mathcal{C}(x,t,u)} \int_0^t \widetilde{L}(\eta(s), \dot{\eta}(s), s, v(\eta(s), s))ds + \phi(\eta(0)).$$

We now deduce from (4.8) that (4.3) is valid.

Fix any $\eta \in \mathcal{C}(x, t, u)$. We may assume that $\phi(\eta(0)) < \infty$. Indeed, otherwise, it is obvious that

$$u(x, t) \leq \int_0^t L(\eta(s), \dot{\eta}(s), u(\eta(s), s))ds + \phi(\eta(0)).$$

Note that $\eta \in \mathcal{C}(\eta(\tau), \tau, u)$ for all $\tau \in (0, t]$ and that $u(\eta(s), s) < \infty$ for a.e. $s \in [0, t]$. By (4.8), which is valid for general (x, t), we have for a.e. $\tau \in (0, t]$,

$$e^{\lambda \tau} u(\eta(\tau), \tau) = v(\eta(\tau), \tau) \leq \int_0^\tau \widetilde{L}(\eta(s), \dot{\eta}(s), s, v(\eta(s), s))ds + \phi(\eta(0))$$

$$= \int_0^\tau e^{\lambda s}[L(\eta(s), \dot{\eta}(s), u(\eta(s), s)) + \lambda u(\eta(s), s)]ds + \phi(\eta(0)).$$

Setting

$$f(\tau) = e^{-\lambda \tau}\left(\int_0^\tau e^{\lambda s}[L(\eta(s), \dot{\eta}(s), u(\eta(s), s)) + \lambda u(\eta(s), s)]ds + \phi(\eta(0))\right),$$

and using the above, we compute that for a.e. $\tau \in (0, t)$,

$$f'(\tau) = -\lambda f(\tau) + L(\eta(\tau), \dot{\eta}(\tau), u(\eta(\tau), \tau)) + \lambda u(\eta(\tau), \tau) \leq L(\eta(\tau), \dot{\eta}(\tau), u(\eta(\tau), \tau)),$$

and moreover,

$$u(x, t) \leq f(t) \leq \int_0^t L(\eta(s), \dot{\eta}(s), u(\eta(s), s))ds + f(0)$$

$$= \int_0^t L(\eta(s), \dot{\eta}(s), u(\eta(s), s))ds + \phi(\eta(0)).$$

Hence, we find that

$$u(x, t) \leq \inf_{\eta \in \mathcal{C}(x, t, u)} \int_0^t L(\eta(s), \dot{\eta}(s), u(\eta(s), s))ds + \phi(\eta(0)).$$

Now, let $\gamma \in \mathcal{C}(x, t, u)$ be a minimizer for the right hand side of (4.8). We claim that for a.e. $\tau \in (0, t)$,

(4.9) $$v(\gamma(\tau), \tau) = \int_0^\tau \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds + \phi(\gamma(0)).$$

Once this is proved, setting

$$f(\tau) = e^{-\lambda \tau}\left(\int_0^\tau e^{\lambda s}[L(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s)) + \lambda u(\gamma(s), s)]ds + \phi(\gamma(0))\right),$$

we argue as above, to find that for a.e. $\tau \in (0, t)$,

$$f'(\tau) = L(\gamma(\tau), \dot{\gamma}(\tau), u(\gamma(\tau), \tau))$$

and

$$u(x, t) = f(t) = \int_0^t L(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s))ds + \phi(\gamma(0)),$$

which shows that (4.3) is valid and there exists a minimizer of the minimization in (4.3).

It remains to prove that (4.9) holds. As before, from (4.8) we find that for a.e. $\tau \in (0, t)$,

$$v(\gamma(\tau), \tau) \leq \int_0^\tau \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds + \phi(\gamma(0)).$$

We only need to show that the above inequalities are in fact equalities. To see this, we suppose to the contrary that for some $\tau \in (0, t)$,

(4.10)
$$v(\gamma(\tau), \tau) < \int_0^\tau \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds + \phi(\gamma(0)).$$

By (4.8), there is a curve $\eta \in \mathcal{C}(\gamma(\tau), \tau, u)$ such that

$$v(\gamma(\tau), \tau) = \int_0^\tau \widetilde{L}(\eta(s), \dot{\eta}(s), s, v(\eta(s), s))ds + \phi(\eta(0)).$$

We define a curve $\zeta \in \mathrm{AC}([0, t], M)$ by setting

$$\zeta(s) = \begin{cases} \eta(s) & \text{for } s \in [0, \tau], \\ \gamma(s) & \text{for } s \in (\tau, t], \end{cases}$$

and note that $\zeta \in \mathcal{C}(x, t, u)$. Observe by (4.8) and (4.10) that

$$
\begin{aligned}
v(x, t) &\leq \int_0^t \widetilde{L}(\zeta(s), \dot{\zeta}(s), s, v(\zeta(s), s))ds + \phi(\zeta(0)) \\
&= \int_\tau^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds + \int_0^\tau \widetilde{L}(\eta(s), \dot{\eta}(s), s, v(\eta(s), s))ds + \phi(\eta(0)) \\
&= \int_\tau^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds + v(\gamma(\tau), \tau) \\
&< \int_\tau^t \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds + \int_0^\tau \widetilde{L}(\gamma(s), \dot{\gamma}(s), s, v(\gamma(s), s))ds + \phi(\gamma(0)) \\
&= v(x, t),
\end{aligned}
$$

which is a contradiction. This shows that (4.9) holds. $\square$

5. Fundamental solultions

Given $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$, if $u = u(x, t)$ is the BJ solution of (3.1), then we write $u(x, t, \phi) := u(x, t)$ for notational clarity. Let $(y, c) \in M \times (\mathbb{R} \cup \{\infty\})$ and define $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$ by

(5.1)
$$\phi_{y,c}(x) = \begin{cases} c & \text{if } x = y, \\ \infty & \text{otherwise.} \end{cases}$$

We set $h(x, t, y, c) = u(x, t, \phi_{y,c})$ for $(x, t) \in M \times [0, T)$. Notice that $h(x, t, y, \infty) = \infty$. We call this function $h(x, t, y, c)$ on $M \times [0, T)$, with parameter $(y, c) \in M \times (\mathbb{R} \cup \{\infty\})$, a *fundamental solution* to $u_t + H(x, D_x u, u) = 0$ in $M \times (0, T)$.

Lemma 5.1. *Assume* (H1)–(H4).

(i) *For any $(x, t, y) \in M \times [0, T) \times M$, the function $c \mapsto h(x, t, y, c)$ is nondecreasing on $\mathbb{R}$ and Lipschitz continuous on $\mathbb{R}$, with Lipschitz bound $e^{\Lambda t}$.*

(ii) *The function h is lower semicontinuous on $M \times [0, T) \times M \times (\mathbb{R} \cup \{\infty\})$.*

(iii) *For any $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$, the function $(x, t, y) \mapsto h(x, t, y, \phi(y))$ is lower semicontinuous on $M \times [0, T) \times M$.*

(iv) *For any $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$, the function $(x, t) \mapsto \inf_{y \in M} h(x, t, y, \phi(y))$ is lower semicontinuous on $M \times [0, T)$.*

Before going into the proof of the above lemma, we recall that, by definition, a neighborhood of $\infty \in (-\infty, \infty] = \mathbb{R} \cup \{\infty\}$ is a subset of $(-\infty, \infty]$ containing a set of the form $(a, \infty] = (a, \infty) \cup \{\infty\}$, with $a \in \mathbb{R}$.

Proof. We begin with assertion (i). Let $(x, t, y) \in M \times [0, T) \times M$ and $c_1, c_2 \in \mathbb{R}$. Let ϕ_{y,c_1} and ϕ_{y,c_2} be the functions defined by (5.1), with $c = c_1, c_2$, respectively. If $c_1 < c_2$, then $\phi_{y,c_1} < \phi_{y,c_2}$ on M, and Theorem 2.1 yields $h(x, t, y, c_1) \geq h(x, t, y, c_2)$ for all $(x, t, y) \in M \times [0, T) \times M$. This assures the desired monotonicity of $h(x, t, y, c)$ in c. Setting $v(x, t) = h(x, t, y, c_1) + |c_1 - c_2| e^{\Lambda t}$, we note that v is a BJ supersolution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, T)$ and that $h(x, 0, y, c_2) \leq v(x, 0)$ for all $x \in M$, and conclude by Theorem 2.1 that $h(x, t, y, c_2) \leq h(x, t, y, c_1) + |c_1 - c_2| e^{\Lambda t}$ for all $(x, t, y) \in M \times [0, T) \times M$. This shows that $c \mapsto h(x, t, y, c)$ is Lipschitz continuous with Lipschitz bound $e^{\Lambda t}$ for any $(x, t, y) \in M \times [0, T) \times M$.

To check (ii), let $(x_0, t_0, y_0, c_0) \in M \times [0, T) \times M \times (\mathbb{R} \cup \{\infty\})$ and $a \in \mathbb{R}$ be such that $h(x_0, t_0, y_0, c_0) > a$. By assertion (i), there is $b < c_0$ such that $h(x_0, t_0, y_0, b) > a$. In view of the proof of Theorem 3.1, choosing a sequence $\{\phi_k\}_{k \in \mathbb{N}} \subset C^1(M, \mathbb{R})$ such that

$$\phi_k \leq \phi_{y_0, b} \quad \text{on } M \quad \text{and} \quad \lim_{k \to \infty} \phi_k(x) = \phi_{y_0, b}(x) \quad \text{for all } x \in M,$$

we have

$$\lim_{k \to \infty} u(x, t, \phi_k) = h(x, t, y_0, b) \quad \text{for all } (x, t) \in M \times [0, T).$$

Fix $k \in \mathbb{N}$ so that

$$u(x_0, t_0, \phi_k) > a.$$

By the continuity of $(x, t) \mapsto u(x, t, \phi_k)$ and ϕ_k, we may select neighborhoods V and W of (x_0, t_0) and (y_0, c_0), respectively, so that

$$u(x, t, \phi_k) > a \quad \text{for all } (x, t) \in V \quad \text{and} \quad \phi_k \leq \phi_{y,c} \quad \text{on } M \quad \text{for all } (y, c) \in W,$$

which imply, together with Theorem 2.1, that

$$h(x, t, y, c) \geq u(x, t, \phi_k) > a \quad \text{for all } (x, t, y, c) \in V \times W.$$

This assures that (ii) is valid.

To prove assertion (iii), we fix $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$. Let $(x_0, t_0, y_0) \in M \times [0, T) \times M$ and $a \in \mathbb{R}$ be such that $h(x_0, t_0, y_0, \phi(y_0)) > a$. By assertion (ii), we can choose a neighborhood V of $(x_0, t_0, y_0, \phi(y_0))$ such that

$$h(x, t, y, c) > a \quad \text{for all } (x, t, y, c) \in V.$$

In view of the monotonicity of $c \mapsto h(x, t, y, c)$, we may assume that $V = W \times (b, \infty]$ for some neighborhood W of (x_0, t_0, y_0). By the semicontinuity of ϕ, we can choose a neighborhood U of y_0 so that $\phi(y) > b$ for all $y \in U$. Then, we have

$$h(x, t, y, \phi(y)) > a \quad \text{for all } (x, t, y) \in W \cap (M \times [0, T) \times U),$$

which shows the lower semicontinuity of $(x, t, y) \mapsto h(x, t, y, \phi(y))$, proving (iii).

Now, we prove (iv). Fix any $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$. We note that for any $(x, t) \in M \times [0, T)$, the function $y \mapsto h(x, t, y, \phi(y))$ is lower semicontinuous on M by assertion (iii), M is compact, and therefore, it attains a minimum at some point in M. Let $(x_0, t_0) \in M \times [0, T)$ and $a \in \mathbb{R}$ be such that $\min_{y \in M} h(x_0, t_0, y, \phi(y)) > a$. Noting that

$$h(x_0, t_0, y, \phi(y)) > a \quad \text{for all } y \in M,$$

in view of assertion (iii), for each $y \in M$ we can choose neighborhoods U_y and V_y of (x_0, t_0) and y, respectively, such that

$$h(x, t, z, \phi(z)) > a \quad \text{for all } (x, t, z) \in U_y \times V_y.$$

Since M is compact, we may select a finite number of $y_i \in M$, with $i = 1, \ldots, J$, so that $M = \bigcup_{i=1}^{J} V_{y_i}$. Then, we set $U = \bigcap_{i=1}^{J} U_{y_i}$, to find that

$$h(x, t, z, \phi(z)) > a \quad \text{for all } (x, t, z) \in U \times M.$$

This shows that for the neighborhood U of (x_0, t_0) and all $(x, t) \in U$,

$$\min_{y \in M} h(x, t, y, \phi(y)) > a,$$

which proves assertion (iv). $\qquad\square$

Theorem 5.2. *Assume* (H1)–(H4). *Let* $\phi \in \mathrm{LSC}(M, \mathbb{R} \cup \{\infty\})$ *and let* $u \in \mathrm{LSC}(M \times [0, T), \mathbb{R} \cup \{\infty\})$ *be the (unique) BJ solution of* (3.1). *Then,*

$$u(x, t) = \min_{y \in M} h(x, t, y, \phi(y)) \quad \textit{for all } (x, t) \in M \times [0, T).$$

Proof. Set

$$v(x, t) = \min_{y \in M} h(x, t, y, \phi(y)) \quad \text{for } (x, t) \in M \times [0, T).$$

Lemma 5.1, (iv) assures that $v \in \mathrm{LSC}(M \times [0, T), \mathbb{R} \cup \{\infty\})$. It is a standard observation that v is a BJ solution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, T)$. Indeed, let $\psi \in C^2(M \times (0, T), \mathbb{R})$ and assume that $v - \psi$ takes a minimum at $(x_0, t_0) \in M \times (0, T)$. Choose $y_0 \in M$ so that

$$v(x_0, t_0) = h(x_0, t_0, y_0, \phi(y_0)),$$

and note that $(x, t) \mapsto h(x, t, y_0, \phi(y_0)) - \psi(x, t)$ takes a minimum at (x_0, t_0), which yields, since h is a fundamental solution,

$$0 = \psi_t(x_0, t_0) + H(x_0, D_x \psi(x_0, t_0), h(x_0, t_0, y_0, \phi(y_0)))$$
$$= \psi_t(x_0, t_0) + H(x_0, D_x \psi(x_0, t_0), v(x_0, t_0)).$$

Hence, v is a BJ solution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, T)$.

For each fixed $y \in M$, we have $\phi(x) \le \phi_{y, \phi(y)}(x)$ for all $x \in M$, and moreover, by Theorem 2.1, $u(x, t) \le h(x, t, y, \phi(y))$ for all $(x, t) \in M \times [0, T)$. Since $y \in M$ is arbitrary, we find that $u(x, t) \le v(x, t)$ for all $(x, t) \in M \times [0, T)$. For fixed $y \in M$, if $\phi(y) < \infty$, then, by the definition of fundamental solutions (see also (3.3)),

$$\phi(y) = \liminf_{t \to 0+} h(y, t, y, \phi(y)),$$

which implies

$$u(y, 0) = \phi(y) \ge \liminf_{t \to 0+} v(y, t).$$

Thus, we have

$$u(x, 0) \ge \liminf_{t \to 0+} v(x, t) \quad \text{for all } x \in M.$$

Applying Theorem 2.1, we find that $u(x, t) \ge v(x, t)$ for all $(x, t) \in M \times [0, T)$. Thus, we have $u = v$ on $M \times [0, T)$. $\qquad \square$

In our generality, the fundamental solution $h(x, t, y, c)$ may take value $+\infty$ at some point $(x, t) \in M \times (0, \infty)$. A simple example is as follows. Consider the case where $M = \mathbb{T}^1$ and $H(x, p, u) = |p|$. In this case we have

$$h(x, t, y, c) = \begin{cases} c & \text{if } d(x, y) \le t, \\ +\infty & \text{otherwise.} \end{cases}$$

Under hypotheses (H1) and (H2), there are constants $C_0 > 0$ and $R > 0$ such that for all $(x, p) \in T^*M$,

$$H(x, 0, 0) \leq C_0 \quad \text{and} \quad H(x, p, 0) \geq C_0 + 1 \quad \text{if} \ \ |p| \geq R.$$

Moreover, if (H3) holds, then for all $(x, p) \in T^*M$,

$$H(x, p, 0) \geq \tfrac{1}{R}|p| + C_0 \quad \text{if} \ \ |p| \geq R.$$

It is now obvious that if (H1)–(H4) hold, then there exist constants $\delta > 0$ and $C_1 > 0$ such that

$$(5.2) \qquad H(x, p, u) \geq \delta|p| - C_1 - \Lambda|u| \quad \text{for all } (x, p, u) \in T^*M \times \mathbb{R}.$$

Proposition 5.3. *Assume* (H1)–(H4). *Let $y \in M$. There exist constants $\delta > 0$ and $C > 0$ such that for all $(x, t) \in M \times [0, \infty)$,*

$$(5.3) \qquad\qquad h(x, t, y, 0) \geq -C(e^{\Lambda t} - 1),$$

$$(5.4) \qquad\qquad h(x, t, y, 0) \leq C(e^{\Lambda t} - 1) \quad \text{if} \ \ d(x, y) \leq \delta t,$$

Proof. To check (5.3), fix a constant $C_0 > 0$ so that

$$H(x, 0, 0) \leq C_0 \quad \text{for all } x \in M,$$

set

$$v(x, t) = -\Lambda^{-1}C_0(e^{\Lambda t} - 1) \quad \text{for } (x, t) \in M \times [0, \infty),$$

and note that v is a subsolution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, \infty)$. Since $v \in C(M \times [0, \infty), \mathbb{R})$ and $v(x, 0) = 0 \leq h(x, 0, y, 0)$ for all $x \in M$, Theorem 2.1 yields that

$$h(x, t, y, 0) \geq v(x, t) = -\Lambda^{-1}C_0(e^{\Lambda t} - 1) \quad \text{for all } (x, t) \in M \times [0, \infty).$$

To show (5.4), fix constants $\delta > 0$ and $C_1 > 0$ so that (5.2) holds and define $w \in \mathrm{LSC}(M \times [0, \infty), \mathbb{R} \cup \{\infty\})$ by

$$w(x, t) = \begin{cases} \Lambda^{-1}C_1(e^{\Lambda t} - 1) & \text{if} \ \ d(x, y) \leq \delta t, \\ +\infty & \text{otherwise.} \end{cases}$$

We show that w is a BJ supersolution of

$$(5.5) \qquad\qquad u_t + H(x, D_x u, u) = 0 \quad \text{in } M \times (0, \infty).$$

To do this, we note that the function $u(x) := d(x, y)$ is a solution of the eikonal equation

$$|Du(x)| = 1 \quad \text{in } M \setminus \{y\}.$$

This is a standard observation, and skip the proof here. From this remark, we find that the function $u(x,t) := d(x,y) - \delta t$ is a solution of

(5.6) $$u_t + \delta|D_x u| = 0 \quad \text{in } (M \setminus \{y\}) \times (0, \infty).$$

Now, we fix $\theta \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$\theta(r) = 0 \quad \text{for } r \leq 0, \quad \theta(r) > 0 \quad \text{for } r > 0, \quad \text{and} \quad \theta'(r) \geq 0 \quad \text{for } r \in \mathbb{R},$$

define $z^k \in C(M \times [0, \infty), \mathbb{R})$ for every $k \in \mathbb{N}$ by

$$z^k(x,t) = k\theta(d(x,y) - \delta t),$$

and observe that for any $k \in \mathbb{N}$, z^k is a solution of (5.6). Moreover, by sending $k \to \infty$, we find that the function $z \in \mathrm{LSC}(M \times [0, \infty), \mathbb{R} \cup \{\infty\})$ given by

$$z(x,t) = \begin{cases} 0 & \text{if } d(x,y) \leq \delta t, \\ +\infty & \text{otherwise}, \end{cases}$$

is a solution (in the BJ sense) of (5.6). Noting that the set $Z := \{(x,t) \in M \times [0, \infty) : d(x,y) \leq \delta t\}$ is a neighborhood of $\{y\} \times (0, \infty)$ and z vanishes on Z, we easily deduce that z is a solution of

$$u_t + \delta|D_x u| = 0 \quad \text{in } M \times (0, \infty).$$

It is now easy to check that the function $w(x,t) = z(x,t) + \Lambda^{-1}(e^{\Lambda t} - 1)$ is a solution of

$$u_t + \delta|D_x u| - C_1 - \Lambda|u| = 0 \quad \text{in } M \times (0, \infty),$$

which assures, together with (5.2), that w is a supersolution of (5.5). Since $w(x,0) = z(x,0) = h(x,0,y,0)$ for all $x \in M$, we conclude by Theorem 2.1 that $h(x,t,y,0) \leq w(x,t)$ for all $(x,t) \in M \times [0, \infty)$, which yields (5.4). $\square$

6. Long-time behavior of solutions

We are concerned with the long-time behavior of the solution $u = u(x,t)$ of problem

(6.1) $$u_t + H(x, D_x u, u) = 0 \quad \text{in } M \times (0, \infty).$$

Theorem 6.1. *Assume* (H1)–(H4). *Let* $u \in \mathrm{LSC}(M \times (0, \infty), \mathbb{R} \cup \{\infty\})$ *be a BJ solution of* (6.1). *Set*

(6.2) $$v(x) = \lim_{r \to 0+} \inf\{u(y,t) : d(y,x) < r, \ t > r^{-1}\} \quad \text{for } x \in M.$$

Assume that $v(z) \in \mathbb{R}$ *for some* $z \in M$. *Then,* $v \in \mathrm{Lip}(M)$ *and* v *is a viscosity solution of* $H(x, Dv, v) = 0$ *in* M.

We remark, as noted in the proof of Lemma 2.3, that, once the Lipschitz continuity of v is known, v is a BJ solution of $H(x, Dv, v) = 0$ in M if and only if it is a viscosity solution, in the Crandall-Lions sense, of $H(x, Dv, v) = 0$ in M.

Notice that the definition of v above is the so-called lower relaxed limit of $u(x, t)$ as $t \to \infty$. In particular, v is lower semicontinuous on M.

An immediate consequence of the theorem above is the following.

Corollary 6.2. *Assume* (H1)–(H4). *Let* $v \in \mathcal{S}_{\mathrm{BJ}}(H, M)$ *If* $v(z) \in \mathbb{R}$ *for some* $z \in M$, *then* $v \in \mathrm{Lip}(M, \mathbb{R})$.

Notice that, thanks to the above corollary, if $v \in \mathcal{S}_{\mathrm{BJ}}(H, M)$ and $v(x) \not\equiv +\infty$, then $v \in \mathrm{Lip}(M, \mathbb{R})$ and, consequently, $v \in \mathcal{S}(H, M)$.

Proof. Set $u(x, t) = v(x)$ for $(x, t) \in M \times (0, \infty)$ and note that $u \in \mathcal{S}_{\mathrm{BJ}}(\partial_t - H, M \times (0, \infty))$. For these functions u and v, the relation (6.2) holds and, hence, Theorem 6.1 assures that $v \in \mathrm{Lip}(M, \mathbb{R})$. $\square$

Proof of Theorem 6.1. We first show that $v(x) \in \mathbb{R}$ for all $x \in M$. For this, we set $\gamma := \inf_M v < \infty$ and pick a sequence $(y_k, t_k) \in M \times (0, \infty)$ such that $\lim_{k \to \infty} t_k = \infty$ and $\lim_{k \to \infty} u(y_k, t_k) = \gamma$. According to Proposition 5.3, there exist constants $\delta > 0$, $C > 0$ such that for all $x, y \in M$, $t \geq 0$,

$$h(x, t, y, 0) \leq C(e^{\Lambda t} - 1) \quad \text{if} \quad d(x, y) \leq \delta t.$$

We fix $T > 0$ so that δT is larger than or equal to the diameter of M, that is,

$$d(x, y) \leq \delta T \quad \text{for all } x, y \in M.$$

Hence, we have

(6.3) $$h(x, T, y, 0) \leq C(e^{\Lambda T} - 1) \quad \text{for all } x, y \in M.$$

By Theorem 5.2, we have

(6.4) $$u(x, t_k + T) = (S_T u(\cdot, t_k))(x) \leq h(x, T, y_k, u(y_k, t_k)) \quad \text{for all } x \in M.$$

Now, we suppose that $\gamma = -\infty$. We may assume without loss of generality that $u(y_k, t_k) \leq -k$ for all $k \in \mathbb{N}$. Combining this with (6.4), Lemma 5.1, Corollary 2.2, and (6.3), we obtain for all $x \in M$ and $k \in \mathbb{N}$,

$$u(x, t_k + T) \leq h(x, T, y_k, -k) \leq -ke^{-\Lambda T} + h(x, T, y_k, 0) \leq -ke^{-\Lambda T} + C(e^{\Lambda T} - 1).$$

This shows that $v(x) = -\infty$ for all $x \in M$, which contradicts that $v(z) \in \mathbb{R}$. Hence, we find that $\gamma \in \mathbb{R}$.

We choose a constant $C_0 > 0$ so that $u(y_k, t_k) \leq C_0$ for all $k \in \mathbb{N}$. We argue similarly to the above by using (6.4), Lemma 5.1, Corollary 2.2, and (6.3), to obtain for all $x \in M$ and $k \in \mathbb{N}$,

$$u(x, t_k + T) \leq h(x, T, y_k, C_0) \leq C_0 e^{\Lambda T} + C(e^{\Lambda T} - 1),$$

which implies

(6.5) $v(x) \leq C_0 e^{\Lambda T} + C(e^{\Lambda T} - 1) \quad \text{for all} \ \ x \in M.$

Thus, we conclude that $v(x) \in \mathbb{R}$.

As a basic property of the lower half-limit, recalling the definition of v and noting that the functions $(x, t) \mapsto u(x, t + s)$, with $s > 0$, are BJ solutions of (6.1), we find that v is a viscosity supersolution, in the Crandall-Lions sense, both of the Hamilton-Jacobi equations $\partial_t u + H(x, D_x u, u) = 0$ and $-\partial_t u - H(x, D_x u, u) = 0$ in $M \times (0, \infty)$, which means that v is a BJ solution of $H(x, Du, u) = 0$ in M.

Since $v \in \mathrm{LSC}(M, \mathbb{R})$, it follows that v is bounded from below on M, which, together with (6.5), assures that v is bounded on M. Let $C_1 > 0$ be a constant such that $|v(x)| \leq C_1$ for all $x \in M$. If we set $w = -v \in \mathrm{USC}(M, \mathbb{R})$, then w is a viscosity subsolution, in the Crandall-Lions sense, of $H(x, -Du, -u) = 0$ in M. Since $H(x, p, u) \geq H(x, p, 0) - \Lambda C_1$ as far as $|u| \leq C_1$, the Hamiltonian $(x, p) \mapsto H(x, p, 0) - \Lambda C_1$ is coercive and v is a subsolution, in the Crandall-Lions sense, of $H(x, -Du, 0) - \Lambda C_1 = 0$ in M, a standard regularity result assures that $v \in \mathrm{Lip}(M, \mathbb{R})$.

The proof is now complete. $\square$

Given a Hamiltonian $H \in C(T^*M \times \mathbb{R}, \mathbb{R})$, we set

$$H^\ominus(x, p, u) = H(x, -p, -u).$$

Note that $(H^\ominus)^\ominus = H$ and that if H satisfies (H1)–(H4), then so does $H^\ominus$.

Under the hypotheses (H1)–(H4), we write S_t and $S_t^\ominus$ for the operators S_t^H and $S_t^{H^\ominus}$, respectively.

Remark 6.3. By [19, Proposition 2.8], for any $\phi \in C(M, \mathbb{R})$, there hold

$$S_t \phi = T_t^- \phi, \quad S_t^\ominus \phi = -T_t^+(-\phi),$$

where T_t^- and T_t^+ denote the backward and forward Lax-Olenik semigroup associated with H, respectively. By using $T_t^\pm$, new progress on viscosity solutions of contact Hamilton-Jacobi equations was achieved ([17, 20, 21]). We also refer the reader to [23] for existence and long time behavior of viscosity solutions of contact Hamilton-Jacobi equation where no monotonicity assumptions is imposed. Besides, by using

the Herglotz variational principle ([4]), some kinds of representation formulae for the viscosity solution of (6.1) on the Cauchy problem were provided in [11].

We establish the following theorems.

Theorem 6.4. *Assume* (H1)–(H4). *For any* $u_0 \in \mathcal{S}(H)$, *the function* $t \mapsto S_t^{\ominus}(-u_0)(x)$ *is nondecreasing on* $[0, \infty)$ *for any* $x \in M$, *and the limit*

$$v_0(x) := \lim_{t \to \infty} S_t(-u_0)(x)$$

exists for any $x \in M$, *and* $v_0 \in \mathcal{S}(H^{\ominus})$. *The convergence above is uniform on* M.

In view of the theorem above, under the hypotheses (H1)–(H4), we may introduce the operators $T_\infty : \mathcal{S}(H^{\ominus}) \to \mathcal{S}(H)$, $T_\infty^{\ominus} : \mathcal{S}(H) \to \mathcal{S}(H^{\ominus})$ by

$$T_\infty v(x) = \lim_{t \to \infty} S_t(-v)(x) \quad \text{for } x \in M \text{ and } v \in \mathcal{S}(H^{\ominus}),$$

$$T_\infty^{\ominus} u(x) = \lim_{t \to \infty} S_t^{\ominus}(-u)(x) \quad \text{for } x \in M \text{ and } u \in \mathcal{S}(H).$$

The monotonicity of $S_t(-v)(x)$ and $S_t^{\ominus}(-u)(x)$ in t yields

(6.6)
$$\begin{cases} T_\infty v \geq -v \quad \text{on } M \text{ for all } v \in \mathcal{S}(H^{\ominus}), \\ T_\infty^{\ominus} u \geq -u \quad \text{on } M \text{ for all } u \in \mathcal{S}(H). \end{cases}$$

Also, the comparison principle implies

(6.7)
$$\begin{cases} v_1, v_2 \in \mathcal{S}(H^{\ominus}), \ v_1 \leq v_2 \implies T_\infty v_1 \geq T_\infty v_2, \\ u_1, u_2 \in \mathcal{S}(H), \ u_1 \leq u_2 \implies T_\infty^{\ominus} u_1 \geq T_\infty^{\ominus} u_2. \end{cases}$$

Let $\mathcal{I}(T_\infty)$ and $\mathcal{I}(T_\infty^{\ominus})$ denote the images of T_∞ and $T_\infty^{\ominus}$, respectively, that is,

$$\mathcal{I}(T_\infty) = \{T_\infty v \, : \, v \in \mathcal{S}(H^{\ominus})\} \quad \text{and} \quad \mathcal{I}(T_\infty^{\ominus}) = \{T_\infty^{\ominus} u \, : \, u \in \mathcal{S}(H)\}.$$

Theorem 6.5. *Assume* (H1)–(H4).

(1) $\mathcal{S}(H) \neq \emptyset$ *if and only if* $\mathcal{S}(H^{\ominus}) \neq \emptyset$.

(2) *For any* $u \in \mathcal{I}(T_\infty)$ *and* $v \in \mathcal{I}(T_\infty^{\ominus})$,

$$T_\infty \circ T_\infty^{\ominus} u = u \quad \text{and} \quad T_\infty^{\ominus} \circ T_\infty v = v.$$

(3) *Let* $u \in \mathcal{I}(T_\infty)$ *and* $\phi \in \mathrm{LSC}(M, \mathbb{R})$. *Assume that there exist a finite number of* $v_1, \ldots, v_k \in \mathcal{S}(H^{\ominus})$ *such that* $T_\infty v_i = u$ *for all* $i = 1, \ldots, k$, *and* $\min_i(-v_i) \leq \phi \leq u$ *on* M. *Then*

$$\lim_{t \to \infty} S_t \phi = u \quad \text{uniformly on } M.$$

For the proof of the theorems above, we need some lemmas. In the following lemmas, we always *assume* (H1)–(H4).

Lemma 6.6. *Let $u_0 \in \mathcal{S}(H)$ and $x \in M$. There exists a curve $\gamma : (-\infty, 0] \to M$ such that $\gamma(0) = x$, $\gamma \in AC([-\tau, 0], M)$ for every $\tau > 0$, and, for all $t > 0$,*

$$u_0(\gamma(0)) = u_0(\gamma(-t)) + \int_{-t}^{0} L(\gamma(s), \dot\gamma(s), u_0(\gamma(s)))ds.$$

The lemma above is a classical observation and follows from Theorem 4.1.

Outline of proof. Since $u_0 \in \mathcal{S}_{BJ}(\partial_t + H)$, by Theorem 4.1, there exists $\{\gamma_k\}_{k \in \mathbb{N}} \subset AC([-1, 0], M)$ such that

$$u_0(\gamma_k(0)) = \int_{-1}^{0} L(\gamma_k(s), \dot\gamma_k(s), u_0(\gamma_k(s)))ds + u_0(\gamma_k(-1)) \quad \text{for all} \ \ k \in \mathbb{N},$$

$$\gamma_1(0) = x, \quad \text{and} \quad \gamma_{k+1}(0) = \gamma_k(-1) \quad \text{for all } k \in \mathbb{N}.$$

Define $\gamma : (-\infty, 0] \to M$ by

$$\gamma(s) = \begin{cases} \gamma_1(s) & \text{for } s \in (-1, 0], \\ \gamma_2(s+1) & \text{for } s \in (-2, -1], \\ \gamma_3(s+2) & \text{for } s \in (-3, -2], \\ \quad \vdots & \quad \vdots \end{cases}$$

It is easily checked that γ has all the required properties in Lemma 6.6. □

Lemma 6.7. *Let $u_0 \in \mathcal{S}(H)$ and set $v_0 = -u_0$. Then, $v_0 \in \mathcal{S}_{BJ}^-(H^{\ominus})$ and*

$$\min_{x \in M}(S_t v_0(x) - v_0(x)) = 0 \quad \text{for all } t > 0.$$

Proof. We set $v(x, t) = S_t v_0(x)$ for $(x, t) \in M \times [0, \infty)$.

Since $u_0 \in \mathrm{Lip}(M, \mathbb{R}) \cap \mathcal{S}^-(H)$, we have

$$v_0 \in \mathcal{S}_{BJ}^-(H^{\ominus}) \subset \mathcal{S}_{BJ}^-(\partial_t + H^{\ominus}),$$

and, by Theorem 2.1,

(6.8) $$v_0(x) \leq v(x, t) \quad \text{for all } (x, t) \in M \times [0, \infty).$$

Fix any x, and, in view of Lemma 6.6, choose a curve γ, with $\gamma(0) = x$, such that for all $t > 0$,

$$\gamma \in AC([-t, 0], M) \quad \text{and} \quad u_0(\gamma(0)) = u_0(\gamma(-t)) + \int_{-t}^{0} L(\gamma(s), \dot\gamma(s), u_0(\gamma(s)))ds.$$

Set $\eta(s) := \gamma(-s)$, to find that for all $t > 0$,

$$u_0(\eta(0)) = u_0(\eta(t)) + \int_0^t L(\eta(s), -\dot\eta(s), u_0(\eta(s)))ds,$$

which reads

$$v_0(\eta(t)) = v_0(\eta(0)) + \int_0^t L(\eta(s), -\dot{\eta}(s), -v_0(\eta(s)))ds.$$

By Theorem 4.1, we have

$$v(x, t) = \inf_{\xi(t)=x} \left(v_0(\xi(0)) + \int_0^t L(\xi(s), -\dot{\xi}(s), -v(\xi(s), s))ds \right).$$

In particular,

$$v(\eta(t), t) \leq v_0(\eta(0)) + \int_0^t L\left(\eta(s), -\dot{\eta}(s), -v(\eta(s), s)\right) ds.$$

Since $u \mapsto L(x, \xi, -u)$ is Lipschitz continuous in $\mathbb{R}$, with Λ as a Lipschitz bound, and, by (6.8), $v(\eta(s), s) \geq v_0(\eta(s))$ for all $s \geq 0$, we have

$$L(\eta(s), -\dot{\eta}(s), -v(\eta(s), s)) \leq L(\eta(s), -\dot{\eta}(s), -v_0(\eta(s))) + \Lambda(v(\eta(s), s) - v_0(\eta(s)).$$

Thus, for all $t > 0$,

$$v(\eta(t), t) \leq v_0(\eta(t)) + \int_0^t \Lambda(v(\eta(s), s) - v_0(\eta(s)))ds,$$

which implies that

$$v(\eta(t), t) - v_0(\eta(t)) \leq 0 \quad \text{for all } t > 0.$$

We conclude that

$$\min_{x \in M}(v(x, t) - v_0(x)) = 0 \quad \text{for all } t \geq 0. \qquad \square$$

Lemma 6.8. *For any $\phi \in \mathcal{S}_{\mathrm{BJ}}^-(H)$, the function $t \mapsto S_t\phi(x)$ is nondecreasing on $[0, \infty)$ for every $x \in M$.*

Proof. Fix any $\phi \in \mathcal{S}_{\mathrm{BJ}}^-(H)$ and set $u(x, t) = S_t\phi(x)$ for $(x, t) \in M \times [0, \infty)$.

By Theorem 2.1, we have

$$\phi(x) \leq u(x, t) \quad \text{for all } (x, t) \in M \times [0, \infty).$$

For any $\delta > 0$,

$$\phi(x) \leq u(x, \delta) \quad \text{for all } x \in M,$$

and hence, again by the comparison priciple,

$$u(x, t) \leq (S_t u(\cdot, \delta))(x) = u(x, \delta + t) \quad \text{for all } (x, t) \in M \times [0, \infty).$$

This shows that the function $t \mapsto S_t\phi(x)$ is nondecreasing on $[0, \infty)$ for any $x \in M$. $\qquad \square$

Proof of Theorem 6.4. Fix any $u_0 \in \mathcal{S}(H)$ and set $\phi = -u_0$. Set

$$v_0(x) := \lim_{r \to 0+} \inf\{S_t^{\ominus}\phi(y) \,:\, d(y,x) < r, \ t > r^{-1}\} \quad \text{for } x \in M.$$

By Lemma 6.7, we have $\phi \in \mathcal{S}_{\mathrm{BJ}}^-(H^{\ominus})$ and

$$\min_{x \in M}(S_t^{\ominus}\phi(x) - \phi(x)) = 0 \quad \text{for all } t > 0,$$

which implies that, since M is compact,

$$v_0(z) < \infty \quad \text{for some } z \in M.$$

Theorem 6.1, with H replaced by $H^{\ominus}$, ensures that $v_0 \in \mathrm{Lip}(M, \mathbb{R})$ and $v_0 \in \mathcal{S}(H^{\ominus})$.

By Lemma 6.8, with $H^{\ominus}$ in place of H, the function $t \mapsto S_t^{\ominus}\phi(x)$ is nondecreasing on $[0, \infty)$ for any $x \in M$. Hence, $S_\tau^{\ominus}\phi \le S_{\tau+t}^{\ominus}\phi$ on M for all $\tau, t > 0$, which shows that $S_\tau^{\ominus}\phi \le v_0$ on M for all $\tau > 0$. On the other hand, by the definition of v_0, we have $\lim_{t \to \infty} S_t^{\ominus}\phi(x) \ge v_0(x)$ for all $x \in M$. Thus, we find that

$$\lim_{\tau \to \infty} S_\tau^{\ominus}\phi(x) = v_0(x) \quad \text{for all } x \in M.$$

The convergence above is uniform in $x \in M$ by the Dini theorem. □

Proof of Theorem 6.5. We first treat (1). It is an immediate consequence of Theorem 6.4 that $\mathcal{S}(H) \ne \emptyset$ implies $\mathcal{S}(H^{\ominus}) \ne \emptyset$. The converse implication also follows from Theorem 6.4, with H replaced by $H^{\ominus}$.

We next consider (2). Fix any $u \in \mathcal{I}(T_\infty)$ and choose $v_0 \in \mathcal{S}(H^{\ominus})$ such that $T_\infty v_0 = u$. Set $v := T_\infty^{\ominus}u$. It follows from (6.6) that $v_0 + u \ge 0$ and $u + v \ge 0$ on M. Hence, by the comparison principle, we find that $S_t^{\ominus}(-u) \le v_0$ and $S_t(-v) \le u$ on M for all $t \ge 0$, which implies that $T_\infty^{\ominus}u \le v_0$ and $T_\infty v \le u$ on M. From the first of these inequalities, together with (6.7), we obtain $T_\infty \circ T_\infty^{\ominus}u \ge T_\infty v_0 = u$ on M, which, combined with the latter of the inequalities above, yields $u \ge T_\infty v = T_\infty \circ T_\infty^{\ominus}u \ge T_\infty v_0 = u$ on M. This shows that $T_\infty \circ T_\infty^{\ominus}u = u$ on M. Replacing H by $H^{\ominus}$ and noting that $(H^{\ominus})^{\ominus}$, we also conclude that $T_\infty^{\ominus} \circ T_\infty v = v$ on M for all $v \in \mathcal{I}(T_\infty^{\ominus})$.

Now, we treat (3). First of all, we show that for all $(x, t) \in M \times [0, \infty)$,

(6.9) $$\min_{1 \le i \le k} S_t(-v_i)(x) = S_t(\min_{1 \le i \le k}(-v_i))(x).$$

Indeed, by Lemma 2.5, we have

$$(x, t) \mapsto \min_{1 \le i \le k} S_t(-v_i)(x) \in \mathcal{S}_{\mathrm{BJ}}(\partial_t + H, M \times (0, \infty)).$$

Moreover, it is easily checked that

$$(x, t) \mapsto \min_{1 \le i \le k} S_t(-v_i)(x) \in \mathrm{LSC}(M \times [0, \infty), \mathbb{R} \cup \{\infty\}),$$

and for all $x \in M$,

$$\min_{i \leq i \leq k}(-v_i)(x) = \liminf_{t \to \infty} \min_{1 \leq i \leq k} S_t(-v_i)(x).$$

Thus, by the definition of the operator S_t, we conclude that (6.9) holds.

By the choice of v_i, we have

$$\lim_{t \to \infty} S_t(-v_i) = u \quad \text{uniformly on } M \quad \text{for all } i \in \{1, \ldots, k\},$$

which assures together with (6.9) that

(6.10) $$\lim_{t \to \infty} S_t(\min_{1 \leq i \leq k}(-v_i))(x) = u(x) \quad \text{uniformly on } M.$$

Since $\min_{1 \leq i \leq k}(-v_i) \leq \phi \leq u$ on M and $u \in \mathcal{S}_{\mathrm{BJ}}(H)$, we deduce by the comparison principle that for all $t \geq 0$,

$$S_t(\min_{1 \leq i \leq k}(-v_i)) \leq S_t\phi \leq S_t u = u \quad \text{on } M,$$

which, combined with (6.10), yields

$$\lim_{t \to \infty} S_t\phi = u \quad \text{uniformly on } M. \qquad \square$$

7. Classification of the solutions of $H = 0$

For $v \in \mathcal{S}(H^{\ominus})$, we define the set $\mathcal{D}_v$ by

$$\mathcal{D}_v := \{w \in \mathcal{S}(H) : T_\infty^\ominus w = v\}.$$

The sets $\mathcal{D}_v$, with $v \in \mathcal{I}(T_\infty^\ominus)$, classifies $\mathcal{S}(H)$ as follows.

Theorem 7.1. *Assume* (H1)–(H4)*. Then:*

(1) $\#\mathcal{I}(T_\infty) = \#\mathcal{I}(T_\infty^\ominus)$*, where $\#A$ denotes the cardinality of set A.*
(2) $\mathcal{S}(H) = \bigsqcup_{v \in \mathcal{I}(T_\infty^\ominus)} \mathcal{D}_v = \bigsqcup_{u \in \mathcal{I}(T_\infty)} \mathcal{D}_{T_\infty^\ominus u}.$
(3) $w \geq u$ *for all $u \in \mathcal{I}(T_\infty)$ and $w \in \mathcal{D}_{T_\infty^\ominus u}$.*

Proof. By (2) of Theorem 6.5, $T_\infty^\ominus$ is a bijection of $\mathcal{I}(T_\infty)$ to $\mathcal{I}(T_\infty^\ominus)$, with the inverse map T_∞. Hence, $\#\mathcal{I}(T_\infty^\ominus) = \#\mathcal{I}(T_\infty)$, which proves (1).

Since $T_\infty^\ominus : \mathcal{S}(H) \to \mathcal{I}(T_\infty^\ominus)$ is a surjection, if we introduce the binary relation $\sim$ on $\mathcal{S}(H)$ by

$$u_1 \sim u_2 \iff T_\infty^\ominus u_1 = T_\infty^\ominus u_2,$$

then this relation is clearly an equivalence relation on $\mathcal{S}(H)$ and the sets $\mathcal{D}_v$, with $v \in \mathcal{I}(T_\infty^\ominus)$, constitute all the equivalence classes in this relation. Consequently,

$$\mathcal{S}(H) = \bigsqcup_{v \in \mathcal{I}(T_\infty^\ominus)} \mathcal{D}_v.$$

Since $T_\infty : \mathcal{I}(T_\infty^\ominus) \to \mathcal{I}(T_\infty)$ is bijective, we have

$$\mathcal{S}(H) = \bigsqcup_{v \in \mathcal{I}(T_\infty^\ominus)} \mathcal{D}_v = \bigsqcup_{u \in \mathcal{I}(T_\infty)} \mathcal{D}_{T_\infty^\ominus u},$$

which proves assertion (2).

To check (3), let $u \in \mathcal{I}(T_\infty)$ and $w \in \mathcal{D}_{T_\infty^\ominus u}$. By (6.6), we have $T_\infty^\ominus w + w \geq 0$ on M and, hence, $w = S_t w \geq S_t(-T_\infty^\ominus w) = S_t(-T_\infty^\ominus u)$ on M for all $t \geq 0$, which implies that $w \geq T_\infty \circ T_\infty^\ominus u = u$ on M. $\square$

In the following remark, we discuss examples of Hamilton-Jacobi equations, with emphasis on the cardinality of the set of solutions. We refer to, e.g., [16] for some examples similar to ours.

Remark 7.2. (1) Under the assumptions (H1)–(H4) and that $\mathcal{S}(H) \neq \emptyset$, if $H = H(x, p, u)$ is strictly monotone in u, then $\#\mathcal{I}(T_\infty) = 1$. More precisely,

(i) if H is strictly increasing in u, then

$$\#\mathcal{I}(T_\infty) = \#\mathcal{S}(H) = 1;$$

(ii) if H is strictly decreasing in u, then

$$\#\mathcal{I}(T_\infty) = 1 \leq \#\mathcal{S}(H).$$

Concerning (ii) above, consider two examples. The first example is about the equation (see [9, Proposition 10])

$$(7.1) \qquad -u + \frac{1}{2}|Du|^2 + \cos x - 1 = 0 \quad \text{in } \mathbb{T}.$$

For (7.1), we have $\#\mathcal{S}(H) = 1$. The other one concerns the equation (see [19, Example 1.1])

$$(7.2) \qquad -u + \frac{1}{2}|Du|^2 = 0 \quad \text{in } \mathbb{T}.$$

For (7.2), the solutions u are given by

$$u(x) = \min_{y \in K} \frac{1}{2} d(x, y)^2,$$

with K being nonempty compact subsets of $\mathbb{T}$. Since the totality of compact subsets of $\mathbb{T}$ is an infinite set, we have $\#\mathcal{S}(H) = \infty$. Moreover, one can choose a Hamiltonian H so that $\#\mathcal{I}(T_\infty) < \#\mathcal{S}(H) < \infty$. The following equation is taken from [22, Proposition 1.14]:

$$(7.3) \qquad -u + \frac{1}{2}|Du|^2 + Du \cdot V(x) = 0 \quad \text{in } \mathbb{T},$$

where $V : \mathbb{T} \to \mathbb{R}$ is a smooth function which has exactly two zeros x_1, x_2 with $V'(x_1) > 0$, $V'(x_2) < 0$. For (7.3), we have

$$\# \mathcal{S}(H) = 2.$$

(2) If H is non-monotone in u, then it may happen that $\# \mathcal{I}(T_\infty) > 1$. For example, consider the equation

$$f(u) + \frac{1}{2}|Du|^2 = 0 \quad \text{in } \mathbb{T},$$

where f is a smooth function satisfying $f(u) = f(-u)$, $f > 0$ on $(-1, 1)$ and $f(u) = u + 1$ for $u \in (-\infty, -\frac{1}{2}]$. In this case, the solutions u are given by either $u(x) = -1$, or

$$u(x) = 1 + \min_{y \in K} \frac{1}{2} d(x, y)^2,$$

where K ranges over all compact subsets of $\mathbb{T}$, and we have

$$\# \mathcal{I}(T_\infty) = 2, \quad \# \mathcal{S}(H) = \infty.$$

The Hamiltonian H associated with (7.3) is self-adjoint in the sense that $H^\ominus = H$.
(3) If H is independent of u and $\mathcal{S}(H) \neq \emptyset$, then

$$\# \mathcal{I}(T_\infty) = \# \mathcal{S}(H) = \infty.$$

Moreover, the structure of $\mathcal{S}(H)$ can be described in terms of static classes in the Aubry set (see [8, Theorem 0.2]).

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Appendix A. A classical existence result

We give here a proof of the following classical existence theorem for a viscosity solution of (3.1).

Theorem A.1. *Assume* (H1)–(H4) *and that* $\phi \in C^1(M, \mathbb{R})$. *Then there exists a unique viscosity solution* $u \in \mathrm{Lip}(M \times [0, T), \mathbb{R})$ *of* (3.1).

We begin with a lemma concerning the Cauchy problem for

(A.1) $$u_t + H(x, D_x u, u) + \gamma |D_x u|^2 = 0 \quad \text{in } M \times (0, T),$$

where γ and T are constants such that $0 \le \gamma < \infty$ and $0 < T \le \infty$.

Lemma A.2. *Assume that H is a bounded and uniformly continuous function on $T^*M \times \mathbb{R}$ and satisfies* (H3) *and* (H4). *Let $v \in \text{USC}(M \times [0, T), \mathbb{R})$ and $w \in \text{LSC}(M \times [0, T), \mathbb{R})$ be a viscosity subsolution and supersolution, respectively, of* (A.1). *Assume that v, w are locally bounded on $M \times [0, T)$ and that $v(x, 0) \le w(x, 0)$ for all $x \in M$. Then $v \le w$ on $M \times [0, T)$.*

The proof below is similar to but simpler than that for Theorem 2.1.

Proof. We may assume without loss of generality that $T < \infty$. If we set

$$\tilde{v}(x, t) = e^{-\Lambda t} v(x, t) \quad \text{and} \quad \tilde{w}(x, t) = e^{-\Lambda t} w(x, t) \quad \text{for } (x, t) \in M \times [0, T),$$

then $\tilde{v}$ and $\tilde{w}$ are a viscosity subsolution and supersolution, respectively, of

$$u_t + \widehat{H}(x, D_x u, t, u) + \gamma e^{\Lambda t} |D_x u|^2 = 0,$$

where $\widetilde{H} \in C(T^*M \times [0, T) \times \mathbb{R}, \mathbb{R})$ is given by

$$\widehat{H}(x, p, t, u) := \Lambda u + e^{-\Lambda t} H(x, e^{\Lambda t} D_x u, e^{\Lambda t} u).$$

We denote by $\widetilde{H}$ the function $(x, p, t, u) \mapsto e^{-\Lambda t} H(x, e^{\Lambda t} p, e^{\Lambda t} u)$ on $T^*M \times [0, T) \times \mathbb{R}$. Note that for any $(x, p, t) \in T^*M \times [0, T)$, the function $u \mapsto \widehat{H}(x, p, t, u)$ is nondecreasing on $\mathbb{R}$ and the function $u \mapsto \widetilde{H}(x, p, t, u)$ is Lipschitz continuous with Lipschitz bound Λ.

We need only to show that $\tilde{v} \le \tilde{w}$ on $M \times [0, T)$, and by contradiction, we suppose that $\sup_{M \times [0, T)}(v - w) > 0$. For ease of notation, we henceforth write v and w for $\tilde{v}$ and $\tilde{w}$, respectively.

For $\varepsilon > 0$, set

$$v^\varepsilon(x, t) = v(x, t) - \frac{\varepsilon}{T - t + \varepsilon^2} \quad \text{for } (x, t) \in M \times [0T).$$

Note that, as $\varepsilon \to 0+$, $\sup_{M \times (T - \varepsilon^2, T)} v^\varepsilon(x, t) \to -\infty$ and $v^\varepsilon(x, t) \to v(x, t)$ uniformly on $M \times [0, S]$ for all $0 < S < T$, and that $v^\varepsilon(x, 0) < w(x, 0)$ for all $x \in M$. We may fix $\varepsilon > 0$ small enough so that $v_\varepsilon - w$ takes a positive maximum at some point $(x_0, t_0) \in M \times (0, T)$. It is easily seen that v^ε is a viscosity subsolution of

$$u_t + \widehat{H}(x, D_x u, t, u) + \gamma e^{\Lambda t} |D_x u|^2 + \frac{\varepsilon}{(T - t + \varepsilon^2)^2} = 0 \quad \text{in } M \times (0, T).$$

Setting $\delta = \varepsilon / (2(T + \varepsilon^2)^2)$, we see immediately that v^ε is a viscosity subsolution of

(A.2) $$u_t + \widehat{H}(x, D_x u, t, u) + \gamma e^{\Lambda t} |D_x u|^2 + 2\delta = 0 \quad \text{in } M \times (0, T).$$

We fix a local coordinates in an open neighborhood V of the point x_0 so that we may regard V as an open subset of $\mathbb{R}^n$ and $T^*V = V \times \mathbb{R}^n$. We introduce a (bump) function $\rho \in C^1(V \times (0, T), \mathbb{R})$ such that

$$(\text{A.3}) \qquad \begin{cases} (\rho - w)(x_0, t_0) > 0, \\ (\rho - w)(x, t) < -1 \quad \text{for all} \ \ (x, t) \in V \times (0, T) \setminus B, \end{cases}$$

where B is a bounded open subset of $\mathbb{R}^{n+1}$ whose closure $\overline{B}$ is included in $V \times (0, T)$.

For $\eta \in (0, 1)$, we set

$$v^{\varepsilon,\eta}(x, t) = (1 - \eta)v^\varepsilon(x, t) + \eta\rho(x, t) \quad \text{for} \ \ (x, t) \in V \times (0, T),$$

and note by (A.3) that the function

$$v^{\varepsilon,\eta}(x, t) - w(x, t) = (1 - \eta)(v^\varepsilon - w)(x, t) + \eta(\rho - w)(x, t)$$

takes a positive maximum at some point $(x_\eta, t_\eta) \in B$ and

$$(\text{A.4}) \quad v^{\varepsilon,\eta}(x, t) - w(x, t) < v^{\varepsilon,\eta}(x_\eta, t_\eta) - w(x_\eta, t_\eta) - \eta \quad \text{for all} \ \ (x, t) \in V \times (0, T) \setminus B.$$

Taking into account the convexity of $p \mapsto \widetilde{H}(x, p, u) + \gamma e^{\Lambda t}|p|^2$ and the Lipschitz property of $\widetilde{H}$, we compute in a slightly formal way that

$$
\begin{aligned}
v_t^{\varepsilon,\eta} + &\widehat{H}(x, D_x v^{\varepsilon,\eta}, t, v^{\varepsilon,\eta}) + \gamma e^{\Lambda t}|D_x v^{\varepsilon,\eta}|^2 \\
&\leq (1 - \eta)(v_t^\varepsilon + \Lambda v^\varepsilon + \widetilde{H}(x, D_x v^\varepsilon, t, v^{\varepsilon,\eta}) + \gamma e^{\Lambda t}|D_x v^\varepsilon|^2) \\
&\quad + \eta(\rho_t + \Lambda\rho + \widetilde{H}(x, D_x\rho, t, v^{\varepsilon,\eta}) + \gamma e^{\Lambda t}|D_x\rho|^2) \\
&\leq (1 - \eta)(v_t^\varepsilon + \Lambda v^\varepsilon + \widetilde{H}(x, D_x v^\varepsilon, t, v^\varepsilon) + \gamma e^{\Lambda t}|D_x v^\varepsilon|^2 + \eta\Lambda(|v^\varepsilon| + |\rho|))) \\
&\quad + \eta(\rho_t + H(x, D_x\rho, \rho) + \gamma|D_x\rho|^2 + \Lambda(|v^\varepsilon| + |\rho|)).
\end{aligned}
$$

Hence, thanks to (A.2), we may choose $\eta \in (0, 1)$ small enough so that $v^{\varepsilon,\eta}$ is a viscosity subsolution of

$$u_t + \widehat{H}(x, D_x u, t, u) + \gamma e^{\Lambda t}|D_x u|^2 + (1 - \eta)\delta = 0 \quad \text{in} \ V \times (0, T).$$

Consider the function

$$v^{\varepsilon,\eta}(x, t) - w(y, s) - \frac{1}{2}\left(\alpha|x - y|^2 + \alpha^2(t - s)^2\right)$$

on $\overline{B} \times \overline{B}$, where $\alpha > 1$, and pick a maximum point $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$ of this function. It is a standard observation that, as $\alpha \to \infty$,

$$(\text{A.5}) \qquad\qquad\qquad \alpha|\hat{x} - \hat{y}|^2 + \alpha^2(\hat{t} - \hat{s})^2 \to 0,$$

and that for any limiting point $(\bar{x}, \bar{t}, \bar{y}, \bar{s}, \bar{v}, \bar{w})$, as $\alpha \to \infty$, of the family

$$\{(\hat{x}, \hat{t}, \hat{y}, \hat{s}, v^{\varepsilon,\eta}(\hat{x}, \hat{t}), w(\hat{y}, \hat{s}))\},$$

we have

(A.6) $\qquad\qquad (\bar{x}, \bar{t}) = (\bar{y}, \bar{s}) \quad \text{and} \quad \bar{v} - \bar{w} = \max_{\overline{B}} (v^{\varepsilon, \eta} - w).$

Fix such a limiting point $(\bar{x}, \bar{t}, \bar{y}, \bar{s}, \bar{v}, \bar{w})$ and a sequence $\{\alpha_j\}_{j \in \mathbb{N}}$ such that, as $j \to \infty$, the sequence of the points

$$(\hat{x}, \hat{t}, \hat{y}, \hat{s}, v^{\varepsilon, \eta}(\hat{x}, \hat{t}), w(\hat{y}, \hat{s})),$$

with $\alpha = \alpha_j$, converges to $(\bar{x}, \bar{t}, \bar{y}, \bar{s}, \bar{v}, \bar{w})$. Because of (A.6), we find that $(\bar{x}, \bar{t}) \in B$ and $\bar{v} > \bar{w}$, and we may assume by passing a subsequence of $\{\alpha_j\}$ if necessary that $(\hat{x}, \hat{t}), (\hat{y}, \hat{s}) \in B$, $v^{\varepsilon, \eta}(\hat{x}, \hat{t}) > w(\hat{y}, \hat{s})$. By the viscosity properties of $v^{\varepsilon, \eta}$ and w, we get for $\alpha = \alpha_j$,

(A.7) $\quad \begin{cases} \alpha^2(\hat{t} - \hat{s}) + \widehat{H}(\hat{x}, \alpha(\hat{x} - \hat{y}), \hat{t}, v^{\varepsilon, \eta}(\hat{x}, \hat{t})) + \gamma e^{\Lambda \hat{t}} |\alpha(\hat{x} - \hat{y})|^2 + (1 - \eta)\delta \leq 0, \\ \alpha^2(\hat{t} - \hat{s}) + \widehat{H}(\hat{y}, \alpha(\hat{x} - \hat{y}), \hat{s}, w(\hat{y}, \hat{s})) + \gamma e^{\Lambda \hat{s}} |\alpha(\hat{x} - \hat{y})|^2 \geq 0. \end{cases}$

Since $v^{\varepsilon, \eta}(\hat{x}, \hat{t}) > w(\hat{y}, \hat{s})$, we have

$$\widehat{H}(\hat{x}, \alpha(\hat{x} - \hat{y}), \hat{t}, v^{\varepsilon, \eta}(\hat{x}, \hat{t})) \geq \widehat{H}(\hat{x}, \alpha(\hat{x} - \hat{y}), \hat{t}, w(\hat{x}, \hat{t})).$$

Hence, from (A.7) we obtain

(A.8) $\quad \begin{aligned} &\widetilde{H}(\hat{x}, \alpha(\hat{x} - \hat{y}), \hat{t}, w(\hat{y}, \hat{s})) - \widetilde{H}(\hat{y}, \alpha(\hat{x} - \hat{y}), \hat{s}, w(\hat{y}, \hat{s})) \\ &\qquad + \gamma(e^{\Lambda \hat{t}} - e^{\Lambda \hat{s}})|\alpha(\hat{x} - \hat{y}|^2 + (1 - \eta)\delta \leq 0. \end{aligned}$

Observe that

$$\begin{aligned} \widetilde{H}(\hat{x}, \alpha(\hat{x} - \hat{y}), \hat{t}, w(\hat{y}, \hat{s})) &\geq \widetilde{H}(\hat{y}, \alpha(\hat{x} - \hat{y}), \hat{t}, w(\hat{y}, \hat{s})) - \omega(|\hat{x} - \hat{y}|) \\ &\geq \widetilde{H}(\hat{y}, \alpha(\hat{x} - \hat{y}), \hat{s}, w(\hat{y}, \hat{s})) - |e^{\Lambda \hat{t}} - e^{\Lambda \hat{s}}||H(\hat{y}, e^{\Lambda \hat{t}} \alpha(\hat{x} - \hat{y}), e^{\Lambda \hat{t}} w(\hat{y}, \hat{s}))| \\ &\quad - \omega(|e^{\Lambda \hat{t}} - e^{\Lambda \hat{s}}|(\alpha|\hat{x} - \hat{y}| + |w(\hat{y}, \hat{s})|)) - \omega(|\hat{x} - \hat{y}|), \end{aligned}$$

where ω denotes the modulus of continuity of the function H, and moreover that

$$\begin{aligned} |e^{\Lambda \hat{t}} - e^{\Lambda \hat{s}}| &\leq \Lambda e^{\Lambda T} |\hat{t} - \hat{s}| \leq \Lambda e^{\Lambda T} \alpha |\hat{t} - \hat{s}|, \\ \alpha|\hat{x} - \hat{y}||\hat{t} - \hat{s}| &\leq \alpha(|\hat{x} - \hat{y}|^2 + |\hat{t} - \hat{s}|^2) \leq \alpha|\hat{x} - \hat{y}|^2 + \alpha^2|\hat{t} - \hat{s}|^2, \\ |\hat{t} - \hat{s}|\alpha^2|\hat{x} - \hat{y}|^2 &\leq \alpha^2(|\hat{x} - \hat{y}|^4 + |\hat{t} - \hat{s}|^2). \end{aligned}$$

Combine these observations with (A.4), we find that if $\alpha = \alpha_j$ is large enough, we have

$$\begin{aligned} &\widetilde{H}(\hat{x}, \alpha(\hat{x} - \hat{y}), \hat{t}, w(\hat{y}, \hat{s})) - \widetilde{H}(\hat{y}, \alpha(\hat{x} - \hat{y}), \hat{s}, w(\hat{y}, \hat{s})) \\ &\qquad + \gamma(e^{\Lambda \hat{t}} - e^{\Lambda \hat{s}})|\alpha(\hat{x} - \hat{y}|^2 > -(1 - \eta)\delta. \end{aligned}$$

This contradicts (A.8), which completes the proof. $\qquad\qquad\qquad\qquad\square$

Proof of Theorem A.1. We may assume that $T < \infty$. Choose a constant $C_1 > 0$ so that

$$|H(x, D\phi(x), \phi(x))| + |D\phi(x)|^2 \leq C_1 \quad \text{for all} \ \ x \in M.$$

Define the functions $f^\pm \in C^1(M \times [0, T), \mathbb{R})$ by

$$f^+(x, t) = \phi(x) + C_1 \Lambda^{-1}(e^{\Lambda t} - 1) \quad \text{and} \quad f^-(x, t) = \phi(x) - C_1 \Lambda^{-1}(e^{\Lambda t} - 1).$$

Choose a constant $C_2 > 0$ so that $-C_2 \leq f^- \leq f^+ \leq C_2$ on $M \times [0, T)$. We define the new Hamiltonians $\widetilde{H}$, $\widetilde{H}_k$, $\widehat{H}_k$ as follows. Set

$$\widetilde{H}(x, p, u) = H\left(x, p, \max\{-C_2, \min\{C_2, u\}\}\right) \quad \text{for} \ \ (x, p, u) \in \mathbb{T}^*M \times \mathbb{R},$$

$$\widetilde{H}_k(x, p, u) = \min\{k, \widetilde{H}(x, p, u)\} \quad \text{for} \ \ (x, p, u) \in T^*M \times \mathbb{R}, \ k \in \mathbb{N},$$

$$\widehat{H}_k(x, p, u) = \widetilde{H}_k(x, p, u) + \tfrac{1}{k}|p|^2 \quad \text{for} \ \ (x, p, u) \in T^*M \times \mathbb{R}, \ k \in \mathbb{N}.$$

It is easily seen that the functions $\widetilde{H}(x, p, u)$, $\widetilde{H}_k(x, p, u)$, $\widehat{H}_k(x, p, u)$ are Lipschitz continuous in u, with Λ as a Lipschitz bound, that for any $f \in C^1(M \times (0, T), \mathbb{R})$, if $|f| \leq C_2$ on $T^*M \times (0, T)$, then $\widetilde{H}(x, D_x f(x, t), f(x, t)) = H(x, D_x f(x, t), f(x, t))$ for all $(x, t) \in M \times (0, T)$, and that $|\widetilde{H}_k(x, D\phi(x), \phi(x))| \leq |\widetilde{H}(x, D\phi(x), \phi(x))|$ and $|\widehat{H}_k(x, D\phi(x), \phi(x))| \leq C_1$ for all $x \in M$ and $k \in \mathbb{N}$.

Compute that

$$f_t^+ + H(x, D_x f^+, f^+) \geq C_1 e^{\Lambda t} + \widehat{H}_k(x, D_x \phi, \phi) - C_1(e^{\Lambda t} - 1) \geq 0,$$

to see that for any $k \in \mathbb{N}$, f^+ is a classical supersolution of

$$(A.9) \qquad\qquad u_t + \widehat{H}_k(x, D_x u, u) = 0 \quad \text{in} \ M \times (0, T).$$

Similarly, we find that for any $k \in \mathbb{N}$, f^- is a classical subsolution of (A.9). Moreover, note that $f^-(x, 0) = f^+(x, 0) = \phi(x)$ for all $x \in M$ and $f^- \leq f^+$ on $M \times [0, T)$. The Perron method yields a Crandall-Lions viscosity solution of (A.9). That is, the formula

$$u^k(x, t) = \sup\{f(x, t) \ : \ f \in \mathcal{S}^-(\partial_t + \widehat{H}_k), \ f^- \leq f \leq f^+ \quad \text{on} \ M \times (0, T)\}$$

gives a solution of (A.9) such that $f^- \leq u^k \leq f^+$ on $M \times (0, T)$. Since $f^\pm(x, 0) = \phi(x)$ for all $x \in M$, we may extend the domain of u^k to $M \times [0, T)$ so that

$$u^k(x, 0) = \phi(x) = \lim_{M \times (0,T) \ni (y,s) \to (x,0)} u(y, s) \quad \text{for all} \ \ x \in M.$$

We note that, thanks to (H2), $\widetilde{H}_k(x, p, u) = k$ if $|p|$ is sufficiently large and, hence, the function $\widetilde{H}_k$ is bounded and uniformly continuous on $T^*M \times \mathbb{R}$. Since the upper and lower semicontinuous envelopes $(u_k)^*$ and $(u_k)_*$ are respectively a viscosity subsolution and supersolution of (A.9), we find by Lemma A.2 that $(u_k)^* \leq (u_k)_*$ on $M \times [0, T)$, which implies that $u_k \in C(M \times [0, T), \mathbb{R})$.

We now show that the family $\{u^k\}_{k\in\mathbb{N}}$ is equi-Lipschitz continuous on $M \times (0, T)$. For this, we show first that the functions $\widehat{H}_k$, with $k \in \mathbb{N}$, are coercive uniformly in k. That is, for any $R > 0$ there exists $Q > 0$, chosen independently of k, such that for any $k \in \mathbb{N}$,

(A.10) $$\widehat{H}_k(x, p, u) > R \quad \text{if} \quad |p| > Q.$$

Indeed, when $R > 0$ is fixed, by (H2) we may choose $Q \geq R$ so that $\widetilde{H}(x, p, u) > R$ if $|p| > Q$. Using the inequality

$$R < k + \tfrac{1}{k}R^2,$$

we find that if $|p| > Q$, then

$$\widehat{H}_k(x, p, u) \geq \min\{k + \tfrac{1}{k}Q^2, \widetilde{H}(x, p, u)\} > R.$$

Hence, (A.10) is valid.

Fix $h > 0$ sufficiently small. Since $u^k \geq f^-$ on $M \times [0, T)$, we find that

$$u^k(x, h) \geq \phi(x) - C_1\Lambda^{-1}(e^{\Lambda h} - 1) \geq \phi(x) - C_1 e^{\Lambda T}h \quad \text{for all} \quad x \in M.$$

Setting $v(x, t) = u^k(x, t) - C_1 h e^{\Lambda(T+t)}$ for $(x, t) \in M \times [0, T)$, we easily observe that v is a viscosity subsolution of (A.9). We apply Lemma A.2, to obtain the inequality $v(x, t) \leq u(x, h + t)$ for all $(x, t) \in M \times [0, T - h)$, which shows that

$$\liminf_{h\to 0+} \frac{u^k(x, t + h) - u^k(x, t)}{h} \geq -C_1 e^{2\Lambda T}.$$

This assures that for any $(x, t) \in M \times (0, T)$ and $(p, q) \in D^+u^k(x, t)$, $q \geq -C_1 e^{2\Lambda T}$ and

$$|q| + \widehat{H}_k(x, p, u^k(x, t)) \leq q + \widehat{H}_k(x, p, u^k(x, t)) + 2C_1 e^{2\Lambda T}.$$

Thus, u^k is a viscosity subsolution of

$$|u_t| + \widehat{H}_k(x, D_x u, u) - 2C_1 e^{2\Lambda T} = 0 \quad \text{in} \quad M \times (0, T),$$

which, together with (A.10), ensures that for some constant $C_3 > 0$, independent of k,

$$|u_t^k| + |D_x u^k| \leq C_3 \quad \text{in} \quad M \times (0, T) \quad \text{in the viscosity sense.}$$

This shows (see [5, Theorem I.14], [15, Proposition 1.14]) that $\{u^k\}_{k\in\mathbb{N}}$ is equi-Lipschitz continuous on $M \times [0, T)$. Recalling that $f^- \leq u^k \leq f^+$ on $M \times [0, T)$ for all $k \in \mathbb{N}$, we find by the Ascoli-Arzela theorem that the family $\{u^k\}_{k\in\mathbb{N}}$ has a subsequence, converging to some u in $C(M \times [0, T), \mathbb{R})$. Since $\{\widehat{H}_k\}_{k\in\mathbb{N}}$ converges to $\widetilde{H}$ in $C(T^*M \times \mathbb{R}, \mathbb{R})$, we find that u is a viscosity solution of $u_t + \widetilde{H}(x, D_x u, u) = 0$ in $M \times (0, T)$. It is obvious that $|u| \leq C_2$ on $M \times [0, T)$, which implies that u is a viscosity solution of $u_t + H(x, D_x u, u) = 0$ in $M \times (0, T)$, u is Lipschitz continuous

on $M \times [0, T)$, and $u(x, 0) = \phi(x)$ for all $x \in M$. Thus, u is a Lipschitz continuous solution of (3.1). The uniqueness assertion of the current theorem is a well-known result and we do not repeat the standard proof here. The uniqueness is also a consequence of Theorem 2.1. It follows as well from Lemma A.2, once the Hamiltonian H is replaced by a bounded and uniformly continuous function, which can be done based on the Lipschitz continuity of given solutions. $\qquad \square$

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