Asymptotic solutions of Hamilton-Jacobi equations with semi-periodic Hamiltonians

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Abstract

We study the long time behavior of viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations in $\mathbb{R}^n$. We prove that if the Hamiltonian $H(x,p)$ is coercive and strictly convex in a mild sense in p and upper semi-periodic in x, then any solution of the Cauchy problem "converges" to an asymptotic solution for any lower semi-almost periodic initial function.

1 Introduction.

In this paper we study the Cauchy problem for Hamilton-Jacobi equations of the form

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\,\cdot\,, 0) = u_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{1}$$

under the following standing assumptions on the Hamiltonian $H = H(x,p)$:

(A1) (continuity) $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0,R))$ for all $R > 0$, where $B(0,R) := \{x \in \mathbb{R}^n \,|\, |x| \le R\}$.

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(A2) (coercivity) $\lim_{R\to+\infty} \inf\{H(x,p) \mid x \in \mathbb{R}^n,\ |p| \geq R\} = +\infty$.

(A3) (convexity) $H(x,p)$ is convex with respect to p for every $x \in \mathbb{R}^n$.

We will show under some additional assumptions that the continuous viscosity solution $u(x,t)$ of (1) for a given initial datum has the convergence

$$u(x,t) + c\,t - \phi(x) \longrightarrow 0 \qquad \text{uniformly on compact subsets of } \mathbb{R}^n, \tag{2}$$

where c and ϕ are some real number and continuous function on $\mathbb{R}^n$, respectively. Note that if (2) holds, then the pair (c,ϕ) should satisfy the following time independent Hamilton-Jacobi equation:

$$H(x, D\phi) - c = 0 \qquad \text{in } \ \mathbb{R}^n. \tag{3}$$

The problem of finding such a pair (c,ϕ) is called the additive eigenvalue problem for H and for any solution (c,ϕ) of the additive eigenvalue problem for H, we call the function $\phi(x) - c\,t$ an *asymptotic solution* of the Cauchy problem (1).

After a pioneering work by Namah-Roquejoffre [16], asymptotic problems of this type have been investigated intensively by several authors. Fathi [7] introduced an approach to asymptotic problems for (1) based on weak KAM theory (see [6, 8, 10]) and obtained general convergence results in the case where the state space is a smooth compact manifold. By using a nice PDE approach, Barles-Souganidis [2, 4] established similar convergence results under much weaker hypotheses on H. Indeed, the results in [2] cover some classes of non-convex Hamiltonians H. Roquejoffre [17] and then Davini-Siconolfi [5] modified and improved the approach due to Fathi [7]. A typical result obtained in these developments is stated as follows in the case when the state space is n-dimensional unit torus $\mathbb{T}^n$: for any initial function $u_0 \in C(\mathbb{T}^n)$ and Hamiltonian on $\mathbb{T}^n \times \mathbb{R}^n$, the convergence (2) is valid if the function $H(x,p)$, regarded as a function periodic in $x \in \mathbb{R}^n$, satisfies (A1), (A2) and

(A3′) $H(x,p)$ is *strictly* convex with respect to p for every $x \in \mathbb{R}^n$.

Regarding this strict convexity assumption, we should remark that the results in [16] do not require any strict convexity of Hamiltonians and that, even in the convex Hamiltoninan case, the results in [2] require only a sort of strict convexity of H near $\{(x,p) \mid H(x,p) = c\}$.

Concerning asymptotic problems in unbounded region, the above (A1), (A2) and (A3′) are insufficient to guarantee the convergence of the form (2). In order to compensate for the unboundedness, one of the authors assumed in his recent paper

[14] the following additional condition:

(C1) $\exists \phi_i \in C^{0+1}(\mathbb{R}^n)$, $\exists \sigma_i \in C(\mathbb{R}^n)$ with $i = 0, 1$ such that for $i = 0, 1$,

$$H(x, D\phi_i(x)) \leq -\sigma_i(x) \qquad \text{a.e. } x \in \mathbb{R}^n,$$
$$\lim_{|x|\to\infty} \sigma_i(x) = \infty, \qquad \lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty.$$

An important role of (C1) is that, although (3) may have infinitely many solutions for a given c, the uniqueness (or comparison) theorem is valid outside a certain compact subset (called the Aubry set) of $\mathbb{R}^n$ in the space Φ_0 of functions ϕ bounded below by ϕ_0 modulo a constant. That is, under (C1), given two solutions in the space Φ_0 of (3) which are identical on the Aubry set are indeed identical on the whole $\mathbb{R}^n$. Roughly speaking, uniqueness of solutions of this type reduces the question whether the convergence (2) is valid or not to that on the Aubry set which is compact under (C1). See [14] for details and Fujita-Ishii-Loreti [11] for related results. We also refer to Barles-Roquejoffre [1] for other recent results on asymptotic problems in unbounded region.

The objective of this paper is to find another type of conditions on H and u_0 which is different from (C1) but ensures the convergence of solutions of (1) to asymptotic solutions. We are particularly interested in the case where H and u_0 possess a kind of almost periodic structure which contains classical periodic cases such as [5]. In the main theorems (Theorem 2.2 and its generalization, Theorem 6.1), we prove that upper semi-periodicity of H guarantees the convergence of solutions of (1) to asymptotic solutions for all obliquely lower semi-almost periodic initial data. The precise definition of semi-periodicity and (obliquely) semi-almost periodicity will be given in the next section. Note that a part of the results presented in this paper has been announced in [12].

It is worth mentioning in connection with weak KAM theory that our results include the case where the Aubry set for $H - c$ is non-compact or empty. The latter case arises in particular when the eigenvalue c is supercritical, that is, when c is greater than the infimum of all the eigenvalues of (3). Such supercritical cases are also investigated in [1]. Remark that our approach in this paper is based on both dynamical and PDE approaches and that any knowledge of Aubry sets is not needed in our proof of the main theorem.

Although the results in [2] cover some cases of non-convex Hamiltonians, it is still interesting to note that, under some basic assumptions on $H(x, p)$ including its convexity in p, our condition (A4) or (A6), a sort of strict convexity assumption on H, is equivalent to the corresponding condition in [2], i.e., (H4) or (H5) in [2].

This paper is organized as follows. In the next section, we fix standing assumptions and state our main results. Section 3 is devoted to establishing several estimates that play a key role throughout this paper. The proof of the main theorem is provided in Section 4. We also give examples in Section 5. In Section 6 we establish a generalization of the main theorem which achives the same generality in regard to "strict convexity" condition as [2]. Some fundamental facts used in this paper are collected in Appendices A and B. In Appendix C we prove two theorems concerning our "strict convexity" condition (A4) or (A6).

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2 Assumptions and main results.

Let $C(\mathbb{R}^n)$ be the totality of continuous functions on $\mathbb{R}^n$ equipped with the topology of locally uniform convergence. We say a family of functions $\{u_j\}_{j\in\mathbb{N}} \subset C(\mathbb{R}^n)$ converges to a function u in $C(\mathbb{R}^n)$ if and only if $u_j(x) \longrightarrow u(x)$ as $j \to +\infty$ uniformly on any compact subset of $\mathbb{R}^n$. We denote by $\mathrm{BC}(\mathbb{R}^n)$ the set of all bounded and continuous functions on $\mathbb{R}^n$. We also use symbols $\mathrm{UC}(\mathbb{R}^n)$, $\mathrm{BUC}(\mathbb{R}^n)$ and $\mathrm{Lip}(\mathbb{R}^n)$ to denote, respectively, the totality of uniformly continuous, bounded uniformly continuous and globally Lipschitz continuous functions.

For any finite closed interval J, the set of all absolutely continuous functions on J with values in $\mathbb{R}^n$ is denoted by $\mathrm{AC}(J, \mathbb{R}^n)$. For given right closed interval J, with right end point T, and $x, y \in \mathbb{R}^n$, we set

$$\mathcal{C}(J; x) := \{\gamma \in C(J, \mathbb{R}^n) \mid \gamma(T) = x,\, \gamma \in \mathrm{AC}([S,T], \mathbb{R}^n) \text{ for all } S \in J\}.$$

The solvability of (1) up to an arbitrarily given $T > 0$ is now classical (see Appendix A for details).

Theorem 2.1. *Let $H = H(x,p)$ be a Hamiltonian satisfying (A1)-(A3) and fix any $T > 0$. Then, for every $u_0 \in \mathrm{UC}(\mathbb{R}^n)$, there exists a unique viscosity solution $u \in \mathrm{UC}(\mathbb{R}^n \times [0,T))$ of*

$$u_t + H(x, Du) = 0 \qquad \text{in } \mathbb{R}^n \times (0,T) \tag{4}$$

satisfying $u(\,\cdot\,,0)=u_0$*. Moreover, for all* $t,s>0$ *and* $x\in\mathbb{R}^n$*, we have*

$$u(x,s+t)=\inf\Big\{\int_{-t}^{0}L(\gamma(r),\dot\gamma(r))\,dr+u(\gamma(-t),s)\;\Big|\;\gamma\in\mathcal{C}([-t,0];x)\Big\},\tag{5}$$

where $L(x,\xi):=\sup\{\xi\cdot p-H(x,p)\mid p\in\mathbb{R}^n\}$*, which is proper and convex with respect to* $\xi\in\mathbb{R}^n$ *for all* $x\in\mathbb{R}^n$*.*

We next introduce the notion of semi-periodicity for H.

Definition 1. We call a Hamiltonian H is lower (resp. upper) semi-periodic if for any sequence $\{y_j\}_{j\in\mathbb{N}}\subset\mathbb{R}^n$, there exist a subsequence $\{z_j\}_{j\in\mathbb{N}}$ of $\{y_j\}$, a sequence $\{\xi_j\}_{j\in\mathbb{N}}\subset\mathbb{R}^n$ converging to zero, and a function $G\in C(\mathbb{R}^n\times\mathbb{R}^n)$ such that for all $(x,p,j)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{N}$, $H(x+z_j+\xi_j,p)\ge G(x,p)$ (resp. $H(x+z_j+\xi_j,p)\le G(x,p)$) and $H(x+z_j,p)\longrightarrow G(x,p)$ in $C(\mathbb{R}^n\times\mathbb{R}^n)$ as $j\to\infty$.

Throughout this paper, we always assume the following:

(A4) For each $a\in\mathbb{R}$ there exists a modulus ω_a satisfying $\omega_a(r)>0$ for all $r>0$ such that for all $(x,p)\in\mathbb{R}^n\times\mathbb{R}^n$ such that $H(x,p)=a$ and for all $\xi\in D_2^-H(x,p)$, $q\in\mathbb{R}^n$,

$$H(x,p+q)\ge H(x,p)+\xi\cdot q+\omega_a((\xi\cdot q)_+),\tag{6}$$

where $D_2^-H(x,p)$ stands for the subdifferential of H with respect to the second variable p and $r_+:=\max\{r,0\}$ for $r\in\mathbb{R}$.

(A5) H is upper semi-periodic.

Actually we do not need the conditions in (A4) for every $a\in\mathbb{R}$, but only the one for a single a which will be specified. We refer as $(\mathrm{A4})_a$ the condition in (A4) for fixed $a\in\mathbb{R}$.

Remark. (a) For Hamiltonian $H\in C(\mathbb{T}^n\times\mathbb{R}^n)$, strict convexity assumption (A3′) implies (A4). As far as convex Hamiltonians H are concerned, condition $(\mathrm{A4})_0$ is equivalent to (H5), with $K=\emptyset$, in [2]. For this, see Appendix C.

(b) Any Hamiltonian of the form

$$H(x,p)=H_0(x,p)-f(x)$$

for some $H_0\in C(\mathbb{R}^n\times\mathbb{R}^n)$ periodic in x and for some non-negative $f\in C(\mathbb{R}^n)$ vanishing at infinity satisfies (A5).

(c) A function is periodic if and only if it is both lower and upper semi-periodic (see Appendix B).

As to the class of initial data, we use the following notion of (obliquely) semi-almost periodicity.

Definition 2. A function $\phi \in C(\mathbb{R}^n)$ is called lower (resp. upper) semi-almost periodic if for any sequence $\{y_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ and any $\varepsilon > 0$, there exist a subsequence $\{z_j\}_{j\in\mathbb{N}}$ of $\{y_j\}$ and a function $\psi \in C(\mathbb{R}^n)$ such that $\phi(\cdot + z_j) \longrightarrow \psi$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $\phi(x+z_j)+\varepsilon > \psi(x)$ (resp. $\phi(x+z_j)-\varepsilon < \psi(x)$) for all $(x, j) \in \mathbb{R}^n \times \mathbb{N}$.

Definition 3. A function $\phi \in C(\mathbb{R}^n)$ is called obliquely lower (resp. upper) semi-almost periodic if for any sequence $\{y_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ and any $\varepsilon > 0$, there exist a subsequence $\{z_j\}_{j\in\mathbb{N}}$ of $\{y_j\}$ and a function $\psi \in C(\mathbb{R}^n)$ such that $\phi(\cdot+z_j)-\phi(z_j) \longrightarrow \psi$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $\phi(x+z_j)-\phi(z_j)+\varepsilon > \psi(x)$ (resp. $\phi(x+z_j)-\phi(z_j)-\varepsilon < \psi(x)$) for all $(x, j) \in \mathbb{R}^n \times \mathbb{N}$.

Remark. (a) It is not difficult to check that a function $\phi \in C(\mathbb{R}^n)$ is almost periodic if and only if it is both lower and upper semi-almost periodic.
(b) If $\phi \in C(\mathbb{R}^n)$ is lower or upper semi-almost periodic, then $\phi \in \mathrm{UC}(\mathbb{R}^n)$.
(c) Definitions 2 and 3 are equivalent if $\phi \in \mathrm{BC}(\mathbb{R}^n)$.

Let us denote by $\mathcal{S}^-_{H-c}$ (resp. $\mathcal{S}^+_{H-c}$) the totality of viscosity subsolutions (resp. supersolutions) of (3). We set $\mathcal{S}_{H-c} := \mathcal{S}^-_{H-c} \cap \mathcal{S}^+_{H-c}$. It is known that, under (A1) and (A2), there exists $c_0 \in \mathbb{R}$ such that $\mathcal{S}_{H-c} \neq \emptyset$ for all $c \geq c_0$. Moreover, $\mathcal{S}^-_{H-c} \subset \mathrm{Lip}(\mathbb{R}^n)$.

For a given $u_0 \in \mathrm{UC}(\mathbb{R}^n)$, we define $u_0^- : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$u_0^-(x) := \sup\{\, \phi(x) \,|\, \phi \in \mathcal{S}^-_{H-c}, \quad \phi \leq u_0 \ \text{ in } \mathbb{R}^n\} \tag{7}$$

if there exists $\phi \in \mathcal{S}^-_{H-c}$ such that $\phi \leq u_0$.

The conditions we impose on $u_0 \in \mathrm{UC}(\mathbb{R}^n)$ are the following:

(B1) There exist a constant $c \in \mathbb{R}$ and functions $\phi_0 \in \mathcal{S}^-_{H-c}$ and $\psi_0 \in \mathcal{S}^+_{H-c}$ such that $\phi_0 \leq u_0 \leq \phi_0 + C_0$ for some $C_0 > 0$ and $u_0^- \leq \psi_0$, where u_0^- is defined by (7).
(B2) u_0 is obliquely lower semi-almost periodic.

We are now in position to state our main results.

Theorem 2.2. *Let H and u_0 satisfy (A1)-(A3), (A4)$_c$, (A5) and (B1)-(B2), respectively. Then,*

$$u(x,t) + c\,t \longrightarrow u_\infty(x) \qquad \text{in } \ C(\mathbb{R}^n) \quad \text{as } t \to \infty, \tag{8}$$

where

$$u_\infty(x) := \inf\{\, \psi(x) \,|\, \psi \in \mathcal{S}_{H-c}, \quad \psi \geq u_0^- \ \ \text{in } \mathbb{R}^n\} \in \mathcal{S}_{H-c}.$$

Remark. (a) For a given $u_0 \in \mathrm{UC}(\mathbb{R}^n)$, the constant c in (B1), if it exists, is uniquely determined.
(b) In view of comparison, we see that the existence of $\psi \in \mathcal{S}^+_{H-c}$ required in (B1) is indeed a necessary condition for the convergence (2).

The following result can be regarded as a particular case of Theorem 2.2.

Corollary 2.3. *Let $c \in \mathbb{R}$ and H satisfy (A1)-(A3), (A4)$_c$, (A5), and assume that (3) has viscosity solutions in the class* $\mathrm{BUC}(\mathbb{R}^n)$*. Then, the convergence (8) holds for any bounded, lower semi-almost periodic initial function u_0.*

Recall here an example due to Barles-Souganidis [3] (see also [9]) with a small variation. Let $n = 1$ and consider the Cauchy problem

$$u_t + \frac{1}{2}|Du|^2 - Du = 0 \ \text{ in } \mathbb{R} \times (0, \infty) \quad \text{and} \quad u(\cdot, 0) = u_0 \ \text{ on } \mathbb{R}. \tag{9}$$

Here we select u_0 to be the function g which is defined as follows. For $k \in \mathbb{N}$, we set $t_k = 4(k+2)^2$, $s_k = (t_{k+1} + t_k)/2$, $a_k = s_k + 1$, and $b_k = s_{k+1} - 1$. We have $t_{k+1} - a_k = 4k + 9$, $b_k - t_{k+1} = 4k + 13$ for all $k \in \mathbb{N}$. Hence we have $a_k < t_{k+1} < b_k$ and $(b_{k+1} - t_{k+1})^2 > (t_{k+1} - a_k)^2 = (4k+9)^2 > 8(k+3)^2 = 2t_{k+1}$ for all $k \in \mathbb{N}$. We set $S = \{\pm s_k \mid k \in \mathbb{N}\} \cup (-s_1, s_1)$ and then set $h(x) = \inf\{|x - y| \mid y \in S\}$ and $g(x) = \min\{h(x), 1\}$ for all $x \in \mathbb{R}$. Note that $g(\pm s_k) = 0$ and $g(x) = 1$ for all $x \in [a_k, b_k]$ and $k \in \mathbb{N}$. By the Hopf-Lax formula, we have

$$u(0, t) = \inf_{y \in \mathbb{R}} \left(g(y) + \frac{|t - y|^2}{2t} \right) \quad \text{for } t \geq 0.$$

It is therefore clear that $u(0, s_k) = 0$ for all $k \in \mathbb{N}$. On the other hand, since $(t_{k+1} - a_k)^2 > 2t_{k+1}$ and $(b_k - t_{k+1})^2 > 2t_{k+1}$, we see that $u(0, t_{k+1}) = 1$ for all $k \in \mathbb{N}$ and conclude that u does not converge to any asymptotic solutions. Thus, roughly speaking, if u_0 "oscillates" slowly at infinity, then the convergence of solutions of (1) to asymptotic solutions may not hold. This observation somehow justifies our assumption in Theorem 2.2 and Corollary 2.3 that initial functions u_0 are semi-almost periodic.

We now choose u_0 to be $-g$. It is easily checked that $-g$ is a lower semi-periodic function, and Corollary 2.3 ensures the convergence of the solution u of (9) to an asymptotic solution.

It is not clear for the authors if one can replace or not condition (A5) in Theorem 2.2 or Corollary 2.3 by the semi-almost periodicity of H. One indication in this regard is the following example in [15]. Let $n = 1$ again and consider the Cauchy problem

$$u_t + |Du|^2 = f(x)^2 \quad \text{in } \mathbb{R} \quad \text{and} \quad u(\cdot, 0) = 0, \tag{10}$$

where $f(x) := 2 + \cos x + \cos(\sqrt{2}x)$. The function f is quasi-periodic and has the properties $\inf_{\mathbb{R}} f = 0$ and $f(x) > 0$ for all $x \in \mathbb{R}$. It is not difficult to see that (10) has a unique non-negative solution u and that the critical eigenvalue for the Hamiltonian $H(x,p) := |p|^2 - f(x)^2$ is zero, but the problem $H(x, D\phi) = 0$ in $\mathbb{R}$ does not have any non-negative solution. These together show that the non-negative solution u of (10) does not converge to any asymptotic solution. On the other hand, we should mention that our assumption (B1) in Theorem 2.2 excludes example (10).

Notice. In order to prove Theorem 2.2, we need only to study the case where $c = 0$ by replacing, if necessary, H and $u(x,t)$ by $H - c$ and $u(x,t) + ct$, respectively. Therefore, from now on, we always assume that $c = 0$.

3 Key estimates.

Let us set $Q := \{(x,p) \in \mathbb{R}^n \times \mathbb{R}^n \mid H(x,p) = 0\}$ and

$$S := \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \in D_2^- H(x,p) \ \text{ for some } (x,p) \in Q\}.$$

The goal of this section is to prove Proposition 3.4 below which plays a key role in the sequel. For this purpose, we establish several lemmas. We use the notation: $P(x,\xi) := \{p \in \mathbb{R}^n \mid \xi \in D_2^- H(x,p)\}$ for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Lemma 3.1. *Let H satisfy (A1)-(A3) and $(A4)_0$. Then,*
(a) Q, $S \subset \mathbb{R}^n \times B(0,R_0)$ for some $R_0 > 0$.
(b) There exist constants $\delta > 0$ and $R_1 > 0$ such that for any $(x,\xi) \in S$ and $\varepsilon \in (0,\delta)$, we have $P(x,(1+\varepsilon)\xi) \neq \emptyset$ and $P(x,(1+\varepsilon)\xi) \subset B(0,R_1)$.

Proof. (a) It follows from coercivity (A2) that there exists a constant $R_1 > 0$ such that $Q \subset \mathbb{R}^n \times B(0,R_1)$. Next, fix any $(x,\xi) \in S$. Then, by the definition of S, we can find $p \in P(x,\xi)$ such that $(x,p) \in Q$. Note that $|p| \le R_1$. By convexity (A3), we have

$$H(x,q) \ge H(x,p) + \xi \cdot (q - p) \quad \text{for all } q \in \mathbb{R}^n.$$

Setting $q = p + \xi/|\xi|$ $(\xi \neq 0)$ yields

$$|\xi| = \xi \cdot (q-p) \le H(x,q) - H(x,p) \le \sup_{\mathbb{R}^n \times B(0,R_1+1)} H - \inf_{\mathbb{R}^n \times B(0,R_1)} H.$$

By virtue of coercivity (A2), we can choose $R_2 > 0$ so that the right-hand side is less than R_2, and therefore $\xi \in B(0, R_2)$. If we set $R_0 := \max\{R_1, R_2\}$, we can conclude that Q, $S \subset \mathbb{R}^n \times B(0, R_0)$.

(b) Let $R_0 > 0$ be a constant chosen as above and set $\delta := \omega_0(1)$, where ω_0 is from $(A4)_0$. In view of coercivity (A2), replacing $R_0 > 0$ by a larger constant if necessary, we may assume that $H(x,p) \geq 1+\omega_0(1)$ for all $(x,p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus B(0,R_0))$.

Fix any $(x,\xi) \in S$, $p \in P(x,\xi)$ and $\varepsilon \in (0,\delta)$. Note that $\xi, p \in B(0,R_0)$. By $(A4)_0$, for all $x \in \mathbb{R}^n$ we have

$$H(x,q) \geq \xi \cdot (q-p) + \omega_0\left((\xi \cdot (q-p))_+\right).$$

We set $V := \{q \in B(p, 2R_0) \mid |\xi \cdot (q-p)| \leq 1\}$. Let $q \in V$ and observe that if $q \in \partial B(p, 2R_0)$ then $|q| \geq R_0$ and hence $H(x,q) \geq 1+\omega(1) > 1+\varepsilon \geq (1+\varepsilon)\xi\cdot(q-p)$, similarly, if $\xi \cdot (q-p) = 1$, then $H(x,q) \geq 1+\omega(1) > 1+\varepsilon \geq (1+\varepsilon)\xi \cdot (q-p)$, and finally, if $\xi \cdot (q-p) = -1$, then $H(x,q) \geq \xi \cdot (q-p) > (1+\varepsilon)\xi \cdot (q-p)$. Thus the function $G(q) := H(x,q) - (1+\varepsilon)\xi \cdot (q-p)$ on $\mathbb{R}^n$ is positive on ∂V while it vanishes at $q = p \in V$, and therefore it attains a minimum over the compact set V at an interior point of V. Hence, $P(x,(1+\varepsilon)\xi) \neq \emptyset$. By the convexity of G, we see easily that $G(q) > 0$ for all $q \in \mathbb{R}^n \setminus V$ and conclude that $P(x,(1+\varepsilon)\xi) \subset B(0, 2R_0)$. □

The following estimate (11) has been shown in Lemma 5.2 of [5] and Proposition 7.1 of [14] for strictly convex Hamiltonians.

Lemma 3.2. *Assume that H satisfies (A1)-(A3), $(A4)_0$. Then, there exist a constant $\delta_1 > 0$ and a modulus ω_1 such that for any $\varepsilon \in [0, \delta_1]$ and $(x,\xi) \in S$,*

$$L(x,(1+\varepsilon)\,\xi) \leq (1+\varepsilon)L(x,\xi) + \varepsilon\,\omega_1(\varepsilon). \tag{11}$$

Proof. Let $R_0 > 0$, $R_1 > 0$ and $\delta > 0$ be the constants from Lemma 3.1. Fix any $(x,\xi) \in S$. Take any $\varepsilon \in [0,\delta)$ and, in view of Lemma 3.1, a $p_\varepsilon \in P(x,(1+\varepsilon)\xi)$. Then we have $|p_\varepsilon - p_0| \leq R_1$, $|\xi| \leq R_0$ and $|\xi \cdot (p_\varepsilon - p_0)| \leq R_0 R_1$. Note that by $(A4)_0$,

$$H(x,p_\varepsilon) \geq \xi \cdot (p_\varepsilon - p_0) + \omega_0\left((\xi \cdot (p_\varepsilon - p_0))_+\right).$$

Thus, we obtain

$$\begin{aligned} L(x,(1+\varepsilon)\,\xi) &= (1+\varepsilon)\,\xi \cdot p_\varepsilon - H(x,p_\varepsilon) \leq (1+\varepsilon)\,\xi \cdot p_\varepsilon \\ &\qquad - \xi \cdot (p_\varepsilon - p_0) - \omega_0\left((\xi \cdot (p_\varepsilon - p_0))_+\right) \\ &\leq (1+\varepsilon)[\xi \cdot p_0 - H(x,p_0)] + \varepsilon\,\xi \cdot (p_\varepsilon - p_0) - \omega_0\left((\xi \cdot (p_\varepsilon - p_0))_+\right) \\ &\leq (1+\varepsilon)L(x,\xi) + \varepsilon \max_{0 \leq r \leq R_0 R_1} \left(r - \frac{1}{\varepsilon}\omega_0(r)\right). \end{aligned}$$

We define the function ω_1 on $[0,\infty)$ by setting $\omega_1(s) = \max_{0\le r\le R_0 R_1}(r - \omega_0(r)/s)$ for $s > 0$ and $\omega_1(0) = 0$ and observe that $\omega_1 \in C([0,\infty))$. We have also $L(x,(1+\varepsilon)\xi) \le (1+\varepsilon)L(x,\xi) + \varepsilon\omega_1(\varepsilon)$ for all $\varepsilon \in (0,\delta)$. Thus (11) holds with $\delta_1 := \delta/2$. □

Let $\phi \in \mathcal{S}_H$. From Theorem 2.1, for any $(x,t) \in \mathbb{R}^n \times [0,\infty)$, we have

$$\phi(x) = \inf\Big\{ \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds + \phi(\gamma(-t)) \;\Big|\; \gamma \in \mathcal{C}([-t,0];x)\Big\}. \tag{12}$$

We denote by $\mathcal{E}((-\infty,0];x;\phi)$ the set of curves $\gamma \in \mathcal{C}((-\infty,0];x)$ satisfying

$$\phi(x) = \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds + \phi(\gamma(-t)) \qquad \text{for all } t > 0. \tag{13}$$

Lemma 3.3. *$\mathcal{E}((-\infty,0];x;\phi) \ne \emptyset$ for every $\phi \in \mathcal{S}_H$ and $x \in \mathbb{R}^n$.*

Proof. Fix $y \in \mathbb{R}^n$. For each $k \in \mathbb{N}$, in view of (12), we may choose a curve $\gamma_k \in \mathcal{C}([-1,0];y)$ so that

$$\phi(\gamma_k(0)) + \frac{1}{k} > \int_{-1}^{0} L(\gamma_k(s),\dot\gamma_k(s))ds + \phi(\gamma_k(-1)).$$

By the dynamic programming principle, we easily check that for any $t \in [0,1]$,

$$\phi(\gamma_k(0)) + \frac{1}{k} > \int_{-t}^{0} L(\gamma_k(s),\dot\gamma_k(s))ds + \phi(\gamma_k(-t)). \tag{14}$$

Define $\psi \in C(\mathbb{R}^n)$ by $\psi(x) = \phi(x) - |x|$ and observe that $\psi \in \mathrm{Lip}(\mathbb{R}^n)$ and therefore that there is a $C > 0$ such that $\psi \in \mathcal{S}^-_{H-C}$. Hence, for any $t \in [0,1]$ we get (see for instance Proposition 2.5 of [14]),

$$\psi(\gamma_k(0)) - \psi(\gamma_k(-t)) \le \int_{-t}^{0} [L(\gamma_k(s),\dot\gamma_k(s)) + C]\,ds.$$

Combining this with (14), we get

$$|\gamma_k(-t)| \le \phi(y) - \psi(y) + C + 1 \quad \text{ for all } t \in [0,1].$$

We now invoke Lemmas 6.3 and 6.4 of [14], to conclude that there is a subsequence of $\{\gamma_k\}_{k\in\mathbb{N}}$ converging to a $\gamma \in \mathrm{AC}([-1,0],\mathbb{R}^n)$ and moreover

$$\int_{-1}^{0} L(\gamma(s),\dot\gamma(s))ds \le \phi(\gamma(0)) - \phi(\gamma(-1)). \tag{15}$$

For each $y \in \mathbb{R}^n$ we fix a curve $\gamma(s)$ in $\mathcal{C}([-1,0];y)$ so that (15) holds and refer it as $\gamma(s;y)$.

Fix $x \in \mathbb{R}^n$. We define the sequence $\{x_k\}_{k\in\mathbb{N}}$ inductively by setting $x_1 = x$ and $x_{k+1} = \gamma(-1;x_k)$ for $k \in \mathbb{N}$, and then define the curve $\gamma \in C((-\infty,0];x)$ by setting $\gamma(s) = \gamma(s+k-1;x_k)$ for $-k \le s \le -k+1$ and $k \in \mathbb{N}$. It is clear from (15) that for all $k \in \mathbb{N}$,

$$\int_{-k}^{0} L(\gamma(s),\dot\gamma(s))ds \le \phi(\gamma(0)) - \phi(\gamma(-k)),$$

from which, together with (12), one easily deduces that $\gamma \in \mathcal{E}((-\infty,0];\phi;x)$ □

Proposition 3.4. *Let H and $u_0 \in \mathrm{UC}(\mathbb{R}^n)$ satisfy (A1)-(A3), (A4)$_0$ and (B1), respectively. Then there exists $\lambda > 1$ such that for any $\phi \in \mathcal{S}_H$, $x \in \mathbb{R}^n$, $\gamma \in \mathcal{E}((-\infty,0];x;\phi)$ and for any t, $\tau > 0$ satisfying $t \ge \lambda\tau$,*

$$u(x,t) - \phi(x) \le u(\gamma(-t),\tau) - \phi(\gamma(-t)) + \frac{t\tau}{t-\tau}\,\omega_1\Big(\frac{\tau}{t-\tau}\Big), \tag{16}$$

where ω_1 is taken from (11).

Proof. Fix $\phi \in \mathcal{S}_H$, $x \in \mathbb{R}^n$ and $\gamma \in \mathcal{E}((-\infty,0];x;\phi)$ arbitrarily. We first check (following [14]) that $(\gamma(s),\dot\gamma(s)) \in S$ for a.e. $s \in (-\infty,0)$. Let $t > 0$. There exists a $q \in L^\infty(-t,0;\mathbb{R}^n)$ such that $q(s) \in \partial_c\phi(\gamma(s))$ for a.e. $s \in (-t,0)$ and

$$\phi(\gamma(0)) - \phi(\gamma(-t)) = \int_{-t}^{0} q(s)\cdot\dot\gamma(s)\,ds,$$

where $\partial_c\phi$ stands for the Clarke differential of ϕ (see Proposition 2.4 of [14] for details). Moreover, since $\phi \in \mathcal{S}_H$, we have $H(\gamma(s),q(s)) = 0$ for a.e. $s \in (-t,0)$, and therefore

$$\begin{aligned}\int_{-t}^{0} q(s)\cdot\dot\gamma(s)\,ds \;&\le \int_{-t}^{0}\{L(\gamma(s),\dot\gamma(s)) + H(\gamma(s),q(s))\}\,ds\\ &\le \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds = \phi(\gamma(0)) - \phi(\gamma(-t)).\end{aligned}$$

In particular, we get

$$q(s)\cdot\dot\gamma(s) = L(\gamma(s),\dot\gamma(s)) + H(\gamma(s),q(s)) \quad \text{a.e. } s \in (-t,0),$$

which implies that $\dot\gamma(s) \in D_2^- H(\gamma(s),q(s))$ for a.e. $s \in (-t,0)$. Thus, we conclude that $(\gamma(s),\dot\gamma(s)) \in S$ for a.e. $s \in (-\infty,0)$.

Now, let $\delta_1 > 0$ be the constant from Lemma 3.2 and set $\lambda := \delta_1^{-1}(1+\delta_1)$. Note that if t and $\tau > 0$ satisfy $t \ge \lambda\tau$, then $\varepsilon := (t-\tau)^{-1}\tau \le \delta_1$. Fix such $\tau, t > 0$ and define a new curve by $\eta(s) := \gamma((1+\varepsilon)s)$. Then, by taking account of (5) and (11), we see

$$\begin{aligned} u(x,t) &\le \int_{-(t-\tau)}^{0} L(\eta(s),\dot\eta(s))\,ds + u(\eta(-(t-\tau)),\tau) \\ &= \int_{-t}^{0} (1+\varepsilon)^{-1} L(\gamma(s),(1+\varepsilon)\dot\gamma(s))\,ds + u(\gamma(-t),\tau) \\ &\le \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds + t\,\varepsilon\,\omega_1(\varepsilon) + u(\gamma(-t),\tau). \end{aligned}$$

Since γ satisfies (13), we can conclude that (16) is valid. □

4 Proof of the main theorem.

This section is devoted to the proof of Theorem 2.2. Let $(T_t)_{t\ge 0}$ be the semigroup on $\mathrm{UC}(\mathbb{R}^n)$ defined by $(T_t u_0)(x) := u(x,t)$, where $u(x,t)$ is the unique viscosity solution of (1) with initial function u_0.

Lemma 4.1. *Let H and $u_0 \in \mathrm{UC}(\mathbb{R}^n)$ satisfy (A1)-(A3) and (B1), respectively. Then, for every $(x,t) \in \mathbb{R}^n \times [0,\infty)$,*

$$(T_t u_0^-)(x) = \inf_{s\ge t} u(x,s) \tag{17}$$

$$u_\infty(x) = \liminf_{s\to\infty} u(x,s). \tag{18}$$

Proof. We set $v(x,t) := \inf_{s\ge t} u(x,s)$ and show that $(T_t u_0^-)(x) = v(x,t)$. Since $u_0^- \in \mathcal{S}_H^-$ and $u_0^- \le u_0$, by comparison, we get $u_0^-(x) \le (T_s u_0^-)(x) \le (T_s u_0)(x)$ for all $(x,s) \in \mathbb{R}^n \times [0,\infty)$. Moreover, by using comparison again, we have $(T_t u_0^-)(x) \le (T_{t+s} u_0)(x)$ for all $(x,s,t) \in \mathbb{R}^n \times [0,\infty) \times [0,\infty)$. Thus, taking infimum over all $s \ge 0$ yields $(T_t u_0^-)(x) \le v(x,t)$.

Observe next that for each $r \ge 0$, the function $v^r(x,t) := u(x,t+r) = (T_t(T_r u_0))(x)$ is a solution of $v_t + H(x,Dv) = 0$ in $\mathbb{R}^n \times (0,\infty)$, and that $v(x,t) = \inf_{r\ge 0} v^r(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Then, convexity (A3) ensures that v is a solution of $v_t + H(x,Dv) = 0$ in $\mathbb{R}^n \times (0,\infty)$. Since the function: $t \mapsto v(x,t)$ is non-decreasing on $[0,\infty)$ for each $x \in \mathbb{R}^n$, we see that $v(\cdot,t) \in \mathcal{S}_H^-$ for all $t \ge 0$. Since $v(\cdot,0) \le u_0$ in $\mathbb{R}^n$, we have $v(\cdot,0) \le u_0^-$ in $\mathbb{R}^n$. Thus by comparison, we get $v(x,t) = (T_t v(\cdot,0))(x) \le (T_t u_0^-)(x)$. Hence we have proved (17).

Set $w(x) := \liminf_{t\to\infty} u(x,t)$. Since $u_0^- \le u_\infty$ in $\mathbb{R}^n$, in view of comparison, we get $T_t u_0^- \le u_\infty$ in $\mathbb{R}^n$ for all $t \ge 0$. By virtue of (17), we obtain $w(x) \le u_\infty(x)$ for all

$x \in \mathbb{R}^n$. On the other hand, we have $w \in \mathcal{S}_H$ and $u_0^- \le T_t u_0^- \le w$ in $\mathbb{R}^n$ for all $t \ge 0$. Consequently, we have $u_\infty \le w$ in $\mathbb{R}^n$. We thus conclude that (18) is valid. □

Proof of Theorem 2.2. Define $u^+ \in \mathrm{UC}(\mathbb{R}^n)$ by $u^+(x) := \limsup_{t\to\infty} u(x,t)$. In view of the previous lemma, we need only to show that $u^+(x) \le u_\infty(x)$ for all $x \in \mathbb{R}^n$.

Fix any $y \in \mathbb{R}^n$ and choose a diverging sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ so that $u^+(y) = \lim_{j\to\infty} u(y,t_j)$. Take any $\gamma \in \mathcal{E}((-\infty,0]; y; u_\infty)$, and set $y_j = \gamma(-t_j)$ for $j \in \mathbb{N}$. From (A5), passing to a subsequence if necessary, we may assume that for some $G \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and $\{\xi_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ converging to zero,

$$H(\cdot + y_j, \cdot\,) \longrightarrow G \quad \text{in } C(\mathbb{R}^n \times \mathbb{R}^n) \quad \text{as } j \to \infty,$$
$$H(x + y_j + \xi_j, p) \le G(x,p) \quad \text{for all } (x,p,j) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}.$$

Fix any $\varepsilon > 0$. We may assume as well that for some function $v_0 \in \mathrm{UC}(\mathbb{R}^n)$,

$$u_0(x + y_j + \xi_j) + \varepsilon \ge v_0(x) + u_0(y_j + \xi_j) \quad \text{for all } (x,j) \in \mathbb{R}^n \times \mathbb{N}, \tag{19}$$
$$u_0(\cdot + y_j + \xi_j) - u_0(y_j + \xi_j) \longrightarrow v_0 \quad \text{in } C(\mathbb{R}^n) \quad \text{as } j \to \infty. \tag{20}$$

We now consider the Cauchy problem

$$\begin{cases} v_t + G(x, Dv) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty), \\ v(\,\cdot\,,0) = v_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{21}$$

and set $(T_t^G v_0)(x) := v(x,t)$, where $v(x,t)$ stands for the unique viscosity solution of (21). We denote by $\mathcal{S}_G^-$, $\mathcal{S}_G^+$ the set of all sub- and supersolutions of

$$G(x, D\phi) = 0 \quad \text{in } \mathbb{R}^n,$$

respectively. Set $\mathcal{S}_G := \mathcal{S}_G^- \cap \mathcal{S}_G^+$.

We first claim that there exists $\phi \in \mathcal{S}_G^-$ such that $\phi \le v_0$ in $\mathbb{R}^n$. For this, observe by (B1) that $-C_0 \le \phi_0 - u_0 \le 0$ for some $\phi_0 \in \mathcal{S}_H^- \subset \mathrm{Lip}(\mathbb{R}^n)$. If we set

$$\overline{\phi}_{0j}(x) := \phi_0(x + y_j + \xi_j) - u_0(y_j + \xi_j) \qquad \text{for } j \in \mathbb{N},$$

then $\{\overline{\phi}_{0j}\}_{j\in\mathbb{N}}$ forms a locally uniformly bounded and equi-Lipschitz family in $C(\mathbb{R}^n)$. Hence, we may assume that $\overline{\phi}_{0j}$ converges to a function $\phi \in \mathrm{Lip}(\mathbb{R}^n)$ in $C(\mathbb{R}^n)$ as $j \to \infty$. Under the notation $H_j(x,p) := H(x + y_j + \xi_j, p)$, we see that $\overline{\phi}_{0j} \in \mathcal{S}_{H_j}^-$ and passing to the limit as $j \to \infty$, we have $\phi \in \mathcal{S}_G^-$. Moreover, since $\phi_0 \le u_0$ in $\mathbb{R}^n$,

$$\overline{\phi}_{0j}(x) \le u_0(x + y_j + \xi_j) - u_0(y_j + \xi_j) \qquad \text{for all } (x,j) \in \mathbb{R}^n \times \mathbb{N}.$$

Letting $j \to \infty$ yields $\phi \leq v_0$ in $\mathbb{R}^n$ by virtue of (20). Hence the claim has been proved.

We next set

$$v_0^-(x) := \sup\{\, \phi(x) \,|\, \phi \in \mathcal{S}_G^-, \ \ \phi \leq v_0 \ \text{ in } \mathbb{R}^n\} \ (> -\infty) \quad \text{for all } x \in \mathbb{R}^n,$$

and show that there exists a $\psi \in \mathcal{S}_G$ such that $\psi \geq v_0^-$ in $\mathbb{R}^n$. In view of (19) and the definition of v_0^-, we have

$$v_0^-(x) + u_0(y_j + \xi_j) - \varepsilon \leq v_0(x) + u_0(y_j + \xi_j) - \varepsilon \leq u_0(x + y_j + \xi_j).$$

Since $v_0^- \in \mathcal{S}_G^- \subset \mathcal{S}_{H_j}^-$, we can see by comparison that

$$\begin{aligned} v_0^-(x) + u_0(y_j + \xi_j) - \varepsilon \ &\leq T_t^{H_j}(v_0^-(\cdot) + u_0(y_j + \xi_j) - \varepsilon)(x) \\ &\leq (T_t^{H_j} u_0(\cdot + y_j + \xi_j))(x) = (T_t^H u_0)(x + y_j + \xi_j) \end{aligned}$$

for all $t > 0$ and $j \in \mathbb{N}$. Thus, $v_0^-(x) + u_0(y_j + \xi_j) - \varepsilon \leq u_\infty(x + y_j + \xi_j)$. Since $\overline{u}_{\infty j}(x) := u_\infty(x + y_j + \xi_j) - u_0(y_j + \xi_j) + \varepsilon \in \mathcal{S}_{H_j} \subset \mathcal{S}_G^+$, we can apply the Perron method to construct $\psi_j \in \mathcal{S}_G$ such that $v_0^- \leq \psi_j \leq \overline{u}_{\infty j}$ in $\mathbb{R}^n$ for all $j \in \mathbb{N}$. In particular, the function

$$v_\infty(x) := \inf\{\, \psi(x) \,|\, \psi \in \mathcal{S}_G, \ \ \psi \geq v_0^- \ \text{ in } \mathbb{R}^n\},$$

is well-defined and moreover $v_0^- \leq v_\infty \leq \overline{u}_{\infty j}$ in $\mathbb{R}^n$ for all $j \in \mathbb{N}$.

Now we apply Proposition 3.4 for $\phi := u_\infty$ to get

$$\begin{aligned} u(y, t_j) - u_\infty(y) \ &\leq u(y_j, \tau) - u_\infty(y_j) + \frac{t_j \tau}{t_j - \tau}\, \omega_1\Big(\frac{\tau}{t_j - \tau}\Big) \\ &\leq u(y_j, \tau) - u_0(y_j + \xi_j) - v_\infty(-\xi_j) + \varepsilon + \frac{t_j \tau}{t_j - \tau}\, \omega_1\Big(\frac{\tau}{t_j - \tau}\Big) \end{aligned} \tag{22}$$

for every $\tau > 0$ and sufficiently large $j \in \mathbb{N}$, where we have used $v_\infty \leq \overline{u}_{\infty j}$ in the second inequality. Remark here that by stability,

$$u(x + y_j + \xi_j, t) - u_0(y_j + \xi_j) \longrightarrow v(x, t) \quad \text{in } C(\mathbb{R}^n \times [0, \infty)) \quad \text{as } j \to \infty.$$

Thus, sending $j \to \infty$ in (22), we have

$$u^+(y) - u_\infty(y) \leq v(0, \tau) - v_\infty(0) + \varepsilon.$$

Letting $\tau = \tau_j \to \infty$ along a sequence $\{\tau_j\}$ such that $\lim_{j \to \infty} v(0, \tau_j) = \liminf_{t \to \infty} v(0, t)$, we finally obtain $u^+(y) \leq u_\infty(y) + \varepsilon$. Since $\varepsilon > 0$ and $y \in \mathbb{R}^n$ are arbitrary, we conclude that $u^+ \leq u_\infty$ in $\mathbb{R}^n$. $\square$

5 Examples.

In this section, we give a couple of examples that satisfy all conditions in Theorem 2.2 or that in Corollary 2.3.

We begin with a simple but typical example of Hamiltonian which is upper semi-periodic.

Example 1. Let $n = 1$ and $f(x) := 1 + \sin x + e^{-|x|}$. Note that $0 < f(x) < 3$ for all $x \in \mathbb{R}$ and $\inf_{\mathbb{R}} f = 0$. We define H by $H(x,p) := |p|^2 - f(x)^2$. Then, H satisfies (A1)-(A4). We shall check the upper semi-periodicity (A5) of H. Let $\{y_j\}_{j\in\mathbb{R}} \subset \mathbb{R}$ be any sequence. If $\sup_j |y_j| < \infty$, then there exists a subsequence $\{z_j\}_{j\in\mathbb{N}} \subset \{y_j\}$ such that $z_j \to z$ for some $z \in \mathbb{R}$ as $j \to \infty$. Set $G(x,p) := H(x+z,p)$ and $\xi_j := z - z_j$. Then,

$$H(x + z_j + \xi_j, p) = H(x + z_j + z - z_j, p) = H(x + z, p) = G(x, p)$$

for all $(x,p,j) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}$, and clearly $H(x + z_j, p) \longrightarrow G(x,p)$ in $C(\mathbb{R} \times \mathbb{R})$.

We next suppose that $\sup_j |y_j| = \infty$. Then, we can find a subsequence $\{z_j\}_{j\in\mathbb{N}} \subset \{y_j\}$ such that $z_j = \zeta_j + \eta_j$, $(\zeta_j, \eta_j) \in (2\pi\mathbb{Z}) \times [0, 2\pi)$, and $\eta_j \longrightarrow \eta$ as $j \to \infty$ for some $\eta \in [0, 2\pi]$. We set $g(x) := 1 + \sin(x + \eta)$, $G(x,p) := |p|^2 - g(x)^2$ and $\xi_j := \eta - \eta_j$. Then,

$$H(x+z_j+\xi_j, p) \le |p|^2 - (1+\sin(x+\eta_j+\eta-\eta_j))^2 = G(x,p) \quad \text{for all } (x,p,j) \in \mathbb{R}\times\mathbb{R}\times\mathbb{N},$$

and moreover $H(x + z_j, p) \longrightarrow G(x,p)$ in $C(\mathbb{R} \times \mathbb{R})$ since $e^{-|x+z_j|} \longrightarrow 0$ as $j \to \infty$. Hence, H is upper semi-periodic.

Let us consider the Hamilton-Jacobi equation

$$u_t + H(x, Du) = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty) \tag{23}$$

with initial condition $u_0(x) := x$. It suffices to check (B1) since (B2) is obvious. Let us define $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\phi(x) := \int_0^x f(y)\,dy = x + 1 - \cos x + \operatorname{sgn}(x)(1 - e^{-|x|}), \qquad x \in \mathbb{R}, \tag{24}$$

where $\operatorname{sgn}(x) := 1$ if $x > 0$ and $\operatorname{sgn}(x) := -1$ if $x < 0$. Then, ϕ is a viscosity solution of

$$H(x, D\phi) = 0 \qquad \text{in } \mathbb{R}. \tag{25}$$

Moreover, we see $\phi(x) - 3 \le u_0(x) \le \phi(x) + 1$ for all $x \in \mathbb{R}$, which implies (B1). Hence, Theorem 2.2 holds with $c = 0$. We remark that the Aubry set $\mathcal{A}$ for H (or for equation (25)) is empty, where $\mathcal{A}$ is defined by

$$\mathcal{A} := \{y \in \mathbb{R} \mid H(y,p) \ge 0 \ \text{ for all } \ \phi \in \mathcal{S}_H^-,\, p \in D^-\phi(y)\}$$

(see [10] and [14] for further information on Aubry sets).

Another observation to be noted is that in this example we cannot take any bounded function as initial function as far as Theorem 2.2 is valid. Indeed, suppose that $u_0 = 0$ in $\mathbb{R}$ for simplicity and let $u(x,t)$ be the solution of (23) satisfying $u(x,0) = u_0(x)$. We deduce a contradiction by assuming that there exist a constant $c \in \mathbb{R}$ and a solution v of

$$H(x, Dv) - c = 0 \qquad \text{in } \mathbb{R} \tag{26}$$

such that $u(x,t) + c\,t - v(x) \longrightarrow 0$ in $C(\mathbb{R})$ as $t \to \infty$.

Let us first show that $c = 0$. Fix any $x \in \mathbb{R}$. For each $k \in \mathbb{N}$, we set $\tau_k := (2k+1)\pi - x$ and define $\eta_k \in \mathcal{C}((-\infty,0];x)$ by

$$\eta_k(s) := \begin{cases} x - s & \text{if } \ s \in [-\tau_k, 0], \\ (2k+1)\pi & \text{if } \ s \in (-\infty, -\tau_k]. \end{cases}$$

Then, in view of $f(\eta_k(-\tau_k)) = e^{-(2k+1)\pi}$ and

$$L(x,\xi) := \sup_{p\in\mathbb{R}}(p\,\xi - H(x,p)) = \frac{1}{4}|\xi|^2 + f(x)^2 \ge 0,$$

we observe that for every $t > \tau_k$,

$$\begin{aligned} 0 \le u(x,t) \ &\le \int_{-\tau_k}^{0} L(\eta_k(s), \dot\eta_k(s))\, ds + \int_{-t}^{-\tau_k} L(\eta_k(s), \dot\eta_k(s))\, ds \\ &\le (\frac{1}{4} + |f|_\infty^2)\tau_k + e^{-(2k+1)\pi}(t - \tau_k). \end{aligned}$$

Thus, we obtain

$$0 \le \limsup_{t\to\infty} \frac{u(x,t)}{t} \le e^{-(2k+1)\pi} \qquad \text{for all } \ k \in \mathbb{N}.$$

Letting $k \to \infty$ infers $\lim_{t\to\infty} t^{-1}u(x,t) = 0$. Hence $c = 0$.

Suppose now that $u(\,\cdot\,,t) \longrightarrow v \in \mathcal{S}_H$ in $C(\mathbb{R})$ as $t \to \infty$. Since $u(x,t) \ge 0$ for all $(x,t) \in \mathbb{R} \times [0,\infty)$, v should be bounded from below, but it is impossible. Indeed, let $\phi \in \mathcal{S}_H$ be the function defined by (24). Remark that $-\phi \in \mathcal{S}_H$. We set

$$w(x) := \min\{\phi(x), -\phi(x)\} \in \mathcal{S}_H.$$

Fix $M > 0$ so that $w(0) + M > v(0)$. Since $\lim_{|x|\to\infty} w(x) = -\infty$, we can find $R > 0$ such that $w(-R)+M \le v(-R)$ and $w(R)+M \le v(R)$. Thus, by comparison (recall that H is convex in p and $f < 0$ in $\mathbb{R}$), we get $w \le v$ on $[-R, R]$. This is a contradiction. Hence, the solution $u(x,t)$ of (23) satisfying $u(\,\cdot\,, 0) = 0$ does not converge to any asymptotic solution.

The next example also satisfies all conditions in Theorem 2.2.

Example 2. Let $n = 1$ and $f(x) := 1 + \sin x$. We set $H(x,p) := |p|^2 - f(x)^2$. Obviously, H satisfies (A1)-(A5) since f is non-negative and $\mathbb{Z}$-periodic.

We consider the Hamilton-Jacobi equation

$$u_t + H(x, Du) = 0 \qquad \text{in } \mathbb{R} \times (0, +\infty)$$

with initial condition $u_0(x) := |x|$. We show that u_0 satisfies (B1). Define $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) := \int_\pi^x f(y)\,dy = x - \pi - (1 + \cos x), \qquad x \in \mathbb{R}.$$

Note that $g \in C^1(\mathbb{R})$, $g(\pi) = 0$ and $g'(\pi) = f(\pi) = 0$. We next set

$$\phi(x) := \max\{g(x), -g(x)\}, \qquad x \in \mathbb{R}.$$

Then, $\phi \in C^1(\mathbb{R})$ and ϕ is a solution of

$$H(x, D\phi) = 0 \qquad \text{in } \mathbb{R}. \tag{27}$$

Moreover, we have

$$\lim_{x\to+\infty} \frac{\phi(x)}{x} = 1, \qquad \lim_{x\to-\infty} \frac{\phi(x)}{x} = -1.$$

Thus, there exists a constant $C > 0$ such that $\phi - C \le u_0 \le \phi + C$ in $\mathbb{R}$. This implies (B1). Clearly, u_0 satisfies (B2). In this case, the Aubry set for H is $\{(2k+1)\pi \mid k \in \mathbb{Z}\} \subset \mathbb{R}$, which is not compact.

We give an example of Corollary 2.3.

Example 3. Let $n = 2$ and set $e_1 := (1, 0)$ and $e_2 := (0, 1)$. Let $f \in C(\mathbb{R}^2)$ be such that $f \ge 0$ and $\operatorname{supp} f \subset B(0, 1)$. We define H by

$$H(x,p) := |p - e_1|^2 - 1 - \sum_{k\in\mathbb{N}} f(x - 2^k e_2).$$

It is easy to check that H satisfies (A1)-(A5).

We now prove that there exists a bounded viscosity solution of

$$H(x, D\phi) = 0 \qquad \text{in } \mathbb{R}^2. \tag{28}$$

Observe first that the Lagrangian L associated with H can be calculated as

$$L(x, \xi) = \frac{1}{4}|\xi + 2e_1|^2 + \sum_{k\in\mathbb{N}} f(x - 2^k e_2) \geq 0.$$

For $x \in \mathbb{R}^2$, we define $\gamma_x \in \mathcal{C}((-\infty, 0]; x)$ by $\gamma_x(s) := x - 2se_1$. Then, there exists at most one number $j \in \mathbb{N}$ such that $\gamma_x((-\infty, 0]) \cap \operatorname{supp} f(\,\cdot\, - 2^j e_2) \neq \emptyset$. Thus, for every $t > 0$,

$$\begin{aligned}\int_{-t}^{0} L(\gamma_x(s), \dot\gamma_x(s))\, ds &= \sum_{k\in\mathbb{N}} \int_{-t}^{0} f(\gamma_x(s) - 2^k e_2)\, ds \\ &= \int_{-t}^{0} f(\gamma_x(s) - 2^j e_2)\, ds \leq \max f.\end{aligned}$$

If we set

$$d(x, y) := \inf\Big\{ \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds \;\Big|\; t > 0,\ \gamma \in \mathcal{C}([-t, 0]; x),\ \gamma(-t) = y \Big\},$$

then, $d(\cdot, y)$ is a viscosity solution of (28) in $\mathbb{R}^2 \setminus \{y\}$. Moreover, for $a > 0$, we see

$$0 \leq d(x, x + ae_1) \leq \int_{-\frac{a}{2}}^{0} L(\gamma_x(s), \dot\gamma_x(s))\, ds \leq \max f.$$

Now, we set $\phi_j(x) := \inf_{r\in\mathbb{R}} d(x, (j, r))$ and $D_j := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < j\}$ for $j \in \mathbb{N}$. Then, $0 \leq \phi(x) \leq d(x, (j, x_2)) \leq \max f$ for all $x \in D_j$ and ϕ_j is a viscosity solution of $H(x, D\phi) = 0$ in D_j. Hence, by taking a subsequence $\{\phi_{j_k}\}_{k\in\mathbb{N}}$ of $\{\phi_j\}_{j\in\mathbb{N}}$ so that $\phi_{j_k} \longrightarrow \phi$ in $C(\mathbb{R}^2)$ for some ϕ as $k \to \infty$, we can conclude in view of stability that ϕ is indeed a bounded viscosity solution of (28).

6 A generalization.

Following [2] we modify $(A4)_0$ and generalize Theorem 2.2.

(A6) There exists a closed set $K \subset \mathbb{R}^n$ having the properties (a) and (b):
(a) $\min_{p\in\mathbb{R}^n} H(x, p) = 0$ for all $x \in K$.

(b) For each $\varepsilon > 0$ there exists a modulus ω_ε satisfying $\omega_\varepsilon(r) > 0$ for all $r > 0$ such that for all $(x,p) \in \mathbb{R}^{2n}$, $\xi \in D_2^- H(x,p)$ and $q \in \mathbb{R}^n$, if $\mathrm{dist}(x,K) \geq \varepsilon$ and $H(x,p) = 0$, then

$$H(x,p+q) \geq \xi \cdot q + \omega_\varepsilon((\xi \cdot q)_+).$$

The following theorem generalizes Theorem 2.2.

Theorem 6.1. *Assume that (A1)-(A3), (A5)-(A6) and (B1)-(B2) hold and that $c = 0$. Then, the convergence (8) holds.*

Proof. Let u_∞ and u^+ be as in the proof of Theorem 2.2. It is enough to show that $u^+(x) \leq u_\infty(x)$ for all $x \in \mathbb{R}^n$. To this end, we fix any $\bar{x} \in \mathbb{R}^n$ and an extremal curve γ for u_∞ such that $\gamma(0) = \bar{x}$.

In the case when

$$\mathrm{dist}(\gamma((-\infty,0]),K) := \inf\{|\gamma(s) - y| \mid s \in (-\infty,0],\, y \in K\} > 0,$$

using the same argument as in the proof of Theorem 2.2, we can show that $u^+(\bar{x}) = u_\infty(\bar{x})$. Thus we may assume henceforth that $\mathrm{dist}(\gamma((-\infty,0]),K) = 0$.

We claim that there exist $\delta > 0$ and $R > 0$ such that $B(0,\delta) \subset \bigcup\{D_2^- H(x,p) \mid p \in B(0,R)\}$ for all $x \in \mathbb{R}^n$. To show this, we first note by (A1) that $\sup\{|H(x,p)| \mid (x,p) \in \mathbb{R}^n \times B(0,r)\} < \infty$ for all $r \geq 0$. We set $A := \sup\{|H(x,0)| \mid x \in \mathbb{R}^n\}$ and choose $R > 0$ in view of coercivity (A2) so that $\inf\{H(x,p) \mid (x,p) \in \mathbb{R}^{2n},\, |p| \geq R\} > 1 + A$. Fix any $\xi \in B(0,1/R)$ and observe that $H(x,0) + \xi \cdot p \leq A + 1$ for all $p \in \partial B(0,R)$. Hence, $H(x,p) > \xi \cdot p + H(x,0)$ for all $(x,p) \in \mathbb{R}^n \times \partial B(0,R)$ and, for each $x \in \mathbb{R}^n$, the function $p \mapsto \xi \cdot p - H(x,p) + H(x,0)$ attains a maximum at an interior point of $B(0,R)$. Therfore, $\xi \in \bigcup\{D_2^- H(x,p) \mid p \in B(0,R)\}$. Thus, setting $\delta := 1/R$, we get $B(0,\delta) \subset \bigcup\{D_2^- H(x,p) \mid p \in B(0,R)\}$.

We fix $\delta > 0$ and $R > 0$ as above. We next claim that there exists an $M > 0$ such that $|L(x,\xi)| \leq M$ for all $(x,\xi) \in \mathbb{R}^n \times B(0,\delta)$. Indeed, let $(x,\xi) \in \mathbb{R}^n \times B(0,\delta)$. Then, by the choice of δ and $R > 0$, we have $\xi \in D_2^- H(x,q)$ for some $q \in B(0,R)$, from which we have $L(x,\xi) = \xi \cdot q - H(x,q)$. Hence we get

$$|L(x,\xi)| \leq |\xi||q| + |H(x,q)| \leq \delta R + \sup\{|H(y,p)| \mid y \in \mathbb{R}^n,\, p \in B(0,R)\}.$$

Setting $M := \delta R + \sup\{|H(y,p)| \mid y \in \mathbb{R}^n,\, p \in B(0,R)\}$, we conclude that $|L(x,\xi)| \leq M$.

Fix any $\varepsilon > 0$. We choose a $\rho > 0$ so that $M\rho < \delta\varepsilon$ and then $y \in K$ and $r \geq 0$ so that $|\gamma(-r) - y| < \rho$. It is important to notice that $L(y,0) = -\max_{p\in\mathbb{R}^n} H(y,p) = 0$.

Since u_∞ is uniformly continuous on $\mathbb{R}^n$, we may assume that $|u_\infty(z) - u_\infty(z')| \le \varepsilon$ if $|z - z'| < \rho$. We define the curve $\eta : (-\infty, 0] \to \mathbb{R}^n$ by $\eta(s) = \gamma(s)$ for $-r \le s \le 0$, $\eta(s) = \mu(s + r)$ for $-r - \rho/\delta \le s \le -r$, $\eta(s) = y$ for $s \le -r - \rho/\delta$, where $\mu \in \mathrm{AC}([-\rho/\delta, 0], \mathbb{R}^n)$ is given by

$$\mu(s) = \gamma(-r) - \frac{\delta s}{\rho}(y - \gamma(-r)).$$

Note that $\mu(0) = \gamma(-r)$, $\mu(-\delta/\rho) = y$ and $|\dot\mu(s)| = |y - \gamma(-r)|\delta/\rho < \delta$, so that $|L(\mu(s), \dot\mu(s))| \le M$ for all $s \in [-\rho/\delta, 0]$.

Next, we set $T := r + \rho/\delta$ and choose $\tau > 0$ so that $u(y, \tau) < u_\infty(y) + \varepsilon$. For any $t > T$ we compute that

$$\begin{aligned}
u(\bar x, t + \tau) &\le \int_{-t}^{0} L(\eta(s), \dot\eta(s))ds + u(\eta(-t), \tau) \\
&\le \int_{-r}^{0} L(\gamma(s), \dot\gamma(s))ds + \int_{-\rho/\delta}^{0} L(\mu(s), \dot\mu(s))ds + u(y, \tau) \\
&< u_\infty(\bar x) - u_\infty(\gamma(-r)) + M\rho/\delta + u_\infty(y) + \varepsilon \\
&\le u_\infty(\bar x) + 2\varepsilon + u_\infty(y) - u_\infty(\gamma(-r)) \le u_\infty(\bar x) + 3\varepsilon,
\end{aligned}$$

from which we conclude that $u^+(\bar x) \le u_\infty(\bar x)$. □

Appendix A: Solvability of the Cauchy problem.

We denote by $\mathrm{USC}(\mathbb{R}^n \times [0, T))$ (resp. $\mathrm{LSC}(\mathbb{R}^n \times [0, T))$) the set of upper (resp. lower) semi-continuous functions on $\mathbb{R}^n \times [0, T)$.

Proposition A.2. *Let $T > 0$ be given and assume that H satisfies (A1) and (A3). Let $u \in \mathrm{USC}(\mathbb{R}^n \times [0, T))$ and $v \in \mathrm{LSC}(\mathbb{R}^n \times [0, T))$ be viscosity sub- and supersolution of (4), respectively.*

If $v(x, t) \ge -C_0(|x| + 1)$ for all $(x, t) \in \mathbb{R}^n \times [0, T)$ for some $C_0 > 0$ and $u \le v$ on $\mathbb{R}^n \times \{0\}$, then $u \le v$ in $\mathbb{R}^n \times [0, T)$.

Proof. Choose a function $g \in C^1(\mathbb{R}^n)$ so that $g(x) = (C_0 + 1)|x|$ for $x \in \mathbb{R}^n \setminus B(0, 1)$. Set $\tilde u(x, t) := u(x, t) + g(x)$, $\tilde v(x, t) := v(x, t) + g(x)$ for $(x, t) \in \mathbb{R}^n \times [0, T)$ and $\widetilde H(x, p) := H(x, p - Dg(x))$ for $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Observe that (A1) and (A3) is valid with $\widetilde H$ in place of H. Observe as well that $\tilde u \in \mathrm{USC}(\mathbb{R}^n \times [0, T))$ and $\tilde v \in \mathrm{LSC}(\mathbb{R}^n \times [0, T))$ are viscosity sub- and supersolution of

$$u_t + \widetilde H(x, Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, T),$$

respectively. Clearly,

$$\lim_{R\to\infty} \inf\{\tilde{v}(x,t) \,|\, (x,t) \in (\mathbb{R}^n \setminus B(0,R)) \times [0,T)\} = \infty$$

and $\tilde{u} \le \tilde{v}$ on $\mathbb{R}^n \times \{0\}$. Thus, replacing H, u, and v with $\widetilde{H}$, $\tilde{u}$, and $\tilde{v}$, respectively, we may assume that

$$\lim_{R\to\infty} \inf\{v(x,t) \,|\, (x,t) \in (\mathbb{R}^n \setminus B(0,R)) \times [0,T)\} = \infty.$$

We choose a constant $C_1 > 0$ so that $H(x,0) \le C_1$ for all $x \in \mathbb{R}^n$. Fix any $A > 0$ and set $w(x,t) = \min\{u(x,t), -C_1t + A\}$ for $(x,t) \in \mathbb{R}^n \times [0,T)$. Note that w is a viscosity subsolution of (4). (To check this, one may apply an argument in which sup-convolutions of u in the t-variable are used to approximate u with Lipschitz subsolutions.)

Since w is bounded above in $\mathbb{R}^n \times [0,T)$, we may choose a constant $R \equiv R(A) > 0$ so that $w \le v$ in $(\mathbb{R}^n \setminus B(0,R)) \times [0,T)$. Since $u \le v$ on $\mathbb{R}^n \times \{0\}$, we have $w \le v$ on $\mathbb{R}^n \times \{0\}$. We apply a standard comparison theorem to see that $w \le v$ in $B(0,R) \times [0,T)$, which guarantees that $w \le v$ in $\mathbb{R}^n \times [0,T)$. Sending $A \to \infty$, we conclude that $u \le v$ in $\mathbb{R}^n \times [0,T)$. □

Proposition A.3. *Let H satisfy (A1)-(A3) and fix any $T > 0$. Then, for every $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$, there exists a viscosity solution $u \in \mathrm{Lip}(\mathbb{R}^n \times [0,T))$ of (4) satisfying $u(\,\cdot\,,0) = u_0$.*

Proof. Set $Q_T = \mathbb{R}^n \times [0,T)$. Choose a constant $C_0 > 0$ so that $|H(x, Du_0(x))| \le C_0$ for all $x \in \mathbb{R}^n$. Set

$$u^+(x,t) = u_0(x) + C_0t \quad \text{and} \quad u^-(x,t) = u_0(x) - C_0t \quad \text{for all } (x,t) \in Q_T,$$

and observe that $u^\pm \in \mathrm{Lip}(Q_T)$, $u^- \le u^+$ in Q_T and $u^-(\,\cdot\,,0) = u^+(\,\cdot\,,0) = u_0$. Moreover, u^- and u^+ are viscosity sub- and supersolutions of

$$u_t + H(x, Du(x)) = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{29}$$

respectively. Thus, by the Perron method, we see that there exists a viscosity solution u of (29) such that $u^- \le u \le u^+$ in Q_T. By the comparison theorem, we see that $u \in C(\mathbb{R}^n \times (0,T))$ and moreover, if we set $u(x,0) = u_0(x)$ for $x \in \mathbb{R}^n$, then $u \in C(\mathbb{R}^n \times [0,T))$.

Fix any $\varepsilon \in (0,T)$ and set $v(x,t) = u(x,t+\varepsilon) + C_0\varepsilon$ for $(x,t) \in Q_{T-\varepsilon} := \mathbb{R}^n \times [0,T-\varepsilon)$. Observe that v is a viscosity solution of $v_t + H(x,Dv) = 0$ in $Q_{T-\varepsilon}$

and that $v(x,0) = u(x,\varepsilon)+C_0\varepsilon \geq u(x,0)$ for all $x \in \mathbb{R}^n$. By the comparison theorem, we get $u(x,t) \leq v(x,t) = u(x,t+\varepsilon) + C_0\varepsilon$ for all $(x,t) \in Q_{T-\varepsilon}$. Consequently, we have $u(x,t+\varepsilon) - u(x,t) \geq -C_0\varepsilon$ for all $\varepsilon \in (0,T)$ and $(x,t) \in Q_{T-\varepsilon}$, which implies that $u_t \geq -C_0$ in Q_T in the viscosity sense. Hence, u is a viscosity subsolution of $H(x,Du) = C_0$ in Q_T, which guarantees that the family $\{u(\cdot,t) \,|\, t \in [0,T)\}$ is equi-Lipschitz continuous in $\mathbb{R}^n$. We now see that $u_t \leq C_1$ in Q_T in the viscosity sense for some $C_1 > 0$, and that the family $\{u(x,\cdot) \,|\, x \in \mathbb{R}^n\}$ is equi-Lipschitz continuous in $[0,T)$. We thus conclude that $u \in \mathrm{Lip}(\mathbb{R}^n \times [0,T))$. □

Proposition A.4. *Let H satisfy (A1)-(A3) and fix any $T > 0$. Then, for every $u_0 \in \mathrm{UC}(\mathbb{R}^n)$, there exists a viscosity solution $u \in \mathrm{UC}(\mathbb{R}^n \times [0,T))$ of (4) with $u(\,\cdot\,,0) = u_0$.*

Proof. For each $\varepsilon \in (0,1)$, there is a function $u_0^\varepsilon \in \mathrm{Lip}(\mathbb{R}^n)$ such that $|u_0^\varepsilon(x) - u_0(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^n$. Fix such a family $\{u_0^\varepsilon \mid \varepsilon \in (0,1)\} \subset \mathrm{Lip}(\mathbb{R}^n)$. Due to the previous proposition, for each $\varepsilon \in (0,1)$ there exists a viscosity solution $u^\varepsilon \in \mathrm{Lip}(\mathbb{R}^n \times [0,T))$ of (4) satisfying $u^\varepsilon(\,\cdot\,,0) = u_0^\varepsilon$. By the comparison theorem, for any $\varepsilon, \delta \in (0,1)$, we get

$$|u^\varepsilon(x,t) - u^\delta(x,t)| \leq \varepsilon + \delta \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T),$$

which shows that the family $\{u_0^\varepsilon \mid \varepsilon \in (0,1)\}$ converges uniformly on $\mathbb{R}^n \times [0,T)$ to a function $u \in \mathrm{UC}(\mathbb{R}^n \times [0,T))$ as $\varepsilon \to 0$. It is easy to check that the function u is a viscosity solution of (4) with $u(\,\cdot\,,0) = u_0$. □

Appendix B: Semi-periodic functions.

Lemma B.1. *Let $f \in C(\mathbb{R}^n)$ be both lower and upper semi-periodic. Let $e \in \mathbb{R}^n$ and $\varepsilon > 0$. Then there exist an integer $m \geq 1$ and a vector $v \in B(0,\varepsilon)$ such that*

$$f(x) = f(x + m(e+v)) \qquad \textit{for all } x \in \mathbb{R}^n. \tag{30}$$

Proof. We note first that any lower (or upper) semi-periodic function is bounded and uniformly continuous on $\mathbb{R}^n$. In particular, we have $f \in \mathrm{BUC}(\mathbb{R}^n)$.

We show that there exist an $m \in \mathbb{N}$ and a $v \in B(0,\varepsilon)$ such that

$$f(x) \geq f(x + m(e+v)) \quad \text{for all } x \in \mathbb{R}^n. \tag{31}$$

For this, we set $y_j = je$ for $j \in \mathbb{N}$. Since f is lower semi-periodic, there are an increasing sequence $\{j_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ converging to zero, and a function $g \in C(\mathbb{R}^n)$ such that $f(x + j_k e) \longrightarrow g(x)$ in $C(\mathbb{R}^n)$ as $k \to \infty$ and $f(x + j_k e + \xi_k) \ge g(x)$ for all $(x, k) \in \mathbb{R}^n \times \mathbb{N}$. In view of upper semi-periodicity of f, we may assume by taking a common subsequence of $\{j_k\}$ that there exists a sequence $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ converging to zero such that $f(x + j_{k+1}e + \eta_k) \le g(x)$ for all $(x, k) \in \mathbb{R}^n \times \mathbb{N}$. Consequently, we have

$$f(x + j_k e + \xi_k) \ge f(x + j_{k+1}e + \eta_k) \quad \text{for all } (x, k) \in \mathbb{R}^n \times \mathbb{N},$$

which reads

$$f(x) \ge f(x + (j_{k+1} - j_k)e + \eta_k - \xi_k) \quad \text{for all } (x, k) \in \mathbb{R}^n \times \mathbb{N}. \tag{32}$$

We select $k \in \mathbb{N}$ so large that $\eta_k - \xi_k \in B(0, \varepsilon)$ and set $m = j_{k+1} - j_k$ and $v = m^{-1}(\eta_k - \xi_k)$. Observing that $m \in \mathbb{N}$ and $v \in B(0, \varepsilon)$, we conclude that (31) is valid.

Next, let $m \in \mathbb{N}$ and $v \in B(0, \varepsilon)$ be such that (31) holds. We argue by contradiction to show that (30) indeed holds. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) > f(x_0 + m(e + v))$. We set $e_0 = m(e + v)$ and choose a constant $\delta > 0$ so that $f(x_0) \ge \delta + f(x_0 + e_0)$. By (31), we find that $f(x_0 + e_0) \ge f(x_0 + je_0)$ for all $j \in \mathbb{N}$. Similarly as in the proof of (32), we can find an integer $j \in \mathbb{N}$ and a vector $w \in \mathbb{R}^n$ such that $\omega(|w|) < \delta$ and $f(x) \ge f(x - je_0 + w)$ for all $x \in \mathbb{R}^n$, where ω denotes the modulus of continuity for f. Hence we have

$$f(x) \le f(x + je_0 - w) \le f(x + je_0) + \omega(|w|) < f(x + je_0) + \delta,$$

and therefore,

$$f(x_0 + je_0) + \delta \le f(x_0 + e_0) + \delta \le f(x_0) < f(x_0 + je_0) + \delta,$$

which is a contradiction. Thus we obtain (30). □

Proposition B.2. *Let $f \in C(\mathbb{R}^n)$ be both lower and upper semi-periodic. Then f is periodic, that is, the linear span,* Span M, *of*

$$M := \{z \in \mathbb{R}^n \mid f(x + z) = f(x) \ \ \textit{for all } x \in \mathbb{R}^n\}$$

equals the space $\mathbb{R}^n$.

Proof. Let $\{e_1, e_2, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$. By the previous lemma, for each $i = 1, 2, ..., n$ and $k \in \mathbb{N}$, there exist an $m_{ik} \in \mathbb{N}$ and a vector $v_{ik} \in B(0, 1/k)$ such that $m_{ik}(e_i + v_{ik}) \in M$. Regarding e_i and v_{ik} as column vectors, we observe that

$$\det(e_1 + v_{1k}, e_2 + v_{2k}, ..., e_n + v_{nk}) \to \det(e_1, e_2, ..., e_n) = 1 \quad \text{as } k \to \infty.$$

This guarantees that for sufficiently large k,

$$\det(e_1 + v_{1k}, e_2 + v_{2k}, ..., e_n + v_{nk}) > 0$$

and therefore vectors $m_{1k}(e_1+v_{1k})$, $m_{2k}(e_2+v_{2k})$, ..., $m_{nk}(e_n+v_{nk})$ in M are linearly independent. Hence, $\operatorname{Span} M = \mathbb{R}^n$. $\square$

Proposition B.3. *Let $f, g \in C(\mathbb{R}^n)$ be obliquely lower semi-almost periodic. Then $f + g$ and $\max\{f, g\}$ are obliquely lower semi-almost periodic.*

Proof. Let $\varepsilon > 0$ and $\{y_j\} \subset \mathbb{R}^n$ be a sequence, and assume that $f(x+y_j)-f(y_j) \longrightarrow \bar{f}(x)$ and $g(x + y_j) - g(y_j) \longrightarrow \bar{g}(x)$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $f(x + y_j) - f(y_j) + \varepsilon \geq \bar{f}(x)$ and $g(x + y_j) - g(y_j) + \varepsilon \geq \bar{g}(x)$ for all $x \in \mathbb{R}^n$. Then it is clear that $f(x + y_j) + g(x + y_j) - f(y_j) - g(y_j) \longrightarrow \bar{f}(x) + \bar{g}(x)$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $f(x + y_j) + g(x + y_j) - f(y_j) - g(y_j) + 2\varepsilon \geq \bar{f}(x) + \bar{g}(x)$ for all $x \in \mathbb{R}^n$. These observations show that $f + g$ is an obliquely lower semi-almost periodic function.

As above, let $\varepsilon > 0$ and $\{y_j\} \subset \mathbb{R}^n$ be a sequence, and assume that $f(x + y_j) - f(y_j) \longrightarrow \bar{f}(x)$ and $g(x + y_j) - g(y_j) \longrightarrow \bar{g}(x)$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $f(x + y_j) - f(y_j) + \varepsilon \geq \bar{f}(x)$ and $g(x + y_j) - g(y_j) + \varepsilon \geq \bar{g}(x)$ for all $x \in \mathbb{R}^n$. We divide our considerations into two cases.

The first case is when $\sup_{j\in\mathbb{N}} |f(y_j) - g(y_j)| < \infty$. We may assume that $c_j := f(y_j) - g(y_j) \longrightarrow c \in \mathbb{R}$ as $j \to \infty$. We may assume as well that $c \leq c_j + \varepsilon$ for all $j \in \mathbb{N}$.

Now, we note that $f(x + y_j) - g(y_j) - c \longrightarrow \bar{f}(x)$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and that $f(x + y_j) - g(y_j) + 2\varepsilon = f(x + y_j) - f(y_j) + c_j + 2\varepsilon \geq \bar{f}(x) + c$ for all $x \in \mathbb{R}^n$. Therefore we see that $\max\{f(x + y_j), g(x + y_j)\} - g(y_j) \longrightarrow \max\{\bar{f}(x) + c, \bar{g}(x)\}$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $\max\{f(x + y_j), g(x + y_j)\} - g(y_j) + 2\varepsilon \geq \max\{\bar{f}(x) + c, \bar{g}(x)\}$ for all $x \in \mathbb{R}^n$.

The remaining case is when $\sup_{j\in\mathbb{N}} |f(y_j) - g(y_j)| = \infty$. We may assume that $c_j := f(y_j) - g(y_j) \longrightarrow \infty$ as $j \to \infty$.

We observe that $g(x+y_j)-f(y_j) = g(x+y_j)-g(y_j)-c_j \longrightarrow -\infty$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and hence that $\max\{f(x+y_j), g(x+y_j)\}-f(y_j) \longrightarrow \bar{f}(x)$ in $C(\mathbb{R}^n)$ as $j \to \infty$. We next observe that $\max\{f(x+y_j), g(x+y_j)\}-f(y_j)+\varepsilon = f(x+y_j)-f(y_j)+\varepsilon \geq \bar{f}(x)$ for all $x \in \mathbb{R}^n$.

Thus we have checked in both cases after passing to a subsequence that for some function $h \in C(\mathbb{R}^n)$ and some sequence $\{a_j\} \subset \mathbb{R}^n$, $\max\{f(x+y_j), g(x+y_j)\}-a_j \longrightarrow h(x)$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $\max\{f(x + y_j), g(x + y_j)\} - a_j + 2\varepsilon \geq h(x)$ for all $x \in \mathbb{R}^n$, and we conclude that $\max\{f, g\}$ is obliquely lower semi-almost periodic. $\square$

Appendix C: Notes on condition (A6) or $(A4)_0$.

We show here that, for any function $H(x,p)$ which is periodic in x and satifies (A1)-(A3), condition (H5) in [2] is equivalent to (A6), which is a generalization of $(A4)_0$. A warning is that we are here assuming the convexity of H while it is not assumed in [2]. Also, we show that, under (A1)-(A3), condition $(A4)_0$ is equivalent to the condition that the corresponding Lagrangian L satisfies (11) for all $\varepsilon \in (0, \delta_1]$ and $(x,\xi) \in S$ and for some modulus ω_1.

Let us recall condition (H5) of [2]:

(H5) There exists a closed set $K \subset \mathbb{R}^n$ having the properties (a) and (b):
(a) $\min_{p\in\mathbb{R}^n} H(x,p) = 0$ for all $x \in K$.
(b) For each $\varepsilon > 0$ there exists a modulus ψ_ε satisfying $\psi_\varepsilon(r) > 0$ for all $r > 0$ such that for all $x, p, q \in \mathbb{R}^n$, if $\mathrm{dist}(x,K) \geq \varepsilon$, $H(x,p) \leq 0$ and $H(x,p+q) \geq \nu$, then

$$H(x, p + (1+t)q) \geq (1+t)H(x, p+q) + t\psi_\varepsilon(\nu) \quad \text{for all } t \geq 0.$$

Theorem C.1. *Let $H(x,p)$ be a function on $\mathbb{R}^{2n}$ periodic in x with period $\mathbb{Z}^n$. Assume that H satisfies (A1)-(A3). Then H satisfies (A6) if and only if it satisfies (H5).*

Proof. To simplify our arguments, we only prove here that two conditions (A6) and (H5), with $K = \emptyset$, are equivalent each other. Then, in (A6) or (H5), the condition $\mathrm{dist}(x,K) = \infty \geq \varepsilon$ is satisfied for any $\varepsilon > 0$. Thus, in what follows, we fix an $\varepsilon > 0$ and write ω and ψ for the moduli ω_ε in (A6) and ψ_ε in (H5), respectively.

We first show that (H5) implies (A6). For this fix $(x,p) \in \mathbb{R}^{2n}$ such that $H(x,p) = 0$. Fix $\xi \in D_2^- H(x,p)$ and $q \in \mathbb{R}^n$. By convexity (A3), we have

$$H(x, p+r) \geq \xi \cdot r + H(x,p) = \xi \cdot r \quad \text{for all } r \in \mathbb{R}^n.$$

Assume that $\xi \cdot q > 0$. Then we have $H(p + q/2) \geq \xi \cdot q/2 > 0$. Therefore, by (H5), we get

$$\begin{aligned} H(x,p+q) &= H(x, p + 2(q/2)) \geq 2H(x, p+q/2) + \psi(H(p+q/2)) \\ &\geq 2\xi \cdot \frac{q}{2} + \psi(\xi \cdot q/2) = \xi \cdot q + \psi(\xi \cdot q/2). \end{aligned}$$

On the other hand, if $\xi \cdot q \leq 0$, then we get immediately $H(x,p+q) \geq \xi \cdot q = \xi \cdot q + \psi((\xi \cdot q)_+/2)$. Thus, setting $\omega(t) := \psi(t/2)$, we conclude that $H(x,p+q) \geq \xi \cdot q + \omega((\xi \cdot q)_+)$ and (H5) holds.

Next, we show that (A6) implies (H5). Thus we suppose that (A6) is satisfied.

We observe that convexity (A3) is equivalent to the condition that

$$H(x, p + (1 + t)q) \geq (1 + t)H(x, p + q) - tH(x, p) \text{ for all } x, p, q \in \mathbb{R}^n,\ t \geq 0. \quad (33)$$

Indeed, the above inequality can be rewritten as

$$H(x, p+q) \leq \frac{1}{1 + t} H(x, p+(1+t)q) + \frac{t}{1 + t} H(x, p) \text{ for all } x, p, q \in \mathbb{R}^n,\ t \geq 0, \quad (34)$$

which is clearly equivalent to (A3).

Fix any $\nu > 0$. We show that there is a constant $\theta \in (0, 1)$ such that for all $(x, p, q) \in \mathbb{R}^n \times \mathbb{R}^{2n}$, if $H(x, p) \leq 0$ and $H(x, p + q) = \nu$, then

$$H(x, p + (1 + t)q) = 2\nu \quad \text{for some } t \in (0, \theta]. \quad (35)$$

We henceforth regard H as a function on $\mathbb{T}^n \times \mathbb{R}^n$, so that $\{(x, p) \mid H(x, p) \leq a\}$ is a compact set for any $a \in \mathbb{R}$. To see (35), we set $W := \{(x, p, q) \in \mathbb{T}^n \times \mathbb{R}^{2n} \mid H(x, p) \leq 0,\ H(x, p + q) = \nu\}$ and observe by coercivity (A2) that W is a compact set. Note by (33) that, for any $(x, p, q) \in W$, we have $H(x, p + (1 + t)q) \geq (1 + t)H(x, p + q) > 2\nu$ for $t > 1$ and $H(x, p + q) = \nu$ and hence $H(x, p + (1 + t)q) = 2\nu$ for some $t \in (0, 1]$. If $W = \emptyset$, then $H(x, p) > 0$ for all (x, p) and we have nothing to prove. We may thus assume that $W \neq \emptyset$. We set

$$\theta = \max\{t \in (0, 1] \mid (x, p, q) \in W,\ H(x, p, +(1 + t)q) = 2\nu\},$$

which is well-defined since W is nonempty and compact. It is clear that $\theta \in (0, 1]$.

We prove that $\theta < 1$, which allows us to conclude the existence of $\theta \in (0, 1)$ for which (35) holds. We argue by contradiction, and thus suppose that $\theta = 1$. Then we can choose a $(x, p, q) \in W$ such that $H(x, p+2q) = 2\nu$. By convexity (A3), there is a $\xi \in D_2^- H(x, p + q)$ and we have

$$H(x, p + q + r) \geq H(x, p + q) + \xi \cdot r \quad \text{for all } r \in \mathbb{R}^n. \quad (36)$$

Plugging $r = q$ and $r = -q$ into this, we find that $\xi \cdot q = \nu$ and $H(x, p) = 0$. Now, by (36), we get $H(x, p + r) \geq \nu + \xi \cdot (r - q) = \xi \cdot r$ for all $r \in \mathbb{R}^n$, which says that $\xi \in D_2^- H(x, p)$. Consequently, using (A6), we find that $\nu = H(x, p + q) \geq H(x, p) + \xi \cdot q + \omega((\xi \cdot q)_+) = \nu + \omega(\nu)$, which is a contradiction. We hence conclude that $\theta < 1$.

Fix $\nu > 0$ and let $\theta \in (0,1)$ be such that (35) holds. We prove that for all $(x,p,q) \in W$, $\tau \in (0,\theta]$ such that $H(x,p+(1+\tau)q) = 2\nu$, and $t \geq 0$,

$$H(x,p+(1+t)(1+\tau)q) \geq (1+t)H(x,p+(1+\tau)q) + \frac{1-\theta}{\theta}\nu t, \tag{37}$$

from which one easily deduces that (H5) holds.

We thus fix $(x,p,q) \in W$ and $\tau \in (0,\theta]$ so that $H(x,p+(1+\tau)q) = 2\nu$. We choose $\xi \in D_2^- H(x,p+(1+\tau)q)$, so that $H(x,p+(1+t)(1+\tau)q) \geq 2\nu + t(1+\tau)\xi \cdot q$ for all $t \in \mathbb{R}$. Inserting $t = -\tau/(1+\tau)$ (i.e., $(1+t)(1+\tau) = 1$) into the above assures that $\xi \cdot q \geq \nu/\tau$. Using this, we observe that for any $t \geq 0$,

$$2\nu + t(1+\tau)\xi \cdot q \geq 2\nu(1+t) + \frac{1-\tau}{\tau}\nu t \geq 2\nu(1+t) + \frac{1-\theta}{\theta}\nu t,$$

which proves that (37) holds. □

Theorem C.2. *Assume that H satisfies (A1)-(A3) and that there exist a constant $\delta_1 > 0$ and a modulus ω_1 such that (11) holds for all $\varepsilon \in [0,\delta_1]$ and $(x,\xi) \in S$. Then $(A4)_0$ holds.*

This theorem together with Lemma 3.2 guarantees that $(A4)_0$ for H is equivalent to condition (11) for L.

Proof. Let $\varepsilon \in [0,\delta_1]$ and $(x,\xi) \in S$. Choose a $p \in \mathbb{R}^n$ so that $\xi \in D_2^- H(x,p)$. By (11), we get

$$L(x,(1+\varepsilon)\xi) \leq (1+\varepsilon)L(x,\xi) + \varepsilon\omega_1(\varepsilon) = (1+\varepsilon)\xi \cdot p + \varepsilon\omega_1(\varepsilon),$$

while we have

$$L(x,(1+\varepsilon)\xi) \geq (1+\varepsilon)\xi \cdot (p+q) - H(x,p+q) \quad \text{for all } q \in \mathbb{R}^n.$$

We combine these, to get

$$H(x,p+q) \geq \xi \cdot q + \varepsilon(\xi \cdot q - \omega_1(\varepsilon)) \quad \text{for all } q \in \mathbb{R}^n.$$

We define $\omega_0 \in C(\mathbb{R})$ by setting $\omega_0(r) = \max_{0 \leq \varepsilon \leq \delta_1} \varepsilon(r - \omega_1(\varepsilon))$ and observe that $\omega_0(r) = 0$ for $r \leq 0$, which implies that $\omega_0(r_+) = 0 = \omega_0(r)$ for $r \leq 0$, and $\omega_0(r) > 0$ for all $r > 0$. Moreover we have $H(x,p+q) \geq \xi \cdot q + \omega_0((\xi \cdot q)_+)$ for all $q \in \mathbb{R}^n$, which says that $(A4)_0$ holds. □

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