

Hamilton-Jacobi Equations with Discontinuous Hamiltonians on Arbitrary Open Sets

By

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0. Introduction

In this paper we investigate the existence and uniqueness properties of viscosity solutions of first-order nonlinear partial differential equations including Hamilton-Jacobi equations.

The notion of viscosity solution for nonlinear first-order partial differential equations was first introduced in [3] by M. G. Crandall-P. L. Lions. They obtained there quite general uniqueness results for equations of the Hamilton-Jacobi type

$$(0.1) \quad H(x, u(x), Du(x)) = 0 \quad \text{in } \Omega$$

and

$$(0.2) \quad u_t(t, x) + H(t, x, u(t, x), Du(t, x)) = 0 \quad \text{in } (0, T) \times \Omega,$$

together with existence results for model equations, P. L. Lions [13, 14] and P. E. Souganidis [16] made efforts to generalize the existence results, and G. Barles [1] finally obtained general existence theorems. Meanwhile L. C. Evans-H. Ishii [6] introduced a localization technique for (0.1) and obtained also an existence result similar to but less general than [1].

Those results are mainly concerned with solutions in the space of bounded, uniformly continuous functions, and it was first observed by the author [9] that the main uniqueness theorems are valid for the class of unbounded solutions which are (roughly speaking) uniformly continuous in the variables x . Later the author [10] obtained independently of work of G. Barles [1] general existence results for (0.1) and (0.2) with $\Omega = \mathbb{R}^N$ in this class of solutions where the assumptions on H are the same as those for the uniqueness assertions (this is the case with the results in [1]). See also M. G. Crandall-P. L. Lions [4] where similar results are obtained independently. We should mention here that there is much classical literature related to existence and uniqueness of generalized solutions for (0.1) or (0.2). See, e.g., A. Douglis [5],

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S. N. Kružkov [11, 12], W. H. Fleming [7], A. Friedman [8], and P. L. Lions [13].

Our purpose here is to prove uniqueness and existence theorems for (0.1) and (0.2) in the class of unbounded solutions mentioned above. One of advances here beyond [9, 10] is the realization of the fact that one does not need to assume a solution of (0.1) or (0.2) to grow at most linearly as $|x| \rightarrow \infty$ for its uniqueness. See [10] where we assumed $\Omega = \mathbb{R}^N$ to deduce that a uniformly continuous (in x) solution grows at most linearly as $|x| \rightarrow \infty$. The other lies in that we do not require H to be continuous. In fact, except some uniform continuity assumptions on H in its last variables and x our requirements on H are a monotonicity condition and the integrability in t . We need, however, to modify the definition of viscosity solution (see Definitions 1.1 and 7.1 below).

The plan of this paper is as follows. In §1 the definition of viscosity solution of (0.1) and some of its basic properties are provided. We present our comparison results for (0.1) in §2. In §3 we give an estimate of the modulus of continuity of viscosity solutions of (0.1). In §4 an existence theorem for (0.1) is proved in the case $\Omega = \mathbb{R}^N$. In §5 a rough observation concerning the asymptotic behavior of viscosity solutions of (0.1) as $|x| \rightarrow \infty$ is presented. Sections 3, 4, and 5 have a nature of preparations for §6, and we present our main existence result for (0.1) in §6.

Sections 7, 8, 9, and 10 are concerned with the Cauchy problem for (0.2). The definition of viscosity solution of (0.2) and some of its properties are presented in §7. Section 8 is devoted to prove a comparison result for (0.2). Section 9 contains estimates of the modulus of continuity of viscosity solutions of (0.2) and a result concerning the asymptotic behavior of viscosity solutions of (0.2) as $|x| \rightarrow \infty$. The final section §10 is devoted to prove a general existence theorem for (0.2).

We will use the following notations. For $x \in \mathbb{R}^N$ and $r \geq 0$, $B(x; r)$ denotes the closed ball in $\mathbb{R}^N$ of radius r and center x . For $\varepsilon > 0$ and $x \in \mathbb{R}^N$, $\langle x \rangle_\varepsilon$ stands for $(|x|^2 + \varepsilon^2)^{1/2}$. For $r \in \mathbb{R}$, r_+ denotes $\max\{r, 0\}$ and, for a function f , f_+ denotes the function: $x \rightarrow f(x)_+$. For $S \subset \mathbb{R}^N$, $BUC(S)$ and $UC(S)$ denote the spaces of all bounded, uniformly continuous functions on S and of all uniformly continuous functions on S , respectively.

1. Viscosity solutions of stationary problems

Let Ω be an open subset of $\mathbb{R}^N$. In this section we consider the stationary problem

$$(1.1) \quad H(x, u(x), Du(x)) = 0 \quad \text{in } \Omega,$$

where $H: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a given function $u: \Omega \rightarrow \mathbb{R}$ is the unknown, and D denotes the gradient operator, $(\partial/\partial x_1, \dots, \partial/\partial x_N)$. The function H is called the *Hamiltonian*.

Let $x_0 \in \Omega$, $u_0 \in \mathbb{R}$, and $p_0 \in \mathbb{R}^N$. We define

$$(1.2) \quad H^+(x_0, u_0, p_0) = \{G \in C(\Omega \times R^N) \mid G(x, p) \geq H(x, u, p) \text{ for all } x \in B(x_0; \delta), u \in B(u_0; \delta), p \in B(p_0; \delta), \text{ and some } \delta > 0\}$$

and

$$(1.3) \quad H^-(x_0, u_0, p_0) = \{G \in C(\Omega \times R^N) \mid G(x, p) \leq H(x, u, p) \text{ for all } x \in B(x_0; \delta), u \in B(u_0; \delta), p \in B(p_0; \delta), \text{ and some } \delta > 0\}.$$

DEFINITION 1.1. Let $u \in C(\Omega)$. (i) u is called a *viscosity subsolution* (resp., *supersolution*) of (1.1) if $G^-(x_0, D\phi(x_0)) \leq 0$ (resp., $G^+(x_0, D\phi(x_0)) \geq 0$) holds for any $\phi \in C^1(\Omega)$, $x_0 \in \Omega$ and $G^- \in H^-(x_0, u_0, p_0)$ (resp., $G^+ \in H^+(x_0, u_0, p)$), where $u_0 = u(x_0)$ and $p_0 = D\phi(x_0)$, such that $u - \phi$ attains a local maximum (resp., minimum) at x_0 . (ii) u is called a *viscosity solution* of (1.1) if it is both a viscosity sub- and supersolution of (1.1).

We mention that, when $H \in C(\Omega \times R \times R^N)$, this definition coincides with the original one in M. G. Crandall-P. L. Lions [3] (see also M. G. Crandall-L. C. Evans-P. L. Lions [2]).

In what follows we use the observation that "local maximum (resp., minimum)" in the above definition can be replaced by "local strict maximum (resp., minimum)". Indeed, for $u \in C(\Omega)$, $\phi \in C^1(\Omega)$, and $\zeta \in C^1(\Omega)$, if $u - \phi$ attains a local maximum (resp., minimum) at $x_0 \in \Omega$ and ζ attains a local strict maximum (resp., minimum) at x_0 , then $u - \phi + \zeta$ attains a local strict maximum (resp., minimum) at x_0 and $D(\phi - \zeta)(x_0) = D\phi(x_0)$. Thus, e.g., if $G^-(x_0, D\phi(x_0)) \leq 0$ for all $x_0 \in \Omega$, $\phi \in C^1(\Omega)$, and $G^- \in H^-(x_0, u(x_0), D\phi(x_0))$ such that $u - \phi$ has a local strict maximum at x_0 , then $G^-(x_0, D\phi(x_0)) \leq 0$ for all $x_0 \in \Omega$, $\phi \in C^1(\Omega)$, and $G^- \in H^-(x_0, u(x_0), D\phi(x_0))$ such that $u - \phi$ attains a local maximum at x_0 .

Viscosity solutions have a nice stability property.

PROPOSITION 1.1. Let H and H_n be functions on $\Omega \times R \times R^N$ for all $n \in N$. Let $u, u_n \in C(\Omega)$ for all $n \in N$. If u_n is a viscosity subsolution (resp., supersolution) of $H_n(x, u_n, Du_n) = 0$ in Ω for all $n \in N$, $u_n(x) \rightarrow u(x)$ uniformly on all compact subsets of Ω and $H_n(x, r, p) \rightarrow H(x, r, p)$ uniformly on compact subsets of $\Omega \times R \times R^N$ as $n \rightarrow \infty$, then u is a viscosity subsolution (resp., supersolution) of (1.1).

Proof. We prove this assertion only for subsolution and omit the proof for supersolution as the proof for supersolution is similar to that for subsolution.

Let $\phi \in C^1(\Omega)$ and assume that $u - \phi$ has a local strict maximum at $x_0 \in \Omega$. Set $u_0 = u(x_0)$ and $p_0 = D\phi(x_0)$, and let $G^- \in H^-(x_0, u_0, p_0)$. By definition there exists a $\delta > 0$ so that

$$G(x, p) \leq H(x, u, p)$$

for all $x \in B(x_0; \delta)$, $p \in B(p_0; \delta)$, and $u \in B(u_0; \delta)$. We see also that $u_n - \phi$ attains a local maximum near x_0 for sufficiently large n , say, at x_n for all $n \geq n_0 \in N$, where $x_n \rightarrow x_0$ as $n \rightarrow \infty$.

Set $K = B(x_0; \delta) \times B(u_0; \delta) \times B(p_0; \delta)$,

$$\alpha_n = \sup_{(x, r, p) \in K} |H_n(x, r, p) - H(x, r, p)|,$$

and $\beta_n = |u(x_0) - u_n(x_n)|$ for $n \geq n_0$. By assumption $\alpha_n \rightarrow 0$ and $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. By (1.2) we have $G(x, p) - \alpha_n \leq H_n(x, u, p)$ for $(x, u, p) \in K$. As $H_n(x, u_n, Du_n) \leq 0$ in Ω in the viscosity sense, we thus find that $G(x_n, D\phi(x_n)) - \alpha_n \leq 0$ if $n \geq n_0$ is so large that $|x_n - x_0| < \delta$, $\beta_n < \delta$, and $|D\phi(x_n) - p_0| < \delta$. Now send $n \rightarrow \infty$, to obtain $G(x_0, D\phi(x_0)) \leq 0$ and to finish the proof. Q.E.D.

We now state a few of basic properties of viscosity solutions without proof.

PROPOSITION 1.2. Assume that H is locally bounded on $\Omega \times \mathbb{R} \times \mathbb{R}^N$. If $u \in C(\Omega)$ is a viscosity subsolution (resp., supersolution) of (1.1) and Ω' is an open subset of Ω , then u is a viscosity subsolution (resp., supersolution) of (1.1) in Ω' .

PROPOSITION 1.3. If $u \in C(\Omega)$ and, for any $x \in \Omega$, there exists an open neighborhood $\Omega_x \subset \Omega$ such that u is a viscosity subsolution (resp., supersolution) of (1.1) in Ω_x , then u is a viscosity subsolution (resp., supersolution) of (1.1) in Ω .

PROPOSITION 1.4. If $u, v \in C(\Omega)$ are viscosity subsolutions (resp., supersolutions) of (1.1), then so is $\max\{u, v\}$ (resp., $\min\{u, v\}$).

2. A comparison theorem for the stationary problem

Here we are concerned with uniqueness of viscosity solutions of the stationary problem

$$(2.1) \quad u(x) + H(x, u(x), Du(x)) = 0 \quad \text{in } \Omega.$$

The next three assumptions will be needed.

(A1) $u \rightarrow H(x, u, p)$ is nondecreasing on $\mathbb{R}$ for all $x \in \Omega$ and $p \in \mathbb{R}^N$.

(A2) $\limsup_{r \downarrow 0} \{|H(x, u, p) - H(x, u, q)| \mid |p - q| \leq r, p, q \in B(0; R), x \in \Omega, u \in \mathbb{R}\} = 0$
for all $R > 0$.

(A3) $\limsup_{r \downarrow 0} \{|H(x, u, p) - H(y, u, p)| \mid |x - y|(1 + |p|) \leq r, x, y \in \Omega, p \in \mathbb{R}^N, u \in \mathbb{R}\} = 0$.

For a given $a > 0$ we denote by $\Gamma(a)$ the set of all nondecreasing functions $\gamma \in C([0, a])$ such that $\gamma(0) = 0$. Σ denotes the set of all functions $\sigma \in C([0, \infty) \times [0, \infty))$ such that $\sigma(\cdot, r) \in \Gamma(a)$ for all $r \geq 0$ and $a > 0$ and $r \rightarrow \sigma(s, r)$ is nondecreasing on $[0, \infty)$ for all $s \geq 0$. For $\sigma \in \Sigma$, $\hat{\sigma}$ will denote the function: $r \rightarrow \sigma(r, r)$ on $[0, \infty)$.

We note that (A2) and (A3) are equivalent to the existence, respectively, of $\sigma \in \Sigma$ such that

$$(2.2) \quad |H(x, u, p) - H(x, u, q)| \leq \sigma(|p - q|, R)$$

for all $p, q \in B(0; R)$, $x \in \Omega$, $u \in R$, and $R > 0$ and of $\gamma \in \Gamma(a)$, with $a > 0$, such that

$$(2.3) \quad |H(x, u, p) - H(y, u, p)| \leq \gamma(|x - y|(1 + |p|))$$

for all $x, y \in \Omega$, $u \in R$, and $p \in R^N$ satisfying $|x - y|(1 + |p|) \leq a$.

In the following we agree that we are given $\sigma \in \Sigma$, $a > 0$, and $\gamma \in \Gamma(a)$ such that (2.2) and (2.3) hold without any reference when we assume (A2) and (A3).

By virtue of (A1) there exist the limits

$$H(x, u \pm 0, p) = \lim_{\epsilon \downarrow 0} H(x, u \pm \epsilon, p)$$

for all $x \in \Omega$, $u \in R$, and $p \in R^N$. Note that $H(\cdot, \cdot \pm 0, \cdot)$ satisfy (A1)-(A3) again and, in particular, $H(\cdot, u \pm 0, \cdot)$ belong to $C(\Omega \times R^N)$ for all $u \in R$.

PROPOSITION 2.1. *Under the assumptions (A1)-(A3), $u \in C(\Omega)$ is a viscosity sub-solution (resp., supersolution) of (2.1) if and only if*

$$(2.4) \quad \begin{aligned} u(x_0) + H(x_0, u(x_0) - 0, D\phi(x_0)) &\leq 0 \\ (\text{resp., } u(x_0) + H(x_0, u(x_0) + 0, D\phi(x_0)) &\geq 0) \end{aligned}$$

for all $\phi \in C^1(\Omega)$ and $x_0 \in \Omega$ such that $u - \phi$ attains a local maximum at x_0 .

Proof. We will prove this assertion only for subsolution. $H^-(x_0, u_0, p_0)$ denotes the set defined by (1.3) for the Hamiltonian: $(x, u, p) \rightarrow u + H(x, u, p)$.

Assume $u \in C(\Omega)$ is a viscosity subsolution of (2.1). Let $\phi \in C^1(\Omega)$ and assume that $u - \phi$ has a local maximum at some $x_0 \in \Omega$. Setting $u_0 = u(x_0)$ and $p_0 = D\phi(x_0)$, we see from (A1)-(A3) that $(x, p) \rightarrow u_0 - \delta + H(x, u_0 - \delta, p)$ belongs to $H^-(x_0, u_0, p_0)$ for any $\delta > 0$. Hence, by the definition of viscosity subsolutions of (2.1), we have

$$u(x_0) - \delta + H(x_0, u(x_0) - \delta, D\phi(x_0)) \leq 0$$

for all $\delta > 0$. Thus, sending $\delta \downarrow 0$, we obtain (2.4).

The "if" part of the assertion follows immediately from the observation that if $G \in H^-(x_0, u_0, p_0)$ for some $(x_0, u_0, p_0) \in \Omega \times R \times R^N$ then $G(x_0, u_0, p_0) \leq u_0 + H(x_0, u_0 - 0, p_0)$.

Q.E.D.

Our comparison result is this

THEOREM 2.1. *Assume (A1)-(A3). If $u, v \in UC(\bar{\Omega})$ are, respectively, a subsolution and a supersolution of (1.1) in the viscosity sense, then*

$$(2.5) \quad \sup_{\bar{\Omega}} (u - v) \leq \max \{0, \sup_{\partial\Omega} (u - v)\}.$$

This theorem yields obviously a uniqueness result for a viscosity solution of (2.1) in the class $UC(\bar{\Omega})$.

As a result of this theorem we have

COROLLARY 2.1. Assume (A1)-(A3). If $u, v \in UC(\bar{\Omega})$, $f, g \in C(\Omega)$, $u + H(x, u, Du) \leq f(x)$, and $v + H(x, v, Dv) \geq g(x)$ in Ω in the viscosity sense, then

$$(2.6) \quad \sup_{\bar{\Omega}} (u - v) \leq \max \left\{ \sup_{\partial\Omega} (u - v), \sup_{\bar{\Omega}} (f - g)_+ \right\}.$$

This comparison result was first obtained by M. G. Crandall-P. L. Lions [3] under the additional assumption that $u, v \in BUC(\bar{\Omega})$ and $H \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N)$ and later by H. Ishii [9] in the case when $\Omega = \mathbb{R}^N$ and $H(x, u, p)$ is independent of u . See also M. G. Crandall-P. L. Lions [4].

Proof. We may assume $\sup_{\partial\Omega} (f - g)_+ < \infty$. Then $w \in UC(\bar{\Omega})$ defined by $w(x) = u(x) - \sup_{\partial\Omega} (f - g)_+$ satisfies $w + H(x, w, Dw) \leq g(x)$ in Ω in the viscosity sense. Thus, by virtue of Theorem 2.1, we have $w - v \leq \max \{0, \sup_{\partial\Omega} (w - v)\}$ on Ω from which follows (2.6).

Q.E.D.

Proof of Theorem 2.1. In order to prove (2.5), we may assume $\sup_{\partial\Omega} (u - v) < \infty$. In view of (A1), replacing u by $u - \sup_{\partial\Omega} (u - v)$ if necessary, we may assume further that $\sup_{\partial\Omega} (u - v) \leq 0$.

Replacing $a > 0$ by a smaller one if necessary, we may assume that there exists an $\omega \in \Gamma(a)$ such that

$$|u(x) - u(y)| \leq \omega(|x - y|) \quad \text{and} \quad |v(x) - v(y)| \leq \omega(|x - y|)$$

for all $x, y \in \bar{\Omega}$ with $|x - y| \leq a$. Set

$$\bar{J}_a = \{(x, y) \in \bar{\Omega} \times \bar{\Omega} \mid |x - y| < a\}.$$

First we intend to prove that the function $u - v$ is bounded above on Ω . To see this, we observe that

$$u(x) - v(x) \leq u(0) - v(0) + 2\omega(a) \left(1 + \frac{|x|}{a} \right)$$

for all $x \in \mathbb{R}^N$ if $\Omega = \mathbb{R}^N$ and that

$$u(x) - v(x) \leq 2\omega(a) \left(1 + \frac{1}{a} \text{dist}(x, \partial\Omega) \right)$$

for all $x \in \Omega$ if $\Omega \neq \mathbb{R}^N$. Thus we have in general

$$(2.7) \quad u(x) - v(y) \leq u(x) - v(x) + \omega(a) \leq C_1(1 + |x|)$$

for all $(x, y) \in \bar{J}_a$ and some constant $C_1 > 0$.

Choose an increasing C^1 function η on $[0, a)$ such that $\eta(r) = r^2$ for $0 \leq r \leq a/2$ and $\lim_{r \uparrow a} \eta(r) = \infty$. Let $\alpha > 0$, $\beta > 0$ and set

$$\Phi(x, y) = u(x) - v(y) - \frac{1}{\alpha} \eta(|x - y|) - \beta|x|^2$$

for $(x, y) \in \bar{J}_a$. By virtue of (2.7), Φ attains its global maximum at some point $(x_0, y_0) \in \bar{J}_a$. That is,

$$(2.8) \quad \Phi(x_0, y_0) \geq \Phi(x, y) \quad \text{for all } (x, y) \in \bar{J}_a.$$

In particular, $\Phi(x_0, y_0) \geq \Phi(x_0, y_0)$ and hence

$$(2.9) \quad \frac{1}{\alpha} \eta(|x_0 - y_0|) \leq v(x_0) - v(y_0) \leq \omega(|x_0 - y_0|).$$

Thus, setting $\alpha_0 = a^2/(4\omega(a))$, we find that if $0 < \alpha < \alpha_0$, then $|x_0 - y_0| \leq a/2$ for all $\beta > 0$. Fix $x^* \in \Omega$. Again (2.8) yields $\Phi(x_0, y_0) \geq \Phi(x^*, x^*)$, and so

$$\beta|x_0|^2 \leq u(x_0) - v(y_0) - u(x^*) + v(x^*) + 2\beta|x^*|^2.$$

Hence, taking (2.7) into account, we obtain

$$(2.10) \quad \beta|x_0| \leq C_2$$

for all $0 < \beta < 1$ and some $C_2 > 0$ independent of β .

Fix $0 < \alpha < \alpha_0$, and set $R_1 = a/\alpha + 2C_2$ and $C_3 = 2\delta(R_1) + \gamma(a) + \omega(a)$. We claim that

$$(2.11) \quad \Phi(x_0, y_0) \leq C_3 \quad \text{for all } 0 < \beta < 1.$$

To see this, we fix $0 < \beta < 1$, suppose

$$(2.12) \quad \Phi(x_0, y_0) > C_3,$$

and will obtain a contradiction. It follows from (2.12) that $x_0, y_0 \in \Omega$. Indeed, supposing $x_0 \in \partial\Omega$ or $y_0 \in \partial\Omega$, we have

$$C_3 < \Phi(x_0, y_0) \leq u(x_0) - v(y_0) \leq \sup_{\partial\Omega} (u - v) + \omega(a) \leq \omega(a);$$

a contradiction. Since $x \rightarrow \Phi(x, y_0)$ has a maximum at $x_0 \in \Omega$ and u is a viscosity sub-solution of (2.1), in view of Proposition 2.1 we have

$$u(x_0) + H(x_0, u(x_0) - 0, \frac{2}{\alpha}(x_0 - y_0) + 2\beta x_0) \leq 0.$$

Similarly

$$v(y_0) + H(y_0, u(x_0) - 0, \frac{2}{\alpha}(x_0 - y_0)) \geq 0.$$

Here we used that $\eta(r) = r^2$ for $0 \leq r \leq a/2$ and $u(x_0) > v(y_0)$. Subtracting the second inequality from the first and using (2.2) and (2.3), we have

$$\begin{aligned} u(x_0) - v(y_0) &\leq 2\delta(R_1) + H(y_0, u(x_0) - 0, 0) - H(x_0, u(x_0) - 0, 0) \\ &\leq 2\delta(R_1) + \gamma(|x_0 - y_0|) \leq 2\delta(R_1) + \gamma(a). \end{aligned}$$

Thus we obtain $\Phi(x_0, y_0) \leq C_3$, a contradiction, proving (2.11). It is easy to see from

(2.11) that $u-v$ is bounded above by C_δ on Ω .

Now we repeat an argument similar to the above to complete the proof. Suppose that (2.5) is false, and choose $0 < \varepsilon_0 < 1$ so that

$$(2.13) \quad \sup_{\bar{\Omega}} (u-v) > 3\varepsilon_0.$$

Let $\alpha > 0$, $0 < \varepsilon < \varepsilon_0$, and choose a point $(x_1, y_1) \in \bar{J}_\alpha$ so that

$$(2.14) \quad \sup_{\bar{J}_\alpha} (u(x) - v(y) - \frac{1}{\alpha} \eta(|x-y|)) - \varepsilon < u(x_1) - v(y_1) - \frac{1}{\alpha} \eta(|x_1 - y_1|).$$

This yields in particular

$$(2.15) \quad \sup_{\bar{\Omega}} (u-v) - \varepsilon < u(x_1) - v(y_1) \leq \min \{u(x_1) - v(x_1), u(y_1) - v(y_1)\} + \omega(|x_1 - y_1|).$$

It follows also from (2.14) that

$$u(x_1) - v(x_1) - \varepsilon < u(x_1) - v(y_1) - \frac{1}{\alpha} \eta(|x_1 - y_1|),$$

and hence

$$\eta(|x_1 - y_1|) < \alpha \{\omega(|x_1 - y_1|) + \varepsilon\}.$$

Thus there exists an $\alpha_1 > 0$ such that $|x_1 - y_1| \leq a/2$ and $\omega(|x_1 - y_1|) < \varepsilon_0$ for all $0 < \alpha < \alpha_1$ and $0 < \varepsilon < \varepsilon_0$. The inequalities (2.13) then imply that $\text{dist}(x_1, \partial\Omega) > \delta$ and $\text{dist}(y_1, \partial\Omega) > \delta$ for some $\delta > 0$.

Let $\zeta \in C_0^1(\mathbb{R}^{2N})$ be a function satisfying

$$\begin{cases} \zeta(x_1, y_1) = 1, & 0 \leq \zeta \leq 1, \text{ supp } \zeta \subset B((x_1, y_1); \delta), \\ \text{and } |D\zeta| \leq C_4 \text{ for some constant } C_4 > 0. \end{cases}$$

For $0 < \varepsilon < a/2$ choose $0 < \alpha_2 < \alpha_1$ so that $\sqrt{\alpha_2(\omega(a) + \varepsilon)} + 2\omega(\sqrt{\alpha_2(\omega(a) + \varepsilon)}) + 2\varepsilon < a$. Fix $0 < \varepsilon < \min\{\varepsilon_0, a/2\}$ and $0 < \alpha < \alpha_2$. Set

$$\Psi(x, y) = u(x) - v(y) - \frac{1}{\alpha} \eta(|x-y|) + \varepsilon \zeta(x, y)$$

on $\bar{J}_\alpha$. According to (2.14) Ψ attains a global maximum at some point $(x_2, y_2) \in \bar{J}_\alpha \cap \text{supp } \zeta$. Notice that $x_2, y_2 \in \Omega$. As usual we have

$$(2.16) \quad \begin{cases} \eta(|x_2 - y_2|) \leq \alpha \{\omega(|x_2 - y_2|) + \varepsilon\}, \\ \frac{1}{\alpha} |x_2 - y_2|^2 \leq \omega(\sqrt{\alpha(\omega(a) + \varepsilon)}) + \varepsilon. \end{cases}$$

Appealing that u and v are, respectively, a subsolution and a supersolution of (2.1) in the viscosity sense, we find that

$$u(x_2) - v(y_2) \leq 2\sigma(C_4\varepsilon, R(\alpha)) + \gamma\left(\left|x_2 - y_2\right|\left(1 + \frac{2}{\alpha}\left|x_2 - y_2\right|\right)\right),$$

where $R(\alpha) = 2\sqrt{(\omega(a) + \varepsilon)/\alpha} + C_4$, and hence

$$\begin{aligned} u(x) - v(x) &\leq \Psi(x, x) \leq \Psi(x_2, y_2) \\ &\leq 2\sigma(C_4\varepsilon, R(\alpha)) + \gamma\left(\left|x_2 - y_2\right|\left(1 + \frac{2}{\alpha}\left|x_2 - y_2\right|\right)\right) + \varepsilon \end{aligned}$$

for all $x \in \Omega$. In view of (2.16), sending $\varepsilon \downarrow 0$ and $\alpha \downarrow 0$ in this order, we obtain

$$u(x) - v(x) \leq 0 \quad \text{for all } x \in \Omega.$$

This contradiction to (2.13) proves (2.5).

Q.E.D.

3. Modulus of continuity of viscosity solutions

We begin this section by defining an auxiliary function.

LEMMA 3.1. *Let $a > 0$ and $\gamma, \gamma_0 \in \Gamma(a)$. Assume that $\gamma_0 \in C^1((0, a])$, $s \rightarrow s + s\gamma_0'(s)$ is nondecreasing on $(0, a]$ and $\gamma_0'(a) = 0$. Let γ_1 be any uniformly continuous extension of γ to R . Then there exists a unique viscosity solution $\phi \in C([0, a])$ of*

$$(3.1) \quad \begin{cases} \phi(s) - \gamma_1(s + s\phi'(s)) - \gamma_0(s) = 0 & \text{in } (0, a), \\ \phi(0) = 0, & \phi(a) = \gamma(a) + \gamma_0(a). \end{cases}$$

Moreover $\phi \in \Gamma(a)$, $\phi \geq \gamma + \gamma_0$ on $[0, a]$, and ϕ is Lipschitz continuous on $[\varepsilon, a]$ for any $0 < \varepsilon < a$. In addition, if γ is increasing on $[0, a]$, then $\phi \in C^1((0, a])$, $s \rightarrow s + s\phi'(s)$ is nondecreasing on $(0, a]$, and $\phi'(a) = 0$.

REMARK 3.1. The proof below shows that ϕ does not depend on the extension γ_1 of γ and is determined only by $\gamma \in \Gamma(a)$.

Proof. The function: $(x, u, p) \rightarrow -\gamma_1(x + xp) - \gamma_0(x)$ satisfies (A1)–(A3) for $\Omega = (0, a)$ and $N = 1$. Thus the uniqueness follows from Theorem 2.1.

To prove the existence of a solution of (3.1), we consider first the case when γ is increasing on $(0, a]$ and show the existence of a classical solution of

$$(3.2) \quad \begin{cases} \phi(s) - \gamma(s + s\phi'(s)) - \gamma_0(s) = 0 & \text{for all } 0 < s \leq a, \\ \phi(a) = \gamma(a) + \gamma_0(a). \end{cases}$$

We observe that (3.2) for $\phi \in C^1((0, a])$ is equivalent to

$$(3.3) \quad \begin{cases} \gamma'(t) + \gamma^{-1}(\gamma(t)) = e^{-t}(1 + \gamma_0'(e^{-t})) & \text{for all } t \geq b, \\ \gamma(b) = \gamma(a) \end{cases}$$

for $\gamma(t) = \phi(e^{-t}) - \gamma_0(e^{-t})$ and $b = -\log a$. We extend γ^{-1} to R so that the extended function (which we denote again by γ^{-1}) is continuous and increasing on R . The problem (3.3), with the extended γ^{-1} , has a unique solution $\gamma \in C^1([b, \infty))$.

Set $f(t) = e^{-t}(1 + \gamma_0'(e^{-t}))$ for $t \geq b$. We now claim that $\gamma^{-1}(y(t)) \geq f(t)$ for all $t \geq b$. To see this, we suppose that there exists a $t_1 > b$ such that $\gamma^{-1}(y(t_1)) < f(t_1)$. Taking into account that $\gamma^{-1}(y(b)) = e^{-b} = f(b)$, we see that there is a $t_0 \in [b, t_1)$ such that $\gamma^{-1}(y(t_0)) = f(t_0)$ and $\gamma^{-1}(y(t)) < f(t)$ for all $t_0 < t \leq t_1$. From this it follows that $y'(t) > 0$ for $t_0 < t \leq t_1$. Hence we obtain $y(t_0) < y(t_1)$ and so $f(t_0) = \gamma^{-1}(y(t_0)) < \gamma^{-1}(y(t_1)) < f(t_1)$. This contradicts the monotonicity of f , which proves that $\gamma^{-1}(y(t)) \geq f(t)$ and, in particular, $y(t) > 0$ for all $t \geq b$.

Thus $y'(t) = f(t) - \gamma^{-1}(y(t)) \leq 0$, and so $0 < y(t) \leq \gamma(a)$ for all $t \geq b$. This implies that y solves the original problem (3.3). We see also that the limit $\alpha = \lim_{t \rightarrow \infty} y(t)$ exists. Moreover it follows that

$$0 = \lim_{t \rightarrow \infty} (y(t+1) - y(t)) = \lim_{t \rightarrow \infty} \int_0^1 \{f(t+s) - \gamma^{-1}(y(t+s))\} ds = \gamma^{-1}(\alpha)$$

since $\lim_{t \rightarrow \infty} f(t) = 0$. Therefore $\lim_{t \rightarrow \infty} y(t) = 0$.

Set $\phi(s) = y(-\log s) + \gamma_0(s)$ for $0 < s \leq a$ and $\phi(0) = 0$. Then $\phi \in C([0, a]) \cap C^1((0, a])$ and

$$\phi'(s) - \gamma_0'(s) = -\frac{1}{s} y'(-\log s) = -\frac{1}{s} \{s + s\gamma_0'(s) - \gamma^{-1}(\phi(s) - \gamma_0(s))\}$$

for all $0 < s \leq a$. This shows that $\phi'(a) - \gamma_0'(a) \geq 0$, $0 \leq s + s'(s) \leq a$, and $\phi(s) = \gamma(s + s\phi'(s)) + \gamma_0(s)$ for all $0 < s \leq a$. It is now obvious that (3.2) is satisfied for this ϕ , $s \rightarrow s + s\phi'(s)$ is nondecreasing on $(0, a]$, $\phi \geq \gamma + \gamma_0$ on $[0, a]$, and $\phi'(a) = 0$.

Before turning to the general case we look into ϕ further. Note first that

$$(3.4) \quad s \leq s + s\phi'(s) \leq a \quad \text{for all } 0 < s \leq a.$$

Integrating the first inequality in (3.3) over $[b, c]$, with $c > b$, and sending $c \rightarrow \infty$, we have

$$\int_b^\infty \gamma^{-1}(y(t)) dt = \phi(a) + a.$$

From this and the fact that $y'(t) \leq 0$ for all $t > b$ it follows that if $y(c) = \gamma(\varepsilon)$ for some $c > b$ and $\varepsilon > 0$ then $\varepsilon(c-b) \leq \phi(a) + a$. This implies that

$$(3.5) \quad 0 \leq \phi(s) - \gamma_0(s) \leq \gamma(\varepsilon)$$

for all $0 \leq s < a \exp\{-(\phi(a) + a)/\varepsilon\}$ and $0 < \varepsilon \leq a$.

The proof of existence in the general case is now carried out by approximating γ , reducing to the previous case, and passing to the limit. We can then use (3.4) and (3.5) to show the compactness of the family of approximate solutions in $C([0, a])$.

Q.E.D.

LEMMA 3.2. Let $a > 0$ and $\gamma \in \Gamma(a)$. Let γ_1 be any uniformly continuous extension of γ to $\mathbb{R}$ and $\phi \in \Gamma(a)$ the unique viscosity solution of (3.1) with $\gamma_0 = 0$. Let $h \in C(\mathbb{R}^{2N})$

satisfy $h(p, -p) = 0$ for all $p \in \mathbb{R}^N$. Define $A_a = \{(x, y) \in \mathbb{R}^{2N} \mid |x - y| < a\}$ and $w \in UC(\bar{A}_a)$ by $w(x, y) = \phi(|x - y|)$. Then

$$(3.6) \quad w - \gamma_1(|x - y|(1 + |D_x w|)) + h(Dw) \geq 0 \quad \text{in } A_a$$

in the viscosity sense, where $D = (D_x, D_y)$.

Proof. Let $\phi \in C^1(A_a)$ and $(x_0, y_0) \in A_a$. Assume that $w - \phi$ has a local minimum at (x_0, y_0) . We want to show that

$$(3.7) \quad w(x_0, y_0) - \gamma_1(|x_0 - y_0|(1 + |D_x \phi(x_0, y_0)|)) + h(D\phi(x_0, y_0)) \geq 0$$

holds.

First we note that $w(x_0 + x, y_0 + x) = w(x_0, y_0)$ for all $x \in \mathbb{R}^N$ and hence the function: $x \rightarrow -\phi(x_0 + x, y_0 + x)$ attains a local minimum at $x = 0$. Hence

$$(3.8) \quad D_x \phi(x_0, y_0) + D_y \phi(x_0, y_0) = 0.$$

This means together with the assumption on h that the last term on the right side of (3.7) always vanishes. Since $\phi(0) = \gamma_1(0) = 0$, if $x_0 = y_0$ then (3.7) holds.

We consider the case $x_0 \neq y_0$ in what follows. Set $e = (x_0 - y_0)/|x_0 - y_0|$ and $r_0 = |x_0 - y_0|$. Let $\xi \in \mathbb{R}^N$ be orthogonal to e . Then $|x_0 - y_0 + r\xi|^2 = r_0^2 + r^2|\xi|^2$ for all $r \in \mathbb{R}$. Recall that ϕ is Lipschitz continuous on $[r_0/2, a]$. Therefore $\frac{d}{dr}\phi((r_0^2 + r^2|\xi|^2)^{1/2}) = 0$ at $r = 0$.

On the other hand, $r \rightarrow (w - \phi)\left(x_0 + \frac{r}{2}\xi, y_0 - \frac{r}{2}\xi\right)$ has a local minimum at $r = 0$. Thus, differentiating this function at $r = 0$, we obtain

$$(D_x \phi(x_0, y_0) - D_y \phi(x_0, y_0)) \cdot \xi = 0.$$

From this, using (3.8), we have

$$(3.9) \quad D_x \phi(x_0, y_0) \cdot \xi = 0$$

for all $\xi \in \mathbb{R}^N$ which is orthogonal to e .

Now we observe that $|x_0 - y_0 + re| = |r_0 - r|$ for all $r \in \mathbb{R}$. Hence $w\left(x_0 + \frac{r}{2}e, y_0 - \frac{r}{2}e\right) = \phi(r_0 + r)$ for all $r \in \mathbb{R}$ with $0 \leq r_0 + r < a$, and so $r \rightarrow \phi(r_0 + r) - \phi\left(x_0 + \frac{r}{2}e, y_0 - \frac{r}{2}e\right)$ has a local minimum at $r = 0$.

That is,

$$\phi(r_0 + r) - \phi\left(x_0 + \frac{r}{2}e, y_0 - \frac{r}{2}e\right) \geq \phi(r_0) - \phi(x_0, y_0)$$

or

$$\phi(r_0 + r) - \phi(r_0) \geq \phi\left(x_0 + \frac{r}{2}e, y_0 - \frac{r}{2}e\right) - \phi(x_0, y_0)$$

if $|r|$ is sufficiently small. Divide the latter by $r < 0$ and let $r \uparrow 0$, to get

$$D_s \phi(x_0, y_0) \cdot e \geq 0 \quad \text{in view of (3.8).}$$

This and (3.9) implies that

$$(3.19) \quad D_s \phi(x_0, y_0) \cdot e = |D_s \phi(x_0, y_0)|.$$

Using the definition of viscosity supersolution of (3.1) and the fact that $r \rightarrow \phi(r) - \phi\left(x_0 + \frac{1}{2}(r-r_0)e, y_0 - \frac{1}{2}(r-r_0)e\right)$ has a local minimum at $r=r_0$, we have

$$\phi(r_0) - r_1(r_0(1 + |D_s \phi(x_0, y_0)|)) \geq 0$$

by virtue of (3.8) and (3.10). As remarked above, this is equivalent to (3.7). *Q.E.D.*

Our main result here is the following

THEOREM 3.1. *Assume (A1)-(A3). Let $r \in \Gamma(a)$ satisfy (2.3) and $u \in UC(\bar{\Omega})$ be a viscosity solution of (2.1). Assume that $|u(x) - u(y)| \leq r(a)$ if $x, y \in \bar{\Omega}$ and $|x - y| \leq a$ and that $|u(x) - u(y)| \leq r(|x - y|)$ if $x, y \in \bar{\Omega}$, $|x - y| \leq a$, and either $x \in \partial\Omega$ or $y \in \partial\Omega$. Let $\phi \in \Gamma(a)$ satisfy (3.1) with $r_0 = 0$. Then*

$$(3.11) \quad |u(x) - u(y)| \leq \phi(|x - y|)$$

for all $x, y \in \bar{\Omega}$ with $|x - y| \leq a$.

Proof. Set $v(x, y) = u(x) - u(y)$ for $x, y \in \bar{\Omega}$ and define $\Delta_a = \{(x, y) \in \Omega^2 \mid |x - y| < a\}$ and $\Delta_a^+ = \{(x, y) \in \Delta_a \mid u(x) > u(y)\}$.

If $(x, y) \in \Delta_a^+$, $p, q \in R^N$, and $|x - y|(1 + |p|) \leq a$, then

$$H(x, u(x) - 0, p) - H(y, u(y) + 0, -q) \geq -r(|x - y|(1 + |p|)) - \sigma(|p + q|, |p| + |q|).$$

Define $r : [0, \infty) \rightarrow [0, \infty)$ and $\tilde{H} : \Delta_a \times R^{2N} \rightarrow R$ by

$$r_1(s) = \begin{cases} r(s) & \text{for } 0 \leq s \leq a, \\ r(a) & \text{for } s > a, \end{cases}$$

and

$$H(x, y, p, q) = -r_1(|x - y|(1 + |p|)) - \sigma(|p + q|, |p| + |q|).$$

By assumption u is a viscosity solution of (2.1) and so, taking into account the preceding observation and that $v \leq r(a)$ on Δ_a , we find that

$$(3.12) \quad v + \tilde{H}(x, y, Dv) \leq 0 \quad \text{in } \Delta_a^+$$

in the viscosity sense, where $D = (D_x, D_y)$.

According to Lemma 3.2, setting $w(x, y) = \phi(|x - y|)$ on $\bar{\Delta}_a$, we have

$$(3.13) \quad w + \tilde{H}(x, y, Dw) \geq 0 \quad \text{in } \Delta_a$$

in the viscosity sense.

By assumption we have $v \leq w$ on Δ_s^+ and therefore, by Theorem 2.1, $v \leq w$ on Δ_s^+ . Thus we have proved (3.11). Q.E.D.

4. An existence theorem in R^N

Our objective in this section is to prove this

THEOREM 4.1. *Let $\Omega = R^N$. Assume (A1)–(A3). Then there exists a (unique) viscosity solution of (2.1) belonging to $UC(R^N)$.*

The uniqueness assertion follows from Theorem 2.1.

We need the following lemmas to prove Theorem 4.1.

LEMMA 4.1. *Let $\Omega = R^N$, and assume (A1)–(A3). Let $u \in UC(R^N)$ be a viscosity solution of (2.1). Then*

$$(4.1) \quad |u(x)| \leq C_\sigma(1+|x|)$$

for all $x \in R^N$ and some constant $C_\sigma > 0$ depending only on σ in (2.2), γ in (2.3), and $H(0, 0, 0)$.

LEMMA 4.2. *Under the same hypotheses as in Lemma 4.1*

$$(4.2) \quad |u(x) - u(y)| \leq C_\sigma(1+|x-y|)$$

holds for all $x, y \in R^N$ and some constant C_σ which depends only on σ in (2.2) and γ in (2.3).

These results are essentially proved in [10]. However, for the reader's convenience, we reproduce here the proofs.

Proof of Lemma 4.1. We may assume that γ is defined on $(0, \infty)$, (2.3) is satisfied for all $x, y, p \in R^N$ and $u \in R$, and $\gamma(r) \leq C_\gamma(1+r)$ for all $r \geq 0$ and some $C_\gamma > 0$.

Let $A, B > 0$ be numbers to be selected later. Let $\varepsilon > 0$ and set $v_\varepsilon(x) = A\langle x \rangle_\varepsilon + B$ for $x \in R^N$. By virtue of (2.2) and (2.3), we have

$$\begin{aligned} v_\varepsilon(x) + H(x, v_\varepsilon(x), Dv_\varepsilon(x)) & \\ & \geq A\langle x \rangle_\varepsilon - \gamma(|x|) + B + H(0, 0, 0) - \delta(A) \\ & \geq (A - C_\gamma)\langle x \rangle_\varepsilon + B + H(0, 0, 0) - C_\gamma - \delta(A) \end{aligned}$$

for all $x \in R^N$. Thus, choosing $A = C_\gamma$ and $B = |H(0, 0, 0)| + C_\gamma + \delta(C_\gamma)$ and sending $\varepsilon \downarrow 0$, we obtain

$$v(x) + H(x, v, Dv) \geq 0 \quad \text{in } R^N$$

in the viscosity sense for $v(x) = A|x| + B$.

By virtue of Theorem 2.1 we now find that

$$(4.3) \quad u(x) \leq C_7|x| + |H(0, 0, 0)| + C_7 + \delta(C_7)$$

for all $x \in R^N$. Similarly we have

$$(4.4) \quad u(x) \geq -C_7|x| - |H(0, 0, 0)| - C_7 - \delta(C_7)$$

for all $x \in R^N$. Thus we conclude (4.1).

Q.E.D.

Proof of Lemma 4.2. As in the proof of Lemma 4.1 we assume that γ is defined on $[0, \infty)$, (2.3) is satisfied for all $x, y, p \in R^N$ and $u \in R$, and $\gamma(r) \leq C_7(1+r)$ for all $r \geq 0$ and some $C_7 > 0$.

Set $v(x, y) = u(x) - u(y)$ for $x, y \in R^N$ and $\Delta^+ = \{(x, y) \in R^{2N} | u(x) > u(y)\}$. For any $(x, y) \in \Delta^+$ and $p, q \in R^N$, by (2.2) and (2.3) we have

$$H(x, u(x) - 0, p) - H(y, u(y) + 0, -q) \geq -\gamma(|x - y|) - \delta(|p|) - \delta(|q|).$$

Therefore, setting $\tilde{H}(x, y, p, q) = -\gamma(|x - y|) - \delta(|p|) - \delta(|q|)$ and using that u is a viscosity solution of (2.1), we have

$$(4.5) \quad v(x, y) + H(x, y, Dv) \leq 0 \quad \text{in } \Delta^+$$

in the viscosity sense, where $D = (D_x, D_y)$. Notice also that $v(x, y) = 0$ on $\partial\Delta^+$.

Let $A, B > 0$ be numbers to be chosen later. For any $\epsilon > 0$ set $w_\epsilon(x, y) = A\langle x - y \rangle_\epsilon + B$ for $x, y \in R^N$. Then

$$w_\epsilon(x, y) + \tilde{H}(x, y, Dw_\epsilon) \geq (A - C_7)\langle x - y \rangle_\epsilon + B - C_7 - 2\delta(A)$$

for all $x, y \in R^N$. Choose now $A = C_7$ and $B = C_7 + 2\delta(C_7)$, and let $\epsilon \downarrow 0$, to see that

$$(4.6) \quad w + \tilde{H}(x, y, Dw) \geq 0 \quad \text{in } R^{2N}$$

in the viscosity sense for $w(x, y) = A|x - y| + B$.

In view of Theorem 2.1, comparing (4.5) and (4.6), we conclude that $v \leq w$ on R^{2N} , from which follows (4.2) for $C_\delta = C_7 + 2\delta(C_7)$. Q.E.D.

Proof of Theorem 4.1. As above we may assume $a = 1$. Define $H_n \in C(R^N \times R \times R^N)$ for $n \in N$ by

$$\begin{aligned} H_n(x, u, p) = & 2^n \left(H\left(x, \frac{k}{2^n}, p\right) - H\left(x, \frac{k-1}{2^n}, p\right) \right) \left(u - \frac{k}{2^n}\right) \\ & + H\left(x, \frac{k-1}{2^n}, p\right) \text{ if } \frac{k}{2^n} \leq u < \frac{k+1}{2^n} \text{ for some } k \in \mathbb{Z}. \end{aligned}$$

Then H_n satisfies (A1)-(A3). Notice that (2.2) and (2.3) with γ and σ replaced, respectively, by 2γ and 2σ are satisfied for $H = H_n$. It follows also that

$$(4.7) \quad H_n(x, u, p) \leq H(x, u, p)$$

and

$$(4.8) \quad H_n(x, u, p) \geq H\left(x, u - \frac{1}{2^{n-1}}, p\right)$$

for all $x, p \in R^N$ and $u \in R$.

Let $u_n \in UC(R^N)$ be the unique viscosity solution of $u + H_n(x, u, Du) = 0$ in R^N (see [10] for its existence). Setting $v_n(x) = u_n(x) - 1/2^{n-1}$ for $x \in R^N$, in view of (4.7) and (4.8) we have

$$(4.9) \quad u_n + H(x, u_n, Du_n) \geq 0 \quad \text{and} \quad v_n + H(x, v_n, Dv_n) \leq 0$$

in R^N in the viscosity sense.

According to Lemma 4.1 $\{u_n\}$ is bounded on any compact subset of R^N . From Lemma 4.2 there exists a constant $C_s > 0$ such that

$$(4.10) \quad |u_n(x) - u_n(y)| \leq C_s \quad \text{if} \quad |x - y| \leq 1 \quad \text{and} \quad n \in N.$$

Replacing γ by a new one if necessary, we may assume $2\gamma(1) \geq C_s$. Recalling that H_n satisfy (2.3) with γ replaced by 2γ , we conclude by virtue of Theorem 3.1 that

$$(4.11) \quad |u_n(x) - u_n(y)| \leq \phi(|x - y|)$$

for all $n \in N$ and some $\phi \in \Gamma(1)$ if $|x - y| \leq 1$.

Therefore $\{u_n\}$ is an equicontinuous family of functions on R^N .

Thus, applying the Ascoli-Arzelà theorem to $\{u_n\}$, we find a subsequence of $\{u_n\}$ (which we denote again by $\{u_n\}$) such that $u_n(x) \rightarrow u(x)$ uniformly on all compact subsets of R^N for some $u \in UC(R^N)$ as $n \rightarrow \infty$. By Proposition 1.1, (4.9) shows that u is a viscosity solution of (1.1). The proof is now complete. Q.E.D.

5. Asymptotic behavior of viscosity solutions

We assume Ω to be unbounded and observe the behavior of a viscosity solution of (2.1) as $|x| \rightarrow \infty$.

Assume (A1) and (A3). For $x \in \Omega$, $u \in R$, and $p \in R^N$ define $H^*(x, u, p) \subset R$ by $H^*(x, u, p) = [H(x, u - 0, p), H(x, u + 0, p)]$. Note that the graph $\{(u, v) | u \in R, v \in H^*(x, u, p)\}$ in R^2 is the unique maximal monotone extension of $\{(u, H(x, u, p)) | u \in R\}$ for any $x \in \Omega$ and $p \in R^N$.

For any $x \in \Omega$ we denote by $u^*(x)$ the unique solution of

$$(5.1) \quad u + H^*(x, u, 0) \ni 0$$

(its existence and uniqueness follows from the fact that $\{(u, v) | u \in R, v \in H^*(x, u, 0)\}$ is a maximal monotone graph in R^2). The function $u^* : \Omega \rightarrow R$ is then uniformly continuous. Indeed, for any $x, y \in \Omega$ such that $|x - y| \leq a$ and $u^*(x) > u^*(y)$, we have

$$u^*(x) - u^*(y) \leq H(y, u^*(y) + 0, 0) - H(x, u^*(x) - 0, 0) \leq \gamma(|x - y|) \quad \text{by (2.3).}$$

Hence

$$|u^*(x) - u^*(y)| \leq r(|x - y|) \quad \text{if } x, y \in \Omega \quad \text{and} \quad |x - y| \leq a.$$

THEOREM 5.1. Assume (A1)–(A3). Let $u \in UC(\Omega)$ be a viscosity solution of (2.1) and $u^* \in UC(\Omega)$ the unique solution of (5.1) in Ω . Then

$$(5.2) \quad \sup_{\bar{\Omega}} |u - u^*| < \infty.$$

Proof. There is a uniformly continuous extension of u to $\Omega_\delta \equiv \{x \in R^N \mid \text{dist}(x, \Omega) < \delta\}$ for some $\delta > 0$. Mollifying it, we find a C^1 function $\tilde{u}$ on Ω such that

$$\sup_{\bar{\Omega}} |u - \tilde{u}| \leq 1 \quad \text{and} \quad \sup_{\bar{\Omega}} |D\tilde{u}| \leq C_0$$

for some $C_0 > 0$. We may assume that there exists an $\omega \in \Gamma(a)$ such that $|\tilde{u}(x) - \tilde{u}(y)| \leq \omega(|x - y|)$ for all $x, y \in \Omega$ with $|x - y| \leq a$.

Fix any $x_0 \in \Omega$ satisfying $\text{dist}(x_0, \Omega^c) > a$. The function: $x \rightarrow u(x) - \tilde{u}(x) - 2|x - x_0|^2/a^2$ takes its global maximum at some point $x_1 \in B(x_0; a)$. By Proposition 2.1 we have

$$u(x_1) + H\left(x_1, u(x_1) - 0, D\tilde{u}(x_1) + \frac{4}{a^2}(x_1 - x_0)\right) \leq 0,$$

and hence

$$u(x_0) - \omega(a) + H(x_0, u(x_0) - \omega(a) - 0, 0) - \delta\left(C_0 + \frac{4}{a}\right) - \gamma(a) \leq 0$$

by (A1), (2.2), and (2.3). Thus, setting $v(x) = u(x) - 2\omega(a) - \delta(C_0 + 4/a) - \gamma(a)$, we obtain

$$(5.3) \quad v(x) + H(x, v(x) - 0, 0) \leq 0 \quad \text{for all } x \in \Omega.$$

Since $u^*(x) + H(x, u^*(x) + 0, 0) \geq 0$ for all $x \in \Omega$ and $r \rightarrow r + H(x, r, 0)$ is increasing for each $x \in \Omega$, we see from (5.3) that $u^*(x) \geq v(x)$ for all $x \in \Omega$.

Similarly we have

$$u^*(x) \leq u(x) + 2\omega(a) + \delta\left(C_0 + \frac{4}{a}\right) + \gamma(a)$$

for all $x \in \Omega$. Thus we conclude (5.2). Q.E.D.

REMARK 5.1. The above proof shows that if $v, w \in UC(\bar{\Omega})$ are, respectively, viscosity sub- and super-solutions of (2.1), then $\sup_{\bar{\Omega}} (v - u^*) < \infty$ and $\sup_{\bar{\Omega}} (u^* - w) < \infty$.

6. An existence theorem on a general open set

Our goal here is to prove the following general existence theorem for (2.1).

THEOREM 6.1. Assume (A1)–(A3) and that there exist $v, w \in UC(\bar{\Omega})$ such that

$$(6.1) \quad \begin{cases} v + H(x, v, Dv) \leq 0 & \text{in } \Omega, \\ w + H(x, w, Dw) \geq 0 & \text{in } \Omega, \\ v = w & \text{on } \partial\Omega \end{cases}$$

in the viscosity sense. Then there exists a (unique) viscosity solution $u \in UC(\bar{\Omega})$ of (2.1) satisfying $u = v$ on $\partial\Omega$.

The following observation is needed for the proof of Theorem 6.1.

LEMMA 6.1. *Under the above hypotheses there exist a viscosity subsolution v^* and a viscosity supersolution w^* of (2.1) such that $v^*, w^* \in UC(\bar{\Omega})$, $v^* = w^* = v$ on $\partial\Omega$, and $\sup_{\bar{\Omega}} (w^* - v^*) < \infty$.*

Note that $w^* \geq v^*$ on Ω by Theorem 2.1.

Proof. Let $u^* \in UC(\bar{\Omega})$ be the solution of $u(x) + H^*(x, u(x), 0) \ni 0$ in Ω . By virtue of Remark 5.1 there exists a $C_{10} > 0$ such that $v \leq u^* + C_{10}$ and $w \geq u^* - C_{10}$ on Ω . Hence

$$(6.2) \quad u^* - C_{10} \leq v = w \leq u^* + C_{10} \quad \text{on } \partial\Omega.$$

Extending u^* to $\Omega_\delta \equiv \{x \in R^N \mid \text{dist}(x, \Omega) < \delta\}$, with $\delta > 0$, and mollifying it, we find a C^1 function u_1^* on $\bar{\Omega}$ such that $\sup_{\bar{\Omega}} |u^* - u_1^*| \leq 1$ and $\sup_{\bar{\Omega}} |Du_1^*| \leq C_{11}$ for some constant $C_{11} > 0$. Setting

$$v_1^*(x) = u_1^*(x) - C_{10} - 1 - \delta(C_{11}) \quad \text{on } \bar{\Omega}$$

and

$$w_1^*(x) = u_1^*(x) + C_{10} + 1 + \delta(C_{11}) \quad \text{on } \bar{\Omega},$$

we see that

$$\begin{aligned} v_1^*(x) + H(x, v_1^*(x) - 0, Dv_1^*(x)) &\leq 0 & \text{in } \Omega, \\ w_1^*(x) + H(x, w_1^*(x) + 0, Dw_1^*(x)) &\geq 0 & \text{in } \Omega, \end{aligned}$$

and $v_1^* \leq v = w \leq w_1^*$ on $\partial\Omega$ by (6.2).

In view of Proposition 1.4 the functions $v^* = \max\{v, v_1^*\}$ and $w^* = \min\{w, w_1^*\}$ have the required properties. Q.E.D.

Proof of Theorem 6.1. In view of Lemma 6.1 we may assume $\sup_{\bar{\Omega}} (w - v) < \infty$. This together with Theorem 2.1 asserts that $v(x) \leq w(x) \leq v(x) + J$ for all $x \in \Omega$ and some constant $J > 0$.

Set $\bar{\Omega}_L = \Omega \cap B(0; L)^\circ$ for any $L > 0$. For $\varepsilon > 0$ and $L > 0$ define $\zeta_{\varepsilon, L} \in BUC(R^N)$ by

$$\zeta_{\varepsilon, L}(x) = \begin{cases} \frac{1}{\varepsilon} \text{dist}(x, \bar{\Omega}_L^\circ) & \text{if } \text{dist}(x, \bar{\Omega}_L^\circ) < \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

For each $L > 0$ choose a function $v_L \in BUC(R^N)$ so that $v_L = v$ on $\bar{\Omega}_{L+1}$. Let $\theta \in BUC(R)$

be the function defined by $\theta(r)=r$ for $|r|\leq J$, $\theta(r)=J$ for $r\geq J$, and $\theta(r)=-J$ for $r\leq -J$.

Now we consider the problem

$$(6.3) \quad u + \zeta_{\varepsilon, L}(x)\theta(H(x, u, Du) + v(x)) - v_L(x) = 0 \quad \text{in } R^N,$$

with $\varepsilon, L > 0$. It is easily checked that the function

$$H_{\varepsilon, L} : (x, u, p) \rightarrow \zeta_{\varepsilon, L}(x)\theta(H(x, u, p) + v(x)) - v_L(x)$$

on $R^N \times R \times R^N$ satisfies (A1)-(A3) for $\Omega = R^N$. By Theorem 4.1 we see that (6.3) has a unique viscosity solution $u_{\varepsilon, L} \in UC(R^N)$ for each $\varepsilon, L > 0$.

Note that v and w are, respectively, a viscosity subsolution and a viscosity supersolution of (6.3) in $\bar{\Omega}_{L+1}$ and also that $u_{\varepsilon, L}(x) = v_L(x)$ for all $x \in \bar{\Omega}_L^c$. Thus

$$(6.4) \quad v(x) \leq u_{\varepsilon, L}(x) \leq w(x) \quad \text{for all } x \in \bar{\Omega}_{L+1},$$

and

$$(6.5) \quad u_{\varepsilon, L}(x) = v(x) \quad \text{for all } x \in \bar{\Omega}_{L+1} \setminus \bar{\Omega}_L.$$

By assumption $v, w \in UC(\bar{\Omega})$ and so we may assume that there exists an $\omega \in \Gamma(a)$ such that $|v(x) - v(y)| \leq \omega(|x - y|)$ and $|w(x) - w(y)| \leq \omega(|x - y|)$ for all $x, y \in \bar{\Omega}$ with $|x - y| \leq a$. Since $v_L = v$ on $\bar{\Omega}_{L+1}$, $H_{\varepsilon, L}$ satisfies (2.2) and (2.3) for $\Omega = \bar{\Omega}_{L+1}$ and some $\gamma = \gamma_\varepsilon \in \Gamma(a)$ independent of $L > 0$. In order to apply Theorem 3.1 to $u_{\varepsilon, L}$ in $\bar{\Omega}_{L+1}$, we observe from (6.4) that $|u_{\varepsilon, L}(x) - u_{\varepsilon, L}(y)| \leq J + \omega(a)$ for all $x, y \in \bar{\Omega}_{L+1}$ with $|x - y| \leq a$, and from (6.4) and (6.5) that $|u_{\varepsilon, L}(x) - u_{\varepsilon, L}(y)| \leq \omega(|x - y|)$ either if $x \in \bar{\Omega}_{L+1}$, $y \in \partial\bar{\Omega}_{L+1}$, and $|x - y| \leq a$ or if $x \in \partial\bar{\Omega}_{L+1}$, $y \in \bar{\Omega}_{L+1}$, and $|x - y| \leq a$. Replacing γ_ε by a new one if necessary, we may assume that $\gamma_\varepsilon(r) \geq \omega(r)$ for $0 \leq r \leq a$ and $\gamma_\varepsilon(a) \geq J + \omega(a)$. Thus, by Theorem 3.1, we conclude that there exists a $\phi_\varepsilon \in \Gamma(a)$ such that

$$(6.6) \quad |u_{\varepsilon, L}(x) - u_{\varepsilon, L}(y)| \leq \phi_\varepsilon(|x - y|) \quad \text{for all } x, y \in \bar{\Omega}_{L+1} \text{ with } |x - y| \leq a.$$

Inequalities (6.4) and (6.6) together with the Ascoli-Arzelà theorem tell that there exist sequences $\{L_n(\varepsilon)\} \subset (0, \infty)$ satisfying $L_n(\varepsilon) \rightarrow \infty$ and functions $u_\varepsilon \in UC(\bar{\Omega})$ for $\varepsilon > 0$ such that, for each $\varepsilon > 0$, $u_{\varepsilon, L}(x) \rightarrow u_\varepsilon(x)$ uniformly on all compact subsets of $\bar{\Omega}$ as $L = L_n(\varepsilon) \rightarrow \infty$. Proposition 1.1 ensures that u_ε is the unique viscosity solution of

$$(6.7) \quad u + \zeta_\varepsilon(x)\theta(H(x, u, Du) + v(x)) - v(x) = 0 \quad \text{in } \Omega,$$

where $\zeta_\varepsilon : \Omega \rightarrow R$ is given by

$$\zeta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} \text{dist}(x, \Omega^c) & \text{if } \text{dist}(x, \Omega^c) < \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

We now follow the method employed in [6]. It follows from (6.4) that $v(x) \leq u_\varepsilon(x) \leq w(x)$ for all $x \in \Omega$ and $\varepsilon > 0$. From (6.7) we find that

$$u_\varepsilon + \theta(H(x, u_\varepsilon, Du_\varepsilon) + v(x)) - v(x) = 0 \quad \text{in } \Omega_\varepsilon.$$

holds in the viscosity sense, where Ω_ε denotes the set $\{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$. Therefore Theorem 2.1 asserts that if $0 < \mu, \nu \leq \varepsilon$, then

$$\sup_{\Omega_\varepsilon} |u_\mu - u_\nu| \leq \max \{0, \sup_{\partial\Omega_\varepsilon} |u_\mu - u_\nu|\}$$

and so

$$\sup_{\Omega} |u_\mu - u_\nu| \leq \max \{0, \sup_{\partial\Omega} (w - v)\}.$$

Thus $\{u_\varepsilon\}_{\varepsilon>0}$ forms a Cauchy net in the supremum norm on Ω as $\varepsilon \downarrow 0$. Setting $u(x) = \lim_{\varepsilon \downarrow 0} u_\varepsilon(x)$ for $x \in \bar{\Omega}$, we see that $u \in UC(\bar{\Omega})$, $u = v$ on $\partial\Omega$, and u solves

$$(6.8) \quad u + \theta(H(x, u, Du) + v(x)) - v(x) = 0 \quad \text{in } \Omega$$

in the viscosity sense. Since $v(x) \leq u(x) \leq v(x) + J$ for $x \in \Omega$, we have $u(x) - v(x) = \theta(u(x) - v(x))$ for all $x \in \Omega$. Observe also that, for $s, t \in \mathbb{R}$, $s + t \leq 0$ (resp., $s + t \geq 0$) if and only if $\theta(s) + \theta(t) \leq 0$ (resp., $\theta(s) + \theta(t) \geq 0$). We now conclude from (6.8) that u solves (2.1) in the viscosity sense. The proof is complete. Q.E.D.

7. Viscosity solutions of the Cauchy problem

In the remaining part of this paper we will discuss existence and uniqueness of viscosity solution of the Cauchy problem for the Hamilton-Jacobi equation

$$(7.1) \quad u_t + H(t, x, u, Du) = 0 \quad \text{in } (0, T) \times \Omega,$$

where $0 < T < \infty$ and $H: (0, T) \times \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ are given, Ω is an open subset of $\mathbb{R}^N$, u is the unknown, and u_t and Du denote, respectively, $\partial u / \partial t$ and $(\partial u / \partial x_1, \dots, \partial u / \partial x_N)$.

We denote by Q_T the set $(0, T) \times \Omega$ and by $\partial_p Q_T$ the parabolic boundary of Q_T , $(0, T) \times \partial\Omega \cup \{0\} \times \bar{\Omega}$.

For any open $Q \subset Q_T$, $(t_0, x_0) \in Q$, and $(u_0, p_0) \in \mathbb{R} \times \mathbb{R}^{N+1}$ we let

$$\begin{aligned} H^+(t_0, x_0, u_0, p_0) &= \{(G, b) \mid G \in C(Q \times \mathbb{R}^N), b \in L^1(0, T), \\ G(t, x, p) + b(t) &\geq H(t, x, u, p) \text{ for all } (x, u, p) \in B((x_0, u_0, p_0); \delta), \\ \text{a. a. } t &\in B(t_0; \delta), \text{ and some } \delta > 0\} \end{aligned}$$

and

$$\begin{aligned} H^-(t_0, x_0, u_0, p) &= \{(G, b) \mid G \in C(Q \times \mathbb{R}^N), b \in L^1(0, T), \\ G(t, x, p) + b(t) &\leq H(t, x, u, p) \text{ for all } (x, u, p) \in B((x_0, u_0, p_0); \delta), \\ \text{a. a. } t &\in B(t_0; \delta), \text{ and some } \delta > 0\}. \end{aligned}$$

DEFINITION 7.1. Let Q be an open subset of Q_T . (i) $u \in C(Q)$ is called a *viscosity subsolution* (resp., *supersolution*) of (7.1) in Q if $\phi_t(t_0, x_0) + G(t_0, x_0, D\phi(t_0, x_0)) \leq 0$ (resp.,

$\phi_t(t_0, x_0) + G(t_0, x_0, D\phi(t_0, x_0)) \geq 0$) holds for any $\phi \in C^1(Q)$, $(t_0, x_0) \in Q$, and $(G, b) \in H^-(t_0, x_0, u_0, p_0)$ (resp., $(G, b) \in H^+(t_0, x_0, u_0, p_0)$), where $u_0 = u(t_0, x_0)$ and $p_0 = D\phi(t_0, x_0)$, such that $(t, x) \rightarrow u(t, x) + \int_0^t b(s) ds - \phi(t, x)$ attains a local maximum (resp., minimum) at (t_0, x_0) .

(ii) $u \in C(Q)$ is called a *viscosity solution* of (7.1) in Q if it is both a viscosity subsolution and supersolution of (7.1) in Q .

It is easy to see that this definition of viscosity sub-, super- and solution coincides with that in M. G. Crandall-P. L. Lions [3] when $H \in C(Q_T \times R \times R^N)$. We mention also that (7.1) has a special form of (1.1) and so we have two concepts of "viscosity solution" of (7.1) and that a viscosity subsolution (resp., supersolution, solution) of (7.1) in Q in the sense of Definition 7.1 is a respective one in the sense of Definition 1.1. The converse is true if $(t, x, p) \rightarrow H(t, x, u, p)$ is continuous for all $u \in R$. Henceforth we focus our attention on viscosity sub-, super-, and solutions of (7.1) in the sense of Definition 7.1.

An explanation of the above definition might be in order. Let $h \in L^1(0, T)$ and consider the ODE $u'(t) + h(t) = 0$ in $(0, T)$, a simple case of (7.1). Decomposing h as $h = g + b$, where $g \in C(0, T)$ and $b \in L^1(0, T)$, we see that an absolutely continuous function u solves $u'(t) + h(t) = 0$ a.e. in $(0, T)$ if and only if $u \in C(0, T)$ and v defined by $v(t) = u(t) + \int_0^t b(s) ds$ is a viscosity solution of $v'(t) + g(t) = 0$ in $(0, T)$ in the sense of [3].

We will need a generalization of Definition 7.1. Indeed we will treat the equation of the form

$$(7.2) \quad \Theta(t, x, u, u_t + H(t, x, u, Du)) = 0 \quad \text{in } Q_T.$$

DEFINITION 7.2. Let $Q \subset Q_T$ be open. (i) $u \in C(Q)$ is called a viscosity subsolution (resp., supersolution) of (7.2) in Q if $\Theta(t_0, x_0, u_0, \phi_t(t_0, x_0) + G(t_0, x_0, p_0)) \leq 0$ (resp., $\Theta(t_0, x_0, u_0, \phi_t(t_0, x_0) + G(t_0, x_0, p_0)) \geq 0$) is satisfied for any $\phi \in C^1(Q)$, $(t_0, x_0) \in Q$, and $(G, b) \in H^-(t_0, x_0, u_0, p_0)$ (resp., $(G, b) \in H^+(t_0, x_0, u_0, p_0)$), where $u_0 = u(t_0, x_0)$ and $p_0 = D\phi(t_0, x_0)$, such that $(t, x) \rightarrow u(t, x) + \int_0^t b(s) ds - \phi(t, x)$ has a local maximum (resp., minimum) at (t_0, x_0) .

(ii) $u \in C(Q)$ is called a viscosity solution of (7.2) in Q if it is both a sub- and supersolution of (7.2).

We present here a stability result.

PROPOSITION 7.1. Let H and H_n be functions on $Q_T \times R \times R^N$ for all $n \in N$. Let $Q \subset Q_T$ and $u_n \in C(Q)$ be viscosity subsolutions (resp., supersolutions) of $u_t + H_n(t, x, u, Du) = 0$ in Q for all $n \in N$. Assume that $H_n - H \rightarrow 0$ in $L^1(0, T; C(K))$ for any compact $K \subset \Omega \times R \times R^N$ and $u_n(t, x) \rightarrow u(t, x)$ uniformly on any compact subset of Q for some $u \in C(Q)$ as $n \rightarrow \infty$. Then u is a viscosity subsolution (resp., supersolution) of (7.1) in Q .

Proof. We will prove only the "subsolution" part and omit the proof of the "supersolution" part since their proofs are similar.

Let $\phi \in C^1(Q)$, $(t_0, x_0) \in Q$, and $(G, b) \in H^-(t_0, x_0, u_0, p_0)$, where $u_0 = u(t_0, x_0)$ and $p_0 = D\phi(t_0, x_0)$. Assume that $(t, x) \rightarrow u(t, x) + \int_0^t b(s) ds - \phi(t, x)$ has a local strict maximum at (t_0, x_0) .

By definition we have

$$G(t, x, p) + b(t) \leq H(t, x, r, p)$$

for all $(x, r, p) \in B((x_0, u_0, p_0); \delta)$, a. a. $t \in B(t_0; \delta)$, and some $\delta > 0$. Therefore, setting

$$b_n(t) = \sup_{(x, r, p) \in B((x_0, u_0, p_0); \delta)} |H_n(t, x, r, p) - H(t, x, r, p)|$$

for $0 < t < T$, we have

$$(7.3) \quad G(t, x, p) - b_n(t) + b(t) \leq H_n(t, x, r, p)$$

for all $(x, r, p) \in B((x_0, u_0, p_0); \delta)$, a. a. $t \in B(t_0; \delta)$, and $n \in \mathbb{N}$.

Since $u_n \rightarrow u$ in $C(Q)$ and $b_n \rightarrow 0$ in $L^1(0, T)$, we see that $(t, x) \rightarrow u_n(t, x) + \int_0^t (b(s) - b_n(s)) ds - \phi(t, x)$ have local maximums near at (t_0, x_0) , say, at (t_n, x_n) for sufficiently large n . We may assume that $(t_n, x_n) \rightarrow (t_0, x_0)$ as $n \rightarrow \infty$. Thus, in view of (7.3) and that u_n are viscosity subsolutions of $u_t + H_n(t, x, u, Du) = 0$ in Q , we obtain

$$\phi_t(t_n, x_n) + G(t_n, x_n, D\phi(t_n, x_n)) \leq 0$$

for sufficiently large n . Sending $n \rightarrow \infty$, we conclude that $\phi_t(t_0, x_0) + G(t_0, x_0, D\phi(t_0, x_0)) \leq 0$. Q. E. D.

REMARK 7.1. Let $\theta \in C(Q_T \times R \times R)$. Then the above result remains true if we change (7.1) for (7.2) and the equation $u_t + H_n(t, x, u, Du) = 0$ for $\theta(t, x, u, u_t + H_n(t, x, u, Du)) = 0$ in the statement.

Propositions 1.2, 1.3, and 1.4 are valid for the viscosity solutions which we are now concerned with. Proposition 1.2 remains true even if the assumption that H is locally bounded on $Q_T \times R \times R^N$ is replaced by the condition that, for any compact $K \subset \Omega \times R \times R^N$, $t \rightarrow \sup_K |H(t, \cdot, \cdot, \cdot)|$ is bounded by an integrable function on $(0, T)$.

Finally we remark that if $Q \subset Q_T$ is an open set and $u \in C^1(Q)$ satisfies

$$u_t(t, x) + H(t, x, u(t, x), Du(t, x)) \leq 0$$

$$(\text{resp.}, u_t(t, x) + H(t, x, u(t, x), Du(t, x)) \geq 0)$$

for all $(t, x) \in Q \cap ((0, T) \setminus Z) \times \Omega$, where Z is a null subset of $(0, T)$, then u is a viscosity subsolution (resp., supersolution) of (7.1) in Q .

8. A comparison theorem for the Cauchy problem

For $a > 0$ we let $\Gamma_T(a)$ denote the set of all functions $\gamma : (0, T) \times [0, a] \rightarrow [0, \infty)$ such that $\gamma(t, \cdot) \in \Gamma(a)$ for all $t \in (0, T)$ and $\gamma(\cdot, s) \in L^1(0, T)$ for all $s \in [0, a]$. Σ_T denotes the set of all functions $\sigma : (0, T) \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ such that $\sigma(t, \cdot, \cdot) \in \Sigma$ for all $0 < t < T$ and $\sigma(\cdot, s, r) \in L^1(0, T)$ for all $s, r \geq 0$. We will denote by ∂ the function: $(t, r) \rightarrow \sigma(t, r, r)$ on $(0, T) \times [0, \infty)$ for any $\sigma \in \Sigma_T$.

Henceforth we make the following assumptions on H .

- (A4) $t \rightarrow H(t, x, u, p)$ is integrable on $(0, T)$ for any $(x, u, p) \in \Omega \times R \times R^N$.
- (A5) $\left\{ \begin{array}{l} \text{There is a } \nu \in R \text{ such that } u \rightarrow H(t, x, u, p) - \nu u \text{ is nondecreasing on } R \text{ for all} \\ (x, p) \in \Omega \times R^N \text{ and a. a. } t \in (0, T). \end{array} \right.$
- (A6) $\left\{ \begin{array}{l} \text{For some } \sigma \in \Sigma_T \text{ the inequality} \\ |H(t, x, u, p) - H(t, x, u, q)| \leq \sigma(t, |p - q|, R) \\ \text{holds for all } (x, u) \in \Omega \times R, p, q \in B(0; R), R > 0, \text{ and a. a. } t \in (0, T). \end{array} \right.$
- (A7) $\left\{ \begin{array}{l} \text{For some } a > 0 \text{ and } \gamma \in \Gamma_T(a) \text{ the inequality} \\ |H(t, x, u, p) - H(t, y, u, p)| \leq \gamma(t, |x - y|(1 + |p|)) \\ \text{holds for all } u \in R, p \in R^N, x, y \in \Omega \text{ with } |x - y|(1 + |p|) \leq a, \text{ and a. a. } t \in (0, T). \end{array} \right.$

Let $Q \subset \bar{Q}_T$ and introduce

$$UC_*(Q) = \{u \in C(Q) \mid \lim_{r \downarrow 0} \sup \{|u(t, x) - u(t, y)| \mid (t, x), (t, y) \in Q_T, |x - y| \leq r\} = 0\}.$$

Our intention in this section is to prove the next theorem.

THEOREM 8.1. Assume (A4), (A5) with $\nu = 0$, (A6), and (A7). Let $v, v \in UC_*(\bar{Q}_T)$ be, respectively, a viscosity subsolution and a viscosity supersolution of (7.1). Then

$$(8.1) \quad \sup_{Q_T} (u - v) \leq \sup_{\partial_p Q_T} (u - v)_+.$$

More generally we have

COROLLARY 8.1. Under the assumptions (A4)–(A7), if $f, g \in BUC(Q_T)$ and $v, v \in UC_*(\bar{Q}_T)$ satisfy, respectively,

$$(8.2) \quad u_t + H(t, x, u, Du) \leq f(t, x) \quad \text{in } Q_T$$

and

$$(8.3) \quad v_t + H(t, x, v, Dv) \geq g(t, x) \quad \text{in } Q_T$$

in the viscosity sense, then

$$(8.4) \quad \sup_{Q_T} e^{\nu t} (u - v) \leq \sup_{\partial_p Q_T} e^{\nu t} (u - v)_+ + \int_0^T e^{\nu s} \sup_{\partial} (f - g)_+(s, \cdot) ds,$$

where ν is a constant for which (A5) holds.

Proof. Let $\nu \in R$ be a number for which (A5) holds. The function $H_\nu : (t, x, u, p) \rightarrow H(t, x, u, p) - \nu u$ on $Q_T \times R \times R^N$ satisfies (A4), (A5) with $\nu=0$, (A6), and (A7). And so does the function $H^\nu : (t, x, u, p) \rightarrow e^{\nu t} H_\nu(t, x, e^{-\nu t} u, e^{-\nu t} p)$ on $Q_T \times R \times R^N$.

The functions $u^\nu : (t, x) \rightarrow e^{\nu t} u(t, x)$ and $v^\nu : (t, x) \rightarrow e^{\nu t} v(t, x)$ solve, respectively, $u_t^\nu + H^\nu(t, x, u^\nu, Du^\nu) \leq e^{\nu t} f$ and $v_t^\nu + H^\nu(t, x, v^\nu, Dv^\nu) \leq e^{\nu t} g$ in Q_T in the viscosity sense. Moreover $w^\nu : (t, x) \rightarrow u^\nu(t, x) - \int_0^t e^{\nu s} \sup_{\partial} (f - g)_+(s, \cdot) ds$ satisfies $w_t^\nu + H^\nu(t, x, w^\nu, Dw^\nu) \leq e^{\nu t} g$ in Q_T in the viscosity sense.

Thus we conclude by virtue of Theorem 8.1 that

$$\sup_{Q_T} (w^\nu - v^\nu) \leq \sup_{\partial_p Q_T} (w^\nu - v^\nu)_+,$$

whence we obtain (8.4). Q.E.D.

We need the following observations.

LEMMA 8.1. *Under the assumptions (A4)–(A7), for any compact $K \subset \Omega \times R^N$ and $u \in R$, $t \rightarrow H(t, \cdot, u, \cdot)$ belongs to $L^1(0, T; C(K))$.*

Proof. For any $\varepsilon > 0$ let $\rho_\varepsilon \in C_0(R)$ be a standard mollifier. Extend H, σ in (A6), and γ in (A7) for $t \leq 0$ and $t \geq T$ by setting them to be zero for $t \leq 0$ and $t \geq T$, and define $H_\varepsilon, \sigma_\varepsilon, \gamma_\varepsilon$ by $H_\varepsilon(\cdot, x, u, p) = \rho_\varepsilon * H(\cdot, x, u, p)$ for $(x, u, p) \in \Omega \times R \times R^N$, $\sigma_\varepsilon(\cdot, s, r) = \rho_\varepsilon * \sigma(\cdot, s, r)$ for $s, r \geq 0$, and $\gamma_\varepsilon(\cdot, r) = \rho_\varepsilon * \gamma(\cdot, r)$ for $0 \leq r \leq a$.

It follows from (A4) that $t \rightarrow H_\varepsilon(t, x, u, p)$ is continuous on $[0, T]$. From (A6) and (A7) we have

$$(8.5) \quad |H_\varepsilon(t, x, u, p) - H_\varepsilon(t, x, u, q)| \leq \sigma_\varepsilon(t, |p - q|, R)$$

for all $(t, x, u) \in Q_T \times R, p, q \in B(0; R)$, and $R > 0$ and

$$(8.6) \quad |H_\varepsilon(t, x, u, p) - H_\varepsilon(t, y, u, p)| \leq \gamma_\varepsilon(t, |x - y|(1 + |p|))$$

for all $(t, u, p) \in (0, T) \times R \times R^N$ and $x, y \in \Omega$ with $|x - y|(1 + |p|) \leq a$.

Note that $\gamma_\varepsilon \in \Gamma_T(a)$ and $\sigma_\varepsilon \in \Sigma_T$. Hence, setting $\tilde{\gamma}_\varepsilon(r) = \sup_{0 \leq t \leq T} \gamma_\varepsilon(t, r)$ and $\tilde{\sigma}_\varepsilon(r, s) = \sup_{0 \leq t \leq T} \sigma_\varepsilon(t, r, s)$, we see that $\tilde{\gamma}_\varepsilon \in \Gamma(a)$ and $\tilde{\sigma}_\varepsilon \in \Sigma$ for all $\varepsilon > 0$. Using (8.5) and (8.6), we have

$$\begin{aligned} |H_\varepsilon(t, x, u, p) - H_\varepsilon(s, y, u, q)| &\leq |H_\varepsilon(t, x, u, p) - H_\varepsilon(s, x, u, p)| \\ &\quad + \tilde{\sigma}_\varepsilon(|p - q|, R) + \tilde{\gamma}_\varepsilon(|x - y|(1 + R)) \end{aligned}$$

for all $0 < t, s < T, u \in R, p, q \in B(0; R)$, and $x, y \in \Omega$ with $|x - y|(1 + R) \leq a$. Thus $H_\varepsilon(\cdot, \cdot, u, \cdot) \in UC(Q_T \times B(0; R))$ for all $u \in R$ and $R > 0$.

Now let K be any compact subset of $\Omega \times R^N$ and fix $u \in R$. Choose an $R > 0$ so that $|p| \leq R$ for all $(x, p) \in K$. For any $\eta > 0$ select a finite set $\{(x_i, p_i)\}_{i=1}^m$, where m depends on η , such that

$$K \subset \bigcup_{i=1}^m B((x_i, p_i); \eta).$$

Assume $\eta(1+R) \leq a$. If $0 < t < T$, $(x, p) \in K$, $(y, q) \in K$, $|x-y| \leq \eta$, and $|p-q| \leq \eta$, then

$$|H_\varepsilon(t, x, u, p) - H_\varepsilon(t, y, u, q)| \leq \sigma_\varepsilon(t, \eta, R) + \gamma_\varepsilon(t, \eta(1+R)).$$

Thus we have

$$\begin{aligned} |H_\varepsilon(t, x, u, p) - H_\delta(t, x, u, p)| &\leq \sum_{i=1}^m |H_\varepsilon(t, x_i, u, p_i) - H_\delta(t, x_i, u, p_i)| \\ &\quad + \sigma_\varepsilon(t, \eta, R) + \sigma_\delta(t, \eta, R) + \gamma_\varepsilon(t, \eta(1+R)) + \gamma_\delta(t, \eta(1+R)) \end{aligned}$$

for any $\varepsilon, \delta > 0$, $0 < t < T$, and $(x, p) \in K$, and so

$$\begin{aligned} (8.7) \quad &\int_0^T \sup_K |H_\varepsilon(t, \cdot, u, \cdot) - H_\delta(t, \cdot, u, \cdot)| dt \\ &\leq \sum_{i=1}^m \int_0^T |H_\varepsilon(t, x_i, u, p_i) - H_\delta(t, x_i, u, p_i)| dt \\ &\quad + 2 \int_0^T (\sigma(t, \eta, R) + \gamma(t, \eta(1+R))) dt \end{aligned}$$

Recall that $H_\varepsilon(\cdot, \cdot, u, \cdot) \in C([0, T] \times K)$ and so $H(\cdot, \cdot, u, \cdot) \in L^1(0, T; C(K))$ for all $\varepsilon > 0$. Notice that

$$\int_0^T (\sigma(t, \eta, R) + \gamma(t, \eta(1+R))) dt \rightarrow 0 \quad \text{as } \eta \downarrow 0$$

and $H_\varepsilon(\cdot, x, u, p) \rightarrow H(\cdot, x, u, p)$ in $L^1(0, T)$ for all $(x, p) \in \Omega \times \mathbb{R}^N$ as $\varepsilon \downarrow 0$. Thus we find from (8.7) that $\{H_\varepsilon(\cdot, \cdot, u, \cdot) | \varepsilon > 0\}$ is a Cauchy net in $L^1(0, T; C(K))$ as $\varepsilon \downarrow 0$. Since $L^1(0, T; C(K))$ is complete, we now conclude that $H(\cdot, \cdot, u, \cdot) \in L^1(0, T; C(K))$ and $H_\varepsilon(\cdot, \cdot, u, \cdot) \rightarrow H(\cdot, \cdot, u, \cdot)$ in $L^1(0, T; C(K))$ as $\varepsilon \downarrow 0$. Q.E.D.

LEMMA 8.2. Assume H satisfies (A4), (A5) with $\nu=0$, (A6), and (A7). Let $u, v \in C(Q_T)$ be, respectively, a viscosity subsolution and a viscosity supersolution of (7.1). Let Q be an open subset of $(0, T) \times \Omega^2$ and $\tilde{H}: (0, T) \times \Omega^2 \times \mathbb{R}^{2N} \rightarrow \mathbb{R}$ satisfy the condition that for any $(t_0, x_0, y_0, p_0, q_0) \in Q \times \mathbb{R}^{2N}$ there exists a $\delta > 0$ such that

$$(8.8) \quad \tilde{H}(t, x, y, p, q) \leq H(t, x, u(t_0, x_0) - \delta, p) - H(t, y, v(t_0, y_0) + \delta, -q)$$

for all $(x, y, p, q) \in B((x_0, y_0, p_0, q_0); \delta)$ and a. a. $t \in B(t_0; \delta)$. Then $w: (t, x, y) \rightarrow u(t, x) - v(t, y)$ solves

$$(8.9) \quad w_t + \tilde{H}(t, x, y, Dw) \leq 0 \quad \text{in } Q$$

in the viscosity sense, where D denotes the $2N$ -dimensional gradient operator, (D_x, D_y) .

Proof. Let $\phi \in C^1(Q)$, $(t_0, x_0, y_0) \in Q$, $G \in C((0, T) \times \Omega^2 \times \mathbb{R}^{2N})$, $b \in L^1(0, T)$, and $\delta > 0$. Let us write $X_0 = (x_0, y_0)$, $p_0 = D_x \phi(t_0, x_0)$, $q_0 = D_y \phi(t_0, x_0)$, and $P_0 = (p_0, q_0)$. Assume that

$$(8.9) \quad G(t, x, y) + b(t) \leq \tilde{H}(t, x, y, p, q)$$

holds for all $(x, y, p, q) \in B((X_0, P_0); \delta)$ and a. a. $t \in B(t_0; \delta)$, that $(t, x, y) \rightarrow w(t, x, y) + \int_0^t b(s) ds - \phi(t, x, y)$ attains a local strict maximum at (t_0, x_0) , and that (8.8) holds for all $(x, y, p, q) \in B((X_0, P_0); \delta)$ and a. a. $t \in B(t_0; \delta)$.

Define $H_\epsilon : Q_T \times R \times R^N \rightarrow R$ for all $\epsilon > 0$ as in the proof of Lemma 8.1 and set $G_\epsilon(t, x, p) = H_\epsilon(t, x, u(t_0, x_0) - \delta, p)$ for all $(t, x, p) \in Q_T \times R^N$ and $\epsilon > 0$. In view of Lemma 8.1, setting

$$b_\epsilon(t) = \sup_{(x, p) \in B((x_0, p_0); \delta)} |G_\epsilon(t, x, p) - H(t, x, u(t_0, x_0) - \delta, p)|$$

for $t \in (0, T)$, we find that $b_\epsilon \rightarrow 0$ in $L^1(0, T)$ as $\epsilon \downarrow 0$.

If $\epsilon > 0$ and $\alpha > 0$ are sufficiently small, then $(t, s, x, y) \rightarrow u(t, x) - v(s, y) - \int_0^t b_\epsilon(\tau) d\tau + \int_0^s (b_\epsilon(\tau) - b_\epsilon(\tau)) d\tau - \phi(t, x, y) - (t-s)^2/\alpha$ attains a local maximum at a point (t_1, s_1, x_1, y_1) near at (t_0, t_0, x_0, y_0) . We may assume that $(t_1, s_1, x_1, y_1) \rightarrow (t_0, t_0, x_0, y_0)$ as $\alpha \downarrow 0$ and $\epsilon \downarrow 0$ in this order.

Note that $G_\epsilon(t, x, p) - b_\epsilon(t) \leq H(t, x, u(t_0, x_0) - \delta, p)$ for all $0 < t < T$ and all $(x, p) \in B((x_0, p_0); \delta)$. Hence

$$(8.10) \quad \phi_1(t_1, x_1, y_1) + \frac{2}{\alpha}(t_1 - s_1) + G_\epsilon(t_1, x_1, p_1) \leq 0$$

for sufficiently small $\epsilon > 0$ and $\alpha > 0$, where $p_1 = D_s \phi(t_1, x_1, y_1)$, as u is a viscosity sub-solution of (7.1). Next we observe from (8.8) that

$$H(s, y, v(t_0, y_0) + \delta, q) \leq G_\epsilon(s, x_1, p_1) + b_\epsilon(s) - G(s, x_1, y, p_1, -q) - b(s)$$

for all $(y, q) \in B((y_0, -q_0); \delta/2)$ and a. a. $s \in B(t_0; \delta)$ if $\epsilon > 0$ and $\alpha > 0$ are small enough.

$$(8.11) \quad -\frac{2}{\alpha}(s_1 - t_1) + G_\epsilon(s_1, x_1, p_1) - G(s_1, x_1, y_1, p_1, q_1) \geq 0$$

for sufficiently small $\epsilon > 0$ and $\alpha > 0$, where $q_1 = D_y \phi(t_1, x_1, y_1)$.

Subtracting (8.11) from (8.10), we obtain

$$\phi_1(t_1, x_1, y_1) + G(s_1, x_1, y_1, p_1, q_1) \leq G_\epsilon(s_1, x_1, p_1) - G_\epsilon(t_1, x_1, p_1)$$

for sufficiently small $\alpha > 0$ and $\epsilon > 0$. Taking into account that $s_1 - t_1 \rightarrow 0$ as $\alpha \downarrow 0$, we have

$$\phi_1(t_0, x_0, y_0) + G(t_0, x_0, y_0, D\phi(t_0, x_0, y_0)) \leq 0$$

by sending $\alpha \downarrow 0$ and $\epsilon \downarrow 0$ in this order. The proof is now complete. Q.E.D.

REMARK 8.1. Let $\Theta \in C(Q_T \times R^2)$ and $F \in C((0, T) \times Q^2 \times R)$. Assume that $\{\Theta(\cdot, \cdot, \cdot, r) \mid r \in R\}$ is equicontinuous on $Q_T \times R$, that

$$F(t, x, y, u-v) \leq \theta(t, x, u, r) - \theta(t, y, v, r)$$

for all $0 < t < T$, $x, y \in \Omega$, and $u, v, r \in R$, and that $r \rightarrow \theta(t, x, u, r)$ is nondecreasing on R for every $0 < t < T$, $x \in \Omega$, and $u \in R$. Then, under the same assumptions and notations as in the Lemma 8.2 except that we now assume u and v to be a viscosity subsolution and a viscosity supersolution of (7.2), respectively, we have the conclusion: w satisfies

$$(8.12) \quad \min \{F(t, x, y, w), w_t - \tilde{H}(t, x, y, Dw)\} \leq 0 \quad \text{in } Q$$

in the viscosity sense.

We can prove this assertion in the same way as above. The new observation needed here is that if $\theta(t, x, u, r) - \theta(t, y, v, s) \leq 0$ and $r > s$, then the monotonicity of $r \rightarrow \theta(t, x, u, r)$ implies that $F(t, x, y, u-v) \leq 0$.

Proof of Theorem 8.1. As in the proof of Theorem 2.1 we may assume that $\sup_{\partial_r Q_T} (u-v)_+ = 0$.

We suppose

$$(8.13) \quad \sup_{Q_T} (u-v) > 0,$$

and will obtain a contradiction.

The first step is to prove that $\sup_{Q_T} (u-v) < \infty$. Set $w(t, x, y) = u(t, x) - v(t, y)$ for $(t, x, y) \in [0, T] \times \bar{\Omega}^2$, and define

$$\mathcal{J}_{a,r} = \{(t, x, y) | 0 \leq t \leq T, x, y \in \bar{\Omega}, |x-y| \leq a\}$$

and

$$\mathcal{J}_{a,r}^+ = \{(t, x, y) | 0 < t < T, x, y \in \Omega, |x-y| < a, w(t, x, y) > 0\}.$$

By Lemma 8.2, (A6), and (A7), we find that w satisfies

$$(8.14) \quad w_t - \partial(t, |D_x w|) - \partial(t, |D_y w|) - r(t, |x-y|) \leq 0 \quad \text{in } \mathcal{J}_{a,r}^+$$

in the viscosity sense.

Choose an increasing function $\eta \in C^1([0, a])$ such that $\eta(r) = r^2$ for $0 \leq r \leq a/2$ and $\lim_{r \uparrow a} \eta(r) = \infty$. Let $\alpha > 0$, $\beta > 0$, $\epsilon > 0$, and $b \in L^1(0, T)$. Write

$$\Phi(t, x, y) = w(t, x, y) - \frac{1}{\alpha} \eta(|x-y|) - \beta |x|^2 - \int_0^t b(s) ds - \frac{\epsilon}{T-t}$$

for any $(t, x, y) \in \mathcal{J}_{a,r}$. As in the proof of Theorem 2.1, we see that

$$w(t, x, y) \leq C_{11}(1 + |x|)$$

for all $(t, x, y) \in \mathcal{J}_{a,r}$ and some constant $C_{11} > 0$. Therefore Φ attains a global maximum at some point $(t_0, x_0, y_0) \in \mathcal{J}_{a,r}$.

The same techniques as in the proof of Theorem 2.1 show that there exists an

$\alpha_0 > 0$ such that

$$(8.15) \quad |x_0 - y_0| \leq \frac{a}{2}$$

for all $0 < \alpha < \alpha_0$ and $\beta > 0$ and that

$$(8.16) \quad \beta |x_0| < C_{12}$$

for all $\alpha > 0$, $0 < \beta < 1$, and some constant $C_{12} > 0$.

Assume below that $0 < \alpha < \alpha_0$ and $0 < \beta < 1$. Replacing $a > 0$ by a smaller one if necessary, we may assume that there is an $\omega \in F(a)$ such that

$$|u(t, x) - u(t, y)| \leq \omega(|x - y|) \quad \text{and} \quad |v(t, x) - v(t, y)| \leq \omega(|x - y|)$$

if $|x - y| \leq a$. Set $R_1 = a/\alpha + C_{12}$, and define $b \in L^1(0, T)$ by

$$b(t) = 2\delta(t, R_1) + \gamma(t, a) \quad \text{for } t \in (0, T).$$

If $\Phi(t_0, x_0, y_0) \leq \omega(a)$, then

$$(8.17) \quad \sup_{Q_T} \left(u - v - \frac{\varepsilon}{T-t} \right) \leq \Phi(t_0, x_0, y_0) + C_{12} + \int_0^T b(s) ds \leq \omega(a) + C_{12} + \int_0^T b(s) ds.$$

Consider next the case when $\Phi(t_0, x_0, y_0) > \omega(a)$. Then $w(t_0, x_0, y_0) > \omega(a)$ and so neither $t_0 = 0$, $x_0 \in \partial\Omega$ nor $y_0 \in \partial\Omega$. Noting that

$$-b(t) \leq -\delta(t, |p|) - \delta(t, |q|) - \gamma(t, |x - y|)$$

for all $(t, x, y, p, q) \in \mathcal{A}_{a,T} \times B(0; R_1) \times B(0; R_1)$ and that

$$\left| D_x \left(\frac{1}{\alpha} \eta(|x - y|) + \beta |x|^2 \right) \right| = \left| \frac{2}{\alpha} (x - y) + 2\beta x \right| < R_1$$

and

$$\left| D_y \left(\frac{1}{\alpha} \eta(|x - y|) + \beta |x|^2 \right) \right| = \left| \frac{2}{\alpha} (x - y) \right| < R_1$$

at $(x, y) = (x_0, y_0)$, in view of (8.14) we have

$$\frac{\varepsilon}{(T-t)^2} \leq 0, \text{ a contradiction.}$$

This contradiction means that the case we are now considering does not occur and hence (8.17) is always valid.

Because of the arbitrariness of $\varepsilon > 0$ we conclude from (8.17) that

$$(8.18) \quad \sup_{Q_T} (u - v) \leq C_{13},$$

where $C_{13} = \omega(a) + C_{12} + \int_0^T (\delta(t, R_1) + \gamma(t, a)) dt$.

We begin the next and final step by defining $\mathcal{A}_{a,T} = \{(t, x, y) | 0 < t < T, x, y \in \Omega, |x - y|$

$\langle a \rangle$ and $\tilde{H}: \Delta_{a,T} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\tilde{H}(t, x, y, x, p, q) = -\min \{ \sigma(t, |p+q|, |p|+|q|) + \gamma(t, |x-y|(1+|p|)), C_{13} + \omega(a) \}.$$

Here and later we may assume $\gamma(t, a) \geq C_{13} + \omega(a)$ for all $t \in (0, T)$. Notice that $\sup_{\Delta_{a,T}} w \leq C_{13} + \omega(a)$. In view of Lemma 8.2 we find that w satisfies

$$(8.19) \quad w_t + \tilde{H}(t, x, y, Dw) \leq 0 \quad \text{in } \Delta_{a,T}^+$$

in the viscosity sense.

Let $b \in L^1(0, T)$ be chosen later. By (8.13), there is an $\varepsilon_0 > 0$ such that

$$(8.20) \quad \sup_{(t,x) \in Q_T} \left(u(t, x) - v(t, x) - \int_0^t b(s) ds - \frac{\varepsilon}{T-t} \right) \geq 3\varepsilon_0$$

provided $0 < \varepsilon < \varepsilon_0$ and $\|b\|_{L^1(0,T)} \leq \varepsilon_0$. We assume below that $\|b\|_{L^1(0,T)} \leq \varepsilon$. Choose a point $(t_1, x_1, y_1) \in \Delta_{a,T}$ for each $\alpha > 0$ and $0 < \beta, \varepsilon < \varepsilon_0$ so that

$$(8.21) \quad \sup_{(t,x,y) \in \Delta_{a,T}} \left(w(t, x, y) - \frac{1}{\alpha} \eta(|x-y|) - \int_0^t b(s) ds - \frac{\varepsilon}{T-t} \right) - \beta \\ < w(t_1, x_1, y_1) - \frac{1}{\alpha} \eta(|x_1-y_1|) - \int_0^{t_1} b(s) ds - \frac{\varepsilon}{T-t_1}$$

This yields, as in the proof of Theorem 2.1,

$$\eta(|x_1-y_1|) < \alpha \{ \omega(|x_1-y_1|) + \beta \}$$

and also $w(t_1, x_1, y_1) > \varepsilon_0$. Thus we see that there are $\alpha_1 > 0$ and $\delta > 0$ such that $t_1 > \delta$, $\text{dist}(x_1, \partial\Omega) > \delta$, and $\text{dist}(y_1, \partial\Omega) > \delta$ if $0 < \alpha < \alpha_1$.

Let $\zeta \in C_0(R^{2N+1})$ be a function satisfying $\zeta(t_1, x_1, y_1) = 1$, $0 \leq \zeta \leq 1$, $\text{supp } \zeta \subset B((t_1, x_1, y_1); \delta)$, and $|D\zeta| \leq C_4$ for some constant $C_4 > 0$. Set

$$\Psi(t, x, y) = w(t, x, y) - \frac{1}{\alpha} \eta(|x-y|) + \beta \zeta(t, x, y) - \int_0^t b(s) ds - \frac{\varepsilon}{T-t}$$

for $(t, x, y) \in \Delta_{a,T}$, $0 < \alpha < \alpha_1$, and $0 < \beta, \varepsilon < \varepsilon_0$. According to (8.21), Ψ attains a global maximum at some point $(t_2, x_2, y_2) \in \Delta_{a,T} \cap \text{supp } \zeta$. Note that $(t_2, x_2, y_2) \in \Delta_{a,T}$.

The techniques used in the proof of Theorem 2.1 show again that we may assume $|x_2 - y_2| \leq a/2$ and that

$$(8.21) \quad \begin{cases} |x_2 - y_2| \leq \sqrt{\alpha(\omega(a) + \beta)} \\ \frac{1}{\alpha} |x_2 - y_2|^2 \leq \omega(\sqrt{\alpha(\omega(a) + \beta)}) + \beta. \end{cases}$$

Taking into account that

$$\left| D_x \left(\frac{1}{\alpha} \eta(|x-y|) - \beta \zeta(t, x, y) \right) \right| \leq 2\sqrt{(\omega(a) + \beta)/\alpha} + \beta C_4$$

and

$$\left| D_v \left(\frac{1}{\alpha} \eta(|x-y|) - \beta \zeta(t, x, y) \right) \right| \leq 2\sqrt{(\omega(a) + \beta)/\alpha} + \beta C_4$$

for $(t, x, y) = (t_2, x_2, y_2)$, we set $R(\alpha) = 4\sqrt{(\omega(a) + \varepsilon_0)/\alpha} + 2\varepsilon_0 C_4$.

We may assume without loss of generality that $r \rightarrow \gamma(t, r)$ and $r \rightarrow \sigma(t, r, s)$ are concave, respectively, on $[0, a]$ and $[0, \infty)$ for all $0 < t < T$ and $s \geq 0$. Then we have

$$|\gamma(t, r) - \gamma(t, s)| \leq \gamma(t, |r - s|)$$

for all $0 < t < T$ and $0 \leq r, s \leq a$ and

$$|\sigma(t, r, R) - \sigma(t, s, R)| \leq \sigma(t, |r - s|, R)$$

for all $0 < t < T$ and $r, s, R \geq 0$.

By an argument similar to the proof of Lemma 8.1, we thus find that $t \rightarrow \gamma(t, \cdot)$ and $t \rightarrow \sigma(t, \cdot, R)$ belong to $L^1(0, T; C([0, a]))$ and $L^1(0, T; C([0, L]))$ for any $L, R > 0$, respectively. Extending γ and σ for $t \leq 0$ and $t \geq T$ by setting zero and mollifying them with respect to the variable t , we define $\gamma_\mu \in C([0, T] \times [0, a])$ and $\sigma_\mu(\cdot, \cdot, R) \in C([0, T] \times [0, \infty))$ for all $\mu > 0$ and $R \geq 0$ such that

$$(8.23) \quad \begin{cases} \int_0^T \sup_{0 \leq r \leq a} |\gamma_\mu(t, r) - \gamma(t, r)| dt \rightarrow 0, \\ \int_0^T \sup_{0 \leq r \leq L} |\sigma_\mu(t, r, R) - \sigma(t, r, R)| dt \rightarrow 0 \end{cases}$$

for all $L, R \geq 0$ as $\mu \rightarrow 0$.

Setting

$$b_\mu(t) = \sup_{0 \leq r \leq a} |\gamma_\mu(t, r) - \gamma(t, r)| + \sup_{0 \leq r \leq \varepsilon_0 C_4} |\sigma_\mu(t, r, R(\alpha)) - \sigma(t, r, R(\alpha))|,$$

we have

$$(8.24) \quad \tilde{H}(t, x, y, p, q) \geq -\sigma_\mu(t, |p + q|, R(\alpha)) - \gamma_\mu(t, |x - y|(1 + |p|)) - b_\mu(t)$$

if $\mu > 0$, $(t, x, y, p, q) \in J_{a, T} \times R^{2N}$, $|p| + |q| \leq R(\alpha)$, $|p + q| \leq \varepsilon_0 C_4$, and $|x - y|(1 + |p|) \leq a$. Assume $\mu > 0$ to be so small that $\|b\|_{L^1(0, T)} \leq \varepsilon_0$, and take b_μ for b in the definition of Ψ . According to (8.19), (8.22), (8.23), and (8.24), we obtain

$$\frac{\varepsilon}{(T - t_1)^2} - \beta \zeta_1(t_1, x_1, y_1) - \sigma_\mu(t_1, \beta C_4, R(\alpha)) - \gamma_\mu(t_1, \omega(\alpha, \beta)) \leq 0,$$

where $\omega(\alpha, \beta) = \sqrt{\alpha(\omega(a) + \beta)} + \omega(\sqrt{\alpha(\omega(a) + \beta)}) + \beta$, if $\omega(\alpha, \beta) < a$. Let $\beta \downarrow 0$ and $\alpha \downarrow 0$ in this order, to get $\varepsilon/T^2 \leq 0$, a contradiction. Q.E.D.

REMARK 8.2. The above proof together with Remark 8.1 shows the following: Let $\Theta \in C(Q_T \times R^2)$ and $\omega \in \Gamma(a)$, with $a > 0$. Assume that $\{\Theta(\cdot, \cdot, \cdot, r) | r \in R\}$ is equicontinuous on $Q_T \times R$, $r \rightarrow \Theta(t, x, u, r)$ is nondecreasing on R for all $(t, x, u) \in Q_T \times R$, and $u - v - \omega(|x - y|) \leq \Theta(t, x, u, r) - \Theta(t, y, v, r)$ holds for all $0 < t < T$, $x, y \in Q$ with $|x - y| \leq a$,

$r \in R$, and $u, v \in R$ with $u \geq v$. Assume (A4), (A5) with $\nu=0$, (A6), and (A7). Let $u, v \in UC_s(\bar{Q}_T)$ be a viscosity subsolution and a viscosity supersolution of (7.2). Then

$$(8.25) \quad \sup_{Q_T} (u-v) \leq \sup_{\partial_p Q_T} (u-v)_+.$$

9. Some properties of solutions of the Cauchy problem

We are first concerned with the continuity of solutions of the Cauchy problem (7.1).

LEMMA 9.1. Let $a > 0$, $\gamma_0 \in \Gamma(a)$, and $\gamma \in \Gamma_T(a)$. Assume that $\gamma_0 \in C^1((0, a])$, $s \rightarrow s + s\gamma'_0(s)$ is nondecreasing on $(0, a]$, and $\gamma'_0(a) = 0$. Suppose that $s \rightarrow \gamma(t, s)$ is concave for all $0 < t < T$. Define $\gamma_1: (0, T) \times R \rightarrow R$ by $\gamma_1(t, s) = \gamma(t, s)$ for $(t, s) \in (0, T) \times [0, a] = 0$ for $(t, s) \in (0, T) \times (-\infty, 0)$, and $= \gamma(t, a)$ for $(t, s) \in (0, T) \times (a, \infty)$. Then there exists a unique viscosity solution $\phi \in C([0, T] \times [0, a])$ of

$$(9.1) \quad \begin{cases} \phi(t, s) - \gamma_1(t, s + s\phi(t, s)) = 0 & \text{in } (0, T) \times (0, a), \\ \phi(t, 0) = 0, \quad \phi(t, a) = \gamma_0(a) + \int_0^t \gamma(\tau, a) d\tau & \text{for } 0 < t < T, \\ \phi(0, s) = \gamma_0(s) & \text{for } 0 < s < a. \end{cases}$$

In addition, $\phi \in \Gamma_T(a)$, $t \rightarrow \phi(t, s)$ is nondecreasing on $[0, T]$ for all $s \in [0, a]$, and $s \rightarrow \phi(t, s)$ is Lipschitz continuous on $[\varepsilon, a]$ for all $0 < \varepsilon < a$ and $t \in [0, T]$.

Proof. By the above concavity assumption, $|\gamma_1(t, r) - \gamma_1(t, s)| \leq \gamma_1(t, |r - s|)$ for all $0 < t < T$ and $r, s \in R$, and so the function: $(t, x, u, p) \rightarrow -\gamma_1(t, x + xp)$ on $(0, T) \times (0, a) \times R^2$ satisfies (A4)–(A7). Lemma 8.1 thus implies that $t \rightarrow \gamma_1(t, \cdot)$ belongs to $L^1(0, T; C([-R, R]))$ for all $R > 0$. We extend γ_1 to be zero for $t \leq 0$ and $t \geq T$, mollify the extended γ_1 with respect to the variable t , denote the resulting function by $\gamma_{1,\varepsilon}$, with $\varepsilon > 0$, and put

$$b_\varepsilon(t) = \sup_{r \in R} |\gamma_{1,\varepsilon}(t, r) - \gamma_1(t, r)| \quad \text{for } 0 < t < T.$$

Notice that $b_\varepsilon(t) \rightarrow 0$ in $L^1(0, T)$ as $\varepsilon \downarrow 0$.

For any fixed $n \in N$ we define $\phi^n_j \in \Gamma(a) \cap C^1((0, a])$ for $j=0, 1, \dots, n$ by setting $\phi^n_0 = \gamma_0$, $h = T/n$, and solving

$$(9.2) \quad \begin{cases} \phi^n_j(s) - h\gamma_{1,\varepsilon}\left(jh, s + s\frac{d}{ds}\phi^n_j(s)\right) - \phi^n_{j-1}(s) = 0 & (0 < s < a), \\ \phi^n_j(a) = h\gamma_{1,\varepsilon}(jh, a) + \phi^n_{j-1}(a) \end{cases}$$

for $j=1, 2, \dots, n$. By virtue of Lemma 3.1 the problem (9.2) has a unique solution for each $j=1, \dots, n$.

In view of (3.4) we have

$$(9.3) \quad \left| \frac{d}{ds} \phi^n_j(s) \right| \leq \frac{a}{s} \quad \text{for all } 0 < s < a.$$

By (3.5) we have

$$\phi^n_j(s) \leq \phi^n_{j-1}(s) + h\gamma_{1*}(jh, \beta)$$

for all $0 \leq s \leq a \exp\{-(\phi^n_j(a) + a)/\beta\}$, $0 < \beta \leq a$, and $j=1, \dots, n$. From this we obtain

$$(9.4) \quad \phi^n_k(s) \leq \gamma_0(s) + h \sum_{j=1}^k \gamma_{1*}(jh, \beta)$$

for all $0 \leq s \leq a \exp\{-(\phi^n_k(a) + a)/\beta\}$, $0 < \beta \leq a$, and $k=1, \dots, n$. From the second inequality in (9.2) we get

$$(9.5) \quad \phi^n_k(a) = \gamma_0(a) + h \sum_{j=1}^k \gamma_{1*}(jh, a) \quad \text{for all } k=1, \dots, n.$$

Now define $\phi^n \in C([0, T] \times [0, a])$ for $n \in N$ by setting

$$\phi^n(t, s) = \frac{t - (j-1)h}{h} \phi^n_j(s) + \frac{jh-t}{h} \phi^n_{j-1}(s)$$

if $(j-1)h \leq t \leq jh$ for some $j \in \{1, \dots, n\}$. We see from the first equality in (9.2) that $0 \leq \phi^n_j(s) - \phi^n_{j-1}(s) \leq h\gamma_{1*}(jh, a)$ for all $0 \leq s \leq a$, $j=1, \dots, n$, and $n \in N$. Hence

$$(9.6) \quad 0 \leq \phi^n(t, s) - \phi^n(\tau, s) \leq h \sum_{j=[t/h]+2}^{[s/h]} \gamma_{1*}(jh, a) \\ + (h([t/h]+1) - \tau) \gamma_{1*}([t/h]+1, a) + (t - h[t/h]) \gamma_{1*}([t/h]+1, a)$$

for all $0 \leq \tau < t \leq T$, $0 \leq s \leq a$, and $n \in N$, where $[x]$ denotes the integral part of $x \in R$. Thus we conclude from (9.3), (9.4), (9.5), and (9.6) that $\{\phi^n | n \in N\}$ is uniformly bounded and equicontinuous on $[0, T] \times [0, a]$.

Applying the Ascoli-Arzelà theorem, we find a subsequence of $\{\phi^n\}$ (which we denote again by $\{\phi^n\}$) and a $\phi^* \in C([0, T] \times [0, a])$ for each $\varepsilon > 0$ such that $\phi^n(t, s) \rightarrow \phi^*(t, s)$ uniformly on $[0, T] \times [0, a]$ as $n \rightarrow \infty$. In view of (9.3), (9.4), (9.5), and (9.6) we have

$$(9.7) \quad |\phi^*(t, s) - \phi^*(\tau, r)| \leq a \max \left\{ \frac{1}{s}, \frac{1}{r} \right\} |s - r|$$

for all $0 < r, s \leq a$,

$$(9.8) \quad \phi^*(t, a) = \gamma_0(a) + \int_0^t \gamma_{1*}(\tau, a) d\tau$$

for all $0 \leq t \leq T$,

$$(9.9) \quad \phi^*(t, s) = \gamma_0(s) + \int_0^t \gamma_{1*}(\tau, \beta) d\tau$$

for all $0 \leq t \leq T$, $0 \leq s \leq a \exp\{-(\phi^*(t, a) + a)/\beta\}$, and $0 < \beta \leq a$; and

$$(6.10) \quad 0 \leq \phi^*(t, s) - \phi^*(\tau, s) \leq \int_\tau^t \gamma_{1*}(\tau, a) d\tau$$

for all $0 \leq \tau < t \leq T$ and $0 \leq s \leq a$.

We want to show that ϕ^* is a viscosity solution of

$$(9.11) \quad \phi_t(t, s) - \gamma_{1*}(t, s + s\phi_s(t, s)) = 0 \quad \text{in } (0, T) \times (0, a).$$

To this end, let $\phi \in C^1((0, T) \times (0, a))$ and assume that $\phi^* - \phi$ attains a local strict maximum at $(t_0, s_0) \in (0, T) \times (0, a)$. Assuming that n is sufficiently large, $\phi^n - \phi$ has a local maximum at some point (t_1, s_1) near at (t_0, s_0) . Here we may assume that $(t_1, s_1) \rightarrow (t_0, s_0)$ as $n \rightarrow \infty$. Choose $j \in \{1, \dots, n\}$ so that $(j-1)h \leq t_1 \leq jh$, where $h = T/n$. Then we have

$$\begin{aligned} 0 &\leq \lim_{\delta \downarrow 0} \frac{1}{\delta} \{(\phi^n(t_1, s_1) - \phi(t_1, s_1)) - (\phi^n(t_1 - \delta, s_1) - \phi(t_1 - \delta, s_1))\} \\ &= \frac{1}{h} (\phi^n_{,j}(s_1) - \phi^n_{,j-1}(s_1)) - \phi_t(t_1, s_1), \end{aligned}$$

and $\phi^n_{,j}(t_1, s_1) = \phi_{,j}(t_1, s_1)$. Hence, using (9.2),

$$\phi_t(t_1, s_1) - \gamma_{1*}(jh, s_1 + s_1\phi_s(t_1, s_1)) \leq 0.$$

Letting $n \rightarrow \infty$, we have

$$\phi_t(t_0, s_0) - \gamma_{1*}(t_0, s_0 + s_0\phi_s(t_0, s_0)) \leq 0.$$

Similarly we find that if $\phi \in C^1((0, T) \times (0, a))$ and $\phi^* - \phi$ has a local strict minimum at some $(t_0, s_0) \in (0, T) \times (0, a)$ then

$$\phi_t(t_0, s_0) - \gamma_{1*}(t_0, s_0 + s_0\phi_s(t_0, s_0)) \leq 0.$$

Thus, in view that $\gamma_{1*} \in C(R \times R)$, we conclude that ϕ^* is a viscosity solution of (9.11) in the sense of Definition 7.1.

It follows from (9.7), (9.8), (9.9), and (9.10) that $\{\phi^* | \varepsilon > 0\}$ is equicontinuous on $[0, T] \times [0, a]$. Sending $\varepsilon \downarrow 0$ along a sequence, we find a function $\phi \in C([0, T] \times (0, a))$ as its limit which satisfies (9.1) in the viscosity sense according to Proposition 7.1. It is clear that $\phi \in \Gamma_T(a)$, $t \rightarrow (t, s)$ is nondecreasing on $[0, T]$ for all $s \in [0, a]$, and $s \rightarrow \phi(t, s)$ is Lipschitz continuous on $[\varepsilon, a]$ for all $0 < \varepsilon < a$ and $t \in [0, T]$. Q.E.D.

REMARK 9.1. For any $\gamma_0 \in \Gamma(a)$ there exists a $\tilde{\gamma}_0 \in \Gamma(a) \cap C^1([0, a])$ such that $s \rightarrow s + s\tilde{\gamma}_0'(s)$ is nondecreasing on $(0, a]$, $\tilde{\gamma}_0'(a) = 0$, and $\tilde{\gamma}_0 \geq \gamma_0$ on $[0, a]$. Indeed, selecting an increasing $\tilde{\gamma}_0 \geq \gamma_0$ on $[0, a]$ if necessary and, in view of Lemma 3.1, solving the problem $\phi'(s) - \tilde{\gamma}_0(s + s\phi'(s)) = 0$ for $0 < s < a$, $\phi(0) = 0$, and $\phi(a) = \gamma_0(a)$, we find a function $\phi \in \Gamma(a)$ having the required properties.

LEMMA 9.2. Let $a > 0$, and let $\gamma_0 \in \Gamma(a)$ and $\gamma \in \Gamma_T(a)$ satisfy the same assumptions as in Lemma 9.1. Define γ_{1*} as in Lemma 9.1. and $A_{a,T}$ by

$$A_{a,T} = \{(t, x, y) | 0 < t < T, x, y \in \mathbb{R}^N, |x - y| < a\}.$$

Let $\phi \in C([0, T] \times [0, a])$ be the viscosity solution of (9.1). Let $h: (0, T) \times \mathbb{R}^{2N} \rightarrow \mathbb{R}$ be such that $h(t, p, -p) = 0$ for all $0 < t < T$ and $p \in \mathbb{R}^N$ and $t \mapsto h(t, \cdot, \cdot)$ belongs to $L^1(0, T; C(K))$ for all compact $K \subset \mathbb{R}^{2N}$. Then $w \in UC(\bar{A}_{a,T})$ defined by $w(t, x, y) = \phi(t, |x - y|)$ solves

$$(9.12) \quad w_t - \gamma_1(|x - y|(1 + |D_x w|)) + h(t, Dw) \geq 0 \quad \text{in } A_{a,T}$$

in the viscosity sense, where $D = (D_x, D_y)$.

Proof. Let $\phi \in C^1(\bar{A}_{a,T})$, $b \in L^1(0, T)$, $G \in C(\bar{A}_{a,T} \times \mathbb{R}^{2N})$, $\delta > 0$, and $(t_0, x_0, y_0) \in A_{a,T}$. Suppose that

$$(t, x, y) \longrightarrow w(t, x, y) + \int_0^t b(s) ds - \phi(t, x, y)$$

has a local minimum at (t_0, x_0, y_0) and

$$(9.13) \quad G(t, x, y, p, q) + b(t) \geq -\gamma_1(|x - y|(1 + |p|)) + h(t, p, q)$$

holds for all $(x, y, p, q) \in B(x_0, y_0, D\phi(t_0, x_0, y_0); \delta)$ and a. a. $t \in B(t_0; \delta)$.

We intend to prove that

$$(9.14) \quad \phi_t(t_0, x_0, y_0) + G(t_0, x_0, y_0, p_0, q_0) \geq 0$$

holds, where $p_0 = D_x \phi(t_0, x_0, y_0)$ and $q_0 = D_y \phi(t_0, x_0, y_0)$. Note that

$$(9.15) \quad p_0 + q_0 = 0$$

by the same reasoning as in the proof of Lemma 3.2.

We consider first the case $x_0 = y_0$. Then $w(t_0, x_0, y_0) = \phi(t_0, 0) = 0$ for all $0 \leq t \leq T$ and hence

$$t \longrightarrow \int_0^t b(\tau) d\tau - \phi(t, x_0, y_0)$$

has a local minimum at $t = t_0$. Therefore

$$\int_{t_0}^t b(\tau) d\tau \geq \phi(t, x_0, y_0) - \phi(t_0, x_0, y_0)$$

and so

$$\int_{t_0}^t (b(\tau) - \phi_t(\tau, x_0, y_0)) d\tau \geq 0$$

if t is sufficiently close to t_0 . Thus, for any $\varepsilon > 0$, there is a sequence $\{\tau_n\} \subset (t_0 - \delta, t_0)$ converging to t_0 such that

$$b(\tau_n) - \phi_t(\tau_n, x_0, y_0) \leq \varepsilon \quad \text{for all } n \in N.$$

Combine this, (9.13) with $t = \tau_n$, $x = x_0$, $y = y_0$, $p = p_0$, and $q = q_0$, and (9.15), to obtain

$$\phi_t(\tau_n, x_0, y_0) + G(\tau_n, x_0, y_0, p_0, q_0) \geq -\varepsilon$$

for all $n \in N$. Sending $n \rightarrow \infty$ and then $\varepsilon \downarrow 0$, we conclude (9.14).

Now we turn to the case $x_0 \neq y_0$. Set $e = (x_0 - y_0)/|x_0 - y_0|$ and $s_0 = |x_0 - y_0|$. As in the proof of Lemma 3.2, we have

$$(9.16) \quad p_0 \cdot e = |p_0|.$$

As observed in the proof of Lemma 3.2,

$$w\left(t, x_0 + \frac{r}{2}e, y_0 - \frac{r}{2}e\right) = \phi(t, |x_0 - y_0 + re|) = \phi(t, |s_0 + r|) \quad \text{for all } 0 \leq t \leq T$$

and $r \in \mathbb{R}$ with $|s_0 + r| \leq a$. Therefore

$$(t, s) \longrightarrow \phi(t, s) + \int_{t_0}^t b(\tau) d\tau - \phi\left(t, x_0 + \frac{s-s_0}{2}e, y_0 - \frac{s-s_0}{2}e\right)$$

attains a local minimum at (t_0, s_0) . On the other hand, defining $g \in C((0, T) \times (0, a) \times \mathbb{R})$ by

$$g(t, s, r) = G\left(t, x_0 + \frac{s-s_0}{2}e, y_0 - \frac{s-s_0}{2}e, re, -re\right),$$

we have

$$g(t, s, r) + b(t) \geq -\gamma_1(t, s(1+r))$$

for all $(s, r) \in B((s_0, |p_0|); \delta)$ and a. a. $t \in B(t_0; \delta)$ by (9.13) and (9.16). Thus, using (9.16), we obtain

$$\phi_1(t_0, x_0, y_0) + g(t_0, s_0, |p_0|) \geq 0$$

as ϕ is a viscosity supersolution of (9.1). Noting that

$$g(t_0, s_0, |p_0|) = G(t_0, x_0, y_0, D_x \phi(t_0, x_0, y_0), D_x \phi(t_0, x_0, y_0))$$

by (9.16) and (9.15), we conclude (9.14). Q.E.D.

THEOREM 9.1. Assume (A4)-(A7). Let $u \in UC_x(\bar{Q}_T)$ be a viscosity solution of (7.1) and $r_0 \in \Gamma(a)$, where $a > 0$ is the number in (A7), satisfy the conditions that $|u(t, x) - u(t, y)| \leq r_0(a)$ if $0 \leq t \leq T$, $x, y \in \bar{\Omega}$, and $|x - y| \leq a$ and that $|u(t, x) - u(t, y)| \leq r_0(|x - y|)$ if $x, y \in \bar{\Omega}$, $0 \leq t \leq T$, and either $x \in \partial\Omega$, $y \in \partial\Omega$ or $t = 0$. Then there exists an $\omega \in \Gamma(a)$ such that

$$(9.17) \quad |u(t, x) - u(t, y)| \leq \omega(|x - y|)$$

for all $x, y \in \bar{\Omega}$ with $|x - y| \leq a$ and $0 \leq t \leq T$.

Proof. Let $A_{a,T} = \{(t, x, y) \in (0, T) \times \Omega^2 \mid |x - y| < a\}$ and $A_{a,T}^+ = \{(t, x, y) \in A_{a,T} \mid u(t, x) > u(t, y)\}$. Set $v(t, x, y) = u(t, x) - u(t, y)$ on $A_{a,T}$.

By using a change of the unknowns we may assume that (A5) is valid for $\nu = 0$. We may assume also that $r_0 \in C^1((0, a])$, $s \rightarrow s + sr_0'(s)$ is nondecreasing on $[0, a]$, $r_0'(a) = 0$, $r(t, a) \geq r_0(a)$ for all $0 < t < T$, and $s \rightarrow r(t, s)$ is concave on $[0, a]$ for all $0 < t < T$. We may assume for $\sigma \in \Sigma_T$ in (A6) that $s \rightarrow \sigma(t, s, r)$ is concave on $[0, \infty)$ for all $0 < t < T$.

and $r \geq 0$ and the inequality $\sigma(t, s, R) \leq \sigma(t, s, r) + 2\sigma(t, R - r, r)$ is satisfied for all $0 < t < T$, $s \geq 0$, and $0 \leq r \leq R$. (Indeed, the function

$$\sigma(t, s, r) = \sup \{ |H(t, x, u, p) - H(t, x, u, q)| \mid x \in \Omega, u \in R, p, q \in B(0; r), |p - q| \leq s \}$$

on $(0, T) \times [0, \infty) \times [0, \infty)$ satisfies the above two conditions after modification on a set of measure zero in $(0, T)$.

Define $\gamma_1: (0, T) \times R \rightarrow R$ as in Lemma 9.1 and $\tilde{H}: \mathcal{A}_{a,T} \times R^{2N} \rightarrow R$ by

$$\tilde{H}(t, x, y, p, q) = -\gamma_1(t, |x - y|(1 + |p|)) - \sigma(t, |p + q|, |p| + |q|).$$

Then $\tilde{H}$ satisfies (A4)-(A7) for $Q_T = \mathcal{A}_{a,T}$.

In view of (A5)-(A7) and Lemma 8.2, we find that

$$v_t + \tilde{H}(t, x, y, Dv) \leq 0 \quad \text{in } \mathcal{A}_{a,T}^+$$

in the viscosity sense, where $D = (D_x, D_y)$. Thus v_+ solves

$$(9.18) \quad \frac{\partial}{\partial t} v_+ + \tilde{H}(t, x, y, Dv_+) \leq 0 \quad \text{in } \mathcal{A}_{a,T}$$

in the viscosity sense.

Now let ϕ be the viscosity solution of (9.1) and set $w(t, x, y) = \phi(t, |x - y|)$ for $(t, x, y) \in \bar{\mathcal{A}}_{a,T}$. By Lemma 9.2 we see that

$$w_t + \tilde{H}(t, x, y, Dw) \geq 0 \quad \text{in } \mathcal{A}_{a,T}$$

in the viscosity sense.

According to Theorem 8.1 we conclude that $v_+(t, x, y) \leq w(t, x, y)$ on $\bar{\mathcal{A}}_{a,T}$. Thus, setting $\omega(r) = \max_{0 \leq t \leq T} \phi(t, r)$ for $0 \leq r \leq a$, we have (9.17).

REMARK 9.2. The function ω constructed above depends only on $\gamma_0 \in \Gamma(a)$ and $r \in \Gamma_T(a)$.

REMARK 9.3. The above method applies to prove the following with the help of Remark 8.1. Let $\Theta \in C(Q_T \times R^2)$. Assume the same hypotheses as in Theorem 9.1 except that u is a solution of (7.1). Assume instead that u is a viscosity solution of (7.2) and that $\{\Theta(\cdot, \cdot, \cdot, r) \mid r \in R\}$ is equicontinuous on $Q_T \times R$, $r \rightarrow \Theta(t, x, u, r)$ is nondecreasing on R for all $(t, x, u) \in Q_T \times R$, and $v - w - \gamma_0(|x - y|) \leq \Theta(t, x, v, r) - \Theta(t, y, w, r)$ holds for all $0 < t < T$, $x, y \in \Omega$ with $|x - y| \leq a$, $r \in R$, and $v, w \in R$ with $v \geq w$. Then (9.17) holds for the same $\omega \in \Gamma(a)$ as in Theorem 9.1.

Now we study the continuity in t of viscosity solutions of (7.1).

THEOREM 9.2. Assume (A4)-(A7). Let $u \in UC_s(\bar{Q}_T)$ be a viscosity solution of (7.1). Let $a > 0$ and $\omega \in \Gamma(a)$ be such that $|u(t, x) - u(t, y)| \leq \omega(|x - y|)$ for all $0 \leq t \leq T$ and $x, y \in \Omega$ with $|x - y| \leq a$. Let $L > 0$. Let $M > 0$ be larger than $\sup \{ |u(t, x)| \mid 0 \leq t \leq T, x \in \Omega \cap B(0; L + 1) \}$ and $\mu \in \Gamma(T)$ satisfy $|u(t, x) - u(s, x)| \leq \mu(|t - s|)$ for all $0 \leq t, s \leq T$

and $x \in B(0; L+1) \cap \partial\Omega$. Let $l \in L^1(0, T)$ satisfy $\max \{H(t, 0, M, 0), -H(t, 0, -M, 0)\} \leq l(t)$ for a. a. $0 < t < T$. Then there exists a $\bar{\mu} \in \Gamma(T)$ depending only on $L, M, \omega, \mu, l, \sigma$ in (A6), and r in (A7) such that

$$(9.19) \quad |u(t, x) - u(s, x)| \leq \bar{\mu}(|t-s|)$$

holds for all $0 \leq t, s \leq T$ and $x \in \Omega \cap B(0; L)$.

Proof. For any $\varepsilon > 0$ choose a standard mollification kernel $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ so that $\text{supp } \rho_\varepsilon \subset B(0; \varepsilon)$ and

$$\int_{\mathbb{R}^N} |D\rho_\varepsilon(x)| dx < C_{14}/\varepsilon$$

for any $\varepsilon > 0$ and some $C_{14} > 0$.

We can extend u to $[0, T] \times \bar{\Omega}_\delta$, where $\Omega_\delta = \{x \in \mathbb{R}^N \mid \text{dist}(x, \Omega) < \delta\}$, in such a way that $\delta > 0$ depends only on ω and the resulting function $\bar{u}$ satisfies $|\bar{u}(t, x) - \bar{u}(t, y)| \leq \bar{\omega}(|x-y|)$ for all $0 \leq t \leq T$, $x, y \in \Omega$ with $|x-y| \leq a/2$, and some $\bar{\omega} \in \Gamma(a/2)$, where $\bar{\omega}$ depends only on ω .

Fix $0 < \varepsilon < \min\{a/2, \delta\}$ and $0 \leq s \leq T$. Define $u_{\varepsilon, \cdot} \in C^\infty(\bar{\Omega})$ by the convolution $u_{\varepsilon, \cdot} = \rho_\varepsilon * \bar{u}(s, \cdot)$. It is easily verified that $|u_{\varepsilon, \cdot}(x) - u(s, x)| \leq \bar{\omega}(\varepsilon)$ and $|Du_{\varepsilon, \cdot}(x)| \leq C_{14}\bar{\omega}(\varepsilon)/\varepsilon$ for all $x \in \Omega$.

Select a C^1 function ϕ on $\mathbb{R}^N$ so that $\phi(x) = 0$ for all $x \in B(0; L)$, $\phi(x) = M$ for all $x \in \mathbb{R}^N$ and some $C_{15} > 0$. We now define $v_{\varepsilon, \cdot} \in UC_s([t, T] \times \bar{\Omega})$ by

$$(9.20) \quad \begin{aligned} v_{\varepsilon, \cdot}(t, x) = & u_{\varepsilon, \cdot}(x) + \bar{\omega}(\varepsilon) + \phi(x) + \mu(t-s) \\ & + \int_s^t \{l(\tau) + \gamma(\tau, L+1) + \theta(\tau, C_{14}\bar{\omega}(\varepsilon)/\varepsilon + C_{15})\} d\tau \end{aligned}$$

It is not hard to check that $v_{\varepsilon, \cdot} \geq u$ on the parabolic boundary of $(s, T) \times (\Omega \cap B(0; L+1))$ and that $v_{\varepsilon, \cdot}$ is a viscosity supersolution of (7.1) in $(s, T) \times (\Omega \cap B(0; L+1))$. Thus we conclude that $u \leq v_{\varepsilon, \cdot}$ on $[s, T] \times (\bar{\Omega} \cap B(0; L+1))$. We have therefore

$$u(t, x) \leq u(s, x) + 2\bar{\omega}(\varepsilon) + \mu(t-s) + \int_s^t \{l(\tau) + \gamma(\tau, L+1) + \theta(\tau, C_{14}\bar{\omega}(\varepsilon)/\varepsilon + C_{15})\} d\tau$$

for all $0 \leq s < t \leq T$, $x \in \bar{\Omega} \cap B(0; L)$, and $0 < \varepsilon < \min\{a/2, \delta\}$.

Similarly we have

$$u(t, x) \geq u(s, x) - 2\bar{\omega}(\varepsilon) - \mu(t-s) + \int_s^t \{l(\tau) + \gamma(\tau, L+1) + \theta(\tau, C_{14}\bar{\omega}(\varepsilon)/\varepsilon + C_{15})\} d\tau$$

for all $0 \leq s < t \leq T$, $x \in \bar{\Omega} \cap B(0; L)$, and $0 < \varepsilon < \min\{a/2, \delta\}$. These two inequalities together imply the existence of $\bar{\mu} \in \Gamma(T)$ for which (9.18) holds. Q.E.D.

REMARK 9.4. The following assertion can be proved by the same method as above with the aid of Remark 8.2. Let $\theta \in C(Q_T \times \mathbb{R}^2)$ and $\bar{\omega} \in \Gamma(a)$, with $a > 0$. Assume that

$\{\theta(\cdot, \cdot, \cdot, r) | r \in R\}$ is equicontinuous on $Q_T \times R$, $r \rightarrow \theta(t, x, u, r)$ is nondecreasing on R for all $(t, x, u) \in Q_T \times R$, and $u - v - \bar{\omega}(|x - y|) \leq \theta(t, x, u, r) - \theta(t, y, v, r)$ holds for all $u, v \in R$ with $u \geq v$, $x, y \in \Omega$ with $|x - y| \leq a$, $0 \leq t \leq T$, and $r \in R$. In addition to these assumptions we assume the hypotheses of Theorem 9.2 where u is now understood to be a viscosity solutions of (7.2) and that $\theta(t, x, u(t, x), J) \geq 0$ and $\theta(t, x, u(t, x), -J) \leq 0$ for all $0 < t < T$ and $x \in \Omega \cap B(0; L+1)$. Then there exists a $\bar{\mu} \in F(T)$ depending only on $J, L, M, \omega, \mu, l, \sigma$ in (A6), and γ in (A7) such that

$$(9.21) \quad |u(t, x) - u(s, x)| \leq \bar{\mu}(|t - s|)$$

holds for all $0 \leq t, s \leq T$ and $x \in \Omega \cap B(0; L)$.

Next we turn our attention to the asymptotic behavior of viscosity solutions of (7.1) as $|x| \rightarrow \infty$.

$$H^*(t, x, u, p) = [H(t, x, u-0, p), H(t, x, u+0, p)]$$

for $(t, x, u, p) \in Q_T \times R \times R^N$. Consider the Cauchy problem

$$(9.22) \quad \begin{cases} u_t(t, x) + H^*(t, x, u(t, x), 0) \ni 0 & \text{a. e. in } (0, T), \\ u(0, x) = u_0(x), \end{cases}$$

where the variables x are regarded as parameters and u_0 is assumed to belong to $UC(\Omega)$. It is not hard to see by standard approximation techniques that the problem (9.22) has a unique absolutely continuous solution $u^*(t, x)$ defined on $[0, T]$ for each $x \in \Omega$.

From (9.22), in view of (A5) and (A7), we have

$$\frac{\partial}{\partial t} |u^*(t, x) - u^*(t, y)| + \nu |u^*(t, x) - u^*(t, y)| \leq \gamma(t, |x - y|)$$

for all $x, y \in \Omega$ with $|x - y| \leq a$ and a. a. $0 < t < T$. Thus

$$e^{\nu t} |u^*(t, x) - u^*(t, y)| \leq |u_0(x) - u_0(y)| + \int_0^t e^{\nu s} \gamma(s, |x - y|) ds$$

for all $0 \leq t \leq T$ and $x, y \in \Omega$ with $|x - y| \leq a$. This means that $u^* \in UC_s([0, T] \times \Omega)$.

LEMMA 9.3. *The function $u^* \in UC_s([0, T] \times \Omega)$ defined as a solution of (9.22) is a viscosity solution of*

$$(9.23) \quad u_t(t, x) + H(t, x, u(t, x), 0) = 0 \quad \text{in } Q_T.$$

Proof. Let $\phi \in C^1(Q_T)$, $(t_0, x_0) \in Q_T$, $b \in L^1(0, T)$, and $G \in C(Q_T \times R^N)$ be such that

$$\Phi : (t, x) \longrightarrow u^*(t, x) + \int_0^t b(s) ds - \phi(t, x)$$

has a local maximum at (t_0, x_0) and that $b(t) + G(t, x, p) \leq H(t, x, r, 0)$ for all $(x, r, p) \in B((x_0, u(t_0, x_0), D\phi(t_0, x_0)); \delta)$, a. a. $t \in B(t_0; \delta)$, and some $\delta > 0$.

Since $t \rightarrow \phi(t, x_0)$ is absolutely continuous on $[0, T]$ and attains a local maximum at t_0 , we see that $[t_0 - \varepsilon, t_0] \cap \{t \mid \phi_t(t, x_0) \geq 0\}$ has positive measure for any $\varepsilon > 0$. This implies that there exists a sequences $\{t_n\} \subset [0, t_0]$ such that $t_n \rightarrow t_0$, as $n \rightarrow \infty$, $\phi_t(t_n, x_0) = u^*_t(t_n, x_0) + b(t_n) - \phi_t(t_n, x_0) \geq 0$, $u^*_t(t_n, x_0) + H(t_n, x_0, u^*(t_n, x_0) - 0, 0) \leq 0$, and

$$b(t_n) + G(t_n, x, p) \leq H(t_n, x, r, 0)$$

for all $(x, r, p) \in B((x_0, u^*(t_0, x_0), D\phi(t_0, x_0); \delta)$. Thus

$$\phi_t(t_n, x_0) + G(t_n, x_0, D\phi(t_0, x_0)) \leq 0$$

for sufficiently large $n \in N$. Sending $n \rightarrow \infty$, we conclude that u^* is a viscosity subsolution of (9.23). Similarly we find that u^* is also a viscosity supersolution of (9.23).

Q. E. D.

Our result on the asymptotic behavior is this

THEOREM 9.3. Assume (A4)-(A7). Let $u, u^* \in UC_s(\bar{Q}_T)$ be a viscosity solution of (7.1) and the viscosity solution of (9.23) with the initial data $u_0 = u(0, \cdot)$. Then

$$(9.24) \quad \sup_{Q_T} |u - u^*| < \infty.$$

Proof. Without any loss of generality we may assume the constant ν in (A5) to be zero and that $|u(t, x) - u(t, y)| \leq \omega(|x - y|)$ and $|u^*(t, x) - u^*(t, y)| \leq \omega(|x - y|)$ hold for some $\omega \in \Gamma(a)$ if $|x - y| \leq a$.

Set $v(t, x, y) = u(t, x) - u^*(t, y)$ on $\Delta_{a,r}^+$, where $\Delta_{a,r}^+$ denotes the set

$$\{(t, x, y) \in (0, T) \times \Omega^2 \mid |x - y| < a, u(t, x) > u^*(t, y)\}.$$

By the same method as in the proof of Lemma 8.2 we see that

$$(9.25) \quad v_t - \gamma(t, |x - y|) - \partial(t, |D_x v|) \leq 0 \quad \text{in } \Delta_{a,r}^+$$

in the viscosity sense.

Set $A = 24\omega(a/2)/a$. We claim that

$$(9.26) \quad v(t, x, x) \leq \omega(a) + \int_0^t \{\gamma(s, a) + \partial(s, A)\} ds$$

for all $(t, x) \in Q_T$ such that $\text{dist}(x, \Omega^c) > a$.

To this end we suppose that (9.26) is false. Taking into account that $v(0, x, x) = 0$ for all $x \in \Omega$, we can then choose a point $(t_0, x_0) \in Q_T$ and an $\varepsilon > 0$ such that $\text{dist}(x_0, \Omega^c) > a$ and

$$t \longrightarrow v(t, x_0, x_0) - \omega(a) - \int_0^t \{\gamma(s, a) + \partial(s, A)\} ds - \frac{\varepsilon}{T-t}$$

attains a positive maximum at $t = t_0$. Note that $u(t, x) \leq u(t, x_0) + \omega(a/2)$ and $u^*(t, x) \leq u^*(t, x_0) + \omega(a/2)$ for all $0 \leq t \leq T$ and $x \in \partial B(x_0; a/2)$ and that $v(t, x, y) \leq \omega(a)$ if $|x - y| \leq a$.

We let $\alpha = 12\omega(a/2)/a^2$ and deduce that

$$(t, x, y) \longrightarrow v(t, x, y) - \omega(a) - \int_0^t \{\gamma(s, a) + \delta(s, A)\} ds - \frac{\varepsilon}{T-t} - \alpha(|x-x_0|^2 + |y-x_0|^2)$$

has a positive maximum at some point $(t_1, x_1, y_1) \in (0, T) \times B(x_0; a/2)^2$. In particular $v(t_1, x_1, y_1) > 0$ and so $(t_1, x_1, y_1) \in \Delta_{\varepsilon, r}^+$.

Since v is a viscosity subsolution of (9.25) and $|D_x(\alpha|x-x_0|^2)| < A$ for $x=x_1$, we find that $\varepsilon/(T-t_1)^2 \leq 0$, a contradiction. From this we conclude (9.26) and hence

$$u(t, x) - u^*(t, x) \leq 3\omega(a) + \int_0^t \{\gamma(s, a) + \delta(s, A)\} ds$$

for all $(t, x) \in Q_r$. Similarly we have

$$u^*(t, x) - u(t, x) \leq 3\omega(a) + \int_0^t \{\gamma(s, a) + \delta(s, A)\} ds$$

for all $(t, x) \in Q_r$. Thus we finished the proof. Q.E.D.

REMARK 9.5. A remark analogous to Remark 5.1 is valid for Theorem 9.3.

10. An existence result for the Cauchy problem

We will prove the following theorem in this section.

THEOREM 10.1. *Assume (A4)-(A7) and that there exist a viscosity subsolution $v \in UC_+(\bar{Q}_T)$ and a viscosity supersolution $w \in UC_-(\bar{Q}_T)$ of (7.1) such that $v=w$ on $[0, T] \times \partial\Omega$ and $v(0, \cdot) \leq w(0, \cdot)$ on Ω . Let $u_0 \in UC(\bar{\Omega})$ satisfy $v(0, \cdot) \leq u_0 \leq w(0, \cdot)$ on Ω . Then there exists a unique viscosity solution $u \in UC_+(\bar{Q}_T)$ of (7.1) such that $u=v$ on $[0, T] \times \partial\Omega$ and $u(0, \cdot) = u_0$ on Ω .*

An easy consequence of this result is the following.

COROLLARY 10.1. *Let $\Omega = R^N$ and assume (A4)-(A7). Then for any $u_0 \in UC(R^N)$ there exists a (unique) viscosity solution $u \in UC_+(\bar{Q}_T)$ of (7.1) satisfying $u(0, \cdot) = u_0$ on R^N .*

Proof. In view of Theorem 10.1 it is enough to prove the existence of a viscosity subsolution and a viscosity supersolution of (7.1) which take, respectively, smaller and larger values than u_0 on R^N for $t=0$.

We may assume that $s \rightarrow \gamma(t, s)$ is defined on $[0, \infty)$ for any $0 < t < T$ and $\gamma(t, s) \leq \gamma_0(t)(1+s)$ is satisfied for all $0 < t < T$, $s \geq 0$, and some $\gamma_0 \in L^1(0, T)$. We can choose a constant $C_{16} > 0$ such that

$$|u_0(x)| \leq C_{16}(1+|x|) \quad \text{for all } x \in R^N.$$

Set

$$f(t) = \int_0^t \gamma_0(\tau) d\tau + C_{16},$$

$$g(t) = \int_0^t (|H(\tau, 0, 0, 0)| + \partial(\tau, f(T)) + r_0(\tau)) d\tau + C_{16},$$

$$v(t, x) = -f(t)\langle x \rangle - g(t) \quad \text{and} \quad w(t, x) = f(t)\langle x \rangle + g(t)$$

for all $0 \leq t \leq T$ and $x \in R^N$. It is easy to check that $v, w \in UC_s(\bar{Q}_T)$, v and w are, respectively, viscosity sub- and supersolutions of (7.1), and $v(0, x) \leq u_0(x) \leq w(0, x)$ for all $x \in R^N$. Q.E.D.

For the proof of Theorem 10.1 we need the next lemma.

LEMMA 10.1. *Under the assumptions of Theorem 10.1 there exist a viscosity sub-solution $v^* \in UC_s(\bar{Q}_T)$ and a viscosity supersolution $w^* \in UC_s(\bar{Q}_T)$ of (7.1) such that $v^* = w^* = v$ on $(0, T) \times \partial\Omega$, $v^*(0, \cdot) \leq u_0 \leq w^*(0, \cdot)$ on Ω , and*

$$(10.1) \quad \sup_{Q_T} (w^* - v^*) < \infty.$$

Proof. By making a change of the unknown we may assume the constant ν in (A5) to be zero.

Fix $0 < \varepsilon \leq a$ and let ρ_ε be a standard mollification kernel on R^N with $\text{supp } \rho_\varepsilon \subset B(0; \varepsilon)$. Set $C_\varepsilon = \sup_{R^N} |D\rho_\varepsilon|$. Let u^* be the solution of (9.22). We may assume that

$$|u^*(t, x) - u^*(t, y)| \leq \omega(|x - y|), \quad |v(t, x) - v(t, y)| \leq \omega(|x - y|),$$

$$\text{and} \quad |w(t, x) - w(t, y)| \leq \omega(|x - y|)$$

for all $x, y \in \Omega$ with $|x - y| \leq a$, $0 \leq t \leq T$, and some $\omega \in \Gamma(a)$. Define $u^\varepsilon = \rho_\varepsilon * u^*$ in $[0, T] \times \Omega_\varepsilon$, where $\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \Omega^c) < \varepsilon\}$, and note that

$$|u^\varepsilon(t, x) - u^\varepsilon(t, y)| \leq \omega(|x - y|)$$

for all $0 \leq t \leq T$ and $x, y \in \Omega$ with $|x - y| \leq a$,

$$\sup_{Q_T} |u^* - u^\varepsilon| \leq \omega(\varepsilon), \quad \text{and} \quad \sup_{Q_T} |Du^\varepsilon| \leq C_\varepsilon \omega(\varepsilon) / \varepsilon.$$

From the first two and (9.16) we have

$$u^*(t, y) - u^*(s, y) \leq - \int_s^t H(\tau, y, u^*(\tau, x) - 2\omega(\varepsilon) - 0, 0) d\tau$$

and

$$u^*(t, y) - u^*(s, y) \geq - \int_s^t H(\tau, y, u^*(\tau, x) + 2\omega(\varepsilon) + 0, 0) d\tau$$

for all $0 \leq s < t \leq T$ and $x, y \in \Omega$ with $|x - y| \leq a$. Multiplying these by $\rho_\varepsilon(x - y)$ and integrating with respect to y , by using (A7) we get

$$u^\varepsilon(t, x) - u^\varepsilon(s, x) \leq - \int_s^t (H(\tau, x, u^\varepsilon(\tau, x) - 2\omega(\varepsilon) - 0, 0) - r(\tau, \varepsilon)) d\tau$$

and

$$u^s(t, x) - u^s(s, x) \geq - \int_s^t (H(\tau, x, u^s(\tau, x) + 2\omega(\varepsilon) + 0, 0) + \gamma(\tau, \varepsilon)) d\tau$$

for all $0 \leq s < t \leq T$ and $x \in \Omega_-$. These imply that $t \rightarrow u^s(t, x)$ is absolutely continuous on $[0, T]$ for each $x \in \Omega_-$, and hence that

$$u^s_t(t, x) + H(t, x, u^s(t, x) - 2\omega(\varepsilon) - 0, 0) - \gamma(t, \varepsilon) \leq 0$$

and

$$u^s_t(t, x) + H(t, x, u^s(t, x) + 2\omega(\varepsilon) + 0, 0) + \gamma(t, \varepsilon) \geq 0$$

hold for a. a. $0 \leq t \leq T$ and each $x \in \Omega_-$. Thus, setting

$$v^s(t, x) = u^s(t, x) - 2\omega(\varepsilon) - \int_0^t (\gamma(\tau, \varepsilon) + \partial(\tau, C_s \omega(\varepsilon)/\varepsilon)) d\tau - 1$$

and

$$w^s(t, x) = u^s(t, x) + 2\omega(\varepsilon) + \int_0^t (\gamma(\tau, \varepsilon) + \partial(\tau, C_s \omega(\varepsilon)/\varepsilon)) d\tau + 1$$

for $(t, x) \in [0, T] \times \Omega_-$, we see that

$$v^s_t(t, x) + H(t, x, v^s(t, x) - 0, p) \leq 0$$

and

$$w^s_t(t, x) + H(t, x, w^s(t, x) + 0, p) \geq 0$$

hold for a. a. $0 \leq t \leq T$, all $p \in B(0; C_s(\varepsilon)/\varepsilon)$ and each fixed $x \in \Omega_-$.

We now follow the method of proof of Lemma 9.2 and conclude that

$$v_t^s + H(t, x, v^s, Dv^s) \leq 0 \quad \text{in } (0, T) \times \Omega_-$$

and

$$w_t^s + H(t, x, w^s, Dw^s) \geq 0 \quad \text{in } (0, T) \times \Omega_-$$

in the viscosity sense. Define

$$v^*(t, x) = \begin{cases} v(t, x) & \text{for } (t, x) \in [0, T] \times (\bar{\Omega} \setminus \Omega_-), \\ \max \{v^s(t, x), v(t, x)\} & \text{for } (t, x) \in [0, T] \times \Omega_- \end{cases}$$

and

$$w^*(t, x) = \begin{cases} w(t, x) & \text{for } (t, x) \in [0, T] \times \bar{\Omega} \setminus \bar{\Omega}_-, \\ \min \{w^s(t, x), w(t, x)\} & \text{for } (t, x) \in [0, T] \times \Omega_- \end{cases}$$

Then in view of Remark 9.5 v^* and w^* have the required properties. Q.E.D.

Proof of Theorem 10.1. By changing the unknown if necessary we may assume $\nu=1$ in (A5). We can rewrite (7.1) in the form

$$(10.2) \quad u + u_t + H(t, x, u, Du) = 0 \quad \text{in } Q_T,$$

where H satisfies (A4), (A5) with $\nu=0$, (A6), and (A7).

By virtue of Lemma 10.1 we may assume that

$$\sup_{Q_T} (w-v) \leq J \quad \text{for some } J > 0.$$

By virtue of Theorem 8.1 we have

$$(10.3) \quad v \leq w \leq v+J \quad \text{on } \bar{Q}_T.$$

Let L , R , α , β , and ε be positive numbers. Let $\tilde{\Omega}_L = \Omega \cap B(0; L)^\circ$ and define $\zeta_{\alpha, L} \in BUC(R^N)$ and $\eta_\beta \in BUC([0, \infty))$ by setting

$$\zeta_{\alpha, L}(x) = \begin{cases} \frac{1}{\alpha} \text{dist}(x, \tilde{\Omega}_L^\circ) & \text{if } \text{dist}(x, \tilde{\Omega}_L^\circ) \leq \alpha, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\eta_\beta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T - \beta, \\ \frac{1}{\beta}(T - t) & \text{for } T - \beta \leq t \leq T, \\ 0 & \text{for } t \geq T. \end{cases}$$

Here and later we assume α and β to be small enough and L to be large enough. Let $\theta \in C(R)$ be the function defined by

$$\theta(r) = \begin{cases} 1 & \text{for } r \geq 1, \\ r & \text{for } |r| \leq 1, \\ -1 & \text{for } r \leq -1, \end{cases}$$

and set $\theta_M(r) = M\theta(r/M)$ for all $r \in R$ and $M > 0$. Define $H_R : Q_T \times R \times R^N \rightarrow R$ by

$$H_R(t, x, u, p) = H(t, x, \theta_R(u), \theta_R(p))$$

and then $H_{R, \cdot} : Q_T \times R \times R^N \rightarrow R$ by extending H_R for $t \leq 0$ and $t \geq T$ and mollifying the resulting function with respect to the variable t as in the same way as the proof of Lemma 8.2. Note that H_R satisfies (A4)–(A7) with the same ν , σ , and γ as those for H and that

$$\sup \{ |H_{R, \cdot}(t, x, u, p) - H_{R, \cdot}(s, y, u, q)| \mid 0 < t, s < T, x, y \in \tilde{\Omega}_L, \\ p, q \in R^N, |t-s| + |x-y| + |p-q| \leq r, u \in R \} \rightarrow 0$$

as $r \downarrow 0$. Finally we restrict v to $[0, T] \times \tilde{\Omega}_{L+1}$ and then extend the result to $[0, \infty) \times R^N$ so that the resulting function v_L belongs to $BUC([0, \infty) \times R^N)$.

Now we treat the approximate problem

$$(10.4) \quad u + \xi_{\alpha, \beta, L}(t, x) \theta_J(u_t + H_{R, \cdot}(t, x, u, Du) + v(t, x)) - v_L(t, x) = 0 \\ \text{in } (0, \infty) \times R^N,$$

where $\xi_{\alpha, \beta, L}$ denotes the function: $(t, x) \rightarrow \eta_\beta(t) \zeta_{\alpha, L}(x)$, with the initial data

$$(10.5) \quad u(0, x) = \zeta_{\alpha, L}(x) u_0(x) + (1 - \zeta_{\alpha, L}(x)) v_L(0, x)$$

for all $x \in R^N$.

We want to apply Theorem 6.1 in order to find a unique viscosity solution of this problem. The function

$$(x_0, x, u, p_0, p) \rightarrow \xi_{\alpha, \beta, L}(x_0, x) \theta_J(p_0 + H_{R, \varepsilon}(x_0, x, u, p) + v(x_0, x)) - v_L(x_0, x)$$

satisfies (A1)–(A3) for $\mathcal{Q} = (0, \infty) \times R^N$. We let $u_{0, \alpha, L}$ denote the function on R^N defined by the right side of (10.5). Let $\tilde{\mathcal{Q}}_{L, 1} = \{x \in R^N \mid \text{dist}(x, \tilde{\mathcal{Q}}_L) < 1\}$ and choose an increasing sequence $\{M_n\}$ of positive numbers such that

$$|H_{R, \varepsilon}(t, x, u, p) + v(t, x)| \leq M_n$$

for all $0 \leq t \leq T$, $x \in \tilde{\mathcal{Q}}_L$, $u \in R$, $p \in R^N$, and $n \in N$ and

$$|v_L(t, x) - v_L(0, x)| \leq \frac{1}{n} + M_n t$$

for all $t \geq 0$, $x \in \tilde{\mathcal{Q}}_{L, 1}$, and $n \in N$. Choose a function $\zeta \in C(R^N)$ such that $0 \leq \zeta \leq 1$ on R^N , $\zeta = 1$ on $\tilde{\mathcal{Q}}_L$ and $\text{supp } \zeta \subset \tilde{\mathcal{Q}}_{L, 1}$. Then the functions $\bar{v}, \bar{w} \in UC([0, \infty) \times R^N)$ defined by

$$\bar{v}(t, x) = \zeta(x) \left\{ u_{0, \alpha, L}(x) - \inf \left\{ \frac{1}{n} + M_n t \mid n \in N \right\} \right\} + (1 - \zeta(x)) v_L(t, x)$$

and

$$\bar{w}(t, x) = \zeta(x) \left\{ u_{0, \alpha, L}(x) + \inf \left\{ \frac{1}{n} + M_n t \mid n \in N \right\} \right\} + (1 - \zeta(x)) v_L(t, x)$$

are, respectively, a viscosity subsolution and a viscosity supersolution of (10.4) by Proposition 1.4. Moreover $\bar{v}(0, x) = \bar{w}(0, x) = u_{0, \alpha, L}(x)$ for all $x \in R^N$. We thus conclude from Theorem 6.1 that the problem (10.4) and (10.5) has a unique viscosity solution in $UC([0, \infty) \times R^N)$ which we denote by $u_{\alpha, L, \beta, R, \varepsilon}$.

Note that $H_{R, \varepsilon}$ satisfies (A4)–(A7) with $\nu = 0$, $\sigma = \sigma_\varepsilon$, and $\gamma = \gamma_\varepsilon$, where σ_ε and γ_ε are defined as in the proof of Lemma 8.1. A review of the proof of Lemma 9.1, Theorem 9.1 and 9.2 shows that the conclusions in Remarks 9.3 and 9.4 are valid for the collection $\{H_{R, \varepsilon} \mid \varepsilon > 0\}$. Thus the family $\{u_{\alpha, L, \beta, R, \varepsilon} \mid \varepsilon > 0\}$ is equicontinuous on $[0, \infty) \times R^N$ and

$$|u_{\alpha, L, \beta, R, \varepsilon}(t, x) - u_{\alpha, L, \beta, R, \varepsilon}(s, y)| \leq \omega_{\alpha, L, \beta, R}(|x - y| + |t - s|)$$

holds for all $(t, x), (s, y) \in [0, \infty) \times R^N$ with $|x - y| + |t - s| \leq a$ and some $\omega_{\alpha, L, \beta, R} \in \Gamma(a)$, while it is obvious that the family is uniformly bounded on $[0, \infty) \times R^N$. Therefore

$$u_{\alpha, L, \beta, R, \varepsilon}(t, x) \rightarrow u_{\alpha, L, \beta, R}(t, x)$$

uniformly on $[0, \infty) \times R^N$ for some $u_{\alpha, L, \beta, R} \in BUC([0, \infty) \times R^N)$ as $\varepsilon \downarrow 0$ along a suitable sequence and, by virtue of Remark 7.1, $u_{\alpha, L, \beta, R}$ is a viscosity solution of

$$(10.6) \quad u + \xi_{\alpha, \beta, L}(t, x) \theta_J(u_t + H_R(t, x, u, Du) + v(t, x)) - v_L(t, x) = 0 \quad \text{in } (0, \infty) \times R^N.$$

Using Remarks 9.3 and 9.4 once again we see that $\{u_{\alpha,L,\beta,R} | R > 0, 0 < \beta < T/2\}$ is equicontinuous on $[0, T] \times R^N$ and moreover

$$|u_{\alpha,L,\beta,R}(t, x) - u_{\alpha,L,\beta,R}(s, y)| \leq \omega_{\alpha,L}(|x - y| + |t - s|)$$

for all $x, y \in R^N$, $t, s \in [0, T]$ with $|x - y| + |t - s| \leq T$ and some $\omega_{\alpha,L} \in \Gamma(T)$. It is obviously uniformly bounded on $[0, T] \times R^N$. Thus

$$u_{\alpha,L,\beta,R}(t, x) \rightarrow u_{\alpha,L}(t, x)$$

uniformly on $[0, T] \times R^N$ for some $u_{\alpha,L} \in BUC([0, T] \times R^N)$ as $\beta \downarrow 0$ and $R \rightarrow \infty$ along a sequence. $u_{\alpha,L}$ now solves

$$(10.7) \quad u + \zeta_{\alpha,L}(x)\theta_J(u_t + H(t, x, u, Du) + v(t, x)) - v_L(t, x) = 0 \quad \text{in } (0, T) \times R^N$$

in the viscosity sense.

It is easily checked that v and w are, respectively, a subsolution and a supersolution of (10.7) in $(0, T) \times \tilde{D}_{L+1}$ in the viscosity sense by (10.3). By Remark 8.2 it follows that $v(t, x) \leq u_{\alpha,L}(t, x) \leq w(t, x)$ for all $(t, x) \in [0, T] \times \tilde{D}_{L+1}$. Applying Remarks 9.3 and 9.4 once more, we see that $\{u_{\alpha,L} | L > M\}$ is equicontinuous on $[0, T] \times \tilde{D}_M$ for each $M > 0$ and $|u_{\alpha,L}(t, x) - u_{\alpha,L}(t, y)| \leq \omega_{\alpha,L}(|x - y|)$ for all $0 \leq t \leq T$, $x, y \in \tilde{D}_L$ with $|x - y| \leq 1$ and some $\omega_{\alpha,L} \in \Gamma(1)$. Clearly $\{u_{\alpha,L} | L > M\}$ is uniformly bounded on $[0, T] \times \tilde{D}_M$ for each $M > 0$, and then we find a viscosity solution $u_{\alpha} \in UC_{\alpha}(Q_T)$ of

$$(10.8) \quad u + \zeta_{\alpha}(x)\theta_J(u_t + H(t, x, u, Du) + v(t, x)) - v(t, x) = 0 \quad \text{in } Q_T.$$

Since v and w are a subsolution and a supersolution of (10.8) in the viscosity sense, respectively, by Remark 8.2 we have

$$v(t, x) \leq u_{\alpha}(t, x) \leq w(t, x) \quad \text{in } Q_T.$$

Thus, from (10.8), we obtain

$$\sup_{(0,T) \times \Omega_{-\delta}} |u_{\alpha} - u_{\beta}| \leq \sup_{(0,T) \times (\Omega \setminus \Omega_{-\delta})} (w - v)$$

for all $0 < \alpha, \beta < \delta$ by Remark 8.2, where $\Omega_{-\delta} = \{x \in \Omega | \text{dist}(x, \Omega^c) > \delta\}$ as usual. This implies that $u_{\alpha} \rightarrow u$ uniformly on $\bar{Q}_T$ for some $u \in UC_{\alpha}(\bar{Q}_T)$ as $\alpha \downarrow 0$. And u solves (10.9) $u + \theta_J(u_t + H(t, x, u, Du) + v(t, x)) - v(t, x) = 0$ in Q_T in the viscosity sense. Clearly $v \leq u \leq w$ in Q_T . Hence $u = v = w$ on $[0, T] \times \partial\Omega$ and $0 \leq u - v \leq J$ in Q_T . From the latter and (10.9) we see that u solves (10.2) in the viscosity sense. Finally notice that $u(0, \cdot) = u_0$. Q.E.D.

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