Almost periodic homogenization of Hamilton-Jacobi equations

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§1 Introduction. We are concerned here with the asymptotic behavior, as $\varepsilon \searrow 0$, of the solution u^ε of the Hamilton-Jacobi equation

$$(1) \qquad u(x) + H(x, x/\varepsilon, Du(x)) = 0 \quad (x \in \mathbf{R}^N),$$

where ε is a positive constant. The effect on solutions of the limiting process as $\varepsilon \searrow 0$ should be mild in the sense that the Hamiltonian $H(x, x/\varepsilon, p)$ stays bounded as $\varepsilon \searrow 0$. This effect is called the homogenization. Equation (1) appears as a fundamental equation in optimal control under oscillatory circumstances or as an equation describing a sort of distance functions in the space where the Riemannian metric is oscillatory.

In this paper, we will study the almost periodic homogenization of Hamilton-Jacobi equations. That is, we will assume that the Hamiltonian $H(x, y, p)$ is almost periodic in y. There are many references concerning the homogenization of Hamilton-Jacobi equations. However, as far as Hamilton-Jacobi equations are concerned, most of references deal with the periodic homogenization, i.e., the case where the function $H(x, y, p)$ is periodic in y. See for this [1, 3, 5, 8, 9, 10].

As is well-known, equation (1) does not have a classical solution in general, and we adapt the notion of viscosity solution as weak solutions of (1) that we are concerned with. We will simply refer viscosity subsolutions, viscosity supersolutions, and viscosity solutions as subsolutions, supersolutions, and solutions, respectively.

§2 Main results. We begin with our assumptions on the Hamiltonian H.

(A0) $H \in \mathrm{C}(\mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N)$.

(A1) $\lim_{R\to\infty} \inf\{H(x, y, p) \mid x, y, p \in \mathbf{R}^N, |p| \geq R\} = \infty$.

(A2) For each $R > 0$ there is a function $\omega_R \in \mathrm{C}([0, \infty))$, with $\omega_R(0) = 0$, such that

$$|H(x, y, p) - H(x, y, q)| \leq \omega_R(|p - q|) \quad (x, y \in \mathbf{R}^N, p, q \in B(0, R)).$$

(A3) $\sup\{|H(x, y, p)| \mid x, y \in \mathbf{R}^N, p \in B(0, R)\} < \infty$.

(A4) For each $R > 0$ the family $\{H(\cdot, \cdot + z, \cdot) \mid z \in \mathbf{R}^N\}$ of functions is relatively compact in $\mathrm{BUC}(B(0, R) \times \mathbf{R}^N \times B(0, R))$.

Throughout this paper we assume that (A0)–(A4) hold. The main results are stated as follows.

Theorem 1. *Let $\varepsilon > 0$. There is a unique solution $u \in \mathrm{BUC}(\mathbf{R}^N)$ of (1). Moreover u is Lipschitz continuous on $\mathbf{R}^N$.*

Theorem 2. *Let $\hat{x},\ \hat{p} \in \mathbf{R}^N$. There is a unique constant $\lambda \in \mathbf{R}$ such that for each $\delta > 0$ there is a solution $w \in \mathrm{BUC}(\mathbf{R}^N)$ of*

$$H(\hat{x}, y, \hat{p} + Dw(y)) \le \lambda + \delta \quad (y \in \mathbf{R}^N), \tag{2}$$

and

$$H(\hat{x}, y, \hat{p} + Dw(y)) \ge \lambda - \delta \quad (y \in \mathbf{R}^N). \tag{3}$$

According to this theorem, we can define the *effective Hamiltonian* $\overline{H} : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ by setting $\overline{H}(\hat{x}, \hat{p}) = \lambda$, where λ is the constant given by Theorem 2. The determination of the constant λ by the inequalities (2) and (3), with arbitrary $\delta > 0$, has been inroduced by [2].

Theorem 3. *We have*

$$\inf_{y \in \mathbf{R}^N} H(x, y, p) \le \overline{H}(x, p) \le \sup_{y \in \mathbf{R}^N} H(x, y, p) \quad (x, p \in \mathbf{R}^N).$$

This theorem says that the coerciveness (A1) of H as well as property (A3) is inherited to $\overline{H}$.

Theorem 4. $\overline{H} \in \mathrm{C}(\mathbf{R}^N \times \mathbf{R}^N)$. *Moreover for each $R > 0$ there is a function $\gamma_R \in \mathrm{C}([0, \infty))$, with $\gamma_R(0) = 0$, such that for all $x \in \mathbf{R}^N$ and $p, q \in B(0, R)$,*

$$|\overline{H}(x, p) - \overline{H}(x, q)| \le \gamma_R(|p - q|). \tag{4}$$

Theorem 5. *Let $u^\varepsilon \in \mathrm{BUC}(\mathbf{R}^N)$ be the solution of (1) and $u \in \mathrm{BUC}(\mathbf{R}^N)$ the solution of*

$$u(x) + \overline{H}(x, Du(x)) = 0 \quad (x \in \mathbf{R}^N). \tag{5}$$

Then $u^\varepsilon(x) \to u(x)$ locally uniformly on $\mathbf{R}^N$ as $\varepsilon \searrow 0$.

We should remark that, thanks to Theorems 3 and 4, there is a unique bounded solution of (5) and it is Lipschitz continuous in $\mathbf{R}^N$.

§3 Proof of the main results. In this section we present a proof of the theorems stated in the previous section. We begin with the following general proposition.

Proposition 6. *Let $G \in \mathrm{C}(\mathbf{R}^N \times \mathbf{R}^N)$ satisfy the condition: for each $R > 0$ there is a function $\nu_R \in \mathrm{C}([0, \infty))$, with $\nu_R(0) = 0$, such that*

$$|G(x, p) - G(x, q)| \le \nu_R(|p - q|) \quad (x \in \mathbf{R}^N,\ p, q \in B(0, R)).$$

Let $\lambda, \mu \in \mathbf{R}$. Suppose that there are a bounded, Lipschitz continuous solution v of $G(x, Dv(x)) \le \lambda$ in $\mathbf{R}^N$ and a bounded solution w of $G(x, Dw(x)) \ge \mu$ in $\mathbf{R}^N$. Then $\mu \le \lambda$.

Proof. We argue by contradiction. Thus we suppose that $\mu > \lambda$. Let v and w be as above. Set $L = \|Dv\|_\infty$. Let $\varepsilon \in (0,1)$, and define the function $u \in C(\mathbf{R}^N)$ by $u(x) = v(x) - \varepsilon(|x|^2+1)^{1/2}$. Then we have $\|Du\|_\infty \leq L+1$, and hence u is a solution of

$$G(x, Du(x)) \leq \lambda + \nu_{L+1}(\varepsilon) \quad (x \in \mathbf{R}^N).$$

Now fix $\varepsilon \in (0,1)$ so that $\lambda + \nu_{L+1}(\varepsilon) < \mu$. Let $\alpha > 0$ and consider the function

$$u(x) - w(y) - \alpha|x-y|^2$$

on $\mathbf{R}^N \times \mathbf{R}^N$. This function attains a maximum at a point $(x_\alpha, y_\alpha) \in \mathbf{R}^N \times \mathbf{R}^N$, and we have

$$G(x_\alpha, 2\alpha(x_\alpha - y_\alpha)) \leq \lambda + \nu_{L+1}(\varepsilon)$$

and

$$G(y_\alpha, 2\alpha(x_\alpha - y_\alpha)) \geq \mu.$$

Sending $\alpha \to \infty$, we see that for some $\hat{x}, \hat{p} \in \mathbf{R}^N$, $\mu \leq G(\hat{x}, \hat{p}) \leq \lambda + \nu_{L+1}(\varepsilon)$. This contradiction proves our proposition. □

Proof of Theorem 1. Let $A = \sup\{|H(x,y,0)| \mid x, y \in \mathbf{R}^N\}$. Then the functions $u(x) := A$ and $v(x) := -A$ are a supersolution and a subsolution of (1), respectively. Thus, using Perron's method, we see that there is a solution $u : \mathbf{R}^N \to \mathbf{R}$ of (1), which is upper semicontinuous, such that $|u(x)| \leq A$ for all $x \in \mathbf{R}^N$. It follows from assumption (A1) that u is Lipschitz continuous on $\mathbf{R}^N$. By the standard comparison results, we see that (1) has only one bounded solution. □

The proof above yields the following theorem.

Theorem 7. *Let $\hat{x}$, $\hat{p} \in \mathbf{R}^N$, and $\alpha > 0$. Then there is a unique solution $v_\alpha \in$ BUC$(\mathbf{R}^N)$ of*

$$\alpha v_\alpha(y) + H(\hat{x}, y, \hat{p} + Dv_\alpha(y)) = 0 \quad (y \in \mathbf{R}^N). \tag{6}$$

The following theorem is the key observation in establishing the existence part of Theorem 2.

Theorem 8. *Let $\hat{x}$, $\hat{p} \in \mathbf{R}^N$, and for each $\alpha > 0$, v_α be the bounded solution of* (6). *Then, as $\alpha \to 0$,*

$$\sup_{y \in \mathbf{R}^N} |\alpha v_\alpha(y) - \alpha v_\alpha(0)| \to 0.$$

Proof. We argue by contradiction. Thus we suppose that there were $\delta > 0$ and sequences $\{\alpha_j\} \subset (0,\infty)$ and $\{y_j\} \subset \mathbf{R}^N$ such that $|\alpha_j v_{\alpha_j}(y_j) - \alpha_j v_{\alpha_j}(0)| \geq \delta \quad (j \in \mathbf{N})$. By assumption (A4) we may assume that there is a function $G \in C(\mathbf{R}^N \times \mathbf{R}^N)$ such that as $j \to \infty$, $H(\hat{x}, y + y_j, p) \to G(y, p)$ uniformly on $\mathbf{R}^N \times B(0,R)$ for all $R > 0$.

In view of (A1), there is a constant $R > 0$ such that $|\hat{p}| + \|Dv_\alpha\|_\infty \leq R$ for all $\alpha > 0$. If j, k are large enough, then

$$|H(\hat{x}, y + y_j, p) - H(\hat{x}, y + y_k, p)| \leq \delta/2 \quad (y \in \mathbf{R}^N, p \in B(0,R)).$$

We may assume by relabeling the sequence $\{y_j\}$ if necessary that this inequality holds for all $j, k \in \mathbf{N}$.

Now, for the function $w_j(y) := v_{\alpha_j}(y + y_j - y_1)$, we have

$$0 = \alpha_j w_j(y) + H(\hat{x}, y + y_j - y_1, \hat{p} + Dw_j(y)) \\ \geq \alpha_j w_j(y) + H(\hat{x}, y, \hat{p} + Dw_j(y)) - \delta/2.$$

Using a comparison theorem, we obtain $\alpha_j w_j(y) - \delta/2 \leq \alpha_j v_{\alpha_j}(y)$ for all $y \in \mathbf{R}^N$, $j \in \mathbf{N}$. That is, $\alpha_j v_{\alpha_j}(y + y_j - y_1) \leq \alpha_j v_{\alpha_j}(y) + \delta/2$. Therefore we have

$$\alpha_j v_{\alpha_j}(y_j) \leq \alpha_j v_{\alpha_j}(y_1) + \delta/2 \leq \alpha_j v_{\alpha_j}(0) + \delta/2 + \alpha_j R|y_1|.$$

An argument similar to the above yields

$$\alpha_j v_{\alpha_j}(y_j) \geq \alpha_j v_{\alpha_j}(y_1) - \delta/2 - \alpha_j R|y_1|.$$

Thus we have $|\alpha_j v_{\alpha_j}(y_j) - \alpha_j v_{\alpha_j}(0)| < \delta$ if j is large enough. This is a contradiction, which completes the proof. □

Proof of Theorem 2. Let $\alpha > 0$ and $v_\alpha \in \mathrm{BUC}(\mathbf{R}^N)$ be the solution of (6). As in the previous proof, there is a constant $R > 0$ such that $|\hat{p}| + \|Dv_\alpha\|_\infty \leq R$ for all $\alpha > 0$. Hence, by (6) we see that the set $\{\alpha_j v_{\alpha_j}(0) \mid \alpha > 0\} \subset \mathbf{R}$ is bounded. Choose a sequence $\alpha_j \searrow 0$ so that $\alpha_j v_{\alpha_j}(0) \to -\lambda$ for some $\lambda \in \mathbf{R}$ as $j \to \infty$. Fix $\delta > 0$. According to Theorem 3, there is a $j \in \mathbf{N}$ such that $|\alpha_j v_{\alpha_j}(y) + \lambda| \leq \delta$ for all $y \in \mathbf{R}^N$. Then the function $w := v_{\alpha_j}$ satisfies (2) and (3) in the viscosity sense. The uniqueness of λ is an immediate consequence of Proposition 6. □

Proof of Theorem 3. Note that $v := 0$ is a solution of

$$H(\hat{x}, y, \hat{p} + Dv(y)) \leq \sup_{y \in \mathbf{R}^N} H(\hat{x}, y, \hat{p}) \quad (y \in \mathbf{R}^N).$$

Therefore, using Proposition 6, we have $\overline{H}(\hat{x}, \hat{p}) \leq \sup_{y \in \mathbf{R}^N} H(\hat{x}, y, \hat{p})$. Similarly we have $\overline{H}(\hat{x}, \hat{p}) \geq \inf_{y \in \mathbf{R}^N} H(\hat{x}, y, \hat{p})$. □

Proof of Theorem 4. We first check that (4) holds for some $\gamma_R \in \mathrm{C}([0, \infty))$ satisfying $\gamma_R(0) = 0$.

To this end, we fix $R > 0$. In view of Theorem 3, we have

$$|\overline{H}(x, p)| \leq M \quad (x \in \mathbf{R}^N, p \in B(0, R)),$$

where $M = \sup\{|H(x, y, p)| \mid x, y, p \in \mathbf{R}^N,\ |p| \leq R\}$. According to (A1), there is a constant $L > 0$ such that

$$H(x, y, p) > M + 1 \quad (x, y, p \in R^N,\ |p| \geq L). \tag{7}$$

By (A2), there is a function $\omega \in \mathrm{C}([0, \infty))$, with $\omega(0) = 0$, such that for all $x, z \in B(0, R)$, $y \in \mathbf{R}^N$, and $p, q \in B(0, L + 2R)$,

$$|H(x, y, p) - H(z, y, q)| \leq \omega(|x - z| + |p - q|). \tag{8}$$

Now, let $p, q \in B(0,R)$ and $x \in \mathbf{R}^N$. Fix $\delta \in (0,1)$, and choose $v \in \mathrm{BUC}(\mathbf{R}^N)$ so that v is a solution of

$$(9) \qquad H(x,y,p+Dv(y)) \le \overline{H}(x,p)+\delta \quad (y \in \mathbf{R}^N).$$

We see from (7) that $\|Dv\|_\infty \le L+R$ and hence $\|q+Dv\|_\infty \le L+2R$. From (8) we see that v is a solution of

$$H(x,y,q+Dv(y)) \le \overline{H}(x,p)+\omega(|p-q|)+\delta \quad (y \in \mathbf{R}^N).$$

By the definition of $\overline{H}(x,q)$, there is a solution $w \in \mathrm{BUC}(\mathbf{R}^N)$ of

$$H(x,y,q+Dw(y)) \ge \overline{H}(x,q)-\delta \quad (y \in \mathbf{R}^N).$$

Hence, we conclude in view of Proposition 6 that

$$\overline{H}(x,p)+\omega(|p-q|)+\delta \ge \overline{H}(x,q)-\delta.$$

This shows that $\overline{H}(x,q)-\overline{H}(x,p) \le \omega(|p-q|)$. Similarly we infer that $\overline{H}(x,p)-\overline{H}(x,q) \le \omega(|p-q|)$. These guarantee that (4) holds with $\gamma_R = \omega$.

Next, we select a function $\nu \in \mathrm{C}([0,\infty))$ satisfying $\nu(0)=0$ so that for all $x, z \in B(0,R)$, $y \in \mathbf{R}^N$, and $p \in B(0,R+L)$,

$$|H(x,y,p)-H(z,y,p)| \le \nu(|x-z|).$$

An argument parallel to the above yields that for all $x, z, p \in B(0,R)$,

$$|\overline{H}(x,p)-\overline{H}(z,p)| \le \nu(|x-z|).$$

Thus we see that $\overline{H} \in \mathrm{C}(\mathbf{R}^N \times \mathbf{R}^N)$. □

Proof of Theorem 5. As we have seen in the proof of Theorem 1, we have

$$\|u^\varepsilon\|_\infty \le A := \sup\{|H(x,y,0)| \mid x,y \in \mathbf{R}^N\} \quad (\varepsilon > 0).$$

Select $L > 0$ so that $H(x,y,p) > A$ for all $x,y,p \in \mathbf{R}^N$ satisfying $|p| > L$. Then we have $\sup_{\varepsilon>0} \|Du^\varepsilon\|_\infty \le L$. Hence the family $\{u^\varepsilon \mid \varepsilon > 0\}$ is relatively compact in $\mathrm{C}(\mathbf{R}^N)$.

Fix any sequence $\varepsilon_j \searrow 0$ so that $u^{\varepsilon_j}(x) \to v(x)$ locally uniformly on $\mathbf{R}^N$ as $j \to \infty$. We intend to show that v is a solution of (5). Once this is done, since v is a bounded function, by the uniqueness of bounded solutions of (5), we conclude that $v = u$, which implies that, as $\varepsilon \to 0$, $u^\varepsilon(x) \to u(x)$ locally uniformly on $\mathbf{R}^N$.

Let $\varphi \in \mathrm{C}^1(\mathbf{R}^N)$ and assume that $v - \varphi$ has a strict maximum at $\hat{x}$. For simplicity we write u_j for u^{ε_j}. It follows that $u_j - \varphi$ has a maximum at some x_j and $x_j \to \hat{x}$ as $j \to \infty$. Define $\varphi_j \in \mathrm{C}^1(\mathbf{R}^N)$ by $\varphi_j(x) = \varphi(x) + (1/j)|x-x_j|^2$, so that $u_j - \varphi_j$ has a strict maximum at x_j.

Fix $\delta \in (0,1)$ and let $w^\delta \in \mathrm{BUC}(\mathbf{R}^N)$ be a solution of

$$H(\hat{x},y,D\varphi(\hat{x})+Dw^\delta(y)) \ge \overline{H}(\hat{x},D\varphi(\hat{x}))-\delta \quad (y \in \mathbf{R}^N).$$

Consider the function: $u_j(x) - \varphi_j(y) - \varepsilon_j w^\delta(y/\varepsilon_j) - \alpha|x-y|^2$ on $\mathbf{R}^N \times \mathbf{R}^N$, and let $(x_\alpha, y_\alpha) \in \mathbf{R}^{2N}$ be one of its maximum points, the existence of which is obviously ensured.

Now, we have

$$u_j(x_\alpha) + H(x_\alpha, x_\alpha/\varepsilon_j, 2\alpha(x_\alpha - y_\alpha)) \leq 0,$$

$$H(\hat{x}, y_\alpha/\varepsilon_j, D\varphi(\hat{x}) - D\varphi_j(y_\alpha) + 2\alpha(x_\alpha - y_\alpha)) \geq \overline{H}(\hat{x}, D\varphi(\hat{x})) - \delta.$$

Since $\|Du_j\|_\infty \leq L$ and hence $2\alpha|x_\alpha - y_\alpha| \leq L$, sending $\alpha \to \infty$ along a sequence, we have

$$u_j(x_j) + H(x_j, x_j/\varepsilon_j, p_j) \leq 0,$$

$$H(\hat{x}, x_j/\varepsilon_j, D\varphi(\hat{x}) - D\varphi(x_j) + p_j) \geq \overline{H}(\hat{x}, D\varphi(\hat{x})) - \delta$$

for some $p_j \in B(0, L)$. Subtracting one of these from the other and sending $j \to \infty$, we get

$$u(\hat{x}) + \overline{H}(\hat{x}, D\varphi(\hat{x})) \leq \delta,$$

from which we see that v is a subsolution of (5).

Arguing in a way similar to the above, we deduce that v is a supersolution of (5) as well. □

Acknowledgment. This research was supported in part by Grant-in-Aid for Scientific Research, No. 0944067, the Ministry of Education, Science, Sports and Culture, Japan.

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