

Remarks on Existence of Viscosity Solutions of Hamilton-Jacobi Equations*

By

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Abstract

We consider the stationary and Cauchy problem for Hamilton-Jacobi equations and prove existence theorems for viscosity solutions. These results ensure their existence under the same hypotheses as for the uniqueness theorems in [7], and so complement the results in [19]. We also give another existence result whose assumptions on Hamiltonians are the same as those for a uniqueness theorem in [13].

1. Introduction

We will study here existence of "viscosity solutions" of Hamilton-Jacobi equations. This notion of weak solution introduced in Crandall-Lions [7] has nice existence and uniqueness properties, and its importance has become decisive in applications, especially to optimal control theory and differential games. See [7], Crandall-Evans-Lions [6], Lions [16], Souganidis [18, 19], Ishii [13] for the definitions of and existence and uniqueness results of viscosity solutions, and Barron-Evans-Jensen [2], Capuzzo Dolcetta [3], Capuzzo Dolcetta-Evans [4], Capuzzo Dolcetta-Ishii [5], Evans-Souganidis [10], Lions [16] and Souganidis [18] for applications to optimal control and differential game theory.

We next recall the definition of viscosity solutions of a first order PDE

$$(1.1) \quad F(y, u, Du) = 0 \quad \text{in } O.$$

Here O is a given open set in R^N , $F: O \times R \times R^N \rightarrow R$ is a given continuous function, $u: O \rightarrow R$ is the unknown, and Du denotes the gradient of u ,

$$Du = \left(\frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_N} \right).$$

Following Crandall-Evans-Lions [6], we say that a function $u \in C(O)$ is a *viscosity sub-*

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solution (resp. a *viscosity supersolution*) of (1.1) provided for every $\phi \in C^1(O)$ if $u - \phi$ attains its local maximum at $y_0 \in O$ then

$$(1.2) \quad F(y_0, u(y_0), D\phi(y_0)) \leq 0$$

(resp. if $u - \phi$ attains its local minimum at y_0 then

$$(1.3) \quad F(y_0, u(y_0), D\phi(y_0)) \geq 0.)$$

A *viscosity solution* of (1.1) is defined as a function which is both a viscosity subsolution and supersolution of (1.1). A useful remark is that if $u \in C^1(O)$ and satisfies either

$$\begin{aligned} F(y, u, Du) &\leq 0 & \text{in } O, \\ F(y, u, Du) &\geq 0 & \text{in } O, \end{aligned}$$

or

$$F(y, u, Du) = 0 \quad \text{in } O$$

then u is, respectively, a viscosity subsolution, supersolution, or solution of (1.1).

We are concerned with the stationary problem

$$(1.4) \quad u + H(x, u, Du) = 0 \quad \text{in } R^N$$

and the Cauchy problem

$$(1.5) \quad \begin{cases} u_t + H(t, x, u, Du) = 0 & \text{in } (0, T) \times R^N \\ u(0, x) = u_0(x) & \text{for } x \in R^N, \end{cases}$$

where T is a given positive number, u_0 is a given function,

$$Du = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right), \quad \text{and} \quad u_t = \frac{\partial u}{\partial t}.$$

We call these equations *Hamilton-Jacobi equations* and the functions H the *Hamiltonians*.

Existence results for (1.4) and (1.5) have been obtained in [7], [16], [18, 19]. See also Kruzkov [14, 15], Fleming [11], and Friedman [12]. Among others the assumptions for existence in [19] are very close to those for uniqueness in [7], but have still a gap from those for uniqueness.

The motivation and the main purpose of this paper is to fill this gap, i.e., to prove existence of viscosity solutions of (1.4) and (1.5) under the same assumptions of uniqueness results in [7]. Another important purpose is to present also an existence theorem for unbounded viscosity solutions, for which uniqueness has been demonstrated in [13].

A main difficulty here is in the proof of the equi-continuity of approximate solutions, and for this we adopt a refinement of the method of the proof of [13, Theorem 3.1].

All our existence results treat the case of the whole space R^N , and we refer to

Lions [16, 17] and Evans-Ishii [9] for the case of bounded domains. We should however remark that our results are useful to produce existence theorems on bounded domains as in [9].

After completing this, the author learned about the work of Barles [1], in which he proved existence theorems for bounded, uniformly continuous viscosity solutions of (1.4) and (1.5). As far as bounded solutions on R^N (or $(0, T) \times R^N$) are concerned, his results are the same as in Theorems 4.1 and 4.2 (see remarks after the proofs) and deal with the problems in arbitrary open subsets Ω of R^N (or in $(0, T) \times \Omega$) as well. Our method of proof is however different from that of [1], and our results include also the existence of unbounded solutions.

This paper is organized as follows. In the next section we prove an estimate of the modulus of continuity of viscosity solutions for stationary problems. In Section 3 we demonstrate first a similar estimate concerning the continuity in the space variables for Cauchy problems and then provide a bound for the modulus of continuity (in the time variable) of viscosity solutions. We prove existence theorems by making use of the method of "vanishing viscosity" in Section 4. The same method is used in Section 5 to prove the existence of a viscosity solution under the same hypotheses as for a uniqueness theorem in [13].

We will use the following notation. For $R > 0$, B_R denotes the open ball in R^N with radius R and center at the origin. We write $\langle x \rangle_\varepsilon$ for $(|x|^2 + \varepsilon^2)^{1/2}$ for $x \in R^N$, $\varepsilon > 0$ and delete ε if $\varepsilon = 1$.

2. Modulus of Continuity of Viscosity Solutions (I)

We consider here the stationary problem

$$(2.1) \quad u + H(x, u, Du) = 0 \quad \text{in } R^N,$$

where $H: R^N \times R \times R^N \rightarrow R$ is continuous.

Our assumptions on H are as follows.

(H1) For each $R > 0$ there exists a nondecreasing $\sigma_R: [0, 2R] \rightarrow [0, \infty)$ such that

$$(2.2) \quad \begin{cases} \lim_{r \searrow 0} \sigma_R(r) = 0, \\ |H(x, u, p) - H(x, u, q)| \leq \sigma_R(|p - q|) \end{cases}$$

for all $x \in R^N$, $u \in R$, $p, q \in \bar{B}_R$.

(H2) For all $x, y, p \in R^N$, $u, v \in R$

$$(2.3) \quad H(x, u, p) \geq H(x, v, p) \quad \text{if } u \geq v.$$

(H3) There exists a nondecreasing $\omega: [0, \infty) \rightarrow [0, \infty)$ such that

$$(2.4) \quad \begin{cases} \lim_{r \searrow 0} \omega(r) = 0, \\ |H(x, u, p) - H(y, u, p)| \leq \omega(|x - y|(1 + |p|)) \end{cases}$$

for all $x, y, p \in R^N$, $u \in R$.

We will write $\sigma(r) = \sigma_{r+1}(r)$ for $r \geq 0$. We may assume that σ is nondecreasing on $[0, \infty)$ and that

$$(2.5) \quad \omega(r) \leq C_1(1+r)$$

for all $r \geq 0$ and some constant $C_1 > 0$. In particular

$$(2.6) \quad |H(x, 0, 0)| \leq C_2(1 + |x|),$$

$$(2.7) \quad |H(x, u, 0) - H(y, u, 0)| \leq C_1(1 + |x - y|)$$

for all $x, y \in R^N$, $u \in R$, with $C_2 \equiv C_1 + |H(0, 0, 0)|$.

We need the following comparison result.

PROPOSITION 2.1. *Let Ω be an open set in R^N , and let $H \in C(R^N \times R \times R^N)$ satisfy (H1), (H2), (H3). If $u, v \in UC(R^N)$ are, respectively, a viscosity subsolution and a viscosity supersolution of (2.1) in Ω and $u \leq v$ on $\partial\Omega$, then*

$$(2.8) \quad u \leq v \quad \text{in } \Omega.$$

This is a generalization of uniqueness results in Crandall-Lions [7], and has been proved in [13] in the case when $\Omega = R^N$ and $H(x, u, p)$ is independent of x, u . The proof in [13] can be used to prove this proposition with minor changes, and we omit the details (see also [13, remark 1.3]).

PROPOSITION 2.2. *Assume $H \in C(R^N \times R \times R^N)$ satisfies (H1), (H2), and (H3). If $u \in UC(R^N)$ is the (unique) viscosity solution of (2.1), then*

$$(2.9) \quad |u(x)| \leq C_2|x| + C_2 + \sigma(C_2) \quad \text{for all } x \in R^N.$$

Proof. Let $\varepsilon > 0$, $A > 0$, $B > 0$, and set

$$v(x) = A\langle x \rangle_\varepsilon + B \quad \text{for } x \in R^N.$$

Using (H2), (H1), and (2.6), we compute that for $x \in R^N$

$$\begin{aligned} H(x, v, Dv) &\geq H(x, 0, Dv) = H\left(x, 0, A\frac{x}{\langle x \rangle_\varepsilon}\right) - H(x, 0, 0) + H(x, 0, 0) \\ &\geq -\sigma(A) - C_2(1 + |x|). \end{aligned}$$

Therefore, choosing $A = C_2$ and $B = C_2 + \sigma(C_2)$, we find that

$$v + H(x, v, Dv) \geq 0 \quad \text{in } R^N.$$

Thus by Proposition 2.1

$$u(x) \leq v(x) = C_2\langle x \rangle_\varepsilon + C_2 + \sigma(C_2).$$

Let $\varepsilon \searrow 0$ to get

$$u(x) \leq C_2|x| + C_2 + \sigma(C_2) \quad \text{for all } x \in R^N.$$

Similarly we conclude

$$u(x) \geq -C_2|x| - C_2 - \sigma(C_2) \quad \text{for all } x \in R^N. \quad \square$$

LEMMA 2.3. Assume $H \in C(R^N \times R \times R^N)$ satisfies (H1), (H2), and (H3). If $u \in UC(R^N)$ is the (unique) viscosity solution of (2.1), then

$$(2.10) \quad |u(x) - u(y)| \leq C_1|x - y| + C_1 + 2\sigma(C_1)$$

for all $x, y \in R^N$.

Proof. The function $v(x, y) = u(x) - u(y)$ satisfies

$$(2.11) \quad v + \tilde{H}(x, y, v, Dv) = 0 \quad \text{in } R^{2N}$$

in the viscosity sense, where $D = (D_x, D_y)$ and

$$(2.12) \quad \tilde{H}(x, y, r, p, q) = H(x, r + u(y), p) - H(y, u(y), -q)$$

for $x, y, p, q \in R^N$, $r \in R$. From (H2), (H1), and (2.7)

$$\tilde{H}(x, y, r, p, q) \geq -\sigma(|p|) - \sigma(|q|) - C_1(1 + |x - y|)$$

for $x, y, p, q \in R^N$, $r \geq 0$. We may assume σ is continuous on $[0, \infty)$. Hence the function v_+ defined by

$$v_+(x, y) = \max\{v(x, y), 0\} \quad \text{on } R^{2N}$$

satisfies

$$(2.13) \quad v_+ - \sigma(|D_x v_+|) - \sigma(|D_y v_+|) - C_1(1 + |x - y|) \leq 0 \quad \text{in } R^{2N}$$

in the viscosity sense.

For $\varepsilon > 0$ we define

$$w(x, y) = A\langle x - y \rangle_\varepsilon + B \quad \text{on } R^{2N},$$

with $A = C_1$ and $B = C_1 + 2\sigma(C_1)$. Then w is a viscosity supersolution of

$$w - \sigma(|D_x w|) - \sigma(|D_y w|) - C_1(1 + |x - y|) = 0 \quad \text{in } R^{2N}$$

and so, by Proposition 2.1,

$$(2.14) \quad u(x) - u(y) \leq C_1\langle x - y \rangle_\varepsilon + C_1 + 2\sigma(C_1)$$

for all $x, y \in R^N$. Here we have used that the Hamiltonian

$$(x, y, r, p, q) \longmapsto -\sigma(|p|) - \sigma(|q|) - C_1(1 + |x - y|)$$

satisfies (H1), (H2), and (H3), with dimension N replaced by $2N$. The estimate (2.10) is immediate from (2.14). $\square$

Our main result in this section is the following

THEOREM 2.4. *Assume that $H \in C(R^N \times R \times R^N)$ satisfies (H1), (H2), (H3). Then there exists a function $\eta: [0, \infty) \rightarrow [0, \infty)$ depending only on C_1 , σ , and ω such that*

$$(2.15) \quad \lim_{r \searrow 0} \eta(r) = 0$$

and if $u \in UC(R^N)$ is the (unique) viscosity solution of (2.1), then

$$(2.16) \quad |u(x) - u(y)| \leq \eta(|x - y|) \quad \text{for all } x, y \in R^N.$$

Proof. Fix $\varepsilon > 0$ arbitrarily. We intend to prove that there exists $\delta > 0$ depending only on $\varepsilon, C_1, \sigma(C_1)$, and ω such that

$$(2.17) \quad |u(x) - u(y)| \leq \varepsilon \quad \text{if } |x - y| \leq \delta.$$

Taking (2.5) and that $\lim_{r \searrow 0} \omega(r) = 0$ into account, we select a constant $A = A(\varepsilon, \omega) > 0$ so that

$$(2.18) \quad \omega(r) \leq \frac{\varepsilon}{2} + Ar \quad \text{for all } r \geq 0.$$

We will write

$$\chi(p, q) = \sigma_{|p|+|q|+1}(|p+q|)$$

for $p, q \in R^N$. Note that $\chi(p, -p) = 0$ for all $p \in R^N$. Moreover we may assume that χ is continuous on R^{2N} , replacing by a larger one if necessary. By virtue of (2.3), (2.4), and (2.18) we find that

$$\begin{aligned} \tilde{H}(x, y, r, p, q) &\geq \tilde{H}(x, y, 0, p, -p) - \chi(p, q) \\ &\geq -\omega(|x - y|(1 + |p|)) - \chi(p, q) \\ &\geq -\frac{\varepsilon}{2} - A|x - y|(1 + |p|) - \chi(p, q) \end{aligned}$$

for all $x, y, p, q \in R^N, r \geq 0$. Therefore we see from (2.11) that $v_+(x, y) = \max\{u(x) - u(y), 0\}$ satisfies

$$(2.19) \quad v_+ - \frac{\varepsilon}{2} - A|x - y|(1 + |D_x v_+|) - \chi(D_x v_+, D_y v_+) \leq 0 \quad \text{in } R^{2N}$$

in the viscosity sense.

Fix $0 < \alpha < 1$, and let $0 < \gamma \leq 1, B > 0$ be numbers to be chosen later. Set

$$w(x, y) = \frac{\varepsilon}{2} + B\langle x - y \rangle_1^2 \quad \text{for } x, y \in R^N$$

and denote

$$\Omega = \{(x, y) \in R^{2N} \mid |x - y| < 1\}.$$

Compute that

$$w - \frac{\varepsilon}{2} - A|x-y|(1+|D_x w|) - \chi(D_x w, D_y w) \geq B\langle x-y \rangle_A^2 - A|x-y|(1+\gamma B\langle x-y \rangle_A^{2-1}) \\ \geq (B-A-\gamma AB)\langle x-y \rangle_A^2 \quad \text{in } \Omega.$$

Now we fix

$$\gamma = \min \left\{ \frac{1}{2A}, 1 \right\}$$

and assume

$$B \geq 2A.$$

Then we obtain

$$(2.20) \quad w - \frac{\varepsilon}{2} - A|x-y|(1+|D_x w|) - \chi(D_x w, D_y w) \geq 0 \quad \text{in } \Omega.$$

By Lemma 2.3

$$v_+ \leq 2C_1 + 2\sigma(C_1) \quad \text{on } \partial\Omega.$$

We choose finally

$$B = \max \{2A, 2C_1 + 2\sigma(C_1)\},$$

so that

$$v_+ \leq w \quad \text{on } \partial\Omega.$$

Since the Hamiltonian

$$(x, y, r, p, q) \mapsto -\frac{\varepsilon}{2} - A|x-y|(1+|p|) - \chi(p, q)$$

on $\Omega \times R \times R^{2N}$ has an extension to $R^{2N} \times R \times R^{2N}$ which satisfies (H1), (H2), (H3) with dimension N replaced by $2N$, by virtue of Proposition 2.1 we have from (2.19) and (2.20)

$$v_+ \leq w \quad \text{in } \Omega.$$

Send $\alpha \searrow 0$ to obtain

$$(2.21) \quad u(x) - u(y) \leq \frac{\varepsilon}{2} + B|x-y|^r \quad \text{for all } x, y \in \Omega.$$

This proves (2.17), with $\delta = \min \{1, (\varepsilon/2B)^{1/r}\}$, and thus completes the proof. $\square$

3. Modulus of Continuity of Viscosity Solutions (II)

We are now concerned with the Cauchy problem

$$(3.1) \quad u_t + H(t, x, u, Du) = 0 \quad \text{in } \Omega_T \equiv (0, T) \times R^N,$$

$$(3.2) \quad u(0, x) = u_0(x) \quad \text{for } x \in R^N,$$

where $T > 0$ is a given number, $H: \bar{\Omega}_T \times R \times R^N \rightarrow R$ is continuous, and $u_0: R^N \rightarrow R$ is a given continuous function.

The assumptions on H here are the same as in §2 except that H involves the new variable t . That is, we assume the following.

(H1)' For each $R > 0$ there exists a nondecreasing $\sigma_R: [0, 2R] \rightarrow [0, \infty)$ such that

$$(3.3) \quad \begin{cases} \lim_{r \searrow 0} \sigma_R(r) = 0, \\ |H(t, x, u, p) - H(t, x, u, q)| \leq \sigma_R(\|p - q\|) \end{cases}$$

for all $(t, x) \in \bar{\Omega}_T$, $u \in R$, $p, q \in \bar{B}_R$.

(H2)' For all $0 \leq t \leq T$, $x, p \in R^N$, $u, v \in R$

$$(3.4) \quad H(t, x, u, p) \geq H(t, x, v, p) \quad \text{if } u \geq v.$$

(H3)' There exists a nondecreasing $\omega: [0, \infty) \rightarrow [0, \infty)$ such that

$$(3.5) \quad \begin{cases} \lim_{r \searrow 0} \omega(r) = 0, \\ |H(t, x, u, p) - H(t, y, u, p)| \leq \omega(\|x - y\|(1 + \|p\|)) \end{cases}$$

for all $0 \leq t \leq T$, $x, y, p \in R^N$, $u \in R$.

As before, we will write $\sigma(r) = \sigma_{r+1}(r)$ for $r \geq 0$ and may assume

$$(3.6) \quad \omega(r) \leq C_1(1 + r)$$

for all $r \geq 0$ and some constant $C_1 > 0$. Simple consequences of this are

$$(3.7) \quad |H(t, x, 0, 0)| \leq C_2(1 + |x|)$$

and

$$(3.8) \quad |H(t, x, u, 0) - H(t, y, u, 0)| \leq C_1(1 + |x - y|)$$

for all $0 \leq t \leq T$, $x, y \in R^N$, and $u \in R$, where

$$C_2 = C_1 + \sup_{0 \leq t \leq T} |H(t, 0, 0, 0)|.$$

Let us consider a very simple example. Let $H(t, x, u, p) = \langle x \rangle$ and $u_0(x) \equiv 0$. Then $u(t, x) = -t\langle x \rangle$ is a viscosity solution of (3.1), (3.2). This shows that one cannot expect a uniformly continuous solution of (3.1) and (3.2) in Ω_T in general.

We will discuss uniform continuity in x of viscosity solutions of (3.1) and then their continuity in t .

We introduce a function space

$$UC_s(\bar{\Omega}_T) = \{u \in C(\bar{\Omega}_T) \mid \limsup_{r \searrow 0} \|u(t, x) - u(t, y)\| \mid x, y \in R^N, |x - y| \leq r, 0 \leq t \leq T\} = 0\}.$$

PROPOSITION 3.1. *Let O be an open set in R^N and O_T denote the set $(0, T) \times O$. Assume $H \in C(\bar{\Omega}_T \times R \times R^N)$ satisfies (H1)', (H2)', (H3)'. If $u, v \in UC_s(\bar{\Omega}_T)$ are, respectively, a viscosity subsolution and a viscosity supersolution of (3.1) and $u \leq v$ on the parabolic boundary $\partial_p O_T = (0, T) \times \partial O \cup \{0\} \times O$, then*

$$(3.9) \quad u \leq v \quad \text{in } O_T.$$

This proposition generalizes comparison results in [7], [6] and [13]. The method of proof of [13, Theorem 2.1] together with the idea in the first part of the proof of [13, Theorem 1.1] can be applied to prove Proposition 3.1, and we omit the details.

PROPOSITION 3.2. Assume $H \in C(\bar{\Omega}_T \times R \times R^N)$ satisfies (H1)', (H2)', (H3)'. If $u \in UC_s(\bar{\Omega}_T)$ is a viscosity solution of (3.1) and

$$(3.10) \quad |u(0, x)| \leq C_2(1 + |x|)$$

for all $x \in R^N$, then

$$(3.11) \quad |u(t, x)| \leq C_2(1 + |x|) + t(C_3 + C_2|x|)$$

for all $(t, x) \in \bar{\Omega}_T$, with $C_3 \equiv C_2 + \sigma(C_2(1 + T))$.

Outline of proof. First prove the fuction

$$v(t, x) = C_2(1 + \langle x \rangle_\epsilon) + t(C_3 + C_2\langle x \rangle_\epsilon),$$

with $\epsilon > 0$, is a viscosity supersolution of (3.1). Then apply Proposition 3.1 to show $u \leq v$ on $\bar{\Omega}_T$. The inequality $u \geq -v$ on $\bar{\Omega}_T$ follows similarly. Finally send $\epsilon \searrow 0$ to get (3.11) $\square$

LEMMA 3.3. Let $u \in C(\Omega_T)$ be a viscosity solution of (3.1), and set $v(t, x, y) = u(t, x) - u(t, y)$ for $0 < t < T$, $x, y \in R^N$. Then v is a viscosity solution of

$$(3.12) \quad v_t + \tilde{H}(t, x, y, v, Dv) = 0 \quad \text{in } (0, T) \times R^{2N},$$

where $D = (D_x, D_y)$ and

$$(3.13) \quad \tilde{H}(t, x, y, r, p, q) = H(t, x, r + u(t, y), p) - H(t, y, u(t, x), -q)$$

for $0 < t < T$, $x, y, p, q \in R^N$, $r \in R$.

Proof. Let $\phi \in C^1((0, T) \times R^{2N})$ and $v - \phi$ attain its local maximum at $(t_0, x_0, y_0) \in (0, T) \times R^{2N}$. For each $\epsilon, \delta > 0$ we consider the function

$$\Phi(t, s, x, y) = u(t, x) - u(s, y) - \phi(t, x, y) - \frac{1}{\epsilon}(t - s)^2 - \delta(|t - t_0|^2 + |s - t_0|^2 + |x - x_0|^2 + |y - y_0|^2)$$

on $(0, T)^2 \times R^{2N}$. For each $\delta > 0$, by continuity, there exists a set of points in $(0, T)^2 \times R^{2N}$

$$\{(t_1, s_1, x_1, y_1) = (t_1(\epsilon), s_1(\epsilon), x_1(\epsilon), y_1(\epsilon)) \mid 0 < \epsilon \leq \epsilon_0\},$$

with $\epsilon_0 = \epsilon_0(\delta) > 0$, such that Φ attains its local maximum at (t_1, s_1, x_1, y_1) and that

$$(t_1, s_1, x_1, y_1) \longrightarrow (t_0, t_0, x_0, y_0) \quad \text{as } \epsilon \searrow 0.$$

By the definition of viscosity solutions, we have

$$\phi_t(t_1, x_1, y_1) + \frac{2}{\epsilon}(t_1 - s_1) + 2\delta(t_1 - t_0) + H(t_1, x_1, u(t_1, x_1), D_x\phi(t_1, x_1, y_1) + 2\delta(x_1 - x_0)) \leq 0$$

and

$$\frac{2}{\varepsilon}(t_1 - s_1) - 2\delta(s_1 - t_0) + H(s_1, y_1, u(s_1, y_1)) - D_y \phi(s_1, x_1, y_1) - 2\delta(y_1 - y_0) \geq 0.$$

Subtracting one from the other, letting $\varepsilon \searrow 0$ and then $\delta \searrow 0$, we obtain

$$\phi_t(t_0, x_0, y_0) + \tilde{H}(t_0, x_0, y_0, v(t_0, x_0, y_0), D\phi(t_0, x_0, y_0)) \leq 0.$$

The other condition for v to be a viscosity solution of (3.12) is demonstrated in a similar manner. $\square$

LEMMA 3.4. *Let $H \in C(\bar{\Omega}_T \times R \times R^N)$ satisfy (H1)', (H2)', (H3)'. Assume $u \in UC_s(\bar{\Omega}_T)$ is a viscosity solution of (3.1) and satisfies*

$$(3.14) \quad |u(0, x) - u(0, y)| \leq C_1(1 + |x - y|) \quad \text{for all } x, y \in R^N.$$

Then

$$(3.15) \quad |u(t, x) - u(t, y)| \leq C_1(1 + |x - y|) + t(C_4 + C_1|x - y|)$$

for all $0 \leq t \leq T$, $x, y \in R^N$, with $C_4 = C_1 + 2\sigma(C_1(1 + T))$.

Outline of proof. Set

$$v(t, x, y) = \max \{u(t, x) - u(t, y), 0\}$$

on $[0, T] \times R^{2N}$, and note that if $v - \phi$, with $\phi \in C^1(\Omega_T)$, attains its local maximum at some point in Ω_T where v vanishes, then ϕ itself attains its local minimum and hence $\phi_t = 0$ there. Take into account this and Lemma 3.3 and use (H1)', (H2)', and (3.8) to check that v is a viscosity subsolution of

$$v_t - C_1(1 + |x - y|) - \sigma(|D_x v|) - \sigma(|D_y v|) = 0 \quad \text{in } \Omega_T.$$

Meanwhile the function

$$w(t, x, y) = C_1(1 + \langle x - y \rangle_\varepsilon) + t(C_4 + C_1 \langle x - y \rangle_\varepsilon) \quad (\varepsilon > 0)$$

is a viscosity supersolution of the above equation. Notice that the Hamiltonian

$$(t, x, y, u, p, q) \longmapsto -C_1(1 + |x - y|) - \sigma(|p|) - \sigma(|q|)$$

satisfies (H1)', (H2)', (H3)' with N replaced by $2N$ (we may assume σ is continuous). Thus we find

$$u(t, x) - u(t, y) \leq w(t, x, y) \quad \text{for all } 0 \leq t \leq T, x, y \in R^N.$$

Sending $\varepsilon \searrow 0$, we have (3.15). $\square$

One of the main results in this section is this

THEOREM 3.5. *Suppose $H \in C(\bar{\Omega}_T \times R \times R^N)$ satisfies (H1)', (H2)', (H3)'. Let $u_0 \in UC(R^N)$ and $\eta_0: [0, \infty) \rightarrow [0, \infty)$ be the modulus of continuity of u_0 , i.e.,*

$$\eta_0(r) = \sup \{ |u_0(x) - u_0(y)| \mid x, y \in R^N, |x - y| \leq r \} \quad (r \geq 0).$$

Then there exists $\eta: [0, \infty) \rightarrow [0, \infty)$ depending only on $C_1, \sigma, \eta_0, \omega$, and T such that

$$(3.16) \quad \lim_{r \searrow 0} \eta(r) = 0$$

and if $u \in UC_s(\bar{\Omega}_T)$ is the (unique) viscosity solution of (3.1) and (3.2), then

$$(3.17) \quad |u(t, x) - u(t, y)| \leq \eta(|x - y|)$$

for all $0 \leq t \leq T$, $x, y \in R^N$.

Outline of proof. Replacing C_1 by a larger number if necessary, we may assume

$$\eta_0(r) \leq C_1(1+r) \quad \text{for all } r \geq 0.$$

By Lemma 3.4 we have

$$(3.18) \quad |u(t, x) - u(t, y)| \leq C_1(1+|x-y|) + t(C_4 + C_1|x-y|)$$

for all $0 \leq t \leq T$, $x, y \in R^N$, with $C_4 = C_1 + 2\sigma(C_1 + C_1T)$.

Now we fix $\varepsilon > 0$ arbitrarily and select $A = A(\varepsilon, T, \omega) > 0$, $B = B(\varepsilon, \eta_0) > 1$ so that

$$(3.19) \quad \omega(r) \leq \frac{\varepsilon}{3T} + Ar \quad \text{for all } r \geq 0$$

and

$$(3.20) \quad \eta_0(r) \leq \frac{\varepsilon}{3} + Br \quad \text{for all } 0 \leq r \leq 1.$$

Using (H2)', (H3)', and (3.19), one can easily see that $v(t, x, y) = \max \{u(t, x) - u(t, y), 0\}$ is a viscosity subsolution of

$$(3.21) \quad v_t - \frac{\varepsilon}{3T} - A|x-y|(|1 + |D_x v||) - \chi(D_x v, D_y v) = 0 \quad \text{in } \Omega_T,$$

where $\chi(p, q) = \sigma_{|p|+|q|+1}(|p+q|)$ for $p, q \in R^N$. We will assume χ is continuous on R^{2N} . Define

$$\Omega = \{(t, x, y) \in (0, T) \times R^{2N} \mid |x - y| < 1\},$$

and

$$w(t, x, y) = \frac{\varepsilon}{3} + B\langle x - y \rangle_\alpha + \frac{\varepsilon t}{3T} + 2At\langle x - y \rangle_\alpha$$

on $[0, T] \times R^{2N}$, with $0 < \alpha < 1$ and

$$r = \frac{1}{B + 2AT}.$$

Then w is a viscosity supersolution of (3.21) in Ω . We may assume in addition

$$B \geq 2C_1 + T(C_4 + C_1).$$

By this and (3.18)

$$v \leq w \quad \text{on } \partial_p \Omega.$$

In view of Proposition 3.1 we thus obtain

$$u(t, x) - u(t, y) \leq w(t, x, y) \quad \text{on } \Omega.$$

Send $\alpha \searrow 0$ to get

$$|u(t, x) - u(t, y)| \leq \frac{2}{3}\varepsilon + (B + 2AT)|x - y|$$

for all $(t, x, y) \in \Omega$. Hence

$$|u(t, x) - u(t, y)| \leq \varepsilon \quad \text{if } |x - y| \leq \min \left\{ 1, \left(\frac{\varepsilon}{3(B + 2AT)} \right)^{1/r} \right\}.$$

This implies the existence of a function η satisfying (3.16) and (3.17). $\square$

Next we turn to the continuity in t of viscosity solutions of (3.1). For brevity of the statement below, we introduce the notations:

$$\begin{aligned} v(R) &= C_2(2 + R) + T(C_3 + C_2(R + 1)) + 1, \\ h(R) &= \sup_{(t, x) \in [0, T] \times B_{R+1}} \|H(t, x, v(R), 0) + |H(t, x, -v(R), 0)|\|, \end{aligned}$$

where $C_3 = C_2 + \sigma(C_2(1 + T))$ as in Proposition 3.2.

THEOREM 3.6. Assume $H \in C(\bar{\Omega}_T \times R \times R^N)$ satisfies (H1)', (H2)', (H3)'. Let $u \in UC_s(\bar{\Omega}_T)$ be a viscosity solution of (3.1) and (3.2) and satisfy (3.10). Define $\eta: [0, \infty) \rightarrow [0, \infty)$ by

$$\eta(r) = \sup \|u(t, x) - u(t, y)\| \mid 0 \leq t \leq T, x, y \in R^N, |x - y| \leq r\} \quad (r \geq 0).$$

Then

$$(3.22) \quad |u(t, x) - u(s, x)| \leq 2\eta(\varepsilon) + |t - s| \left[h(R) + \sigma \left(C \frac{\eta(\varepsilon)}{\varepsilon} + 2v(R) \right) \right]$$

for all $0 \leq t, s \leq T$, $x \in B_R$, $\varepsilon > 0$, and $R > 0$, where C is a positive constant depending only on N .

Proof. For simplicity we prove (3.22) for $s = 0$. Let $\rho \in C_0^\infty(R^N)$ satisfy

$$\rho \geq 0, \quad \int_{R^N} \rho(x) dx = 1, \quad \text{supp } \rho \subset B_1,$$

and set $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon^N$ for $\varepsilon > 0$. Define $u^\varepsilon \in C^\infty(R^N)$ by the convolution $u^\varepsilon = \rho_\varepsilon * u_0$. Since $|u_0(x) - u_0(y)| \leq \eta(|x - y|)$ for all $x, y \in R^N$, we find that

$$(3.23) \quad |u^\varepsilon(x) - u_0(x)| \leq \eta(\varepsilon),$$

$$(3.24) \quad |Du^\varepsilon(x)| \leq C \frac{\eta(\varepsilon)}{\varepsilon}$$

for all $x, y \in \mathbb{R}^N$, $\varepsilon > 0$, with

$$C = \int_{\mathbb{R}^N} |D\rho(x)| dx.$$

Fix $\varepsilon > 0$, $R > 0$, and $x_0 \in B_R$. We observe first that the function

$$(3.25) \quad w(t, x) = u^\varepsilon(x) + \eta(\varepsilon) + 2v(R)\langle x - x_0 \rangle_\alpha + t \left[h(R) + \sigma \left(C \frac{\eta(\varepsilon)}{\varepsilon} + 2v(R) \right) \right]$$

is a viscosity supersolution of (3.1) in $(0, T) \times B_{R+1}$ for each $\alpha > 0$. For this we note that by the definition of $v(R)$ and Proposition 3.2

$$|u(t, x)| \leq v(R)$$

and in particular

$$u_0(x) \geq -v(R)$$

for all $(t, x) \in [0, T] \times B_{R+1}$. Consequently by (3.23)

$$w(t, x) \geq u_0(x) \geq -v(R)$$

and hence by the definition of $h(R)$ and (H2)'

$$H(t, x, w(t, x), 0) \geq H(t, x, -v(R), 0) \geq -h(R)$$

for all $(t, x) \in [0, T] \times B_{R+1}$. Therefore, using (H1)' and (3.24), we see that

$$(3.26) \quad w_t + H(t, x, w, Dw) \geq h(R) + \sigma \left(C \frac{\eta(\varepsilon)}{\varepsilon} + 2v(R) \right) - \sigma(|Dw|) + H(t, x, w, 0) \geq 0$$

for all $(t, x) \in [0, T] \times B_{R+1}$. Thus w is a viscosity supersolution of (3.1) in $(0, T) \times B_{R+1}$.

As $u \leq w$ on $[0, T] \times \partial B_{R+1} \cup \{0\} \times B_{R+1}$, applying Proposition 3.1 to u, w and letting $\alpha \searrow 0$ we have

$$(3.27) \quad u(t, x) \leq u^\varepsilon(x) + \eta(\varepsilon) + 2v(R)|x - x_0| + t \left[h(R) + \sigma \left(C \frac{\eta(\varepsilon)}{\varepsilon} + 2v(R) \right) \right]$$

for all $0 \leq t \leq T$, $x \in B_{R+1}$. Plug $x = x_0$ into (3.27) and use (3.23) to get

$$(3.28) \quad u(t, x) \leq u_0(x) + 2\eta(\varepsilon) + t \left[h(R) + \sigma \left(C \frac{\eta(\varepsilon)}{\varepsilon} + 2v(R) \right) \right]$$

for all $0 \leq t \leq T$, $x \in B_R$.

In a similar manner we obtain

$$(3.29) \quad u(t, x) \geq -u_0(x) - 2\eta(\varepsilon) - t \left[h(R) + \sigma \left(C \frac{\eta(\varepsilon)}{\varepsilon} + 2v(R) \right) \right]$$

for all $0 \leq t \leq T$, $x \in B_R$. □

REMARK. Instead of (3.7) if we assume

$$(3.7)' \quad |H(t, x, 0, 0)| \leq C_2 \quad \text{for all } (t, x) \in \Omega_T$$

in the above theorem, then we have

$$(3.22)' \quad |u(t, x) - u(s, x)| \leq 2\eta(s) + |t - s| \left(C_2 + \sigma \left(C \frac{\eta(s)}{\varepsilon} \right) \right)$$

for all $0 \leq t, s \leq T$, $x \in R^N$, $\varepsilon > 0$. This observation is important to prove the existence of a (bounded,) uniformly continuous viscosity solution of (3.1), (3.2).

4. Existence of Viscosity Solutions

The objective of this section is to prove the following existence theorems.

THEOREM 4.1. *Let H satisfy (H1), (H2), and (H3). Then the problem*

$$(2.1) \quad u + H(x, u, Du) = 0 \quad \text{in } R^N$$

has a (unique) viscosity solution in the class $UC(R^N)$.

THEOREM 4.2. *Assume $H \in C(\bar{Q}_T \times R \times R^N)$ satisfies (H1)'-(H3)' and $u_0 \in UC(R^N)$. Then there exists a (unique) viscosity solution $u \in UC_s(\bar{Q}_T)$ of*

$$(3.1) \quad u_t + H(t, x, u, Du) = 0 \quad \text{in } \Omega_T \equiv (0, T) \times R^N$$

and

$$(3.2) \quad u(0, x) = u_0(x) \quad \text{for } x \in R^N.$$

REMARK. In this theorem, the assumption (H2)' can be replaced by the following.

(H2)'' There is $\gamma \in R$ for which

$$H(t, x, u, p) - H(t, x, v, p) \geq \gamma(u - v)$$

for all $0 \leq t \leq T$, $x, p \in R^N$, $u, v \in R$, with $u \geq v$. Indeed, $u \in UC_s(\bar{Q}_T)$ solves (3.1) in the viscosity sense if and only if $v = e^{\gamma t} u \in UC_s(\bar{Q}_T)$ satisfies

$$v_t + \hat{H}(t, x, v, Dv) = 0 \quad \text{in } \Omega_T$$

in the viscosity sense, where

$$\hat{H}(t, x, r, p) = e^{\gamma t} H(t, x, e^{-\gamma t} r, e^{-\gamma t} p) - \gamma r.$$

If H satisfies (H1)', (H2)'', (H3)', then $\hat{H}$ satisfies (H1)', (H2)', (H3)'. It is therefore enough to assume (H2)'' instead of (H2)' in Theorem 4.2.

Proof of Theorem 4.1. For each $s > 0$ choose $\theta = \theta_s \in C^\infty(R)$ so that

$$(4.1) \quad \begin{cases} \theta(r) = r & \text{if } |r| \leq s \\ 0 \leq \theta' \leq 1, & \text{and } |\theta| \leq 2s \text{ on } R. \end{cases}$$

For $\delta > 0$ let $\rho_\delta \in C_0^\infty(R^N \times R \times R^N)$ be a standard molifier. We set

$$H_\delta(x, u, p) = \theta_\delta(H(x, u, p)) \quad \text{on } R^N \times R \times R^N$$

and define

$$H_{s,\delta} = H_s * \rho_\delta.$$

Note that, by (4.1), H_s satisfies (H1), (H2), (H3) with the same functions σ_R and ω and so does $H_{s,\delta}$ with the same σ_R and ω replaced by $r \mapsto \omega((1+\delta)r)$. Moreover $H_{s,\delta} \in C^\infty(R^N \times R \times R^N)$ is bounded on $R^N \times R \times R^N$ with its derivatives, and

$$(4.2) \quad H_{s,\delta} \longrightarrow H$$

uniformly on compact subsets of $R^N \times R \times R^N$ as $s \rightarrow \infty$, $\delta \searrow 0$.

Now we consider the following approximate problem

$$(4.3) \quad -\varepsilon \Delta u + u + H_{s,\delta}(x, u, Du) = 0 \quad \text{in } R^N,$$

with $\varepsilon > 0$. By the standard elliptic PDE theory we see that this problem has a unique solution $u = u_{s,\delta,\varepsilon} \in C^\infty(R^N) \cap L^\infty(R^N)$. As results of application of the maximum principle, we have

$$(4.4) \quad |u_{s,\delta,\varepsilon}| \leq \sup |H_{s,\delta}| \leq 2s \quad \text{on } R^N$$

and

$$(4.5) \quad |Du_{s,\delta,\varepsilon}| \leq \sup |D_s H_{s,\delta}| \quad \text{on } R^N.$$

In view of the Ascoli-Arzelà theorem, these estimates allow us to pass to the limit of $\{u_{s,\delta,\varepsilon}\}$ along a sequence $\varepsilon = \varepsilon_i \searrow 0$ for each $s, \delta > 0$, where the convergence is uniform on compact subsets of R^N . We denote by $u_{s,\delta}$ the limit function. Then by (4.4) and (4.5)

$$(4.6) \quad u_{s,\delta} \in W^{1,\infty}(R^N) \cap C(R^N)$$

and by [6, Theorem 1.4] $u = u_{s,\delta}$ solves

$$(4.7) \quad u + H_{s,\delta}(x, u, Du) = 0 \quad \text{in } R^N$$

in the viscosity sense.

We apply now Proposition 2.2 and Theorem 2.4 to see that

$$(4.8) \quad |u_{s,\delta}(x)| \leq C(1 + |x|)$$

for all $x \in R^N$, $s > 0$, $0 < \delta < 1$ and some constant $C > 0$ independent of s, δ and that the set $\{u_{s,\delta} | 0 < \delta < 1, s > 0\}$ is equi-continuous on R^N . Thus, by the Ascoli-Arzelà theorem, we conclude that for some sequence $\{(s_i, \delta_i)\}$ with $\lim s_i = \infty$ and $\lim \delta_i = 0$ and some $u \in UC(R^N)$

$$u_{s_i, \delta_i}(x) \longrightarrow u(x)$$

uniformly on compact subsets of R^N . By virtue of [6, Theorem 1.4] this together with (4.2) proves that u is a viscosity solution of (2.1). $\square$

REMARK. In Theorem 4.1 assume in addition

$$|H(x, 0, 0)| \leq C \quad \text{for all } x \in R^N$$

and some $C > 0$. Then the solution of (2.1) is bounded on R^N .

Proof of Theorem 4.2. First of all we extend H to $R \times R^N \times R \times R^N$ by setting

$$H(t, x, u, p) = \begin{cases} H(T, x, u, p) & \text{if } t > T, \\ H(0, x, u, p) & \text{if } t < 0. \end{cases}$$

By composing H, u_0 with functions satisfying (4.1) and mollifying the resulting functions, we construct sequences $\{H_k\} \subset C^\infty(R \times R^N \times R \times R^N)$, $\{u_{0k}\} \subset C^\infty(R^N)$ such that

$$(4.9) \quad H_k, D_x H_k \in L^\infty(R \times R^N \times R \times R^N) \quad (k=1, 2, \dots),$$

$$(4.10) \quad H_k \text{ satisfies (H1)'--(H3)' for some } \sigma_R, \omega \text{ independent of } k,$$

$$(4.11) \quad H_k \longrightarrow H$$

uniformly on compact subsets of $\bar{\Omega}_T \times R \times R^N$ as $k \rightarrow \infty$,

$$(4.12) \quad u_{0k}, Du_{0k} \in L^\infty(R^N) \quad (k=1, 2, \dots),$$

$$(4.13) \quad |u_{0k}(x) - u_{0k}(y)| \leq \eta_0(\|x - y\|) \quad (k=1, 2, \dots)$$

for all $x, y \in R^N$ and some $\eta_0: [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\lim_{r \searrow 0} \eta_0(r) = 0,$$

and

$$(4.14) \quad u_{0k}(x) \longrightarrow u_0(x)$$

uniformly on compact subsets of R^N as $k \rightarrow \infty$.

By the standard parabolic PDE theory, the Cauchy problem

$$(4.15) \quad \begin{cases} u_t - \varepsilon \Delta u + H_k(t, x, u, Du) = 0 & \text{in } \Omega_T, \\ u(0, x) = u_{0k}(x) & \text{for } x \in R^N \end{cases}$$

has a unique solution $u = u_{k,\varepsilon} \in C^\infty(\bar{\Omega}_T) \cap L^\infty(\Omega_T)$. An application of the maximum principle shows also that $\{u_{k,\varepsilon} | \varepsilon > 0\}$ is bounded in $W^{1,\infty}(\Omega_T)$ for each $k=1, 2, \dots$. By the Ascoli-Arzelà theorem

$$u_{k,\varepsilon_i} \longrightarrow u^k$$

uniformly on compact subsets of $\bar{\Omega}_T$ as $i \rightarrow \infty$ for some $u^k \in W^{1,\infty}(\Omega_T) \cap C(\bar{\Omega}_T)$ and some sequence $\varepsilon_i \searrow 0$. As before we find that

$$(4.16) \quad \begin{cases} u_t^k + H_k(t, x, u^k, Du^k) = 0 & \text{in } \Omega_T, \\ u^k(0, x) = u_{0k}(x) & \text{for } x \in R^N \end{cases}$$

in the viscosity sense ($k=1, 2, \dots$).

In view of (4.10), (4.11), (4.13), and (4.14), Proposition 3.2, Theorems 3.5 and 3.6 assert that $\{u^k\}$ is uniformly bounded and equi-continuous as a sequence of functions on $[0, T] \times B_R$ for each $R > 0$. Thus we conclude that

$$u^{k_i} \longrightarrow u$$

uniformly on compact subsets of $\bar{Q}_T$ for some $k_i \rightarrow \infty$ and some $u \in C(\bar{Q}_T)$ and that u solves (3.1) and (3.2) in the viscosity sense. It is obvious from Theorem 3.5 applied to u_k that $u \in UC_s(\bar{Q}_T)$. $\square$

REMARK. If we add the assumption that

$$|H(t, x, 0, 0)| \leq C$$

for all $(t, x) \in \bar{Q}_T$ and some $C > 0$, then the solution u of (3.1) and (3.2) belongs to $UC(\bar{Q}_T)$ as remarked after Theorem 3.6, and moreover $u \in BUC(\bar{Q}_T)$ provided $u_0 \in BUC(R^N)$.

5. An Existence Theorem for Locally Lipschitz Hamiltonians

The method used in §§ 2-4 is effective also for proving existence theorems corresponding to the uniqueness results in [13]. We will explain in this section how to modify the foregoing proof to prove such theorems by outlining the proof of Theorem 5.1 below.

We consider accordingly the existence of a viscosity solution of

$$(5.1) \quad u + H(x, Du) = 0 \quad \text{in } R^N,$$

where $H: R^N \times R^N \rightarrow R$ is supposed to satisfy the following conditions.

(A1) There exist constants $A > 0$, $B > 0$ such that

$$(5.2) \quad |H(x, p) - H(x, q)| \leq (A|x| + B)|p - q|$$

for all $x, p, q \in R^N$.

(A2) For each $R > 0$ there exists $\omega_R: [0, \infty) \rightarrow [0, \infty)$ such that

$$(5.3) \quad |H(x, p) - H(y, p)| \leq \omega_R(|x - y|(1 + |p|))$$

for all $x, y \in B_R$, $p \in R^N$.

(A3) For some $0 < m < 1/A$

$$(5.4) \quad H(\cdot, 0) \in Q_m(R^N),$$

where

$$Q_m(R^N) = \left\{ f \in C(R^N) \mid \sup_{x \in R^N} \frac{|f(x)|}{1 + |x|^m} < \infty \right\} \quad (m > 0).$$

THEOREM 5.1. *Under the assumptions (A1)–(A3) there exists a (unique) viscosity solution $u \in Q_m(R^N)$ of (5.1).*

The uniqueness follows from [13, Theorem 1.5]. The main difference in the proof here from the preceding problems is illustrated by the following lemmas. Before stating them, we note that by (5.4)

$$(5.5) \quad |H(x, 0)| \leq C_s(1 + |x|^m)$$

for all $x \in R^N$ and some $C_s > 0$.

LEMMA 5.2. *There exists a constant $C_s > 0$ depending only on m, A, B , and C_s such that if $u \in Q_m(R^N)$ is a viscosity solution of (5.1), then*

$$(5.6) \quad |u(x)| \leq C_s(1 + |x|^m) \quad \text{for all } x \in R^N.$$

Proof. Let $E = (C_s + 1)/(1 - mA)$ and set

$$v(x) = E\langle x \rangle^m + F \quad \text{on } R^N,$$

with $F > 0$. Use (5.2) and (5.5) to obtain

$$v + H(x, Dv) \geq \langle x \rangle^m - mBE\langle x \rangle^{m-1} - C_s + F$$

for $x \in R^N$. We now fix $F = F(m, B, E, C_s)$ so that the right hand side of the above inequality is nonnegative for all $x \in R^N$. Then by the comparison theorem [13, Theorem 1.5]

$$u(x) \leq E\langle x \rangle^m + F \quad \text{for all } x \in R^N$$

Similarly, for the above E, F , we have

$$u(x) \geq -E\langle x \rangle^m - F \quad \text{on } R^N. \quad \square$$

LEMMA 5.3. *For each $R > 0$ there exists $\eta_R: [0, 2R) \rightarrow [0, \infty)$ depending only on A, B, C_s, m, R , and ω_R such that*

$$\lim_{r \searrow 0} \eta_R(r) = 0$$

and if $u \in Q_m(R^N)$ is a (unique) solution of (5.1) then

$$(5.7) \quad |u(x) - u(y)| \leq \eta_R(|x - y|) \quad \text{for all } x, y \in B_R.$$

Outline of proof. Note first that $v(x, y) = u(x) - u(y)$ solves

$$(5.8) \quad v + \tilde{H}(x, y, Dv) = 0 \quad \text{in } R^{2N}$$

in the viscosity sense, where $D = (D_x, D_y)$ and

$$\tilde{H}(x, y, p, q) = H(x, p) - H(y, -q).$$

Fix $\varepsilon > 0$, $0 < \alpha < 1$, and μ so that $m < \mu < 1/A$. Let us set

$$w(x, y) = \frac{\epsilon}{4} + E\langle x - y \rangle_a^I + \beta(\langle x \rangle^\mu + \langle y \rangle^\mu + F)$$

on R^{2N} , with constants $0 < \beta < 1$, $\gamma > 0$, $E > 0$, and $F > 0$ which will be chosen later. Using (5.2) and (5.3), we find that for $x, y \in B_R$

$$(5.9) \quad \begin{aligned} w + \tilde{H}(x, y, Dw) &\geq \frac{\epsilon}{4} + E\langle x - y \rangle_a^I + \beta(\langle x \rangle^\mu + \langle y \rangle^\mu + F) \\ &\quad - (A|x| + B)\beta\mu\langle x \rangle^{\mu-1} - (A|y| + B)\beta\mu\langle y \rangle^{\mu-1} - \omega_R(|x - y|(1 + \gamma E\langle x - y \rangle_a^{I-1})) \\ &\geq \beta((1 - \mu A)(\langle x \rangle^\mu + \langle y \rangle^\mu) - B\mu(\langle x \rangle^{\mu-1} + \langle y \rangle^{\mu-1}) - F) \\ &\quad + \frac{\epsilon}{4} + E\langle x - y \rangle_a^I - \omega_R(|x - y|(1 + \gamma E\langle x - y \rangle_a^{I-1})). \end{aligned}$$

We choose now $F = F(\mu, A, B)$ so large that the expression in the braces above is nonnegative for all $x, y \in R^N$. Then we fix $R > 0$, and choose $\beta = \beta(\epsilon, R, \mu, F)$ so that

$$(5.10) \quad \beta(\langle x \rangle^\mu + \langle y \rangle^\mu + F) < \frac{\epsilon}{4} \quad \text{for } x, y \in B_R.$$

An argument similar to the proof of Lemma 5.2 shows that the function

$$z(x, y) = C_7(\langle x \rangle^\mu + \langle y \rangle^\mu) + C_8$$

is a viscosity supersolution of (5.8) for some $C_7 = C_7(m, A, B, C_8)$ and $C_8 = C_8(m, A, B, C_8)$. Since $\mu > m$, one can select $R_1 = R_1(\beta, \mu, F, m, C_7, C_8)$ such that $R_1 > R$ and

$$\beta(\langle x \rangle^\mu + \langle y \rangle^\mu + F) > z(x, y) \quad \text{for } (x, y) \in R^{2N} \setminus B_{R_1}^2.$$

We use the same technique as in the proof of Theorem 2.4 to find $\gamma = \gamma(\epsilon, \omega_{R_1})$ and $E = E(\epsilon, R_1, \omega_{R_1})$ such that

$$\frac{\epsilon}{4} + E\langle x - y \rangle_a^I - \omega_{R_1}(|x - y|(1 + \gamma E\langle x - y \rangle_a^{I-1})) \geq 0$$

for all $x, y \in B_{R_1}$. This together with (5.9) and the choice of F yields

$$(5.11) \quad w + \tilde{H}(x, y, Dw) \geq 0 \quad \text{in } B_{R_1} \times B_{R_1}.$$

Thus we know ([6, Prop. 1.4]) that

$$W = \begin{cases} \min\{w, z\} & \text{on } B_{R_1} \times B_{R_1}, \\ z & \text{on } R^{2N} \setminus B_{R_1}^2, \end{cases}$$

is a viscosity supersolution of (5.8) in R^{2N} , and hence

$$v \leq w \quad \text{in } B_R^2.$$

Here we have used that $\tilde{H}$ on $R^{2N} \times R^{2N}$ satisfies (A1), (A2), (A3) with N replaced by $2N$. This and (5.10) give

$$|u(x) - u(y)| \leq \frac{\varepsilon}{2} + E|x - y|^r \quad \text{for all } x, y \in B_R,$$

and consequently

$$|u(x) - u(y)| < \varepsilon \quad \text{if } x, y \in B_R \quad \text{and } |x - y| < \left(\frac{\varepsilon}{2E}\right)^{1/r}.$$

This completes the proof. $\square$

Proof of Theorem 5.1. The same proof as Theorem 4.1 with Lemmas 5.2 and 5.3 instead of Proposition 2.2 and Theorem 2.4 applies to prove Theorem 5.1 and we omit the details. $\square$

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