METASTABILITY FOR PARABOLIC EQUATIONS WITH DRIFT: PART I

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Abstract. We study the exponentially long time behavior of solutions to linear uniformly parabolic equations which are small perturbations of transport equations with vector fields having a globally stable (attractive) equilibrium in the domain. The result is that the solutions converge to a constant, which is either the initial value at the stable point or the boundary value at the minimum of the associated quasi-potential. Problems of this type were considered by Freidlin and Wentzell and Freidlin and Koralov using probabilistic arguments related to the theory of large deviations. Our approach, which is self-contained, relies entirely on pde arguments and is flexible to the extent that allows us to study a class of semilinear equations of similar structure. This note also prepares the ground for the forthcoming Part II of this work where we consider general quasilinear problems.

1. Introduction

In this paper we provide a self-contained analysis, based entirely on pde methods, of the long time behavior (at scale $\exp \lambda/\varepsilon$), as $\varepsilon \to 0$, of the solution $u^\varepsilon = u^\varepsilon(x,t)$ of the parabolic equation

$$u^\varepsilon_t = L_\varepsilon u^\varepsilon \quad \text{in } \ Q := \Omega \times (0,\, \infty), \tag{1.1}$$

with the initial-boundary condition

$$u^\varepsilon = g \ \text{ on } \ \partial_{\mathrm{p}} Q := (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,\infty)), \tag{1.2}$$

where Ω is an open subset of $\mathbb{R}^n$ and, for $\varepsilon > 0$, $x \in \Omega$ and ϕ smooth, the elliptic operator L_ε is given by

$$L_\varepsilon \phi(x) := \varepsilon \,\mathrm{tr}[a(x) D^2\phi(x)] + b(x)\cdot D\phi(x).$$

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Here $a(x) = (a_{ij}(x))_{1\le i,j\le n} \in \mathbb{S}^n$, the space of $n\times n$ symmetric matrices, is positive, "tr" and "$\cdot$" denote the trace of square matrices and the inner product in Euclidean spaces respectively and the vector field b has some $x_0 \in \Omega$ as an asymptotically stable equilibrium. Exact assumptions are stated below.

Roughly speaking the result states that there exists $m_0 > 0$ and some $x^* \in \partial\Omega$ such that, as $\varepsilon \to 0$ and locally uniformly in Ω,

$$u^\varepsilon(x, \mathrm{e}^{\lambda/\varepsilon}) \to g(x_0) \ \text{ if } \ \lambda < m_0 \ \text{ and } \ u^\varepsilon(x, \mathrm{e}^{\lambda/\varepsilon}) \to g(x^*) \ \text{ if } \ \lambda > m_0.$$

To make precise statements as well as to provide an interpretation of the results in terms of the metastability properties of random perturbations of some ordinary differential equations (ode for short), we introduce next the assumptions (A1)–(A6) which will hold throughout. In what follows $B_r(x)$ is the open ball in $\mathbb{R}^n$ centered at the x with radius r, $B_r := B_r(0)$ and $\mathrm{Lip}(\mathcal{X};\mathcal{Y})$ is the set of Lipshitz continuous functions defined on $\mathcal{X}$ and values in $\mathcal{Y}$; when $\mathcal{Y} = \mathbb{R}$, we may write $\mathrm{Lip}(\mathcal{X})$. We assume that:

(A1) (Regularity) $a \in \mathrm{Lip}(\mathbb{R}^n;\mathbb{S}^n)$ and $b \in \mathrm{Lip}(\mathbb{R}^n;\mathbb{R}^n)$.

(A2) (Uniform ellipticity) There exists $\theta \in (0,\, 1)$ such that, for all $x \in \mathbb{R}^n$,

$$\theta I \le a(x) \le \theta^{-1} I.$$

(A3) The set Ω is a bounded, open, connected subset of $\mathbb{R}^n$ with C^1-boundary.

We consider the dynamical system generated by the ode

$$\dot{X} = b(X), \tag{1.3}$$

where $\dot{X}$ denotes the derivative of the function $t \mapsto X(t)$. The solution of (1.3) with initial condition $X(0) = x \in \mathbb{R}^n$ is denoted by $X(t;x)$. The assumptions on b are:

(A4) (Global asymptotic stability) For any $x \in \mathbb{R}^n$ $\lim_{t\to\infty} X(t;x) = x_0$ and for any $\delta > 0$ there exists $r > 0$ such that, if $x \in B_r(x_0)$, then $X(t;x) \in B_\delta(x_0)$ for all $t \ge 0$.

(A5) $b(x)\cdot\nu(x) < 0$ on $\partial\Omega$, where $\nu(x)$ is the exterior unit normal at $x \in \partial\Omega$.

To simplify the notation throughout the paper we take $x_0 = 0$. For the convenience of the reader we write this convention as an additional assumption namely

(A6) $x_0 = 0 \in \Omega$.

We remark that (A4) and (A6) imply that

$$b(0) = 0 \ \text{ and } \ b \ne 0 \text{ in } \mathbb{R}^n \setminus \{0\},$$

and that (A5) ensures that Ω (resp. $\overline{\Omega}$) is positively invariant under the flow $X : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, that is, for all $(x,t) \in \Omega \times [0,\, \infty)$ (resp. $(x,t) \in \overline{\Omega}\times[0,\, \infty)$),

$$X(t;x) \in \Omega \ \ (\text{resp. } X(t;x) \in \overline{\Omega}).$$

This behavior is depicted in Figure 1 below.

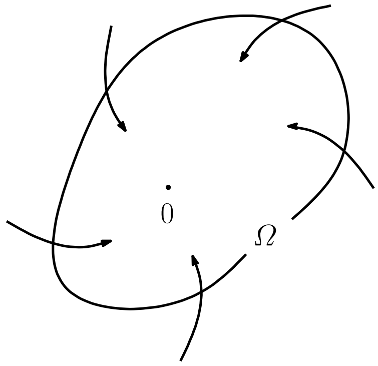

FIGURE 1. Portrait of the flow generated by the vector field b.

The asymptotic behavior of the u^ε's, as $\varepsilon \to 0$, is closely related to that of the the stochastic differential equation

$$\mathrm{d}\, X_t^\varepsilon = b(X_t^\varepsilon)\,\mathrm{d}\, t + \sqrt{2\varepsilon}\sigma(X_t^\varepsilon)\,\mathrm{d}\, W_t, \tag{1.4}$$

which is a random perturbation of (1.3); here $(W_t)_{t\in\mathbb{R}}$ is a standard n-dimensional Brownian motion and the matrix $\sigma \in \mathbb{S}^n$ is the square root of a, that is, $a = \sigma^2$ and $\sigma \geq 0$, which, in view of (A1) and (A2), is Lipschitz continuous in $\mathbb{R}^n$. As in the case of (1.3), we write $X^\varepsilon(t;x)$ for the solution at time t of the solution of (1.4) starting at $t = 0$ at x. Finally, $\mathbb{P}(A)$ denotes the probability of the event A.

The connection is made in terms of the asymptotic behavior of the first exit time τ_x^ε of $X^\varepsilon(t;x)$ from Ω, that is

$$\tau_x^\varepsilon := \inf\{t \geq 0 \,:\, X^\varepsilon(t;x) \notin \Omega\}.$$

To formulate the result we consider the Hamilton-Jacobi equation

$$H(x, Du) = 0 \quad \text{in } \ \Omega, \tag{1.5}$$

where, for $x, p \in \mathbb{R}^n$,

$$H(x,p) = a(x)p \cdot p + b(x) \cdot p, \tag{1.6}$$

its maximal subsolution $V \in C(\overline{\Omega})$ satisfying $V(0) = 0$, and we set $m_0 = \min_{\partial\Omega} V$ —throughout the paper when we refer to solutions of Hamilton-Jacobi and "viscous" Hamilton-Jacobi equations we mean viscosity solutions.

For $a = I$ the results of [11, Chap. 4] (see also [9, 10] for a general case) state roughly that, in probability, for any $x \in \Omega$ and as $\varepsilon \to 0$,

$$\tau_x^\varepsilon \approx \mathrm{e}^{m_0/\varepsilon} \ \text{ and } X^\varepsilon(t;x) \text{ exits from } \ \Omega \text{ near } \arg\min(V|\partial\Omega), \tag{1.7}$$

where

$$\arg\min(V|\partial\Omega) := \{x \in \partial\Omega : V(x) = \min_{\partial\Omega} V\}.$$

A simple example that gives an idea of what is happening is to take $a = I$, $b(x) = -x$ and $\Omega = B_1$. In this case $u(x) = |x|^2/2$ obviously satisfies $H(x, Du(x)) = |x|^2 - x \cdot x = 0$ and, indeed, $V(x) = |x|^2/2$. It also follows from elementary stochastic calculus considerations that, for every $x \in B_1$, $X^\varepsilon(t;x) = x\exp(-t) + \sqrt{2\varepsilon}\int_0^t \exp(s-t)dW_s$.

Given H as in (1.6), let $L \in C(\mathbb{R}^{2n})$ be its convex conjugate, that is

$$L(x,\xi) := \frac{1}{4}a(x)^{-1}(\xi - b(x))\cdot(\xi - b(x)),$$

where $a(x)^{-1}$ denotes the inverse matrix of $a(x)$.

Following Freidlin-Wentzell [11], we introduce the quasi-potential V_Ω on $\overline{\Omega} \times \overline{\Omega}$ given by

$$\begin{aligned} V_\Omega(x,y) := \inf\Big\{ \int_0^T L(X(t), \dot X(t))\,\mathrm{d}\,t \,:\, T > 0,\ X \in \mathrm{Lip}([0,\,T]; \overline{\Omega}), \\ X(0) = x,\ X(T) = y\Big\}. \end{aligned} \tag{1.8}$$

Next we define the function V and the constant m_0 by

$$V(y) := V_\Omega(0,y) \quad \text{and} \quad m_0 = \min_{\partial\Omega} V.$$

Our main theorem is as follows.

Theorem 1. *Assume* (A1)–(A6) *and* $g \in C(\overline{\Omega})$. *For each* $\varepsilon > 0$, *let* $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ *be the solution of* (1.1), (1.2).

(i) *Fix* $\lambda \in (0,\, m_0)$. *For any compact subset* K *of* Ω *and* $\sigma(\varepsilon) > 0$ *such that* $\sigma(\varepsilon) \le \exp(\lambda/\varepsilon)$ *and* $\lim_{\varepsilon\to 0+}\sigma(\varepsilon) = \infty$,

$$\lim_{\varepsilon\to 0+} u^\varepsilon(\cdot, t) = g(0) \quad \textit{uniformly on} \ \ K \times [\sigma(\varepsilon),\, \mathrm{e}^{\lambda/\varepsilon}]. \tag{1.9}$$

(ii) *Assume that* $g = g(0)$ *on* $\arg\min(V|\partial\Omega)$. *For any compact subset* K *of* $\Omega \cup \arg\min(V|\partial\Omega)$ *and* $\sigma(\varepsilon) > 0$ *such that* $\lim_{\varepsilon\to 0+}\sigma(\varepsilon) = \infty$,

$$\lim_{\varepsilon\to 0+} u^\varepsilon(\cdot, t) = g(0) \quad \textit{uniformly on}\ K \times [\sigma(\varepsilon), \infty). \tag{1.10}$$

(iii) *Fix* $\lambda \in (m_0,\, \infty)$ *and assume that* $g = g_0$ *on* $\arg\min(V|\partial\Omega)$ *for some constant* g_0. *Then, for every compact subset* K *of* $\Omega \cup \arg\min(V|\partial\Omega)$,

$$\lim_{\varepsilon\to 0+} u^\varepsilon(\cdot, t) = g_0 \quad \textit{uniformly on} \ \ K \times [\mathrm{e}^{\lambda/\varepsilon},\, \infty). \tag{1.11}$$

The theorem above has been obtained via probabilistic arguments in [11, 9, 10]. We present, in this paper, a proof of the theorem based on entirely pde arguments.

A probabilistic consequence of the above theorem for the general random perturbation (1.4), which may justify (1.7), is stated in the following theorem.

Theorem 2. *For any* $\delta \in (0,\, m_0)$, *any compact* $K \subset \Omega$ *and a neighborhood* U, *relative to* $\partial\Omega$, *of* $\arg\min(V|\partial\Omega)$,

$$\lim_{\varepsilon\to 0+} \mathbb{P}(\mathrm{e}^{(m_0-\delta)/\varepsilon} \le \tau_x^\varepsilon \le \mathrm{e}^{(m_0+\delta)/\varepsilon},\, X^\varepsilon(\tau_x^\varepsilon; x) \in U) = 1 \quad \textit{uniformly for} \ \ x \in K.$$

Next we explain heuristically how Theorem 2 follows from Theorem 1. To this end, recall that the solution u^ε of (1.1), (1.2) is given by $u^\varepsilon(x,t) = \mathbb{E}\, g(X^\varepsilon(\min(t,\tau_x^\varepsilon);x))$, where $\mathbb{E}$ denotes the expectation. We assume here that Theorem 1 and the above formula for u^ε are valid for discontinuous g, something not true in general. Moreover we remark that below we do not repeat the qualifier that all the limits hold as $\varepsilon \to 0$.

Fix a small neighborhood U of $\arg\min(V|\partial\Omega)$, relative to $\partial\Omega$. It follows from (1.11), with $g = \mathbf{1}_U$, that, for any $\delta > 0$ and any compact $K \subset \Omega$,

$$\mathbb{E}(\mathbf{1}_U(X^\varepsilon(\min(t,\tau_x^\varepsilon);x)) \to 1 \quad \text{uniformly on } \; K \times [\mathrm{e}^{(m_0+\delta)/\varepsilon}, \infty), \tag{1.12}$$

and, since obviously

$$\mathbb{E}(\mathbf{1}_U(X^\varepsilon(\min(t,\tau_x^\varepsilon);x)) = \mathbb{P}(t \geq \tau_x^\varepsilon,\, X^\varepsilon(\tau_x^\varepsilon;x) \in U),$$

$$\mathbb{P}(\tau_x^\varepsilon \leq \mathrm{e}^{(m_0+\delta)/\varepsilon},\, X^\varepsilon(\tau_x^\varepsilon;x) \in U) \to 1 \quad \text{uniformly for } x \in K. \tag{1.13}$$

Similarly (1.9), with $g = \mathbf{1}_{\partial\Omega}$, yields that, for any $\delta \in (0,\, m_0)$ and compact $K \subset \Omega$,

$$\mathbb{E}\,\mathbf{1}_{\partial\Omega}(X^\varepsilon(\min(t,\tau_x^\varepsilon);x)) \to 0 \quad \text{uniformly for } \; (x,t) \in K \times [\mathrm{e}^{\delta/\varepsilon},\, \mathrm{e}^{(m_0-\delta)/\varepsilon}],$$

which implies that

$$\mathbb{P}(X^\varepsilon(\min(\mathrm{e}^{(m_0-\delta)/\varepsilon},\tau_x^\varepsilon);x) \in \Omega) = \mathbb{P}(\tau_x^\varepsilon > \mathrm{e}^{(m_0-\delta)/\varepsilon}) \to 1 \quad \text{uniformly for } \; x \in K.$$

The above limit and (1.13) show that, for any $\delta \in (0,\, m_0)$ and compact $K \subset \Omega$,

$$\mathbb{P}(\mathrm{e}^{(m_0-\delta)/\varepsilon} < \tau_x^\varepsilon \leq \mathrm{e}^{(m_0+\delta)/\varepsilon},\, X^\varepsilon(\tau_x^\varepsilon;x) \in U) \to 1 \quad \text{uniformly for } \; x \in K.$$

We continue with a heuristic description of the steps of the proof of Theorem 1. To simplify the presentation we write $a \approx b$ to denote that a and b are, in an appropriate sense, close to each other. For example, for a function f, $f(0) \approx a$ means that $f - a$ is small in a neighborhood of 0. Moreover, since all the asymptotic statements below hold for ε small, we do not repeat this and $f^\varepsilon(0) \approx a$ means that, in a "small" neighborhood of 0 unless otherwise specified, $f^\varepsilon - a$ is uniformly close for small ε.

The first two steps are the observations that there exists some $\delta > 0$ such that, if $g(0) \approx 0$, then $u^\varepsilon(0,t) \approx 0$ for $t \leq \exp(\delta/\varepsilon)$, and, if $\lambda > 0$ is such that $\{V \leq \lambda\} \subset \{g \approx 0\}$, then $u^\varepsilon(0,t) \approx 0$ for $t \leq \exp(\lambda/\varepsilon)$. Both estimates can be shown by constructing appropriate barriers to (1.1) using the quasi potential. Then we show (third step), using again barriers constructed from the quasi potential, that, for each $\delta > 0$, if $g \approx 0$ on $\partial\Omega$, then $u^\varepsilon(\cdot,t) \approx 0$ on $\bar{\Omega} \times [\exp(m_0+\delta)/\varepsilon, \infty)$.

Next we consider (forth step) the solution v^ε of the stationary (time independent) version of (1.1), that is the boundary value problem (5.1) and show that, if $g \approx 0$ on $\arg\min(V|\partial\Omega)$, then $v^\varepsilon(0) \approx 0$.

Since the limits of the u^ε's satisfy the transport equation $u_t + b \cdot Du = 0$, it is then possible to show (step 5), using the properties of the vector field b, that, for each compact $K \subset \Omega$, there exists $T_K > 0$ such that, if $g(0) \approx 0$, then $u^\varepsilon(\cdot, T_K) \approx 0$.

In step 6 we establish that, for every $\delta \in (0, m_0)$ and compact $K \subset \Omega$ there exists $T_K > 0$ such that, if $g(0) \approx 0$, then $u^\varepsilon(\cdot,t) \approx 0$ uniformly on $K \times [T_K, \exp(m_0-\delta)/\varepsilon)$, and in the final (seventh) step we prove that, for every $\delta > 0$ and compact $K \subset \Omega$, if $g \approx 0$ on $\arg\min(V|\partial\Omega)$, then $u^\varepsilon(\cdot,t) \approx 0$ on $K \times [\exp(m_0+\delta)/\varepsilon, \infty)$.

The paper is organized as follows: The first seven sections are devoted to the proofs of the seven steps outlined above. In Section 2 we discuss the basic properties of the quasi-potnetial as well (2.2) and construct, for each $r > 0$, a smooth approximation $w_r \in C^2(\overline{\Omega})$ of $V = V_\Omega(0,\cdot)$ such that $H(x, Dw_r) \leq -\eta$ in $\Omega \setminus B_r$ for some $\eta > 0$. This approximation is used in Section 3 to establish steps 1 and 2, that is to prove in Theorem 8. In Section 4 we show (Proposition 10) that, for $\lambda > m_0$, there exists a semiconcave function $W \in \mathrm{Lip}(\overline{\Omega})$ such that $0 < \min_{\overline{\Omega}} W \leq \max_{\overline{\Omega}} W < \lambda$ and $H(x, -DW) \geq \eta$ for some $\eta > 0$, and we prove Theorem 9, which corresponds to step 3. Steps 4 and 5 are to establish Theorems 11 and 13, which are respectively main topics of Sections 5 and 6. The proof (steps 6 and 7) of the main theorem, Theorem 1, is given in Section 7. The proof of Theorem 2 is given in Section 8. In Section 9 we present a generalization of Theorem 1 to a class of semilinear parabolic equations. Finally, in the Appendix we present a new existence and uniqueness result of viscosity solutions for the class of the semilinear equations considered in Section 9.

Notation and terminology. We write $\overline{B}_R(y,s)$ for the closure of $B_R(y,s)$. We denote by $a \vee b$ and $a \wedge b$ the larger and smaller of a, $b \in \mathbb{R}$ respectively and, for $a \in \mathbb{R}$, $a_+ := a \vee 0$ and $a_- := (-a)_+$. For any $f : A \to B$ we write $\|f\|_{\infty,A}$ for the $\sup_{x\in A} |f(x)|$ and, if $B = \mathbb{R}$, $\{f < \alpha\}$ (resp. $\{f \leq \alpha\}$) for $\{x : f(x) < \alpha\}$ (resp. $\{x : f(x) \leq \alpha\}$). We write USC(U) and LSC(U) for the sets of upper- and lower-semicontinuous functions defined on U. Let $f : A \to B$, and let $\{f_\varepsilon\}_{\varepsilon>0}$ and $\{K_\varepsilon\}_{\varepsilon>0}$ be collections of functions $f_\varepsilon : A \to B$ and of subsets $K_\varepsilon \subset A$. We say that $\lim_{\varepsilon\to 0+} f_\varepsilon = f$ uniformly on K_ε, if $\lim_{\varepsilon\to 0+} \|f_\varepsilon - f\|_{\infty,K_\varepsilon} = 0$. Finally ω_f stands for the modulus of continuity of the uniformly continuous function f.

Throughout the paper subsolutions and supersolutions should be taken to be in the Crandall-Lions viscosity sense. That is, given an open $S \subset \mathbb{R}^n$ and $u : S \to \mathbb{R}^n$ and $F, G : S \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, we say that

$$F(x, u, Du, D^2u) \leq G(x, u, Du, D^2u) \quad \text{in } S$$

holds in the (viscosity) subsolution(resp. supersolution) sense, if we have

$$F(x, u(x), D\phi(x), D^2\phi(x)) \leq G(x, u(x), D\phi(x), D^2\phi(x))$$

(resp.

$$F(x, u(x), D\phi(x), D^2\phi(x)) \geq G(x, u(x), D\phi(x), D^2\phi(x)))$$

for all $(x, \phi) \in S \times C^2(S)$ such that $u - \phi$ takes a maximum (resp. minimum) at x. We also use the term "in the (viscosity) subsolution sense" or "in the (viscosity) supersoluton sense" for strict inequalities, reversed inequalities and sequences of inequalities.

2. The quasi-potential and a smooth approximation

Here we recall some classical facts about the quasi-potential and then we construct a smooth approximation, which plays an important role in the rest of the analysis.

It is well-known from the theory of viscosity solutions ([16, 3, 1, 5]) as well the weak KAM theory ([7, 8], [17, Prop. 7.2]) that $V_\Omega : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ is given by

$$V_\Omega(x,y) = \sup\{\psi(x) - \psi(y) : \psi \in \mathcal{S}^-(\Omega)\}, \tag{2.1}$$

where $\mathcal{S}^-(\Omega) := \{\psi \in C(\overline{\Omega}) : H(x, -D\psi) \le 0 \text{ in } \Omega\}$; note that the coercivity of the Hamiltonian implies that $\mathcal{S}^-(\Omega) \subset \mathrm{Lip}(\overline{\Omega})$ and recall that $\psi \in \mathcal{S}^-(\Omega)$ if and only if $H(x, -D\psi) \le 0$ a.e..

It is obvious from (1.8) that, for all $x, y \in \overline{\Omega}$ and $t \ge 0$, $V_\Omega(x,y) \ge 0$ and $V_\Omega(x, X(t;x)) = 0$. Moreover, letting $t \to \infty$, it follows that, for all $x \in \overline{\Omega}$, $V_\Omega(x, 0) = 0$. It is also easily seen from the definition of V_Ω that, for all $x, y, z \in \overline{\Omega}$,

$$V_\Omega(x,y) \le V_\Omega(x,z) + V_\Omega(z,y).$$

For any $y \in \overline{\Omega}$, let

$$u(x) := V_\Omega(x,y) \quad \text{and} \quad v(x) := V_\Omega(y,x).$$

It is immediate that $u \in \mathcal{S}^-(\Omega)$ and v is a subsolution of $H(x, Dv) \le 0$ in Ω. Finally, in the viscosity sense,

$$H(x, -Du) = 0 \quad \text{in } \Omega \setminus \{y\} \quad \text{and} \quad H(x, Dv) = 0 \quad \text{in } \Omega \setminus \{y\}. \tag{2.2}$$

Next we state a technical fact that we need for the construction of the above mentioned auxiliary function.

Proposition 3. *Assume* (A1)–(A6). *There exists* $\psi \in C(\overline{\Omega} \setminus \{0\})$ *such that, for all* $r > 0$,

$$\psi \in \mathrm{Lip}(\overline{\Omega} \setminus B_r), \ b \cdot D\psi = -1 \ \text{ a.e. in } \Omega \setminus \{0\}, \ \text{ and } \lim_{x \to 0} \psi(x) = -\infty.$$

Before proving the proposition, we show in the next lemma a localization property of the flow $X(t;x)$.

Lemma 1. *Assume* (A1)–(A6). *For any* $0 < r < R$, *there exists* $T = T(r,R) > 0$ *such that, for all* $x \in B_R$ *and* $t \ge T$, $X(t;x) \in B_r$.

Proof. In view of the asymptotic stability of the origin, there exists $\delta > 0$ such that, for all $x \in B_\delta$ and $t \ge 0$, $X(t;x) \in B_r$, while the global asymptotic stability yileds, for each $x \in \overline{B}_R$, a $t_x > 0$ such that $X(t_x;x) \in B_\delta$. Moreover, the continuous dependence of the flow with respect the initial value implies that $X(t_x;y) \in B_\delta$ for all y in a neighborhood of x. Finally, using the compactness of $\overline{B}_R$, we find some $T > 0$ such that, for each $x \in \overline{B}_R$, there exists $\bar{t}_x \in [0, T]$ such that $X(\bar{t}_x;x) \in B_\delta$. It then follows that, for all $t \ge \bar{t}_x$, $X(t;x) \in B_r$, and, hence, $X(t;x) \in B_r$ for all $x \in \overline{B}_R$ and $t \ge T$. □

We continue with the

Proof of Proposition 3. Fix $R > 0$ so that $\overline{\Omega} \subset B_R$, select $f \in \mathrm{Lip}(\mathbb{R}^n)$ such that $f \ge 0$, $f = 1$ on $\overline{\Omega}$ and $f = 0$ in $\mathbb{R}^n \setminus \overline{B}_R$, and consider the transport equation

$$b \cdot D\psi = -f \quad \text{in } B_R \setminus \{0\}.$$

Lemma 1 yields that, for each $r \in (0, R)$, there exists $T(r,R) > 0$ such that, for all $x \in \mathbb{R}^n \setminus B_r$ and $t \geq T(r,R)$,

$$X(-t;x) \in \mathbb{R}^n \setminus B_R.$$

Next define $\psi : B_R \setminus \{0\} \to \mathbb{R}$ by

$$\psi(x) := -\int_0^\infty f(X(-t;x))\,\mathrm{d}\,t,$$

and note that, if r and $T(r,R)$ are as above, then, for all $x \in B_R \setminus B_r$,

$$\psi(x) = -\int_0^{T(r,R)} f(X(-t;x))\,\mathrm{d}\,t.$$

It follows that ψ is Lipschitz continuous on any compact subset of $B_R \setminus \{0\}$ and

$$b \cdot D\psi = -f \quad \text{a.e. in } \ B_R \setminus \{0\}.$$

Since $b(0) = 0$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$, there exists $L > 0$ such that $|b(x)| \leq L|x|$ for all $x \in \mathbb{R}^n$. This implies that, for all $t \geq 0$,

$$|x| = |X(t;X(-t;x))| \leq |X(-t;x)|\,\mathrm{e}^{Lt},$$

and, hence, $\lim_{x\to 0}\psi(x) = -\infty$. □

Next we establish some technical consequences of Proposition 3 which are used later in the paper.

Corollary 4. *Assume* (A1)–(A6). *For each* $r > 0$ *there exist* $\psi_r \in \mathrm{Lip}(\overline{\Omega})$ *and* $\eta > 0$ *such that*

$$H(x, D\psi_r) \leq -\eta \ \text{ a.e. in } \ \Omega \setminus B_r \ \text{ and } \ H(x, D\psi_r) \leq 0 \text{ a.e. in } \ B_r.$$

Proof. Let $\psi \in C(\overline{\Omega} \setminus \{0\})$ be the function constructed in Proposition 3. Fix $r > 0$ and select $R > 0$ so that $\min_{\overline{\Omega}\setminus B_r} \psi > -R$, define $\chi_r \in \mathrm{Lip}(\overline{\Omega})$ by

$$\chi_r(x) := \begin{cases} -R & \text{if } \ x = 0, \\ \max\{\psi(x),\, -R\} & \text{otherwise,} \end{cases}$$

and observe that

$$D\chi_r = \begin{cases} D\psi & \text{a.e. in } \ \Omega \setminus B_r, \\ D\psi \quad \text{or} \quad 0 & \text{a.e. in } \ B_r. \end{cases}$$

Let $\lambda > 0$ be a constant to be fixed later, set $\psi_r := \lambda\chi_r$ and note that, for a.e. $x \in \Omega \setminus B_r$, if $C > 0$ is a Lipschitz bound of χ_r, then

$$H(x, D\psi_r) \leq \theta^{-1}|D\psi_r|^2 + b(x) \cdot D\psi_r \leq \lambda(\theta^{-1}C\lambda - 1).$$

If $\lambda := \theta/(2C)$, then

$$H(x, D\psi_r) \leq -\lambda/2 \quad \text{a.e. in } \ \Omega \setminus B_r,$$

and, similarly, it is easy to check that

$$H(x, D\psi_r) \leq 0 \quad \text{a.e. in } \ B_r \cap \Omega.$$

□

Corollary 5. *Assume* (A1)–(A6). *For all* $y \in \overline{\Omega} \setminus \{0\}, \quad V_\Omega(0, y) > 0.$

Proof. Fix any $y \in \overline{\Omega} \setminus \{0\}$, choose $r > 0$ so that $y \notin \overline{B}_r$ and let $\psi_r \in \mathrm{Lip}(\overline{\Omega})$ be as in the proof of Corollary 4, so that

$$H(x, D\psi_r) \leq 0 \ \text{ a. e. in } \ \Omega \ \text{ and } \ \psi_r(0) < \psi_r(x).$$

Set $\phi := -\psi_r$ and observe that

$$H(x, -D\phi) \leq 0 \ \text{ a.e. in } \ \Omega \ \text{ and } \ \phi(0) - \phi(y) > 0.$$

It follows, in view of (2.1), that

$$V_\Omega(0, y) \geq \phi(0) - \phi(y) > 0. \qquad \square$$

The aim of the rest of this section is to construct a smooth approximation of $V := V_\Omega(0, \cdot)$ which is a strict subsolution of the above Hamilton-Jacobi equation away from 0 while it remains a subsolution in the whole domain.

Proposition 6. *Assume* (A1)–(A6). *Let* $V = V_\Omega(0, \cdot)$. *For any* $r > 0$ *there exist* $v_r \in \mathrm{Lip}(\overline{\Omega})$ *and* $\eta > 0$ *such that,*

$$H(x, Dv_r) \leq -\eta \ \text{ a.e. in } \ \Omega \setminus B_r, \ H(x, Dv_r) \leq 0 \ \text{ a.e. in } \ B_r \ \text{ and } \ \|v_r - V\|_{\infty,\Omega} < r.$$

Proof. Fix $r > 0$, let $\psi_r \in \mathrm{Lip}(\overline{\Omega})$ and $\eta > 0$ be as in Corollary 4 and $\delta \in (0, 1)$ a constant to be fixed later, define $v_r \in \mathrm{Lip}(\overline{\Omega})$ by $v_r := (1 - \delta)V + \delta\psi_r$ and observe that

$$\begin{cases} H(x, Dv_r) \leq (1 - \delta)H(x, DV) + \delta H(x, D\psi_r) \leq -\eta\delta & \text{a.e. in } \Omega \setminus B_r, \\ H(x, Dv_r) \leq 0 \ \text{ a.e. in } \ B_r, \\ |V - v_r| \leq \delta|\psi_r - V| \quad \text{on } \ \overline{\Omega}. \end{cases}$$

The claim follows if $\delta > 0$ is so small that $\delta\|V - \psi_r\|_{\infty,\Omega} < r$. $\square$

Theorem 7. *Assume* (A1)–(A6). *Let* $V = V_\Omega(0, \cdot)$. *For any* $r > 0$ *there exist* $w_r \in C^2(\overline{\Omega})$ *and* $\eta \in (0, 1)$ *such that*

$$H(x, Dw_r) \leq -\eta \ \text{ in } \ \Omega \setminus B_r, \ H(x, Dw_r) \leq 1 \ \text{ in } \ B_r, \ \text{ and } \ \|w_r - V\|_{\infty,\Omega} < r.$$

Proof. Fix $r > 0$ and let $v_r \in \mathrm{Lip}(\overline{\Omega})$ and $\eta > 0$ be as in Proposition 6 and $\delta > 0$. In view of the C^1-regularity of $\partial\Omega$, there exists a C^1-diffeomorphism $\Phi_\delta : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\Phi_\delta(\overline{\Omega}) \subset \Omega, \ \|D\Phi_\delta - I\|_{\infty,\mathbb{R}^n} < \delta \text{ and } \Phi(x) = x \ \text{ for all } \ x \in B_r.$$

Let $v_{r,\delta} := v_r \circ \Phi_\delta$, observe that $v_{r,\delta} \in \mathrm{Lip}(U_\delta)$ where $U_\delta := \Phi_\delta^{-1}(\Omega)$, and fix $\delta > 0$ sufficiently small so that

$$\begin{cases} H(x, Dv_{r,\delta}) < -\eta/2 & \text{a.e. in } \ U_\delta \setminus B_r, \\ H(x, Dv_{r,\delta}) \leq 0 & \text{a.e. in } \ B_r \cap U_\delta, \\ \|v_{r,\delta} - V\|_{\infty,\Omega} < 2r. \end{cases}$$

Next let ρ be a standard mollifier in $\mathbb{R}^n$ with $\operatorname{supp}\rho \subset B_1$, and choose $\gamma \in (0, r/2)$ small enough so that $B_\gamma(x) \subset U_\delta$ for all $x \in \overline{\Omega}$. Hence, $w := \rho_\gamma * v_{r,\delta}$ is well-defined in $\overline{\Omega}$, where $\rho_\gamma(x) := \gamma^{-n}\rho(\gamma^{-1}x)$.

Let $L > 0$ and Ω_H be respectively a Lipschitz bound of $v_{r,\delta}$ and the modulus of continuity of H on $\overline{\Omega} \times B_L$, fix any $x \in \Omega \setminus B_{2r}$, note that

$$-\frac{\eta}{2} \geq H(x-y, Dv_{r,\delta}(x-y)) \geq H(x, Dv_{r,\delta}(x-y)) - \omega_H(\gamma) \quad \text{for a.e.} \quad y \in B_\gamma,$$

and observe that, by Jensen's inequality,

$$H(x, Dw(x)) \leq \int_{B_\gamma} H(x, Dv_{r,\delta}(x-y))\rho_\gamma(y)\,\mathrm{d}\,y \leq \omega_H(\gamma) - \frac{\eta}{2}.$$

Similarly, we find that, for any $x \in \Omega$,

$$H(x, Dw) \leq \omega_H(\gamma).$$

Thus, for $\gamma > 0$ small enough,

$$\begin{cases} H(x, Dw) < -\eta/3 & \text{in } \ x \in \Omega \setminus B_{3r}, \\ H(x, Dw) \leq 1 & \text{in } \ x \in B_{3r} \cap \Omega, \\ \|w - V\|_{\infty,\Omega} < 3r. \end{cases}$$

The function w has all the properties required for w_{3r} and, since $r > 0$ is arbitrary, this completes the proof.

□

3. Asymptotics in a smaller time scale

Fix $r > 0$ and $\mu > 0$ and let $w_r \in C^2(\overline{\Omega})$ and $\eta \in (0, 1)$ be given by Theorem 7. For $\varepsilon > 0$ and $x \in \overline{\Omega}$ set

$$v^\varepsilon(x) := \exp\left(\frac{w_r(x) - \mu}{\varepsilon}\right), \tag{3.1}$$

and note that

$$L_\varepsilon v^\varepsilon = \frac{v^\varepsilon}{\varepsilon}\left(H(x, Dw_r) + \varepsilon\,\mathrm{tr}[aD^2 w_r]\right).$$

Select $C > 0$ and $\varepsilon_0 > 0$ so that $\varepsilon_0 C < \eta$ and, for all $x \in \overline{\Omega}$, $|\,\mathrm{tr}[a(x)D^2 w_r(x)]| \leq C$. It follows that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$L_\varepsilon v^\varepsilon \leq \frac{v^\varepsilon}{\varepsilon}\left(H(x, Dw_r) + \varepsilon C\right) < \begin{cases} 0 & \text{in } \ \Omega \setminus B_r, \\ \dfrac{2v^\varepsilon}{\varepsilon} & \text{in } \ B_r. \end{cases} \tag{3.2}$$

Set $R_\varepsilon := \frac{2}{\varepsilon}\|v^\varepsilon\|_{\infty, B_r \cap \Omega}$ and, for $(x,t) \in \overline{Q}$,

$$w^\varepsilon(x,t) := v^\varepsilon(x) + R_\varepsilon t. \tag{3.3}$$

It follows that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$w_t^\varepsilon > L_\varepsilon w^\varepsilon \quad \text{in } \ Q. \tag{3.4}$$

The main result of this section is about the behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ of (1.1), (1.2) in $B_r \times [0, \exp \lambda/\varepsilon]$, for λ as in the statement below.

Theorem 8. *Assume* (A1)–(A6). *If $\lambda > 0$ is such that $\{V \le \lambda\} \subset \{g \le 0\}$, then, for each $\delta > 0$, there exists $r > 0$ such that*

$$\lim_{\varepsilon \to 0+} (u^\varepsilon - \delta)_+ = 0 \quad \text{uniformly on } \ B_r \times [0, \mathrm{e}^{\lambda/\varepsilon}].$$

Proof. Choose $r > 0$ so small that $B_r \subset \Omega$, let w_r, ε_0, v^ε and w^ε be as above, fix $\delta > 0$ and set $U^\varepsilon := u^\varepsilon - \delta$ and $G := g - \delta$. Since $\{G \le 0\}$ is a neighborhood of $\{g \le 0\}$, we may choose $\gamma > \lambda$ such that

$$\{V \le \gamma\} \subset \{G \le 0\}.$$

It follows from the maximum principle that $\sup_Q U^\varepsilon \le \sup_\Omega G$ and, hence,

$$U^\varepsilon \le M := \|G\|_{\infty,\Omega} \quad \text{on } \ \overline{Q}.$$

Fix $\mu > 0$ in (3.1) (the definition of v^ε) so that $\lambda < \mu < \gamma$, and, if needed, select $r > 0$ even smaller so that $\gamma - r - \mu > 0$, which ensures that

$$v^\varepsilon > \exp\left(\frac{\gamma - r - \mu}{\varepsilon}\right) \quad \text{in } \ \{G > 0\}.$$

Taking, if necessary, $\varepsilon_0 > 0$ even smaller, we may assume that, if $0 < \varepsilon < \varepsilon_0$, then

$$\exp\left(\frac{\gamma - r - \mu}{\varepsilon}\right) > M.$$

Hence, for $\varepsilon \in (0, \varepsilon_0)$,

$$v^\varepsilon > M \ge G \text{ in } \{G > 0\} \ \text{ and } \ v^\varepsilon > 0 \ge G \ \text{ in } \ \{G \le 0\},$$

and, accordingly, $w^\varepsilon \ge G = U^\varepsilon$ on $\partial_{\mathrm{p}} Q$. Using the maximum principle we get, for all $(x,t) \in \overline{\Omega} \times [0, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$,

$$U^\varepsilon(x,t) \le w^\varepsilon(x,t) = v^\varepsilon(x) + R_\varepsilon t. \tag{3.5}$$

Since $V \in \mathrm{Lip}(\overline{\Omega})$ and $V(0) = 0$, there exists $C_0 > 0$ such that $|w_r| \le C_0 r$ in B_r and, therefore,

$$v^\varepsilon \le \exp\left(\frac{C_0 r - \mu}{\varepsilon}\right) \quad \text{in } \ B_r.$$

Next assume that r is even smaller so that $\tilde{\mu} := \mu - C_0 r > \lambda$, which implies that

$$R_\varepsilon = \frac{2\|v^\varepsilon\|_{\infty,B_r}}{\varepsilon} < \frac{2\,\mathrm{e}^{-\tilde{\mu}/\varepsilon}}{\varepsilon},$$

and, for all $x \in B_r$ and $0 \le t \le \mathrm{e}^{\lambda/\varepsilon}$,

$$w^\varepsilon(x,t) \le \mathrm{e}^{-\tilde\mu/\varepsilon} + \frac{2\,\mathrm{e}^{-\tilde\mu/\varepsilon}}{\varepsilon}\ \mathrm{e}^{\lambda/\varepsilon} = \mathrm{e}^{-\tilde\mu/\varepsilon} + \frac{2\,\mathrm{e}^{(\lambda-\tilde\mu)/\varepsilon}}{\varepsilon}.$$

Hence,

$$\lim_{\varepsilon\to 0+} U_+^\varepsilon = 0 \ \text{ uniformly on } \ B_r \times [0,\, \mathrm{e}^{\lambda/\varepsilon}]. \qquad \square$$

4. Asymptotics in a larger time scale

The main theorem here concerns the behavior of the solutions $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ of

$$u_t^\varepsilon = L_\varepsilon u^\varepsilon \ \text{in} \ Q, \quad u^\varepsilon = 0 \text{ on } \ \partial_p Q \ \text{ and } \ \sup_{\varepsilon>0} \|u^\varepsilon\|_{\infty,Q} < \infty. \tag{4.1}$$

We have:

Theorem 9. *Assume* (A1)–(A6). *Fix $\lambda > m_0$ and, for $\varepsilon > 0$, assume that $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ solves* (4.1). *Then*

$$\lim_{\varepsilon\to 0+} u^\varepsilon = 0 \qquad \textit{uniformly on } \ \overline{\Omega} \times [\mathrm{e}^{\lambda/\varepsilon},\, \infty).$$

The following proposition is a key observation needed to prove Theorem 9. Its proof is presented later in the section.

Proposition 10. *Let $\lambda > m_0$. There exists $W \in \mathrm{Lip}(\overline{\Omega})$ and $\eta > 0$ such that*

$$0 < \min_{\overline{\Omega}} W \le \max_{\overline{\Omega}} W < \lambda, \tag{4.2}$$

and, in the viscosity supersolution sense,

$$H(x, -DW) \ge \eta \ \ \textit{in} \ \ \Omega \ \textit{and} \ \ \eta\,\mathrm{tr}[a(x)D^2W(x)] \le 1 \ \textit{in} \ \ \Omega. \tag{4.3}$$

Proof of Theorem 9. Since u^ε and $-u^\varepsilon$ both solve (4.1), it is enough to show that, for any $\lambda > m_0$,

$$\lim_{\varepsilon\to 0+} u_+^\varepsilon = 0 \ \text{ uniformly on } \ \overline{\Omega} \times [\mathrm{e}^{\lambda/\varepsilon},\, \infty). \tag{4.4}$$

Moreover, multiplying the u^ε's by a positive constant if necessary, we may assume that

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{\infty,Q} \le 1.$$

Fix $\lambda > m_0$, let $W \in \mathrm{Lip}(\overline{\Omega})$, let $\eta > 0$ be as in Proposition 9, set

$$\delta := \min_{\overline{\Omega}} W, \ \text{ and } \ \mu := \max_{\partial\Omega} W,$$

and, for $\varepsilon \in (0,\, \eta^2/2)$,

$$v^\varepsilon(x) := \exp\left(-\frac{W(x)}{\varepsilon}\right) \qquad \text{for } \ x \in \overline{\Omega},$$

and observe, using (4.3), that, in the subsolution sense,

$$\begin{cases} \dfrac{\varepsilon^2}{(v^\varepsilon)^2}\, aDv^\varepsilon \cdot Dv^\varepsilon + \dfrac{\varepsilon}{v^\varepsilon}\, b \cdot Dv^\varepsilon \geq \eta & \text{in } \Omega, \\ -\dfrac{\varepsilon}{v^\varepsilon}\, \operatorname{tr}(aD^2 v^\varepsilon) + \dfrac{\varepsilon}{(v^\varepsilon)^2}\, aDv^\varepsilon \cdot Dv^\varepsilon \leq \dfrac{1}{\eta} & \text{in } \Omega, \end{cases}$$

and, consequently,

$$L_\varepsilon v^\varepsilon \geq \frac{v^\varepsilon}{\varepsilon}\left(-\frac{\varepsilon}{\eta} + \eta\right) \geq \frac{\eta v^\varepsilon}{2\varepsilon} \quad \text{in } \Omega.$$

Note also that

$$\mathrm{e}^{-\mu/\varepsilon} \leq v^\varepsilon \leq \mathrm{e}^{-\delta/\varepsilon} \text{ in } \overline{\Omega}.$$

Next we fix some $\gamma \in (0,\, \eta]$, set, for $(x,t) \in \overline{Q}$,

$$w^\varepsilon(x,t) := 1 + \mathrm{e}^{-\delta/\varepsilon} - v^\varepsilon(x) - \frac{\gamma}{2\varepsilon}\, \mathrm{e}^{-\mu/\varepsilon}\, t,$$

and observe that

$$w^\varepsilon_t - L_\varepsilon w^\varepsilon = -\frac{\gamma}{2\varepsilon}\, \mathrm{e}^{-\mu/\varepsilon} + L_\varepsilon v^\varepsilon \geq -\frac{\gamma}{2\varepsilon}\, \mathrm{e}^{-\mu/\varepsilon} + \frac{\eta v^\varepsilon}{2\varepsilon} \geq 0 \quad \text{in } Q,$$

and

$$\begin{cases} w^\varepsilon(x,0) \geq 1 & \text{for all } x \in \overline{\Omega}, \\ w^\varepsilon(x,t) \geq 1 - \dfrac{\gamma}{2\varepsilon}\, \mathrm{e}^{-\mu/\varepsilon}\, t & \text{for all } (x,t) \in \partial\Omega \times [0,\, \infty), \\ w^\varepsilon(x,t) \leq 1 + \mathrm{e}^{-\delta/\varepsilon} - \dfrac{\gamma}{2\varepsilon}\, \mathrm{e}^{-\mu/\varepsilon}\, t & \text{for all } (x,t) \in \overline{\Omega} \times [0,\, \infty). \end{cases}$$

Then for $T := \frac{2\varepsilon}{\gamma}\, \mathrm{e}^{\mu/\varepsilon}$, we have

$$u^\varepsilon \leq w^\varepsilon \quad \text{on } (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,\, T)),$$

and, by the comparison principle,

$$u^\varepsilon \leq w^\varepsilon \quad \text{on } \overline{\Omega} \times [0,\, T],$$

and, in particular,

$$u^\varepsilon(x,T) \leq \mathrm{e}^{-\delta/\varepsilon} \quad \text{for all } x \in \Omega.$$

Since $\gamma \in (0,\, \eta]$ is arbitrary, it follows that

$$u^\varepsilon \leq \mathrm{e}^{-\delta/\varepsilon} \text{ on } \overline{\Omega} \times [(2\varepsilon/\eta)\, \mathrm{e}^{\mu/\varepsilon},\, \infty),$$

from which we conclude that (4.4) holds. □

The proof of Proposition 10 requires a number of technical facts which we state and prove first. To this end, we introduce the function U on $\overline{\Omega}$ defined by

$$U(x) = \min\{V_\Omega(x,y) \,:\, y \in \partial\Omega\}. \tag{4.5}$$

By the coercivity of H and (2.2), the collection $\{V_\Omega(\cdot\,,y) : y \in \partial\Omega\}$ is equi-Lipschitz continuous on $\overline{\Omega}$ and the fucntion U is Lipschitz continuous on $\overline{\Omega}$. It is a standard observation in weak KAM theory, the main idea of which goes back to Barron and Jensen [4] (see also [2]), that U is a solution of

$$H(x, -DU) = 0 \quad \text{in} \ \ \Omega. \tag{4.6}$$

Lemma 2. *The function U given by* (4.5) *is the maximal subsolution of*

$$\begin{cases} H(x, -Du) = 0 & \textit{in} \ \ \Omega, \\ u = 0 & \textit{on} \ \ \partial\Omega, \end{cases} \tag{4.7}$$

and

$$0 \le U \le U(0) = m_0 \quad \textit{on} \ \ \overline{\Omega}.$$

Proof. Let $u \in C(\overline{\Omega})$ be a subsolution of (4.7). By (2.1), we have

$$u(x) \le V_\Omega(x, y) \quad \text{for all} \ \ x \in \overline{\Omega},\ y \in \partial\Omega,$$

and, hence,

$$u \le U \ \ \text{on} \ \ \overline{\Omega}.$$

It is obvious that $U = 0$ on $\partial\Omega$, and, therefore, U is a solution of (4.7), Thus, the above inequality yields the first part of the claim.

Moreover,

$$U(0) = \inf\{V_\Omega(0, y) \, : \, y \in \partial\Omega\} = \min_{\partial\Omega} V = m_0.$$

Next we recall that $V_\Omega(x, 0) = 0$ for all $x \in \overline{\Omega}$. Hence,

$$V_\Omega(x, y) \le V_\Omega(x, 0) + V_\Omega(0, y) = V_\Omega(0, y) \quad \text{for all} \ \ y \in \overline{\Omega}.$$

Taking infimum over all $y \in \partial\Omega$, we find $U \le U(0)$ on $\overline{\Omega}$, and the proof is now complete. □

Lemma 3. *For each $\gamma > 0$, there exists a unique solution $u \in \mathrm{Lip}(\overline{\Omega})$ of*

$$\begin{cases} H(x, -Du) = \gamma & \textit{in} \ \ \Omega, \\ u = 0 & \textit{on} \ \ \partial\Omega. \end{cases} \tag{4.8}$$

Proof. Choose $M > 0$ such that

$$H(x, p) \ge \gamma \ \ \text{for all} \ \ (x, p) \in \overline{\Omega} \times (\mathbb{R}^n \setminus B_M).$$

It is easy to check that $f(x) := M\mathrm{dist}(x, \partial\Omega)$ is a supersolution of (4.8). It is also obvious that 0 is a subsolution of (4.8). Perron's method now implies that there exists a solution $u \in \mathrm{Lip}(\overline{\Omega})$ of (4.8).

Note that $H(x, 0) = 0 < \gamma$ for all $x \in \Omega$ and recall that $p \mapsto H(x, p)$ is convex for any $x \in \Omega$. Under these conditions, the uniqueness follows from a well known comparison (see e.g. [1, 3, 13]) which we state below as a separate lemma without proof.

□

Lemma 4. *Let $\gamma > 0$. If $u \in C(\overline{\Omega})$ (resp. $v \in C(\overline{\Omega})$) is a subsolution (resp. supersolution) of $H(x, -Dw) = \gamma$ in Ω and $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.*

We continue with

Lemma 5. *For each* $\gamma > 0$ *let* $u_\gamma \in \mathrm{Lip}(\overline{\Omega})$ *be the solution of* (4.8) *and* $U \in \mathrm{Lip}(\overline{\Omega})$ *be the fucntion given by* (4.5). *Then*

$$\lim_{\gamma\to 0} u_\gamma = U \quad \textit{uniformly on} \quad \overline{\Omega}. \tag{4.9}$$

Proof. Note that, if $0 < \gamma_1 < \gamma_2$, then u_{γ_1} is a subsolutionof (4.8) with $\gamma = \gamma_2$. Therefore the comparison yields $u_{\gamma_1} \le u_{\gamma_2}$ on $\overline{\Omega}$

Observe also that the u_γ's, with $\gamma \in (0,1)$, are subsolutions to (4.8) with $\gamma = 1$ and, therefore, the collection $\{u_\gamma\}_{\gamma\in(0,1)}$ is equi-Lipschitz on $\overline{\Omega}$. It follows that there exists some $u \in \mathrm{Lip}(\overline{\Omega})$ such that

$$\lim_{\gamma\to 0} u_\gamma = u \ \text{ uniformly on } \ \overline{\Omega} \ \text{ and } \ u = 0 \ \text{ on } \ \partial\Omega. \tag{4.10}$$

We see by the stability of viscosity solutions that u is a solution of (4.7), and, moreover, by the maximality of U, that $U \ge u$ on $\overline{\Omega}$. Note also that U is a subsolution of (4.8) with $\gamma > 0$. Hence, $U \le u_\gamma$ on $\overline{\Omega}$ and, therefore, and $U \le u$ on $\overline{\Omega}$. Thus we conclude that $u = U$ on $\overline{\Omega}$ and (4.9) holds. □

We are now in a position to present the

Proof of Proposition 10. Fix $\gamma > 0$. It follows from Lemma 5 that, if $\mu \in (0,\gamma)$ is sufficiently small, then the solution $u_\mu \in \mathrm{Lip}(\overline{\Omega})$ of (4.8), with γ replaced by μ, satisfies

$$\|U - u_\mu\|_{\infty,\Omega} < \gamma,$$

and, moreover, $0 \le u_\mu(x) < m_0 + \gamma$ for all $x \in \overline{\Omega}$.

For $x \in \overline{\Omega}$ set

$$W(x) := u_\mu(x) + \gamma,$$

fix any $\delta > 0$ and choose a C^1-diffeomorphism $\Phi_\delta : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$\Phi_\delta(\overline{\Omega}) \subset \Omega, \ \|D\Phi_\delta - I\|_{\infty,\mathbb{R}^n} < \delta \text{ and } \Phi(0) = 0,$$

Let

$$W_\delta := W \circ \Phi_\delta,$$

and note that $\Phi_\delta^{-1}(\Omega)$ is an open neighborhood of $\overline{\Omega}$. It follows that, if $\delta > 0$ is sufficiently small, then

$$H(x, -DW_\delta) \ge \mu/2 \ \text{ in } \ \Phi_\delta^{-1}(\Omega) \ \text{ and } \gamma \le W_\delta \le m_0 + 2\gamma \ \text{ on } \ \overline{\Omega}.$$

For $\alpha > 0$ small, we introduce the inf-convolution $W_{\delta,\alpha}$ of W_δ, given, for $x \in \mathbb{R}^n$, by

$$W_{\delta,\alpha}(x) := \inf\{W_\delta(y) + \frac{1}{\alpha}|x-y|^2 \,:\, y \in \Phi_\delta^{-1}(\Omega)\},$$

which is is semi-concave in $\Phi_\delta^{-1}(\Omega)$ (see, for example, [3]), that is there exists $C_{\delta,\alpha}$ depending on δ, α such that

$$\max\{D^2W_{\delta,\alpha}(x)\xi\cdot\xi \,:\, \xi \in B_1\} \le C_{\delta,\alpha} \quad \text{in } \Phi_\delta^{-1}(\Omega)$$

holds in the supersolution sense, and, if $\alpha > 0$ is sufficiently small, then

$$\|W_\delta - W_{\delta,\alpha}\|_{\infty,\Omega} < \gamma/2 \quad \text{and} \quad H(x, -DW_{\delta,\alpha}) \geq \mu/4 \ \text{ in } \Omega,$$

where the latter inequality holds in the supersolution sense. It is then easily checked that $W_{\delta,\alpha}$ satisfies, in the supersolution sense,

$$\operatorname{tr}\left[aD^2W_{\delta,\alpha}\right] \leq C_{\delta,\alpha} \operatorname{tr} a \quad \text{in } \ \Omega.$$

Thus, noting that

$$\gamma/2 \leq W_{\delta,\alpha} \leq m_0 + 3\gamma \quad \text{in } \ \Omega,$$

and choosing $\gamma > 0$ and $\eta > 0$ so small that $m_0 + 3\gamma < \lambda$, $\eta C_{\delta,\alpha}\| \operatorname{tr} a\|_{\infty,\Omega} \leq 1$ and $\eta \leq \mu/4$, we conclude that $W := W_{\delta,\alpha}$ and η have the required properties. □

5. The stationary problem

We consider the Dirichlet problem

$$\begin{cases} L_\varepsilon v^\varepsilon = 0 & \text{in } \ \Omega, \\ v^\varepsilon = g & \text{on } \ \partial\Omega, \end{cases} \tag{5.1}$$

for

$$g \in C(\overline{\Omega}) \ \text{ such that } g = 0 \ \text{ on } \ \arg\min(V|\partial\Omega). \tag{5.2}$$

The next result is an essential part of a classical observation obtained by Freidlin-Wentzell [11], Devinatz-Friedman [6], Kamin [14, 15], Perthame [18], etc..

Theorem 11. *Assume* (A1)–(A6) *and* (5.2). *Then* $\lim_{\varepsilon\to 0+} v^\varepsilon(0) = 0$.

Proof. Since the equation is linear, it is enough to show that, for any $\delta > 0$, there exits $r > 0$ such that

$$\lim_{\varepsilon\to 0+} (v^\varepsilon - \delta)_+ = 0 \quad \text{uniformly on } \ B_r.$$

The function v^ε depends on g only through its restriction on $\partial\Omega$. Hence we may replace g by a new function $\tilde g \in C(\overline{\Omega})$ as long as $\tilde g = g$ on $\partial\Omega$.

Let $Z := \{x \in \overline{\Omega} \,:\, V \leq m_0\}$, note that $g = 0$ on $Z \cap \partial\Omega$, and set

$$\tilde g(x) := \begin{cases} g(x)\dfrac{\operatorname{dist}(x,Z)}{\operatorname{dist}(x,\partial\Omega)} & \text{for } \ x \in \Omega, \\ g(x) & \text{for } \ x \in \partial\Omega; \end{cases}$$

it is easily checked that $\tilde g \in C(\overline{\Omega})$. Moreover it is obvious that $\tilde g = g$ on $\partial\Omega$ and $\tilde g = 0$ on Z. Thus, by replacing g by $\tilde g$ if necessary, we may assume that $g = 0$ on Z.

Now, fix $\delta > 0$, note that $\{x \in \overline{\Omega} \,:\, g(x) < \delta\}$ is a neighborhood, relative to $\overline{\Omega}$, of Z, and choose $\lambda > m_0$ so that

$$\{V \leq \lambda\} \subset \{g < \delta\}.$$

Let u^ε be the solution of (1.1), (1.2), We apply Theorem 8, with $g - \delta$ in place of g, to find that there exists $r > 0$ such that

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - 2\delta)_+ = 0 \quad \text{uniformly on } \ B_r \times [0, \mathrm{e}^{\lambda/\varepsilon}].$$

Next we set $w^\varepsilon(x,t) := v^\varepsilon(x) - u^\varepsilon(x,t)$ for $(x,t) \in \overline{Q}$ and note that $w^\varepsilon \equiv 0$ on $\partial_p Q$, and $\|w^\varepsilon\|_{\infty,Q} \le \|v^\varepsilon\|_{\infty,\Omega} + \|u^\varepsilon\|_{\infty,Q} \le 2\|g\|_{\infty,\Omega}$.

Applying Theorem 9 to w^ε we get

$$\lim_{\varepsilon\to 0+} w^\varepsilon = 0 \quad \text{uniformly on } \overline{\Omega} \times [\mathrm{e}^{\lambda/\varepsilon}, \infty).$$

It is now immediate that

$$\lim_{\varepsilon\to 0+} (v^\varepsilon(x) - 2\delta)_+ = 0 \quad \text{uniformly on } B_r.$$

□

We have indeed shown the following:

Theorem 12. *Assume* (A1)–(A6) *and* (5.2)*. For any $\delta > 0$ there exists $r > 0$ such that*

$$\lim_{\varepsilon\to 0} (v^\varepsilon - \delta)_+ = 0 \quad \textit{uniformly on } B_r.$$

6. Asymptotic constancy

In this section we state precisely the claim that the limit, as $\varepsilon \to 0$, of (1.1) is the transport equation $u_t = b \cdot Du$ and provide its proof where (A5) plays a critical role.

Theorem 13. *Assume* (A1)–(A6)*, let $\tau(\varepsilon) > 0$ be such that $\lim_{\varepsilon\to 0+} \tau(\varepsilon) = \infty$ and, for each $\varepsilon > 0$, let $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a solution of* (1.1)*. Assume that, for some $r > 0$,*

$$\lim_{\varepsilon\to 0+} u^\varepsilon = 0 \quad \textit{uniformly on } B_r \times [0, \tau(\varepsilon)), \tag{6.1}$$

and

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{\infty,\, \Omega\times[0,\,\tau(\varepsilon))} < \infty.$$

There exists $T = T(r) > 0$ such that, for any compact subset K of Ω and any $\tau_0 > 0$,

$$\lim_{\varepsilon\to 0+} u^\varepsilon = 0 \quad \textit{uniformly on } K \times [T, \tau(\varepsilon) - \tau_0).$$

As before we prove a slightly generalized, one-sided version of the above theorem, which readily yields the claim.

Theorem 14. *Assume* (A1)–(A6)*, let $\tau(\varepsilon) > 0$ be such that $\lim_{\varepsilon\to 0+} \tau(\varepsilon) = \infty$, and, for each $\varepsilon > 0$, consider a solution $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ to* (1.1)*. Fix $r > 0$ so that $B_r \subset \Omega$ and let N be a (possibly empty) open subset of $\partial\Omega$. Assume that*

$$\lim_{\varepsilon\to 0+} u^\varepsilon_+ = 0 \quad \textit{uniformly on } (B_r \,\cup\, N) \times [0, \tau(\varepsilon)) \tag{6.2}$$

and

$$\limsup_{\varepsilon\to 0+} \|u^\varepsilon\|_{\infty,\, \Omega\times[0,\,\tau(\varepsilon))} < \infty. \tag{6.3}$$

There exists $T = T(r) > 0$ such that, for any compact subset K of $\Omega \cup N$ and any $\tau_0 > 0$,

$$\lim_{\varepsilon \to 0+} u_+^\varepsilon = 0 \quad \text{uniformly on } \; K \times [T,\, \tau(\varepsilon) - \tau_0).$$

The following lemma plays an important role in the proof of Theorem 14.

Lemma 6. *Let $u \in \mathrm{USC}(\overline{Q})$ be a subsolution of $u_t = b \cdot Du$ in Q and, for $(x,t) \in Q$, set $X(s) := X(s;x)$. The function $s \mapsto u(X(s), t-s)$ is nondecreasing on $[0,\, t]$.*

Proof. Note that (A5) yields that, for all $x \in \Omega$, $s \geq 0$, $X(s;x) \in \Omega$. For $s \in [0,\, t]$, set $v(s) := u(X(s), t-s)$. We show that, in the subsolution sense, $v' \geq 0$ in $(0,\, t)$ which implies that v is nondecreasing on $[0,\, t]$.

Let $\phi \in C^1([0,\, t])$ and assume that $v - \phi$ attains a strict maximum at a point $\hat{s} \in (0,\, t)$. For $\alpha > 0$ consider the map

$$(y, s) \mapsto u(y, t-s) - \phi(s) - \alpha |y - X(s)|^2$$

on $\overline{\Omega} \times [0,\, t]$ and let (y_α, s_α) be a maximum point. It is easy to see that, as $\alpha \to \infty$, $(y_\alpha, s_\alpha) \to (\hat{s}, X(\hat{s}))$ and $\alpha|y_\alpha - X(s_\alpha))|^2 \to 0$. Fix a sufficiently large α so that $(y_\alpha, s_\alpha) \in Q$. Noting that $\psi(y,s) := u(y, t-s)$ is a subsolution of $-\psi_s = b \cdot D\psi$ in $\Omega \times (0,\, t)$, we find that

$$-\phi'(s_\alpha) + 2\alpha(y_\alpha - X(s_\alpha)) \cdot \dot{X}(s_\alpha) \leq 2\alpha b(y_\alpha) \cdot (y_\alpha - X(s_\alpha)),$$

from which we get

$$\phi'(s_\alpha) \geq 2\alpha(y_\alpha - X(s_\alpha)) \cdot (b(X(s_\alpha)) - b(y_\alpha)) \geq -2\alpha L |y_\alpha - X(s_\alpha)|^2,$$

where $L > 0$ is a Lipschitz bound of b. Sending $\alpha \to \infty$ yields $\phi'(\hat{s}) \geq 0$ and the proof is complete. □

We continue with the

Proof of Theorem 14. We introduce the upper relaxed limit $U \in \mathrm{USC}(\overline{Q})$ given by

$$U(x,t) := \lim_{\lambda \to 0+} \sup\{u^\varepsilon(y,s)_+ : (y,s) \in \overline{Q},\; |y-x| + |s-t| \leq \lambda,\; 0 < \varepsilon < \lambda\},$$

and recall the standard observation that U is a subsolution of $U_t = b \cdot DU$ in Q.

According to Lemma 1, we may choose $T = T(r) > 0$ such that, for all $(x,t) \in \overline{\Omega} \times [T, \infty)$, $X(s;x) \in B_r$. From Lemma 6 it follows that, for any $(x,t) \in Q$ and $s \in [0,\, t]$, $U(X(s;x), t-s) \geq U(x,t)$. Hence, for any $(x,t) \in Q$ with $t \geq T$, we have $X(T;x) \in B_r$ and

$$U(x,t) \leq U(X(T;x), t-T) \leq 0. \tag{6.4}$$

Next we show that

$$U = 0 \;\text{ on }\; N \times (T, \infty). \tag{6.5}$$

Fix $(y,s) \in N \times (T,\, \infty)$ and, in view of (A5), choose $R > 0$ so small that

$$y + \lambda b(y) \in \Omega \;\text{ for all }\; \lambda \in (0,\, R), \quad s > R + T, \quad \text{and} \quad \overline{B}_R(y) \cap \partial\Omega \subset N.$$

Reformulating the last observation in terms of $l(y,s) := \{(y,s)+\lambda(b(y),-1) \,:\, \lambda > 0\}$, a half-line in $\mathbb{R}^{n+1}$ with vertex at (y,s), we have

$$\begin{cases} \overline{B}_R(y,s) \cap l(y,s) \subset \Omega \times (T,\,\infty), \\ \overline{B}_R(y,s) \subset \mathbb{R}^n \times (T,\,\infty), \\ \overline{B}_R(y,s) \cap (\partial\Omega \times \mathbb{R}) \subset N \times (T,\,\infty). \end{cases} \tag{6.6}$$

For any $\gamma \in (0,1)$, we consider the open convex cone C_γ in $\mathbb{R}^{n+1}$ with vertex at the origin given by

$$C_\gamma := \bigcup_{\lambda>0} \lambda\left((b(y),-1)) + B_\gamma\right),$$

and we set

$$C_\gamma(y,s) := (y,s) + C_\gamma.$$

From (A5) again, we may choose $\gamma \in (0,\,1/2)$ small enough so that

$$\overline{B}_R(y,s) \cap C_{2\gamma}(y,s) \subset \Omega \times (T,\,\infty), \tag{6.7}$$

which strengthens the first inclusion of (6.6). Noting that C_γ is an open neighborhood of $(b(y),-1)$, we may also choose $\rho \in (0,\,R)$ so that

$$(b(x),-1) \subset C_\gamma \quad \text{for all } \; x \in B_\rho(y),$$

which ensures that

$$(b(x),-1) \subset C_\gamma \quad \text{for all } \; (x,t) \in B_\rho(y,s). \tag{6.8}$$

Define next $d,\,\phi \,:\, \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$d(x,t) := \operatorname{dist}((x,t),\,C_\gamma(y,s)) \quad \text{ and } \quad \phi(x,t) := d^2(x,t),$$

and recall the well-known facts that $\phi \in C^1(\mathbb{R}^{n+1})$, $D\phi \in \operatorname{Lip}(\mathbb{R}^{n+1})$ and $D\phi(x,t)$ is in the (negative) dual cone of C_γ, that is, for all $(\xi,\tau) \in C_\gamma,\; (x,t) \in \mathbb{R}^{n+1}$

$$D\phi(x,t) \cdot (\xi,\tau) \le 0.$$

Combining the above remark with (6.8) yields

$$b \cdot D\phi \le \phi_t \;\text{ in }\; B_\rho(y,s). \tag{6.9}$$

Next we compare u^ε and ϕ on the set

$$Q(y,s) := B_\rho(y,s) \cap Q,$$

and note that

$$\partial Q(y,s) \subset (\partial B_\rho(y,s) \cap Q) \cup (\overline{B}_\rho(y,s) \cap \partial Q).$$

In view of (6.7), we may choose $\lambda > 0$ so that

$$(\overline{C}_\gamma(y,s) + \overline{B}_\lambda) \cap \partial B_\rho(y,s) \subset \Omega \times (T,\,\infty).$$

Set

$$K := (\overline{C}_\gamma(y,s) + \overline{B}_\lambda) \cap \partial B_\rho(y,s),$$

which is clearly a compact subset of $\Omega \times (T,\,\infty)$, and fix any $\delta > 0$.

Note that (6.3) and (6.4) imply that there exist $\varepsilon_0 > 0$ and $M > 0$ such that, for all $\varepsilon \in (0,\, \varepsilon_0)$,

$$\overline{Q}(y,s) \subset \overline{\Omega} \times (T,\, \tau(\varepsilon)) \;\text{ and } u^\varepsilon \le \delta \;\text{ in }\; K \;\text{ and }\; u^\varepsilon \le M \quad\text{in }\; \overline{Q}(y,s). \tag{6.10}$$

Set $A := M/\lambda^2$. Then, for $\varepsilon \in (0,\, \varepsilon_0)$,

$$u^\varepsilon \le M = A\lambda^2 \le A\phi \text{ in }\; \overline{Q}(y,s) \cap \{d \ge \lambda\}. \tag{6.11}$$

Since $\overline{B}_\rho(t,s) \cap \partial Q$ is a compact subset of $N \times (T,\, \infty)$, in view of (6.2), we may assume, replacing, if needed, ε_0 by a smaller positive number, that, for all $\varepsilon \in (0,\, \varepsilon_0)$,

$$u^\varepsilon \le \delta \;\text{ in }\; \overline{B}_\rho(t,s) \cap \partial Q. \tag{6.12}$$

Fix $(x,t) \in \partial Q(y,s)$ and $\varepsilon \in (0,\, \varepsilon_0)$. If $(x,t) \in \partial B_\rho(y,s) \cap Q = \partial B_\rho(y,s) \cap (\Omega \times (T,\, \infty))$ and $(x,t) \notin K$, then $d(x,t) \ge \lambda$ and, by (6.11), $u^\varepsilon \le A\phi$. Otherwise, that is, if $(x,t) \in \partial B_\rho(y,s) \cap Q \cap K$, (6.10) gives $u^\varepsilon(x,t) \le \delta$.

Moreover, if $(x,t) \in \overline{B}_\rho(y,s) \cap \partial Q$, then, by (6.12), we have $u^\varepsilon(x,t) \le \delta$, and, therefore, for all $\varepsilon \in (0,\, \varepsilon_0)$,

$$u^\varepsilon \le \delta + A\phi \;\text{ on }\; \partial Q(y,s).$$

Since, for each t, $D\phi(\cdot,t) \in \mathrm{Lip}(\mathbb{R}^n)$, there exists some $C > 0$ so that, in the supersolution sense, $\mathrm{tr}[aD_x^2\phi] \le C$ in $Q(y,s)$, Hence, using (6.9), we see that $\psi(x,t) := \delta + A\phi(x,t) + \varepsilon ACt$ is a supersolution to

$$\psi_t \ge L_\varepsilon \psi \;\text{ in }\; Q(y,s).$$

Thus, by comparison, we get

$$u^\varepsilon \le \psi \;\text{ in }\; Q(y,s),$$

which yields

$$U(y,s) \le \delta + A\phi(y,s) = \delta,$$

and, after letting $\delta \to 0$, $U(y,s) = 0$. This proves (6.5).

Since we have shown that $U = 0$ on $(\Omega \cup N) \times (T,\, \infty)$, it follows that, for any compact subset K of $(\Omega \cup N) \times (T,\, \infty)$,

$$\lim_{\varepsilon \to 0+} u_+^\varepsilon = 0 \;\text{ uniformly on }\; K.$$

To complete the proof, let $T > 0$ be as above, fix any $\tau_0 > 0$, choose $\varepsilon_0 > 0$ so that $T + \tau_0 < \tau(\varepsilon)$ for all $\varepsilon \in (0,\, \varepsilon_0)$, set, for $(x,t) \in \overline{\Omega} \times [0,\, T + \tau_0]$ and $\varepsilon \in (0,\, \varepsilon_0)$,

$$v^\varepsilon(x,t) := \sup\{u^\varepsilon(x, t+s) \,:\, 0 \le s < \tau(\varepsilon) - T - \tau_0\}$$

and

$$U(x,t) := \lim_{\lambda \to 0+} \sup\{v_+^\varepsilon(y,s) \,:\, (y,s) \in \overline{\Omega} \times [0,\, T_0 + \tau_0], \\ |y - x| + |s - t| < \lambda,\, 0 < \varepsilon < \lambda\},$$

and note that, for any $0 < \varepsilon < \varepsilon_0$, v^ε is a subsolution to $v_t^\varepsilon = L_\varepsilon v^\varepsilon$ in $\Omega \times (0,\, T + \tau_0)$.

It follows, as above, that $U = 0$ in $\Omega \times [T,\, T + \tau_0)$ and $U = 0$ in $N \times (T,\, T + \tau_0)$.

Let K be a compact subset of $\Omega \cup N$. Then $K \times [T+\tau_0/3,\, T+2\tau_0/3]$ is a compact subset of $N \times (T,\, T+\tau_0)$ and, thus,

$$\lim_{\varepsilon\to 0+} v_+^\varepsilon = 0 \quad \text{uniformly on } \; K \times [T+\tau_0/3,\, T+2\tau_0/3],$$

which yields

$$\lim_{\varepsilon\to 0+} u_+^\varepsilon = 0 \quad \text{uniformly on } \; K \times [T+\tau_0/3,\, \tau(\varepsilon)-\tau_0/3].$$

The proof is now complete. □

We close the section with the following generalization of Theorem 11.

Theorem 15. *Under the hypotheses of Theorem 11, if K is a compact subset of $\Omega \cup \arg\min(V|\partial\Omega)$, then $\lim_{\varepsilon\to 0+} v^\varepsilon = 0$ uniformly on K.*

Proof. Fix a compact $K \subset \Omega \cup \arg\min(V|\partial\Omega)$ and $\delta > 0$. Theorem 14 applied to $u^\varepsilon(x,t) := v^\varepsilon(x) - \delta$ with $N = \{y \in \partial\Omega : g(y) < \delta\}$, gives $\lim_{\varepsilon\to 0+}(v^\varepsilon - \delta)_+ = 0$ uniformly on K, and, hence, $\lim_{\varepsilon\to 0+} v_+^\varepsilon = 0$ uniformly on K. Similarly, $\lim_{\varepsilon\to 0+} v_-^\varepsilon = 0$ uniformly on K. □

7. The proof of the main theorem

We are now ready to prove the main theorem. Note that, in view of the linearity of pde (1.1), it is enough to show that the following holds.

Theorem 16. *Let $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be the solution of* (1.1), (1.2) *and fix $\delta > 0$.*

(i) *There exists $T = T(\delta, g) > 0$ such that, for any $\lambda \in (0,\, m_0)$ and any compact subset K of Ω,*

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - \delta)_+ = 0 \;\; \textit{uniformly on } \; K \times [T,\, \mathrm{e}^{\lambda/\varepsilon}]. \tag{7.1}$$

(ii) *Assume that $g = g(0)$ on $\arg\min(V|\partial\Omega)$. There exists $T = T(\delta, g) > 0$ such that, for any compact subset K of $\Omega \cup \arg\min(V|\partial\Omega)$,*

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - \delta)_+ = 0 \;\; \textit{uniformly on } K \times [T,\, \infty). \tag{7.2}$$

(iii) *Assume that $g = g_0$ on $\arg\min(V|\partial\Omega)$ for some $g_0 \in \mathbb{R}$. Then, for any $\lambda \in (m_0,\, \infty)$ and any compact subset K of $\Omega \cup \arg\min(V|\partial\Omega)$,*

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g_0 - \delta)_+ = 0 \;\; \textit{uniformly on } \; K \times [\mathrm{e}^{\lambda/\varepsilon},\, \infty). \tag{7.3}$$

Proof. We begin with (i). Fix any $\lambda \in (0,\, m_0)$ and $\delta > 0$, recall that $V(0) = 0$ and $V > 0$ in $\overline{\Omega} \setminus \{0\}$, choose a $\gamma = \gamma(\delta, g) > 0$ so that

$$\{V \le \gamma\} \subset \{g - g(0) - \delta \le 0\},$$

and recall that Theorem 8 yields some $r = r(\gamma) > 0$ such that

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - \delta)_+ = 0 \quad \text{uniformly on } \; B_r \times [0,\, \mathrm{e}^{\gamma/\varepsilon}].$$

Next we use Theorem 14 to select $T = T(r) > 0$ such that, for any compact subset K of Ω and $\tau_0 > 0$,

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - \delta)_+ = 0 \quad \text{uniformly on } \ K \times [T, \mathrm{e}^{\gamma/\varepsilon} - \tau_0].$$

Fix $\mu \in (\lambda,\, m_0)$. The above convergence, for $K = \{V \le \mu\}$, ensures that there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\, \varepsilon_0)$,

$$u^\varepsilon(\cdot, T) - g(0) - 2\delta \le 0 \ \text{ on } \ \{V \le \mu\},$$

that is, for all $\varepsilon \in (0,\, \varepsilon_0)$,

$$\{V \le \mu\} \subset \{u^\varepsilon(\cdot\,, T) - g(0) - 2\delta \le 0\}.$$

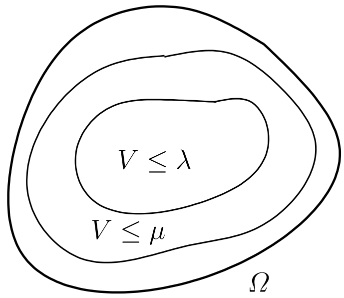

FIGURE 2. The inclusion between two sublevel sets of V.

Noting that $\{V \le \mu\}$ is a neighborhood of $\{V \le \lambda\}$, we may select $G \in C(\overline{\Omega})$ so that

$$G = 0 \ \text{ in } \ \{V \le \lambda\}, \ G \ge 0 \ \text{ on } \ \overline{\Omega}, \text{ and } \ \max_{\overline{\Omega}} g - g(0) - 2\delta \le G \ \text{ in } \ \{V > \mu\},$$

and observe that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$u^\varepsilon(\cdot, T) - g(0) - 2\delta \le G \ \text{ on } \ \overline{\Omega} \quad \text{and} \quad \{V \le \lambda\} \subset \{G \le 0\}.$$

Let $U^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be the solution of (1.1) with initial-boundary condition $U^\varepsilon = G$ on $\partial_{\mathrm{p}} Q$. The maximum principle implies that, for all $(x, t) \in \overline{Q}$,

$$u^\varepsilon(x, T + t) - g(0) - 2\delta \le U^\varepsilon(x, t),$$

while, in view of Theorem 8 and Theorem 14, there exist $r_1 = r_1(\delta, G) > 0$ and $T_1 = T_1(r_1) > 0$ respectively such that, for any compact $K \subset \Omega$,

$$\lim_{\varepsilon\to 0+} (U^\varepsilon - \delta)_+ = 0 \ \text{ uniformly on } \ B_{r_1} \times [0,\, \mathrm{e}^{\lambda/\varepsilon}],$$

and

$$\lim_{\varepsilon\to 0+} (U^\varepsilon - \delta)_+ = 0 \quad \text{uniformly on } \ K \times [T_1,\, \mathrm{e}^{\lambda/\varepsilon} - T],$$

and, hence,

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - 3\delta)_+ = 0 \quad \text{uniformly on } \ K \times [T + T_1,\, \mathrm{e}^{\lambda/\varepsilon}],$$

and the proof of (i) is complete.

Next we prove (iii). Fix any $\lambda > m_0$ and $\delta > 0$ and let $v^\varepsilon \in \mathrm{Lip}(\overline{\Omega})$ be the solution of

$$L_\varepsilon v^\varepsilon = 0 \ \text{ in } \ \Omega \quad \text{and} \quad v^\varepsilon = g \ \text{ on } \ \partial\Omega.$$

Theorem 12 yields $r = r(\delta) > 0$ such that

$$\lim_{\varepsilon\to 0+} (v^\varepsilon - g_0 - \delta)_+ = 0 \quad \text{uniformly on } \ B_r.$$

Set $N = \{x \in \partial\Omega \,:\, g(x) < g_0 + \delta\}$ and note that

$$\lim_{\varepsilon\to 0+} (v^\varepsilon - g_0 - \delta)_+ = 0 \quad \text{uniformly on } \ B_r \cup N.$$

Hence, by Theorem 14, for any compact subset K of $\Omega \cup N$,

$$\lim_{\varepsilon\to 0+} (v^\varepsilon - g_0 - \delta)_+ = 0 \quad \text{uniformly on } \ K.$$

Let $w^\varepsilon : \bar{Q} \to \mathbb{R}$ be given by $w^\varepsilon := u^\varepsilon - v^\varepsilon$, and note that $w^\varepsilon = 0$ on $\partial\Omega \times [0, \infty)$. Then Theorem 9 yields

$$\lim_{\varepsilon\to 0+} w^\varepsilon = 0 \quad \text{uniformly on } \ \overline{\Omega} \times [\mathrm{e}^{\lambda/\varepsilon}, \infty),$$

and we may conclude that, for any compact subset K of $\Omega \cup N$,

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g_0 - \delta)_+ = 0 \quad \text{uniformly on } \ K \times [\mathrm{e}^{\lambda/\varepsilon}, \infty),$$

which completes the proof of (iii).

To prove (ii) fix any $\delta > 0$, and, as in the proof of (i), choose $r = r(\delta, g) > 0$ and $\gamma > 0$ such that

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - \delta)_+ = 0 \quad \text{uniformly on } \ B_r \times [0, \mathrm{e}^{\gamma/\varepsilon}].$$

Moreover, as in the proof of (iii), set $N = \{x \in \partial\Omega : g(x) - g(0) - \delta < 0\}$ and use Theorem 14 to find $T = T(\delta, g) > 0$ such that, for any compact subset K of $\Omega \cup N$,

$$\lim_{\varepsilon\to 0+} (u^\varepsilon(\cdot, T) - g(0) - \delta)_+ = 0 \quad \text{uniformly on } \ K.$$

We choose now $\lambda > m_0$ such that $\{V \le \lambda\} \subset \Omega \cup N$. It follows that there exists $\varepsilon_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$u^\varepsilon(\cdot, T) - g(0) - 2\delta \le 0 \ \text{ on } \ \{V \le \lambda\}.$$

Fix a $\mu \in (m_0, \lambda)$ and select $G \in C(\overline{\Omega})$ as in proof of (iii) so that

$$G = 0 \ \text{ in } \ \{V \le \mu\}, \ G \ge 0 \ \text{ on } \ \overline{\Omega}, \text{ and } G \ge \max_{\overline{\Omega}}(u^\varepsilon(\cdot, T) - g(0) - 2\delta) \ \text{ in } \ \{V > \lambda\}.$$

Let $U^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be the solution of (1.1) with $U^\varepsilon = G$ on $\partial_{\mathrm{p}} Q$. It follows from the maximum principle that $u^\varepsilon(x, t+T) - g(0) - 2\delta \le U^\varepsilon(x, t)$ for all $(x, t) \in \overline{Q}$, and $\{V \le \mu\} \subset \{G \le 0\}$. Combining Theorem 8 and Theorem 14, as in the proof of (i), we deduce that there exists $T_1 = T_1(\delta, G) > 0$ such that, for any compact subset K of $\Omega \cup N$,

$$\lim_{\varepsilon\to 0+} (U^\varepsilon - \delta)_+ = 0 \quad \text{uniformly on } \ K \times [T_1, \mathrm{e}^{\mu/\varepsilon} - T].$$

Note that $\arg\min(V|\partial\Omega) \subset \{V \le \mu\}$ and, hence, $G = 0 = G(0)$ on $\arg\min(V|\partial\Omega)$. Using assertion (iii), we see that, for any compact subset K of $\Omega \cup \arg\min(V|\partial\Omega)$,

$$\lim_{\varepsilon\to 0+} (U^\varepsilon - \delta)_+ = 0 \quad \text{uniformly on} \;\; K \times [\mathrm{e}^{\mu/\varepsilon},\, \infty).$$

Combining these two observations, we conclude that, for any compact subset K of $\Omega \cup \arg\min(V|\partial\Omega)$,

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - 3\delta)_+ = 0 \quad \text{uniformly on} \;\; K \times [T + T_1,\, \infty);$$

and the proof is complete. □

8. The proof of Theorem 2

We note that there exists a family $\{\Omega_\delta\}_{\delta\in(0,\,1)}$ of bounded, open, connected subsets of $\mathbb{R}^n$ with C^1-boundary such that

$$\overline{\Omega} \subset \Omega_\delta \subset \Omega^\delta \;\; \text{and} \;\; b\cdot\nu_\delta < 0 \;\; \text{on} \;\; \partial\Omega_\delta,$$

where $\Omega^\delta := \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(x,\Omega) < \delta\}$ and $\nu_\delta(x)$ denotes the exterior unit normal of $\partial\Omega_\delta$ at $x \in \partial\Omega_\delta$. Indeed let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω, that is, $\Omega = \{x \in \mathbb{R}^n \,:\, \rho(x) < 0\}$ and $D\rho \ne 0$ if $\rho = 0$; its existence is guaranteed by the assumed regularity of the boundary of Ω. We may assume moreover that $\lim_{|x|\to\infty} \rho(x) = \infty$.

For each $\delta \in (0,\,1)$, we choose $\gamma \in (0,\,\delta)$ small enough and set $\Omega_\delta = \{x \in \mathbb{R}^n \,:\, \rho(x) < \gamma\}$. Then Ω_δ is a bounded, open, connected subset of $\mathbb{R}^n$. Furthermore, we have $\overline{\Omega} \subset \Omega_\delta \subset \Omega^\delta$ and $b\cdot\nu_\delta < 0$ on $\partial\Omega_\delta$.

Recall that if we write $S^-(W)$ for the set of all subsolutions $\phi \in C(\overline{W})$ of $H(x, D\phi(x)) = 0$ in W, then

$$V(y) = \sup\{\phi(y) - \phi(0) \,:\, \phi \in S^-(\Omega)\} \quad \text{for all} \;\; y \in \overline{\Omega}.$$

For each $\delta \in (0,\,1)$, we set

$$V_\delta(y) = \sup\{\phi(y) - \phi(0) \,:\, \phi \in S^-(\Omega_\delta)\} \quad \text{for all} \;\; y \in \overline{\Omega}_\delta$$

and $m_\delta = \min_{\partial\Omega_\delta} V_\delta$.

Proposition 17. *For any* $\delta \in (0,\,1)$*,* $m_\delta \ge m_0$*. Furthermore,* $\lim_{\delta\to 0+} m_\delta = m_0$.

Proof. Since $V_\delta \in S^-(\Omega_\delta)$ implies that $V_\delta \in S^-(\Omega)$, we have

$$V_\delta(y) \le V(y) \quad \text{for all} \;\; y \in \overline{\Omega}. \tag{8.1}$$

Fix $m \in (0,\,m_0)$ and set

$$v := \begin{cases} \min\{V,\, m\} & \text{in } \Omega, \\ m & \text{in } \Omega_\delta \setminus \Omega. \end{cases}$$

Noting that $u \equiv m$ is a solution of $H(x, Du) = 0$ in Ω_δ and that $m < V$ on $\partial\Omega$, we see that v is a subsolution of $H(x, Dv) = 0$ in Ω, and, hence, in Ω_δ. It follows from the maximality of V_δ, that $V_\delta \ge v$ on Ω_δ, and, hence,

$$m_\delta \ge m_0. \tag{8.2}$$

Fix any $\kappa > 0$, choose a function $\zeta \in C^1(\overline{\Omega}^1)$ so that $\zeta(x_0) < m_0 + \kappa$ for some point $x_0 \in \arg\min(V|\partial\Omega)$ and $\zeta \geq V$ on $\partial\Omega$, and $R > 0$ so that

$$H > 1 \quad \text{on} \quad \overline{\Omega}^1 \times (\mathbb{R}^n \setminus B_R) \quad \text{and} \quad |D\zeta| \leq R \quad \text{on} \quad \overline{\Omega}^1.$$

We may assume by relabeling the family $\{\Omega_\delta\}_{\delta\in(0,1)}$ if needed that

$$\inf\{|D\rho(x)| \,:\, x \in \bigcup_{\delta\in(0,1)} \Omega_\delta \setminus \Omega\} > 0,$$

and we choose $A > 0$ so that

$$A \inf\{|D\rho(x)| \,:\, x \in \bigcup_{\delta\in(0,1)} \Omega_\delta \setminus \Omega\} > 2R.$$

Fix any $\delta \in (0, 1)$ and recall that $\Omega_\delta = \{x \in \mathbb{R}^n \,:\, \rho(x) < \gamma\}$ for some $\gamma \in (0, \delta)$. For each $\mu \in (0, \gamma)$ we select a C^1-function $\theta \,:\, [0, \gamma) \to [0, \infty)$ so that $\theta(r) = Ar$ for $r \in [0, \mu]$, $\theta' \geq A$ in $[0, \gamma)$ and $\lim_{r\to\gamma-} \theta(r) = \infty$.

Set $w := \zeta + \theta \circ \rho$ on $\Omega_\delta \setminus \Omega$ and observe that $w \in C^1(\Omega_\delta \setminus \Omega)$,

$$|Dw| \geq \theta'(\rho)|D\rho| - |D\zeta| \geq 2R - R \geq R \quad \text{on} \quad \Omega_\delta \setminus \Omega,$$

and, hence, $H(x, Dw) > 0$ on $\Omega_\delta \setminus \Omega$. Furthermore, we have $V_\delta \leq V \leq w$ on $\partial\Omega$ and $\lim_{\Omega_\delta \ni x\to\partial\Omega_\delta} w(x) = \infty$.

We use next the fact that V_δ is a subsolution of $H(x, DV_\delta) = 0$ in Ω_δ, to conclude that $V_\delta \leq w$ in $\Omega_\delta \setminus \Omega$. Indeed, otherwise $V_\delta - w$ attains a positive maximum at a point $\xi \in \Omega_\delta \setminus \Omega$ since we have either $(V_\delta - w)(x) \leq 0$ or $\lim_{\Omega_\delta\setminus\Omega\ni y\to x}(V_\delta - w)(y) = -\infty$ for all $x \in \partial(\Omega_\delta \setminus \Omega)$, and obtain $H(\xi, Dw(\xi)) \leq 0$, which is a contradiction. Sending $\mu \to \gamma$ in the inequality $V_\delta \leq \zeta + \theta \circ \rho$ in $\Omega_\delta \setminus \Omega$, we find that $V_\delta \leq \zeta + A\rho$ in $\Omega_\delta \setminus \Omega$, which implies that $m_\delta \leq \min_{\partial\Omega_\delta} \zeta + A\delta$, and, hence, $\limsup_{\delta\to 0+} m_\delta \leq m_0 + \kappa$. Since $\kappa > 0$ is arbitrary, we thus get $\limsup_{\delta\to 0+} m_\delta \leq m_0$, which, together with (8.2), implies that $\lim_{\delta\to 0+} m_\delta = m_0$. □

Theorem 18. *Assume* (A1)–(A6). *For any $\delta \in (0, m_0)$ and compact $K \subset \Omega$, there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then, for all $(x, t) \in K \times [\mathrm{e}^{\delta/\varepsilon}, \mathrm{e}^{(m_0-\delta)/\varepsilon}]$,*

$$\mathbb{P}(\tau_x^\varepsilon > t,\, X^\varepsilon(t; x) \in B_\delta) > 1 - \delta.$$

Proof. Select $g \in C(\overline{\Omega})$ so that

$$g(0) = 1 \quad \text{and} \quad g \leq \mathbf{1}_{B_\delta} \quad \text{on} \quad \overline{\Omega},$$

and fix $\delta \in (0, m_0)$ and a compact $K \subset \Omega$. Theorem 1 implies that there exists $\varepsilon_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$|u^\varepsilon - 1| < \delta \quad \text{on} \quad K \times [\mathrm{e}^{\delta/\varepsilon}, \mathrm{e}^{(m_0-\delta)/\varepsilon}].$$

The observation that, for any $(x, t) \in K \times [\mathrm{e}^{\delta/\varepsilon}, \mathrm{e}^{(m_0-\delta)/\varepsilon}]$ and $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} 1 - \delta &< u^\varepsilon(x, t) = \mathbb{E}\, g(X^\varepsilon(t \wedge \tau_x^\varepsilon; x)) \leq \mathbb{E}\, \mathbf{1}_{B_\delta}(X^\varepsilon(t \wedge \tau_x^\varepsilon; x)) \\ &= \mathbb{P}(X^\varepsilon(t \wedge \tau_x^\varepsilon; x) \in B_\delta) = \mathbb{P}(\tau_x^\varepsilon > t,\, X^\varepsilon(t; x) \in B_\delta) \end{aligned}$$

completes the proof. □

Lemma 7. *Assume* (A1)–(A6). *Let* $\delta > 0$ *and* $K \subset \Omega$ *compact. There exists* $\varepsilon_0 > 0$ *such that, for all* $x \in K$ *and* $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{P}(\tau_x^\varepsilon < \mathrm{e}^{(m_0+\delta)/\varepsilon}) > 1 - \delta.$$

Proof. Let $\{\Omega_\delta\}_{\delta\in(0,1)}$ and $\{m_\delta\}_{\delta\in(0,1)}$ be as before, and, for a fixed $\delta > 0$, we may assume that $m_\delta < m_0 + \delta$. We also choose $g \in C(\overline{\Omega}_\delta)$ so that $g \le \mathbf{1}_{\Omega_\delta\setminus\Omega}$ on Ω_δ and $g \equiv 1$ on $\partial\Omega_\delta$ and consider the solution U^ε of

$$\begin{cases} U_t^\varepsilon = L_\varepsilon U^\varepsilon & \text{in } \ Q_\delta := \Omega_\delta \times (0, \infty), \\ U^\varepsilon = g & \text{on } \ \partial_{\mathrm{p}} Q_\delta. \end{cases}$$

Theorem 1 gives some $\varepsilon_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$U^\varepsilon > 1 - \delta \ \text{ in } \ K \times [\mathrm{e}^{(m_0+\delta)/\varepsilon}, \infty).$$

Fix $x \in K$ and $\varepsilon \in (0, \varepsilon_0)$ and observe that, if $\tau_x^{\delta,\varepsilon}$ denotes the first exit time of $X^\varepsilon(\cdot; x)$ from Ω_δ,

$$\begin{aligned} 1 - \delta &< U^\varepsilon(x, \mathrm{e}^{(m_0+\delta)/\varepsilon}) = \mathbb{E}\, g(X^\varepsilon(\tau_x^{\delta,\varepsilon} \wedge \mathrm{e}^{(m_0+\delta)/\varepsilon}; x)) \le \mathbb{E}\, \mathbf{1}_{\overline{\Omega}_\delta\setminus\Omega}(X^\varepsilon(\tau_x^{\delta,\varepsilon} \wedge \mathrm{e}^{(m_0+\delta)/\varepsilon}; x)) \\ &= \mathbb{P}(X^\varepsilon(\tau_x^{\delta,\varepsilon} \wedge \mathrm{e}^{(m_0+\delta)/\varepsilon}; x) \in \overline{\Omega}_\delta \setminus \Omega) \le \mathbb{P}(\tau_x^\varepsilon < \mathrm{e}^{(m_0+\delta)/\varepsilon}), \end{aligned}$$

and the proof is now complete. □

Proof of Theorem 2. Let $\widetilde{W}$ be an open relative to $\overline{\Omega}$ neighborhood of $\arg\min(V|\partial\Omega)$ such that $\widetilde{W} \cap \partial\Omega \subset W$, choose $g \in C(\overline{\Omega})$ so that $g \equiv 1$ on $\arg\min(V|\partial\Omega)$ and $g \le \mathbf{1}_{\widetilde{W}}$ and let u^ε be the solution of

$$\begin{cases} u_t^\varepsilon + H(x, Du^\varepsilon) = 0 & \text{in } \ Q, \\ u^\varepsilon = g & \text{on } \ \partial_{\mathrm{p}} Q. \end{cases}$$

Let $T(\varepsilon) := \mathrm{e}^{(m_0+\delta)/\varepsilon}$ and $t(\varepsilon) := \mathrm{e}^{(m_0-\delta)/\varepsilon}$. Theorem 1 gives some $\varepsilon_0 > 0$ such that

$$u^\varepsilon(x, T(\varepsilon)) > 1 - \delta \quad \text{for all } \ (x, \varepsilon) \in K \times (0, \varepsilon_0), \tag{8.3}$$

while, in view of Theorem 18 and Lemma 7, we may assume that

$$\mathbb{P}(t(\varepsilon) \le \tau_x^\varepsilon \le T(\varepsilon)) > 1 - \delta \quad \text{for all } (x, \varepsilon) \in K \times (0, \varepsilon_0). \tag{8.4}$$

Fix any $(x, \varepsilon) \in K \times (0, \varepsilon_0)$ and observe that

$$\begin{aligned} u^\varepsilon(x, T(\varepsilon)) &\le \mathbb{E}\, g(X^\varepsilon(T(\varepsilon) \wedge \tau_x^\varepsilon; x)) \le \mathbb{E}\, \mathbf{1}_{\widetilde{W}}(X^\varepsilon(T(\varepsilon) \wedge \tau_x^\varepsilon; x)) \\ &= \mathbb{P}(X^\varepsilon(T(\varepsilon) \wedge \tau_x^\varepsilon; x) \in \widetilde{W}) \\ &\le \mathbb{P}(\tau_x^\varepsilon \le T(\varepsilon),\, X^\varepsilon(\tau_x^\varepsilon; x) \in W) + \mathbb{P}(\tau_x^\varepsilon > T(\varepsilon)) \\ &\le \mathbb{P}(t(\varepsilon) \le \tau_x^\varepsilon \le T(\varepsilon),\, X^\varepsilon(\tau_x^\varepsilon; x) \in W) + \mathbb{P}(\tau_x^\varepsilon < t(\varepsilon) \ \text{ or } \ \tau_x^\varepsilon > T(\varepsilon)). \end{aligned}$$

Combining the above with (8.3) and (8.4) yields

$$\mathbb{P}(t(\varepsilon) \le \tau_x^\varepsilon \le T(\varepsilon),\, X^\varepsilon(\tau_x^\varepsilon; x) \in W) > 1 - 2\delta \quad \text{for all } \ (x, \varepsilon) \in K \times (0, \varepsilon_0). \qquad \square$$

9. A semilinear parabolic equation

For $f_\varepsilon \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ we consider here the semilinear parabolic equation

$$u_t^\varepsilon = L_\varepsilon u^\varepsilon + f_\varepsilon(x, u^\varepsilon, Du^\varepsilon) \ \text{ in } Q. \tag{9.1}$$

In addition to (A1)–(A6), throughout this section we assume that

(A7) for each $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that, for all $(x, u, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$,

$$|f_\varepsilon(x, u, p)| \le M(\varepsilon)|p|, \ \lim_{\varepsilon \to 0+} M(\varepsilon) = 0, \quad \text{and } u \mapsto f_\varepsilon(x, u, p) \ \text{ is nonincreasing;}$$

note that it is immediate from (A7) that, for all $(x, u) \in \overline{\Omega} \times \mathbb{R}$, $f_\varepsilon(x, u, 0) = 0$.

In what follows, for $\phi \in C^2(\Omega)$, we set

$$\mathcal{L}_\varepsilon \phi := L_\varepsilon \phi + f_\varepsilon(x, \phi, D\phi) \quad \text{and} \quad \mathcal{L}_\varepsilon^+ \phi := L_\varepsilon \phi + M(\varepsilon)|D\phi|,$$

and remark that any subsolution u^ε of (9.1) is also a subsolution of $u_t^\varepsilon = \mathcal{L}_\varepsilon^+ u^\varepsilon$ in Q.

It is possible to deal with (9.1) with the nonlinear term f_ε which depends further on the second derivatives in x of u^ε, but, to make the presentation simple and to avoid technicalities, we restrict ourselves here to study the semilinear pde (9.1).

Theorem 19. *Assume* (A1)-(A7). *The assertions of Theorem 1 hold for the solution* $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ *of* (9.1) *satisfying, for* $g \in C(\overline{\Omega})$*, the initial-boundary value condition* (1.2)*.*

It is not clear to the authors whether the initial-boundary value problem (9.1), (1.2) has a classical solution in $C(\overline{Q}) \cap C^{2,1}(Q)$. It is, hence, worthwhile stating an existence and uniqueness result for viscosity solutions of (9.1), (1.2). For this we may replace (A1) by the weaker assumption:

(A1w) a is Hölder continuous on $\overline{\Omega}$ with exponent $\gamma > 1/2$ and b is continuous on $\overline{\Omega}$.

We have:

Theorem 20. *Assume* (A1w), (A2), (A3), (A6) *and* (A7)*. Then there exists a unique viscosity solution* $u^\varepsilon \in C(\overline{Q})$ *of* (9.1), (1.2)*.*

We present the proof of Theorem 20, which is rather long and technical, in the Appendix. Here we continue with Theorem 19, which actually holds also for viscosity solutions of (9.1), (1.2). Indeed we have:

Theorem 21. *Assume* (A1)–(A7) *and* $g \in C(\overline{\Omega})$*. The assertions of Theorem 19 hold for the (viscosity) solution* $u^\varepsilon \in C(\overline{Q})$ *of* (9.1), (1.2)*.*

In view of the facts that, for any $\varepsilon > 0$, $-f_\varepsilon(x, -u, -p)$ satisfies condition (A7) if f_ε does and, if $u^\varepsilon \in C(Q)$ is a solution of (9.1) then $v^\varepsilon := -u^\varepsilon$ is a solution of

$$v_t^\varepsilon = L_\varepsilon v^\varepsilon - f_\varepsilon(x, -v^\varepsilon, -D_x v^\varepsilon) \quad \text{in } \ Q,$$

Theorem 20 is an easy consequence of the following version of Theorem 16.

Theorem 22. *Assume* (A1)–(A7) *and* $g \in C(\overline{\Omega})$*. For each* $\varepsilon > 0$*, let* $u^\varepsilon \in C(\overline{Q})$ *be a subsolution of* (9.1), (1.2)*. Fix* $\delta > 0$*.*

(i) *There exists* $T = T(\delta, g) > 0$ *such that, for any* $\lambda \in (0, m_0)$ *and any compact subset* K *of* Ω,

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - \delta)_+ = 0 \quad \textit{uniformly on} \ \ K \times [T, \mathrm{e}^{\lambda/\varepsilon}].$$

(ii) *Assume that* $g = g(0)$ *on* $\arg\min(V|\partial\Omega)$. *There exists* $T = T(\delta, g) > 0$ *such that, for any compact subset* K *of* $\Omega \cup \arg\min(V|\partial\Omega)$,

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g(0) - \delta)_+ = 0 \quad \textit{uniformly on } K \times [T, \infty).$$

(iii) *Assume that* $g = g_0$ *on* $\arg\min(V|\partial\Omega)$ *for some* $g_0 \in \mathbb{R}$. *Then, for any* $\lambda \in (m_0, \infty)$ *and any compact subset* K *of* $\Omega \cup \arg\min(V|\partial\Omega)$,

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - g_0 - \delta)_+ = 0 \quad \textit{uniformly on} \ \ K \times [\mathrm{e}^{\lambda/\varepsilon}, \infty).$$

The proof of Theorem 22 parallels that of Theorem 16. Instead of giving the full detailed proof, we indicate here its major differences from that of Theorem 16.

Choose r, W_r, μ, η and C as those at the beginning of Section 3, let $v^\varepsilon \in C^2(\overline{\Omega})$ be the function defined by (3.1) and observe that

$$f_\varepsilon(x, v^\varepsilon, D_x v^\varepsilon) \le M(\varepsilon)|D_x v^\varepsilon| \le \frac{v^\varepsilon}{\varepsilon} M(\varepsilon)|DW_r|$$

and

$$\mathcal{L}_\varepsilon v^\varepsilon \le \frac{v^\varepsilon}{\varepsilon}\left(H(x, DW_r) + \varepsilon C + M(\varepsilon)\|DW_r\|_{\infty,\Omega}\right).$$

Select $\varepsilon_0 > 0$ so that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon C + M(\varepsilon)\|DW_r\|_{\infty,\Omega} \le \eta,$$

and observe that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\mathcal{L}_\varepsilon v^\varepsilon(x) \le \begin{cases} 0 & \text{in } \ \Omega \setminus B_r, \\ \dfrac{2v^\varepsilon}{\varepsilon} & \text{in } \ B_r. \end{cases}$$

Then $w^\varepsilon \in C^2(\overline{Q})$ defined by $w^\varepsilon(x,t) := v^\varepsilon(x) + R_\varepsilon t$, with $R_\varepsilon := (2/\varepsilon)\|v^\varepsilon\|_{\infty,B_r}$, satisfies $w^\varepsilon_t \ge \mathcal{L}_\varepsilon w^\varepsilon$ in Q.

The next assertion (Theorem 23) is similar to Theorem 8. Its proof follows by a straightforward adaptation of the proof of Theorem 8 with the above choice of function w^ε.

Theorem 23. *Assume* (A1)–(A7). *For each* $\varepsilon > 0$ *let* $u^\varepsilon \in \mathrm{USC}(\overline{Q})$ *be a subsolution of* (9.1), (1.2) *and let* $\lambda > 0$ *be such that* $\{V \le \lambda\} \subset \{g \le 0\}$. *For any* $\delta > 0$ *there exists* $r > 0$ *such that*

$$\lim_{\varepsilon\to 0+} (u^\varepsilon - \delta)_+ = 0 \quad \textit{uniformly on} \ \ B_r \times [0, \mathrm{e}^{\lambda/\varepsilon}].$$

Theorem 9 can be reformulated for subsolutions of (9.1) as follows.

Theorem 24. *Assume* (A1)–(A7). *Fix* $\lambda > m_0$ *and, for each* $\varepsilon > 0$, *let* $u^\varepsilon \in \mathrm{USC}(\overline{Q})$ *be a subsolution of* (9.1). *Assume that* $u^\varepsilon \le 0$ *on* $\partial\Omega\times[0,\infty)$ *and* $\sup_{\varepsilon>0}\|u^\varepsilon\|_{\infty,\overline{Q}} < \infty$. *Then*

$$\lim_{\varepsilon\to 0+} u^\varepsilon_+ = 0 \quad \textit{uniformly on } \ \overline{\Omega}\times[\mathrm{e}^{\lambda/\varepsilon},\,\infty).$$

Let W, η, δ, μ and v^ε be as in the proof Theorem 9. We deduce, following the arguments in the proof of Theorem 9, that, in the subsolution sense,

$$\mathcal{L}_\varepsilon v^\varepsilon \ge \frac{v^\varepsilon}{\varepsilon}\Big(-\frac{\varepsilon}{\eta}+\eta - M(\varepsilon)\|DW\|_{\infty,\,\Omega}\Big) \ \text{ in } \Omega.$$

Fix $\varepsilon_0 > 0$ so that, for all $\varepsilon \in (0,\,\varepsilon_0)$,

$$\frac{\varepsilon}{\eta} + M(\varepsilon)\|DW\|_{\infty,\,\Omega} < \frac{\eta}{2}, \quad \text{ and, hence, } \quad \mathcal{L}_\varepsilon v^\varepsilon \ge \frac{\eta v^\varepsilon}{2\varepsilon} \quad \text{ in } \Omega,$$

define $w^\varepsilon \in \mathrm{Lip}(\overline{Q})$ as in the proof of Theorem 9, that is, for $\gamma \in (0,\,\eta]$, set

$$w^\varepsilon(x,t) := 1 + \mathrm{e}^{-\delta/\varepsilon} - v^\varepsilon(x) - \frac{\gamma}{2\varepsilon}\,\mathrm{e}^{-\mu/\varepsilon}\, t,$$

and then follow the proof of Theorem 9 with w^ε as above to conclude.

A review of the proof of Theorem 14 shows that, with a minor modification of the function ψ, the assertion of Theorem 14 holds true for subsolutions $u^\varepsilon \in \mathrm{USC}(\overline{Q})$ of (9.1). To prove the first claim of Theorem 22, we just need to follow the proof of part (i) of Theorem 16, with Theorem 8 replaced by Theorem 23 and with Theorem 14 replaced by the corresponding assertion for subsolutions $u^\varepsilon \in \mathrm{USC}(\overline{Q})$ of (9.1).

Now we discuss a version of Theorem 12 for subsolutions of

$$\begin{cases} \mathcal{L}_\varepsilon v^\varepsilon = 0 & \text{in } \ \Omega, \\ v^\varepsilon = g & \text{on } \ \partial\Omega, \end{cases} \tag{9.2}$$

with $g \in C(\overline{\Omega})$. The existence and uniqueness of a solution in $C(\overline{\Omega})$ of (9.2) follow similarly to the case of Theorem 19.

Following the proof of Theorem 11 we obtain:

Theorem 25. *Assume* (A1)–(A7). *For each* $\varepsilon > 0$ *let* $v^\varepsilon \in \mathrm{USC}(\overline{Q})$ *be a subsolution of* (9.2). *Assume that* $g \le g_0$ *on* $\arg\min(V|\partial\Omega)$ *for some constant* g_0. *Then, for any* $\delta > 0$, *there exists* $r > 0$ *such that*

$$\lim_{\varepsilon\to 0+}(v^\varepsilon - g_0 - \delta)_+ = 0 \quad \textit{uniformly on } \ B_r.$$

When following the proof of Theorem 11, it is necessary to replace G and u^ε respectively by $G := g - g_0 - \delta$ and the solution $u^\varepsilon \in C(\overline{Q})$ of

$$u^\varepsilon_t = \mathcal{L}^+_\varepsilon u^\varepsilon \ \text{ in } \ Q \ \text{ with } u^\varepsilon = h \ \text{ on } \ \partial_{\mathrm{p}} Q, \tag{9.3}$$

where $h \in C(\overline{\Omega})$ is chosen as in the proof of Theorem 11 with the present choice of G. Once it is shown that $w^\varepsilon := v^\varepsilon - u^\varepsilon$ is a subsolution of (9.3), the rest of the argument goes exactly as in the proof of Theorem 11. Thus, the following lemma completes the proof of Theorem 25.

Lemma 8. *Fix $\varepsilon > 0$. If $v \in \mathrm{USC}(Q)$ and $u \in \mathrm{LSC}(Q)$ are respectively a subsolution and a supersolution of* (9.3)*, then $w := v - u$ is a subsolution of* (9.3)*.*

Proof. Let $\phi \in C^2(Q)$ and $(\hat{x}, \hat{t}) \in Q$ be such that $w - \phi$ achieves a strict maximum at $(\hat{x}, \hat{t})$. We need to show that $\phi_t \le \mathcal{L}_\varepsilon^+ \phi$ at $(\hat{x}, \hat{t})$.

We argue by contradiction and thus assume that this inequality does not hold. In this case we may choose $r > 0$ so that $Q_r := B_r(\hat{x}) \times (\hat{t} - r, \hat{t} + r) \subset Q$ and

$$\phi_t > \mathcal{L}_\varepsilon^+ \phi \ \text{ in } \ Q_r := B_r(\hat{x}) \times (\hat{t} - r,\, \hat{t} + r).$$

It is easily seen that $v - \phi$ is a subsolution of (9.3) in Q_r. Moreover, there is a comparison between $v - \phi$ and u (see the comparison principle at the beginning of the proof of Theorem 20 below), that is, we have

$$\max_{\overline{Q}_r}(v - u - \phi) \le \max_{\partial_{\mathrm{p}} Q_r}(v - u - \phi),$$

which is a contradiction since $w - \phi = v - u - \phi$ has a strict maximum at $(\hat{x}, \hat{t}) \in Q_r$. □

The proof of part (iii), (ii) of Theorem 22 follows as that of part (iii), (ii) of Theorem 16 once v^ε is chosen as the solution of

$$\mathcal{L}_\varepsilon^+ v^\varepsilon = 0 \ \text{ in } \ \Omega \quad \text{with} \quad v^\varepsilon = g(x) \ \text{ on } \ \partial\Omega,$$

and Theorems 12, 14, 9 and 8 are replaced by those for subsolutions of (9.1) and (9.2).

10. Appendix: The well posedness of the semilinear problem

We begin with the comparison principle which is a special parabolic version of (i) of Theorem III.1 in [12] and, hence, we omit its proof. A useful comment here is that the proof of (i) of Theorem III.1 in [12] works even when the constant C_R in the assumption (3.2) there replaced by $C_R(1 + |p|^\gamma)$, for some $\gamma \in (0, 1)$.

Lemma 9. *If $v \in \mathrm{USC}(\overline{Q})$ and $w \in \mathrm{LSC}(\overline{Q})$ are, respectively, a subsolution and a supersolution of* (9.1) *and $v \le w$ on $\partial_{\mathrm{p}} Q$, then $u \le w$ in Q.*

For the existence of the solutions we will need the following lemma; its proof is postponed for later.

Lemma 10. *There exists a constant $\lambda_0 > 0$ such that, for all $y \in \partial\Omega$ and $\lambda \in (0,\, \lambda_0)$, $y + \lambda\nu(y) \in \mathbb{R}^n \setminus \overline{\Omega}$. Moreover, if $\delta(\lambda) := \min_{y \in \partial\Omega} \mathrm{dist}(y + \lambda\nu(y), \Omega)$, then $\lim_{\lambda \to 0+} \delta(\lambda)/\lambda = 1$.*

Outline of the proof of Theorem 20. The uniqueness follows from Lemma 9, while the existence of a solution follows from Perron's method provided we construct appropriate subsolutions and supersolutions of (9.1) in Q.

To this end, let $\lambda_0 \in (0, 1)$ and $\delta : (0,\, \lambda_0) \to (0,\, \lambda_0)$ be as in Lemma 10. For each $y \in \partial\Omega$ and $\lambda \in (0,\, \lambda_0)$ set $z := y + \lambda\nu(y)$ and, for $\alpha = \alpha(\lambda) > 0$ to be fixed later, define $u_{\mathrm{b}}, v_{\mathrm{b}} \in C^\infty(\overline{\Omega})$ by

$$u_{\mathrm{b}}(x) := u_{\mathrm{b}}(x; y, \lambda) := \mathrm{e}^{-\alpha(|x - z|^2 - \delta(\lambda)^2)} \quad \text{and} \quad v_{\mathrm{b}}(x) := v_{\mathrm{b}}(x; y, \lambda) := 1 - u_{\mathrm{b}}(x),$$

and observe that, if $d := \operatorname{diam}(\Omega)$, then for all $x \in \overline{\Omega}$,

$$\delta(\lambda) \le |x - z| \le d + 1.$$

Next we estimate $\mathcal{L}_\varepsilon v_{\mathrm{b}}$ from above to find

$$\begin{aligned}\mathcal{L}_\varepsilon v_{\mathrm{b}}(x) &\le u_{\mathrm{b}}(x)\big\{\varepsilon\left(2\alpha \operatorname{tr} a(x) - 4\alpha^2 a(x)(x-z)\cdot(x-z)\right) + 2\alpha M(\varepsilon)|x-z| \\ &\qquad + 2\alpha|x-z|\|b\|_{\infty,\Omega}\big\} \\ &\le \alpha u_{\mathrm{b}}(x)\{\varepsilon\left(2n\theta^{-1} - 4\alpha\theta\delta(\lambda)^2\right) + 2(d+1)(M(\varepsilon) + \|b\|_{\infty,\Omega})\}.\end{aligned}$$

Fix $\Lambda > 0$ and $\alpha = \alpha(\lambda) > 0$ so that

$$\varepsilon\left(2n\theta^{-1} - 4\theta\Lambda\right) + 2(d+1)\left(M(\varepsilon) + \|b\|_{\infty,\Omega}\right) = 0 \quad \text{and} \quad \alpha\delta(\lambda)^2 = \Lambda,$$

note that, with this choice,

$$\mathcal{L}_\varepsilon v_{\mathrm{b}} \le 0 \quad \text{on} \quad \overline{\Omega}, \tag{10.1}$$

and observe that

$$\begin{aligned} v_{\mathrm{b}}(y) &= 1 - \exp\left(-\alpha\left(\lambda^2 - \delta(\lambda)^2\right)\right) = 1 - \exp\left(-\Lambda\left(\lambda^2/\delta(\lambda)^2 - 1\right)\right), \\ v_{\mathrm{b}}(x) &\ge 1 - \exp\left(-\alpha\left(\delta(\lambda)^2 - \delta(\lambda)^2\right)\right) = 0 \quad \text{for all} \ \ x \in \overline{\Omega},\end{aligned}$$

and, for any $x \in \overline{\Omega} \setminus B_{3\lambda}(y)$,

$$\begin{aligned} v_{\mathrm{b}}(x) &\ge 1 - \exp\left(-\alpha\left((|x-y| - |y-z|)^2 - \delta(\lambda)^2\right)\right) \ge 1 - \exp\left(-\alpha\left(4\lambda^2 - \delta(\lambda)^2\right)\right) \\ &> 1 - \exp\left(-\alpha\delta(\lambda)^2\right) = 1 - \mathrm{e}^{-\Lambda}. \end{aligned}$$

Lemma 10 together with the first observation above yields

$$\lim_{\lambda \to 0+} v_{\mathrm{b}}(y; y, \lambda) = 0. \tag{10.2}$$

Next let ω denote the modulus of continuity of g, choose $A > 0$ so that $A\left(1 - \mathrm{e}^{-\Lambda}\right) > \omega(d)$, and observe that, for any $x \in \overline{\Omega}$ and $y \in \partial\Omega$,

$$g(x) \le g(y) + \omega(d) \le g(y) + A v_{\mathrm{b}}(x; y, \lambda) \quad \text{ if } \ x \notin B_{3\lambda}(y),$$

and

$$g(x) \le g(y) + \omega(3\lambda) \quad \text{ if } \ x \in B_{3\lambda}(y).$$

Hence, for all $x \in \overline{\Omega}$,

$$g(x) \le g(y) + \omega(3\lambda) + A v_{\mathrm{b}}(x; y, \lambda),$$

and thus, setting, for $x \in \overline{\Omega}$,

$$w_{\mathrm{b}}(x) := \inf\{g(y) + \omega(3\lambda) + A v_{\mathrm{b}}(x; y, \lambda) : \lambda \in (0, \lambda_0), \, y \in \partial\Omega\}$$

and recalling (10.1) and (10.2), we deduce that $w_{\mathrm{b}} \in \mathrm{USC}(\overline{\Omega})$ is a supersolution of $\mathcal{L}_\varepsilon w_{\mathrm{b}} = 0$ in Ω, $w_{\mathrm{b}} \ge g$ on $\overline{\Omega}$ and $w_{\mathrm{b}} = g$ on $\partial\Omega$.

Next let $\gamma > 0$, choose $B = B(\gamma) > 0$ so that $B\gamma^2 \ge \omega(d)$, for $y \in \overline{\Omega}$, define $v_{\mathrm{i}} = v_{\mathrm{i}}(\cdot\,, y, \gamma) \in C^\infty(\overline{\Omega})$ by

$$v_{\mathrm{i}}(x) := g(y) + B|x-y|^2 + \omega(\gamma),$$

and observe that

$$v_{\rm i} \geq g \quad \text{on} \quad \overline{\Omega} \quad \text{and} \quad v_{\rm i}(y) = g(y) + \omega(\gamma).$$

Choose $C(\gamma) > 0$ so that

$$\mathcal{L}_\varepsilon v_{\rm i} \leq C(\gamma) \quad \text{in} \quad \overline{\Omega} \quad \text{for all } v_{\rm i} = v_{\rm i}(\cdot\,; y, \gamma) \quad \text{with} \quad y \in \overline{\Omega},$$

set, for $(x,t) \in \overline{Q}$,

$$w_{\rm i}(x,t) := \inf\{v_{\rm i}(x; y, \gamma) + C(\gamma)\, t \,:\, y \in \overline{\Omega},\, \gamma > 0\},$$

and observe that $w_{\rm i} \in \mathrm{USC}(\overline{\Omega})$ is a supersolution of (9.1), $g \leq w_{\rm i}$ on $\overline{Q}$ and $w_{\rm i}(\cdot, 0) = g$ on $\overline{\Omega}$.

Now, for $(x,t) \in \overline{Q}$, let

$$w(x,t) := \min\{w_{\rm b}(x),\, w_{\rm i}(x,t)\};$$

it is immediate that $w \in \mathrm{USC}(\overline{Q})$ is a supersolution of (9.1), and, in addition, $w = g$ on $\partial_{\rm p} Q$ and $w \geq g$ on $\overline{Q}$.

Similarly, we can build a subsolution $z \in \mathrm{LSC}(\overline{Q})$ of (9.1) such that $z = g$ on $\partial_{\rm p} Q$ and $z \leq g$ on $\overline{Q}$. Perron's method together with the comparison claim mentioned at beginning of the ongoing proof yields a solution $u \in C(\overline{Q})$ of (9.1) such that $z \leq u \leq w$ on $\overline{Q}$. The last inequality implies that $u = g$ on $\partial_{\rm p} Q$. □

We present now the

Proof of Lemma 10. Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω.

Since, for any $y \in \partial\Omega$, there exists $\theta_0 \in (0,\, 1)$ such that

$$\rho(y + \lambda\nu(y)) = \lambda D\rho(y + \theta_0 \lambda \nu(y)) \cdot \nu(y)$$

we deduce that there exists $\lambda_0 > 0$ such that, for all $y \in \partial\Omega$ and $\lambda \in (0,\, \lambda_0)$,

$$y + \lambda\nu(y) \in \mathbb{R}^n \setminus \overline{\Omega}.$$

To show that $\lim_{\lambda\to 0+} \delta(\lambda)/\lambda = 1$, we first note that $\delta(\lambda) \leq \lambda$ and assume by contradiction that $\liminf_{\lambda\to 0+} \delta(\lambda)/\lambda < 1$. It follows that there exist $\delta_0 \in (0,\, 1)$ and a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\, \lambda_0)$ such that, as $j \to \infty$, $\lambda_j \to 0$ and $\delta(\lambda_j)/\lambda_j \leq \delta_0$ for all j. Moreover, for each $j \in \mathbb{N}$ there are $y_j,\, \xi_j \in \partial\Omega$ such that

$$\delta(\lambda_j) = |y_j + \lambda_j \nu(y_j) - \xi_j|;$$

note that we may assume by passing, if needed, to a subsequence, that, as $j \to \infty$, $y_j \to y_0$ for some $y_0 \in \partial\Omega$. It is then clear that $\xi_j \to y_0$ as $j \to \infty$.

Since ξ_j is a closest point of $\partial\Omega$ to $y_j + \lambda_j \nu(y_j)$, we have

$$\xi_j + \delta(\lambda_j)\nu(\xi_j) = y_j + \lambda_j \nu(y_j).$$

Hence, noting that, for some $\theta_j,\, \tilde\theta_j \in (0,\, 1)$,

$$\begin{cases} \rho(y_j + \lambda_j \nu(y_j)) = \lambda_j D\rho(y_j + \theta_j \lambda_j \nu(y_j)) \cdot \nu(y_j), \\ \rho(\xi_j + \delta(\lambda_j)\nu(\xi_j)) = \delta(\lambda_j) D\rho(\xi_j + \tilde\theta_j \delta(\lambda_j)\nu(\xi_j)) \cdot \nu(\xi_j), \end{cases}$$

we find

$$\lambda_j D\rho(y_j + \theta_j \lambda_j \nu(y_j)) \cdot \nu(y_j) = \delta(\lambda_j) D\rho(\xi_j + \tilde{\theta}_j \delta(\lambda_j)\nu(\xi_j)) \cdot \nu(\xi_j),$$

which, in the limit as $j \to \infty$, yields

$$\lim_{j\to\infty} \frac{\delta(\lambda_j)}{\lambda_j} = \lim_{j\to\infty} \frac{D\rho(y_j + \theta_j \lambda_j \nu(y_j)) \cdot \nu(y_j)}{D\rho(\xi_j + \tilde{\theta}_j \delta(\lambda_j)\nu(\xi_j)) \cdot \nu(\xi_j)} = 1,$$

a contradiction to the inequality $\delta(\lambda_j)/\lambda_j \le \delta_0 < 1$ for $j \in \mathbb{N}$. □

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