Hamilton-Jacobi equations and viscosity solutions

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Example 1

Consider the *eikonal equation*

$$|u'(x)| = 1 \quad \text{in } (-1,1),$$

with boundary condition $u(-1) = u(1) = 0$. No C^1 solution.

This is a *Hamilton-Jacobi equation*.

This appears in geometric optics and describes the wave front. In the above case, the light sources are located at $x = \pm 1$ and the speed of light is assumed to be one.

The right solution should be $u(x) = 1 - |x| = \min\{x - 1, 1 - x\} = \text{dist}\,(x, \{\pm 1\})$. The set $\{x : u(x) = a\}$ is the set of points where the light arrives after time a coming from $\{\pm 1\}$.

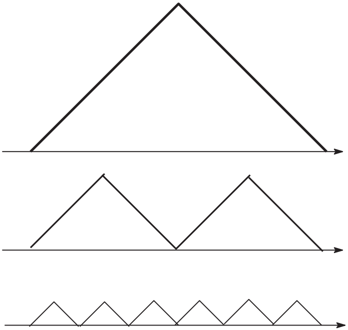
In view of the theory of differential equations, this gives a big problem.

No classical solution, but ∃ a right solution.

What is a good generalised (weak) solution?

People tried to find a good notion of generalized solutions in the class of Lipschitz functions which satisfy the given equation in the almost everywhere sense.

$$|u'(x)| = 1 \quad \text{a.e. } (-1,1) \quad \text{and} \quad u(-1) = u(1) = 0.$$



Some a.e. solutions

- Semi-concave a.e. solutions: Kruzkov (after entropy solutions for conservation laws by Oleinik, Douglis) $\longrightarrow$ No downward pointing corner.

  The existence of solutions can be a problem in general.

- *Viscosity solutions:* Crandall-Lions, Crandall-Evans-Lions

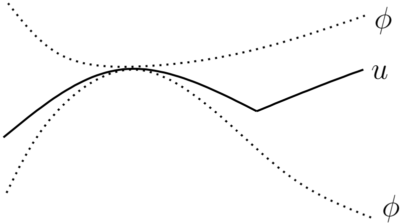
  Based on the maximum principle: if $u, \phi \in C^1$ and $u - \phi$ takes a maximum (or minimum) at x, then $u'(x) = \phi'(x)$.

Definition 2 (Preliminary)

$u \in C(-1,1)$ is a (viscosity) subsolution of $|u'| = 1$ (or $|u'| \leq 1$) in $(-1,1)$ if, whenever $\phi \in C^1(-1,1)$ and $(u-\phi)(\hat{x}) = \max(u-\phi)$, we have

$$|\phi'(\hat{x})| \leq 1.$$

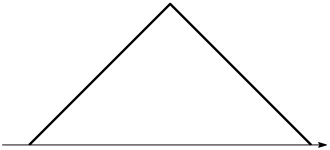
For the definition of (viscosity) supersolution, we replace $(\max, \leq)$ by $(\min, \geq)$. (Viscosity) solution is defined as a function which has both sub and super solution properties.



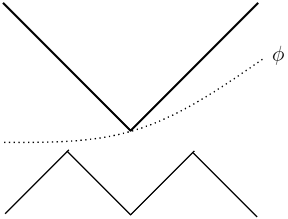
$\max(u-\phi)=0$

$\min(u-\phi)=0$

Let $u = \operatorname{dist}(x, \{\pm 1\})$ and $\phi \in C^1(-1,1)$. Assume that $\max(u-\phi) = (u-\phi)(\hat{x})$ for some $\hat{x}$. If $\hat{x} \neq 0$, then $u'(\hat{x}) = \phi'(\hat{x})$ and $|\phi'(\hat{x})| = |u'(\hat{x})| = 1$. If $\hat{x} = 0$, then $|\phi'(\hat{x})| \leq 1$.



Instead, if $\min(u-\phi) = (u-\phi)(\hat{x})$, then $\hat{x} \neq 0$ and $|\phi'(\hat{x})| = 1$.



- For classical smooth solutions,

$$|u'| = 1 \iff -|u'| = -1.$$

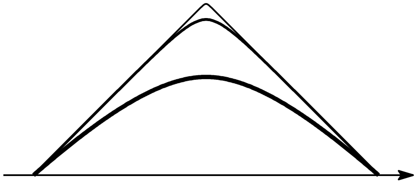
This is not true for viscosity solutions. For instance, $u = \text{ dist } (x, \{\pm 1\})$ (resp., $u = -\text{ dist } (x, \{\pm 1\})$) is a viscosity solution to $|u'| = 1$ (resp., $-|u'| = -1$), but not to $-|u'| = -1$ (resp., $|u'| = 1$).

- The vanishing viscosity method: when "right" solutions may have singularities, a classical argument to pick up a "right"solution (physically meaning solution) is to introduce an artificial viscosity to the equation. In our example, we consider

$$-\varepsilon u''(x) + |u'| = 1 \text{ in } (-1, 1), \text{ and } u(\pm 1) = 0, \text{ with } \varepsilon > 0.$$

This has a C^2 solution

$$u_\varepsilon(x) = 1 + \varepsilon e^{-\frac{1}{\varepsilon}} - |x| - \varepsilon e^{-\frac{|x|}{\varepsilon}}.$$



$\uparrow$ $\varepsilon = 1/2,\ 1/10,\ 1/100$

$$\text{dist}\,(x, \{\pm 1\}) = \lim_{\varepsilon \to 0^+} u_\varepsilon(x);$$ "viscosity" solution.

Example 3

Given two functions $f : \mathbb{R}^n \times \mathsf{C} \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathsf{C} \to \mathbb{R}^n$,

$$\dot{X}(t) = g(X(t), \alpha(t)), \quad X(0) = x,$$
$$J(x, \alpha) = \int_0^\infty e^{-\lambda t} f(X(t), \alpha(t))dt$$

Here, $X(t)$ is the solution of the Cauchy problem for the ODE given by g, $J(x, \alpha)$ is the *cost functional*, which gives the criteria for the choice of the control α. The constant $\lambda > 0$ is the so-called *discount factor*, and the effect of the *running cost* f is decreasing with the factor $e^{-\lambda t}$ as the time proceeds.

We assume that C is a compact subset of $\mathbb{R}^m$, the functions f, g are continuous on $\mathbb{R}^n \times \mathsf{C}$, and there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n$, $c \in \mathsf{C}$,

$$|f(x, c)| \vee |g(x, c)| \leq C,$$
$$|f(x, c) - f(y, c)| \vee |g(x, c) - g(y, c)| \leq C|x - y|.$$

The set of all measurable functions $\alpha : [0, \infty) \to \mathsf{C}$ is denoted by $\mathcal{C}$. For any $\alpha \in \mathcal{C}$, the Cauchy problem

$$\dot{X}(t) = g(X(t), \alpha(t)), \quad X(0) = x \in \mathbb{R}^n$$

has a unique solution $X(t) = X(t; x, \alpha)$, and the cost functional $J(x, \alpha)$ is well defined.

The *value function* V on $\mathbb{R}^n$ is defined by

$$V(x) = \inf_{\alpha \in \mathcal{C}} J(x, \alpha).$$

Note:

$$|J(x, \alpha)| \leq \int_0^\infty e^{-\lambda t} |f(X(t), \alpha(t))| dt \leq C/\lambda,$$

and

$$|V(x)| \leq C/\lambda.$$

Since

$$|X(t;x,\alpha) - X(t;y,\alpha)| \le |x-y|e^{Ct},$$

we have

$$\begin{aligned}|J(x,\alpha) - J(y,\alpha)| &\le \int_0^T e^{-\lambda t + Ct} C|x-y|dt + 2C\int_T^\infty e^{-\lambda t}dt \\ &\le O(|x-y|e^{CT} + e^{-\lambda T}) \ \ \forall T > 0.\end{aligned}$$

If we choose $T > 0$ so that $|x-y|e^{CT} = e^{-\lambda T}$ (i.e., $e^T = |x-y|^{-1/(C+\lambda)}$), the O term becomes $O(|x-y|^{\lambda/(C+\lambda)})$. The value function V is in $\mathrm{BUC}(\mathbb{R}^n)$.

Optimal control theory:

- Find $\alpha \in \mathcal{C}$ such that $V(x) = J(x,\alpha)$. optimal control!
- Find the value of V.

Bellman equation The Bellman equation should characterize the value function V.

$$\max_{c\in C}(\lambda u(x) - g(x,c)\cdot Du(x) - f(x,c)) = 0 \quad \text{in } \mathbb{R}^n.$$

($Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ gardient of u.) If we write

$$\begin{aligned} H(x,p,r) &= \max_{c\in C}(\lambda r - g(x,c)\cdot p - f(x,c)) \\ &= \lambda r + \max_{c\in C}(-g(x,c)\cdot p - f(x,c)), \end{aligned}$$

then the above equation reads $H(x, Du(x), u(x)) = 0$.

If $C = \overline{B}_1(0) \subset \mathbb{R}^n$, $g(x,c) = c$, $f(x,c) = 1$ and $\lambda = 0$ (against to the tentative assumption), then

$$H(x,p,r) = H(p) = |p| - 1 \;\; (|Du(x)| - 1 = 0).$$

Similarly, if $C = \overline{B}_1(0) \subset \mathbb{R}^n$, $g(x,c) = g(x)c$, $f(x,c) = f(x)$ and $\lambda = 0$, then

$$H = |g(x)||p| - f(x) \;\; (|g(x)||p| - f(x) = 0).$$

Removing the compactness assumption on C, if $C = \mathbb{R}^n$, $g = c$, $f = |c|^2/2 + 1$, and $\lambda = 0$, then

$$H = \tfrac{1}{2}|p|^2 - 1 \;\; (\tfrac{1}{2}|Du|^2 - 1 = 0).$$

A remark is: the Hamiltonians $H(x, p, r)$ for Bellman equations are convex in p.

Assume that $\mathsf{C} = \{c\}$ (a singleton). Write $f(x) = f(x, c)$, $g(x) = g(x, c)$. Assume evrything are smooth. Then, for $\tau > 0$,

$$\begin{aligned} V(x) &= \int_0^\tau e^{-\lambda t} f(X(t))dt + \int_\tau^\infty e^{-\lambda t} f(X(t))dt \\ &= \int_0^\tau e^{-\lambda t} f(X(t))dt + e^{-\lambda\tau} \int_0^\infty e^{-\lambda t} f(X(t+\tau))dt \\ &= \int_0^\tau e^{-\lambda t} f(X(t))dt + e^{-\lambda\tau} V(X(\tau)), \end{aligned}$$

and

$$\begin{aligned}
0 &= \int_0^\tau e^{-\lambda t} f(X(t))dt + e^{-\lambda\tau} V(X(\tau)) - V(X(0)) \\
&= \int_0^\tau \left(e^{-\lambda t} f(X(t)) + \frac{d}{dt}\left(e^{-\lambda t} V(X(t))\right)\right) dt \\
&= \int_0^\tau e^{-\lambda t} \left(f(X(t)) - \lambda V(X(t)) + DV(X(t)) \cdot g(X(t))\right) dt.
\end{aligned}$$

It follows that

$$\lambda V(x) - g(x) \cdot DV(x) - f(x) = 0 \ \ \forall x \in \mathbb{R}^n.$$

If we start with this PDE, the formula of V is a consequence of the so-called characteristic method applied to this PDE.

Existence, uniqueness and stability of viscosity solutions I

Consider the first-order PDE

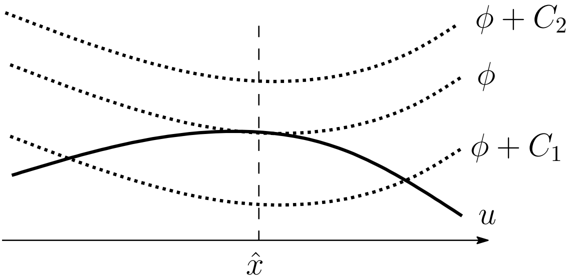
$$F(x, Du(x), u(x)) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{1}$$

Definition 1

Let Ω be an open set $\subset \mathbb{R}^n$ and $F \in C(\Omega \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$. Let $u \in C(\Omega, \mathbb{R})$. We call u a (viscosity) subsolution (resp., supseroslution) of (1) if for any $(\phi, x) \in C^1(\Omega, \mathbb{R}) \times \Omega$ such that $\max(u - \phi) = (u - \phi)(x)$ (resp., $\min(u - \phi) = (u - \phi)(x)$),

$$F(x, D\phi(x), u(x)) \leq 0 \quad \text{(resp., } \; F(x, D\phi(x), u(x)) \geq 0).$$

When u is both a (viscosity) sub and supersolution of (1), we call u a (voscosity) solution of (1).



$\boldsymbol{u} - \boldsymbol{\phi}$ **max** at $\hat{\boldsymbol{x}}$

$\Updownarrow$

$\boldsymbol{\phi} - \boldsymbol{u}$ **min** at $\hat{\boldsymbol{x}}$

$\boldsymbol{u}$ is tested from above by $\boldsymbol{\phi}$ at $\hat{\boldsymbol{x}}$; $\boldsymbol{\phi}$ is an upper tangent to $\boldsymbol{u}$ at $\hat{\boldsymbol{x}}$; $\boldsymbol{u}$ is touched from above by $\boldsymbol{\phi}$ at $\hat{\boldsymbol{x}}$,. . .

- Subsolution for $\boldsymbol{u} \in \mathbf{USC}(\boldsymbol{\Omega}, \mathbb{R} \cup \{-\infty\})$; supersolution for $\boldsymbol{u} \in \mathbf{LSC}(\boldsymbol{\Omega}, \mathbb{R} \cup \{\infty\})$.
- $\boldsymbol{\phi} \in \boldsymbol{C}^{\infty}(\boldsymbol{\Omega})$.
- $\mathbf{max}$, $\mathbf{min}$ $\longrightarrow$ strict $\mathbf{max}$, strict $\mathbf{min}$.

Remark 2

1) In general, when u is a (viscosity) solution of $F(x, Du, u) = 0$, u may not be a (viscosity) solution of $-F(x, Du, u) = 0$. Reverse inequalities.

2) In general, when u is a (viscosity) solution of $F(x, Du, u) = 0$, $v := -u$ may not be a (viscosity) solution of $F(x, -Dv, -v) = 0$. Testing from the reverse side.

3) Set $v := -u$. Then u is a (viscosity) solution of $F(x, Du, u) = 0$ if and only if v is a (viscosity) solution of $-F(x, -Dv, -v) = 0$.

Let $\phi \in C^1$, $\psi := -\phi$, and $\hat{x} \in \Omega$.

$$(u - \phi)(\hat{x}) = \max(u - \phi) \iff (v + \phi)(\hat{x}) = \min(v + \phi)$$
$$\iff (v - \psi)(\hat{x}) = \min(v - \psi),$$

and

$$F(\hat{x}, D\phi(\hat{x}), u(\hat{x})) \leq 0 \iff -F(\hat{x}, -D\psi(\hat{x}), -v(\hat{x})) \geq 0.$$

Theorem 1

The value function V defined above is a viscosity solution of

$$(2) \quad \lambda u + \max_{c \in C}(-g(x,c) \cdot Du - f(x,c)) = 0 \quad \text{in } \mathbb{R}^n.$$

Theorem 2 (DPP)

Let $x \in \mathbb{R}^n$ and $\tau : \mathcal{C} \to [0,\infty]$ be a mapping. Then

$$V(x) = \inf_{\alpha \in \mathcal{C}} \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))dt + e^{-\lambda\tau} V(X(\tau)).$$

We write

$$H(x,p,r) = \lambda r + \max_{c \in C}(-g(x,c) \cdot p - f(x,c)).$$

Proof of Theorem 2:

$$J(x,\alpha) = \int_0^\tau e^{-\lambda t} f(X(t),\alpha(t))dt$$
$$+ e^{-\lambda\tau}\int_0^\infty e^{-\lambda t} f(X(\tau+t),\alpha(\tau+t))dt,$$

$$J(x,\alpha) \geq V(x),$$

$$\int_0^\infty e^{-\lambda t} f(X(\tau+t),\alpha(\tau+t))dt = J(X(\tau),\alpha(\tau+\cdot))$$
$$\geq V(X(\tau)).$$

Proof of Theorem 1: Since C is compact and f, g are continuous, H is continuous. We only check the supersolution property by a contradiction argument. Let $\phi \in C^1$ and $\min(V-\phi) = (V-\phi)(\hat{x})$ for some $\hat{x} \in \mathbb{R}^n$. Suppose that

$$H(\hat{x}, D\phi(\hat{x}), V(\hat{x})) < 0.$$

Replacing ϕ by $\phi + \min(V-\phi)$, we may assume that $\min(V-\phi) = 0$. That is, $V(\hat{x}) = \phi(\hat{x})$.

$$V(x) = \inf_{\alpha \in \mathcal{C}} \int_0^{\tau} e^{-\lambda t} f(X(t), \alpha(t))dt + e^{-\lambda \tau} V(X(\tau)).$$

PROOF Set

$$W(x) = \inf_{\alpha \in \mathcal{C}} \int_0^{\tau} e^{-\lambda t} f(X(t), \alpha(t))dt + e^{-\lambda \tau} V(X(\tau)).$$

Choose $\alpha \in \mathcal{C}$ so that

$$V(x) \approx J(x, \alpha),$$

and compute

$$\begin{aligned} J(x, \alpha) &= \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t), \alpha(t))dt + \int_{\tau(\alpha)}^{\infty} e^{-\lambda t} f(X(t), \alpha(t))dt \\ &= \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t), \alpha(t))dt \end{aligned}$$

$$
\begin{aligned}
&+ e^{-\lambda\tau(\alpha)} \int_0^\infty e^{-\lambda s} f(X(s+\tau(\alpha)), \alpha(s+\tau(\alpha))ds \\
&= \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t), \alpha(t))dt \\
&\quad + e^{-\lambda\tau(\alpha)} J(X(\tau(\alpha)), \alpha(\tau(\alpha)+\cdot)) \\
&\geq \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t), \alpha(t))dt + e^{-\lambda\tau(\alpha)} V(X(\tau(\alpha))) \\
&\geq W(x).
\end{aligned}
$$

Hence,

$$V(x) \geq W(x).$$

Choose $\alpha \in \mathcal{C}$ so that

$$W(x) \approx \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t), \alpha(t))dt + e^{-\lambda\tau(\alpha)} V(X(\tau(\alpha))).$$

Choose $\beta \in \mathcal{C}$ so that

$$V(X(\tau(\alpha))) \approx J(X(\tau(\alpha)), \beta).$$

Then

$$
\begin{aligned}
W(x) &\approx \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t),\alpha(t))dt + e^{-\lambda\tau(\alpha)} J(X(\tau(\alpha)),\beta) \\
&= \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t),\alpha(t))dt \\
&\quad + e^{-\lambda\tau(\alpha)} \int_0^{\infty} e^{-\lambda t} f(X(t, X(\tau(\alpha)),\beta),\beta(t))dt \\
&= \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t),\alpha(t))dt \\
&\quad + e^{-\lambda\tau(\alpha)} \int_{\tau(\alpha)}^{\infty} e^{-\lambda(s-\tau(\alpha))} \times \\
&\qquad \times f(X(s-\tau(\alpha), X(\tau(\alpha)),\beta),\beta(s-\tau(\alpha)))ds \\
&= \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t),\alpha(t))dt \\
&\quad + \int_{\tau(\alpha)}^{\infty} e^{-\lambda t} f(X(t-\tau(\alpha), X(\tau(\alpha)),\beta),\beta(t-\tau(\alpha)))dt
\end{aligned}
$$

Set

$$\gamma(t) = \begin{cases} \alpha(t) & \text{for } t \in [0, \tau(\alpha)) \\ \beta(t - \tau(\alpha)) & \text{for } t \in [\tau(\alpha), \infty), \end{cases}$$

and note that

$$X(t, x, \gamma) = \begin{cases} X(t, x, \alpha) & \text{for } t \in [0, \tau(\alpha)), \\ X(t - \tau(\alpha), X(\tau(\alpha)), \beta) & \text{for } t \in [\tau(\alpha), \infty), \end{cases}$$

to find that

$$\begin{aligned} W(x) &\approx \int_0^{\tau(\alpha)} e^{-\lambda t} f(X(t, x, \gamma), \gamma(t))dt \\ &\quad + \int_{\tau(\alpha)}^{\infty} e^{-\lambda t} f(X(t, x, \gamma), \gamma(t))dt \\ &= J(x, \gamma) \geq V(x). \end{aligned}$$

Thus, $W(x) \geq V(x)$. The proof is complete. $\square$

By continuity, for some $r > 0$,

$$H(x, D\phi(x), \phi(x)) < 0 \quad \forall x \in \overline{B}_r(\hat{x}).$$

Define $\tau : \mathcal{C} \to [0, \infty]$ by

$$\tau = \tau(\alpha) := \inf\{t \geq 0 : X(t; \hat{x}, \alpha) \in \partial B_r(\hat{x})\}.$$

By DPP, for each $\varepsilon > 0$, $\exists \alpha \in \mathcal{C}$ such that

$$V(\hat{x}) + \varepsilon > \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))dt + e^{-\lambda\tau} V(X(\tau)).$$

Note that

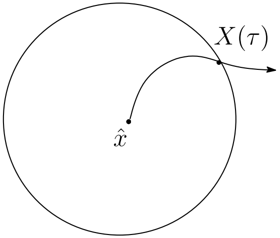
$$V(\hat{x}) = \phi(\hat{x}), \quad V(X(\tau)) \geq \phi(X(\tau)),$$

and, since $|\dot{X}| = |g(X)| \leq C$,

$$\tau \geq \tfrac{r}{C},$$

which implies

$$\int_0^\tau e^{-\lambda t} dt \geq \int_0^{\frac{r}{C}} e^{-\lambda t} dt.$$


$X(\tau)$
$\hat{x}$

We replace ε by

$$\varepsilon \int_0^{\frac{r}{C}} e^{-\lambda t} dt,$$

to obtain

$$\phi(\hat{x}) + \varepsilon \int_0^{\tau} e^{-\lambda t} dt > \int_0^{\tau} e^{-\lambda t} f(X(t), \alpha(t)) dt + e^{-\lambda \tau} \phi(X(\tau)),$$

and, if $0 < \varepsilon \ll 1$,

$$0 < \int_0^{\tau} e^{-\lambda t} \Big(\varepsilon - f(X(t), \alpha(t)) + \lambda \phi(X(t)) - g(X(t), \alpha(t)) \cdot D\phi(X(t)) \Big) dt$$

$$\leq \int_0^{\tau} e^{-\lambda t} \Big(\varepsilon + H\big(X(t), D\phi(X(t)), \phi(X(t))\big) \Big) dt < 0.$$

Hence, a contradiction.

Theorem 1 is an existence theorem.

If we write

$$H(x,p) = \max_{c \in C}(-g(x,c) \cdot p - f(x,c)),$$

then

$$|H(x,p) - H(y,p)| \le C|x-y|(|p|+1),$$
$$|H(x,p) - H(x,q)| \le C|p-q|.$$

Under the above hypotheses on a general H, consider the HJ equation

(2) $$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n.$$

Theorem 3 (Comparison theorem)

Let $v, w \in \mathbf{BC}(\mathbb{R}^n)$ be sub and super solutions of (2), respectively. Then, $v \le w$ in $\mathbb{R}^n$.

The value function V is a unique solution in the class $\mathbf{BC}(\mathbb{R}^n)$.
A PDE characterization of value functions.

1) Fix any $\varepsilon > 0$. Set $v_\varepsilon(x) = v(x) - \varepsilon\langle x\rangle$, where $\langle x\rangle = (|x|^2 + 1)^{1/2}$. Note:

$$\begin{aligned}\lambda v_\varepsilon + H(x, Dv_\varepsilon) &\le \lambda v + H\left(x, Dv - \varepsilon\frac{x}{\langle x\rangle}\right) \\ &\le \lambda v + H(x, Dv) + C\varepsilon.\end{aligned}$$

Replace v_ε by $v_\varepsilon = v - \varepsilon(\langle x\rangle + \lambda^{-1}C)$, to get

$$\lambda v_\varepsilon + H(x, Dv_\varepsilon) \le \lambda v - \varepsilon C + H(x, Dv) + \varepsilon C \le 0.$$

Enough to show that $v_\varepsilon \le w$ in $\mathbb{R}^n$ for all $\varepsilon > 0$ $(0 < \varepsilon \ll 1)$.

2) Fix $\varepsilon > 0$. Since v, w are bounded,

$$\lim_{|x|\to\infty}(v_\varepsilon - w)(x) = -\infty.$$

Choose $R > 0$ so that

$$(v_\varepsilon - w)(x) < 0 \ \ \forall x \in \mathbb{R}^n \setminus B_R.$$

3) To complete the proof, we argue by contradiction. Suppose:

$$\sup_{\mathbb{R}^n}(v_\varepsilon - w) > 0,$$

which implies

$$S := \sup_{B_R}(u_\varepsilon - w) > 0.$$

4) If we have $w \in C^1$, by chance, then, by the viscosity properties,

$$\lambda v_\varepsilon(x)+H(x,Dw(x)) \leq 0, \text{ and } \lambda w(x)+H(x,Dw(x)) \geq 0$$

at any maximum point x of $v_\varepsilon - w$. (v_ε is tested by w from above and w is tested by w itself from below.) Subtracting one from the other yields

$$\lambda(v_\varepsilon - w)(x) \leq 0 \text{ at any maximu point } x \text{ of } v_\varepsilon - w.$$

This is a contradiction: $\lambda S \leq 0$.

5) In the general situation, a standard technique to overcome the lack of regularity is the so-called doubling variable method. For $k \in \mathbb{N}$, consider the function

$$\Phi_k(x,y) = v_\varepsilon(x) - w(y) - k|x-y|^2$$

on $K := \overline{B}_R \times \overline{B}_R$. Let (x_k, y_k) be a maximum point of this function.

6) Observe that

$$\max_K \Phi_k \geq \max_{x \in \overline{B}_R} \Phi_k(x, x) = \max_{\overline{B}_R}(v_\varepsilon - w) = S,$$

and hence,

$$S \leq \Phi_k(x_k, y_k) = v_\varepsilon(x_k) - w(y_k) - k|x_k - y_k|^2 \leq C_1 - k|x_k - y_k|^2.$$

We may assume by passing to a subsequence that for some $(x_0, y_0) \in K$,

$$\lim_k (x_k, y_k) = (x_0, y_0).$$

Since $\{k|x_k - y_k|^2\}_k$ is bounded, we find that

$$x_0 = y_0,$$

and, moreover, from the above,

$$S \leq v_\varepsilon(x_0) - w(x_0) - \limsup_k k|x_k - y_k|^2,$$

which implies that

$$(v_\varepsilon - w)(x_0) = S \quad \text{and} \quad \lim_k k|x_k - y_k|^2 = 0.$$

The first identity above implies that $x_0 \in B_R$ (interior point). Passing to a subsequence, we may assume that

$$x_k, y_k \in B_R \ \ \forall k.$$

Note that the functions

$$\begin{aligned} x &\mapsto \Phi_k(x, y_k) = v_\varepsilon(x) - k|x - y_k|^2 - w(y_k), \\ y &\mapsto -\Phi_k(x_k, y) = w(y) + k|y - x_k|^2 - v_\varepsilon(x_k) \end{aligned}$$

take, respectively, a max at $x = x_k$ and min at $y = y_k$. By the viscosity properties,

$$\begin{aligned} \lambda v_\varepsilon(x_k) + H(x_k, 2k(x_k - y_k)) \le 0, \\ \lambda w(y_k) + H(y_k, -2k(y_k - x_k)) \ge 0. \end{aligned}$$

Hence,

$$\begin{aligned} 0 &\ge \lambda(v_\varepsilon(x_k) - w(y_k)) + H(x_k, 2k(x_k - y_k)) - H(y_k, 2k(x_k - y_k)) \\ &\ge \lambda S - C|x_k - y_k|(2k|x_k - y_k| + 1). \end{aligned}$$

In the limit $k \to \infty$, $\lambda S \le 0$, a contradiction.

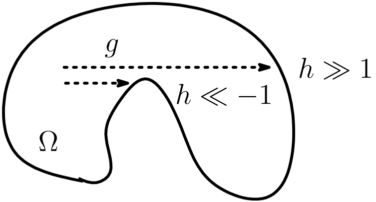
• Dirichlet problem. Let $\Omega \subset \mathbb{R}^n$ be an open set. Let f, g be as above. We introduce a function h on $\partial\Omega$, which is called the *pay-off* in the framework of optimal control. The cost functional is:

$$J(x, \alpha) = \int_0^\tau e^{-\lambda t} f(X(t), \alpha(t))dt + e^{-\lambda\tau} h(X(\tau)),$$

where $\tau = \inf\{t \geq 0 : X(t) \in \mathbb{R}^n \setminus \Omega\}$, called the *exit time.* The value function V is given by

$$V(x) = \inf_{\alpha \in \mathcal{C}} J(x, \alpha).$$

The continuity of V can be a big issue.



When everything goes fine, $u = V$ satisfies the Dirichlet problem

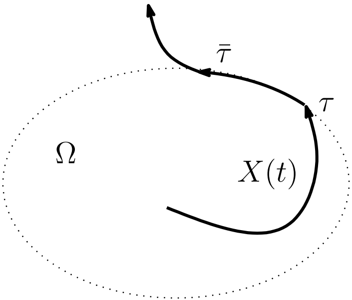
$$\begin{cases} \lambda u + \max_{c \in \mathsf{C}}(-g(x,c) \cdot Du - f(x,c)) = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

In the above choice of τ, X have to stop at the first hitting time to $\partial\Omega$.

Another possible choice of τ is:

$$\bar{\tau} = \inf\{t \geq 0 : X(t) \in \mathbb{R}^n \setminus \overline{\Omega}\}.$$

Here X stays in $\overline{\Omega}$ until it first exits from $\overline{\Omega}$.



Existence, uniqueness and stability of viscosity solutions II

Consider the time-evolution problem

$$(1) \qquad u_t + H(x, D_x u) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

If we set $F(x,t,p,q) := q + H(x,p)$ for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, $(p,q) \in \mathbb{R}^n \times \mathbb{R}$, then the above time-evolution PDE can be written as $F(z, Du) = 0$. The previous definition of viscosity solutions makes sense for the current problem.

If H is given as before by

$$H(x,p) = \max_{c \in \mathsf{C}} (-g(x,c) \cdot p - f(x,c)),$$

then our PDE can be written as

$$\max_{c \in \mathcal{C}} (-g(x,c) \cdot D_x u - (-1) u_t - f(x,c)) = 0.$$

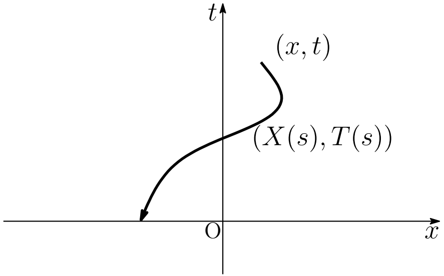
In view of optimal control, the dynamics is described by

$$\dot{X}(s) = g(X(s), \alpha(s)), \ \dot{T}(s) = -1, \quad X(0) = x, \ T(0) = t,$$

and the cost functional is:

$$J(x,t,\alpha) = \int_0^t f(X(s),\alpha(s))ds + h(X(t)),$$

where $h \in \mathbf{BC}(\mathbb{R}^n)$.



A kind of the Dirichlet problem: $\tau = t$.
The value function is now:

$$V(x,t) = \inf_{\alpha \in \mathcal{C}} J(x,t,\alpha). \tag{2}$$

Theorem 1

Assume that f, g satisfy the Lipschitz condition as before and that $h \in \mathbf{BC}(\mathbb{R}^n)$. Then,

- for any $0 < T < \infty$, the value function V, given by (2), is bounded and continuous on $\mathbb{R}^n \times [0, T]$.
- $u = V$ is a (viscosity) solution of the Cauchy problem

$$(3) \qquad u_t + H(x, D_x u) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

$$(4) \qquad u(\cdot, 0) = h \qquad \text{on } \mathbb{R}^n,$$

where $H(x, p) = \max_{c \in \mathsf{C}}(-g(x, c) \cdot p - f(x, c))$.

This can be regarded as an existence result for the Cauchy problem (3) – (4). Here h is the *initial data.*

We have a comparison theorem which covers the above Cauchy problem, and the consequence is that V is a unique solution of (3)–(4).

Let H be a (general) continuous function on $\mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n$ such that for some constant $C > 0$,

$$|H(x,t,p) - H(x,t,q)| \leq C|p-q|,$$
$$|H(x,t,p) - H(y,s,p)| \leq C(|x-y| + |t-s|)(|p|+1).$$

Let $0 < T \leq \infty$. Consider the HJ equation

(5) $$u_t + H(x,t,D_x u) = 0 \quad \text{in } \mathbb{R}^n \times [0,T).$$

Theorem 2

Under the above assumptions on H, let $v, w \in \mathrm{BC}(\mathbb{R}^n \times [0,T))$ be, respectively, a sub and supersolution of (5). Assume moreover that $v(x,0) \leq w(x,0)$ for all $x \in \mathbb{R}^n$. Then, $v \leq w$ in $\mathbb{R}^n \times [0,T)$.

Proof.

1) Enough to show that for any $0 < S < T$, $v \le w$ on $\mathbb{R}^n \times [0, S)$. Fix any $S > 0$.

2) Fix any $\varepsilon > 0$. Set $v_\varepsilon(x,t) = v(x,t) - \varepsilon\langle x\rangle$, where $\langle x\rangle = (|x|^2+1)^{1/2}$. Enough to show that $v_\varepsilon \le w$ on $\mathbb{R}^n \times [0, S)$. Note that

$$v_{\varepsilon,t} + H(x,t,D_x v_\varepsilon) \le v_t + H(x,t,D_x v) + C\varepsilon.$$

Replace v_ε by $v_\varepsilon(x,t) = v(x,t) - \delta\langle x\rangle - C\varepsilon t$, and note that

$$v_{\varepsilon,t} + H(x,t,D_x v_\varepsilon) \le v_t - C\varepsilon + H(x,t,D_x v) + C\varepsilon \le 0.$$

Replace again v_ε by $v(x,t) - \varepsilon\langle x\rangle - C\varepsilon t - \frac{\varepsilon}{S-t}$, and note that

$$v_{\varepsilon,t} + H(x,t,Dv_\varepsilon) \le v_t - \frac{\varepsilon}{(S-t)^2} - C\varepsilon + H(x,t,Dv) + C\varepsilon \le -\eta,$$

where $\eta = \varepsilon S^{-2}$.

Enough to show that $v_\varepsilon \le w$ on $\mathbb{R}^n \times [0, S)$.

5) We argue by contradiction: suppose that $\sup(v_\varepsilon - w) > 0$ and will get a contradiction. Since

$$\lim_{|x|\to\infty} (v_\varepsilon - w)(x,t) = -\infty \quad \text{uniformly in } t,$$

$$\lim_{t\to S-} (v_\varepsilon - w)(x,t) = -\infty \quad \text{uniformly in } x,$$

$$(v_\varepsilon - w)(x,0) < 0 \quad \text{for all } x \in \mathbb{R}^n,$$

$\exists R > 0, \delta > 0$ such that

$$(v_\varepsilon - w)(x,t) < 0 \quad \text{for all } (x,t) \in (\mathbb{R}^n \times [0,S)) \setminus \big(B_R \times (\delta, S-\delta)\big).$$

In particular,

$$\max_{\overline{B}_R \times [\delta, S-\delta]} (v_\varepsilon - w) = \max_{B_R \times (\delta, S-\delta)} (v_\varepsilon - w) > 0.$$

6) If $w \in C^1$, then, at any maximum point of $v_\varepsilon - w$,

$$w_t + H(x,t,Dw) \le -\eta,$$

$$w_t + H(x,t,Dw) \ge 0,$$

which yields a contradiction.

In the general case, we use the doubling variable method, to obtain a contradiction.

$$\Phi_k(x,t,y,s) := v_\varepsilon(x,t) - w(y,s) - k(|x-y|^2 + |t-s|^2).$$

(x_k, t_k, y_k, s_k) a max point of Φ_k.

$$\begin{aligned}
&\lim_{k\to\infty}(x_k, t_k, y_k, s_k) = (x_0, x_0, t_0, t_0),\\
&(v_\varepsilon - w)(x_0, t_0) = \max(v_\varepsilon - w),\\
&\lim_{k\to\infty} k(|x_k - y_k|^2 + |t_k - s_k|^2) = 0,\\
&2(t_k - s_k) + H(x_k, t_k, 2k(x_k - y_k)) \le -\eta,\\
&2(t_k - s_k) + H(y_k, s_k, 2k(x_k - y_k) \ge 0.
\end{aligned}$$

$$\begin{aligned}
-\eta &\ge H(x_k, t_k, \ldots) - H(y_k, s_k, \ldots)\\
&\ge -C(|x_k - y_k| + |t_k - s_k|)(2k|x_k - y_k| + 1) \to 0\\
&\qquad\qquad (k \to \infty).
\end{aligned}$$

Existence, uniqueness and stability of viscosity solutions III

- Stability:
  Well-posedness (Hadamard) = existence, uniqueness, stability.
  Consider the general first-oder PDE

$$F(x, Du, u) = 0 \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $F \in C(\Omega \times \mathbb{R}^n \times \mathbb{R})$.

Theorem 1

Let $\{u_k\}$ be a sequence of continuous functions on Ω converging to a fucntion u in $C(\Omega)$. If every u_k is a (viscosity) subsolution (resp., supersolution, solution) of (1), then so is the function u.

PROOF. Only the subsolution case. Let $\phi \in C^1(\Omega)$ and assume that $\max(u - \phi) = (u - \phi)(\hat{x})$. By adding the function $|x - \hat{x}|^2$ to ϕ (notice that $D|x - \hat{x}|^2 = 0$ at $x = \hat{x}$), we may assume that $\max$ is a strict $\max$.

Choose $0 < r \ll 1$ so that $\overline{B}_r(\hat{x}) \subset \Omega$. Let x_k be a maximum point of $(u_k - \phi)|_{\overline{B}_r(\hat{x})}$. Because of the uniform convergence on $\overline{B}_r(\hat{x})$ and the strict $\max$,

$$\lim_k x_k = \hat{x}.$$

We may assume that $x_k \in B_r(\hat{x})$ (interior point). Since u_k is a subsolution, we have

$$F(x_k, D\phi(x_k), u_k(x_k)) \leq 0.$$

Sending $k \to \infty$ yields

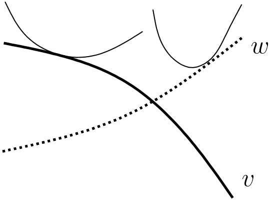
$$F(\hat{x}, D\phi(\hat{x}), u(\hat{x})) \leq 0.$$

The following is a straightforward generalization of the above theorem.

Theorem 2

Let $\{u_k\}$ be a sequence of continuous functions on Ω converging to a fucntion u in $C(\Omega)$. Let $\{F_k\}$ be a sequence of continuous functions on $\Omega \times \mathbb{R}^n \times \mathbb{R}$ converging to a function F in $C(\Omega \times \mathbb{R}^n \times \mathbb{R})$. If each u_k is a (viscosity) subsolution (resp., supersolution, solution) of $F_k(x, Du, u) = 0$ in Ω, then u is a (viscosity) subsolution (resp., supersolution, solution) of $F(x, Du, u) = 0$ in Ω,

Let $v, w \in C(\Omega)$ be subsolutions of (1) and consider the function $v \vee w = \max\{v, w\}$. This function $v \vee w$ is also a subsolution of (1).



Let $\mathcal{F}$ be a family of subsolutions of (1). In general,

$$w(x) := \sup\{v(x) : v \in \mathcal{F}\}$$

does not define a continuous function on Ω. $w(x)$ can be $+\infty$. Given a function f on Ω which is locally bounded (above), we define the upper semicontinuous envelope f^* by

$$\begin{aligned} f^*(x) &:= \inf\{g(x) : g \in C(\Omega), f \leq g \text{ on } \Omega\} \\ &= \lim_{r\to 0^+} \sup\{f(y) : |y-x| < r\}. \end{aligned}$$

Similarly, the lower semicontinuous envelope f_* of f is defined by

$$\begin{aligned} f_*(x) &:= \sup\{g(x) : g \in C(\Omega), f \geq g \text{ on } \Omega\} \\ &= \lim_{r\to 0^+} \inf\{f(y) : |y-x| < r\}. \end{aligned}$$

It follows

$$f^* \in \mathrm{USC}(\Omega), \quad f_* \in \mathrm{LSC}(\Omega), \quad f_* \leq f \leq f^*.$$

Definition 1

Let $u : \Omega \to \mathbb{R}$ be a locally bounded function. We call u a (viscosity) subsolution (resp., supersolution) of (1) if u^* (resp., u_*) satisfies the requirement of being a subsolution (resp., supersolution) of (1). We call u a solution if it is both a subsolution and a supersolution of (1).

Theorem 3

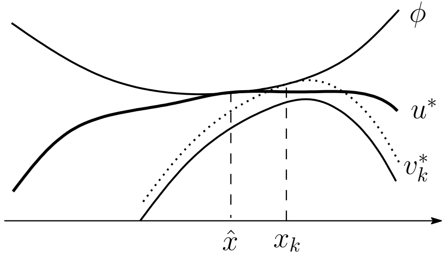
Let $\mathcal{F}$ be a family of subsolutions of (1). Set

$$u(x) = \sup\{v(x) : v \in \mathcal{F}\} \quad \text{for } x \in \Omega.$$

Assume that u is locally bounded in Ω. Then u is a subsolution of (1).

- An assertion parallel to the above for supersolutions holds.
- If u is a subsolution of (1), then $v = -u$ is a supersolution of $-F(x, -Dv, -v) = 0$ in Ω, and vice versa.

Pictorial proof:



Theorem 4

Let $\{v_k\}_{k\in\mathbb{N}} \subset \mathbf{USC}(\Omega)$ and locally uniformly bounded in Ω. Let v_k be a subsolution of (1) for any k. Assume $v_k \geq v_{k+1}$ on Ω for all k. Set

$$v(x) = \lim_k v_k(x) = \inf_k v_k(x) \quad \text{for } x \in \Omega.$$

Then, v is a subsolution of (1).

$$\begin{aligned} &\phi(\hat{x}) = u^*(\hat{x}), \\ &\phi(x) \geq u^*(x) + |x - \hat{x}|^2, \\ &(v_k^* - \phi)(x_k) = \max(v_k^* - \phi), \\ &v_k^*(\hat{x}) > u^*(\hat{x}) - \tfrac{1}{k}, \\ &v_k^* \leq u^*. \end{aligned}$$

$$\begin{array}{l} (v_k^* - \phi)(x_k) \leq (u^* - \phi)(x_k) \leq -|x_k - \hat{x}|^2, \\ \qquad\qquad \| \\ (v_k^* - \phi)(x_k) \geq (v_k^* - \phi)(\hat{x}) > -\tfrac{1}{k}. \end{array}$$

Hence,

$$\lim_k x_k = \hat{x}, \qquad \lim_k v_k^*(x_k) = \phi(\hat{x}) = u^*(\hat{x}).$$

$$F(x_k, D\phi(x_k), v_k^*(x_k)) \leq 0 \implies F(\hat{x}, D\phi(\hat{x}), u^*(\hat{x})) \leq 0.$$

Correction of the previous slide

The choice of v_k (and y_k):

$$\lim y_k = \hat{x}, \qquad v_k^*(y_k) > \phi(\hat{x}) - \tfrac{1}{k}.$$

$$\begin{cases} \phi(\hat{x}) = u^*(\hat{x}), \\ \phi(x) \geq u^*(x) + |x - \hat{x}|^2, \\ (v_k^* - \phi)(x_k) = \max(v_k^* - \phi), \\ v_k^* \leq u^*. \end{cases}$$

$$\begin{array}{l} (v_k^* - \phi)(x_k) \leq (u^* - \phi)(x_k) \leq -|x_k - \hat{x}|^2, \\ \qquad \| \\ (v_k^* - \phi)(x_k) \geq (v_k^* - \phi)(y_k) \gtrapprox -\tfrac{1}{k}. \end{array}$$

Hence,

$$\lim_k x_k = \hat{x}, \qquad \lim_k v_k^*(x_k) = \phi(\hat{x}) = u^*(\hat{x}).$$

$$F(x_k, D\phi(x_k), v_k^*(x_k)) \leq 0 \implies F(\hat{x}, D\phi(\hat{x}), u^*(\hat{x})) \leq 0.$$

PROOF. Let $\phi \in C^1(\Omega)$ and

$$\max(v - \phi) = (v - \phi)(\hat{x}) = 0 \quad \text{(a strict max)}.$$

Then, $\sup(v_k - \phi) \downarrow 0$ as $k \to \infty$. Look at $(v_k - \phi)_+$, which is in $\mathrm{USC}(\Omega)$ and $\downarrow 0$ as $k \to \infty$. Dini's lemma implies that the convergence is locally uniformly on Ω. The situation is now same as in the first stability theorem.

Theorem 5 (Barles-Perthame, half-relaxed limits)

Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence of functions on Ω, which is locally uniformly bounded in Ω. Let v_k be a subsolution of (1) for any k. Set

$$v(x) = \lim_{r\to 0+} \sup\{v_k(y) : k > \tfrac{1}{r},\ |y-x| < r\} \quad \text{for } x \in \Omega.$$

Then, v is a subsolution of (1).

PROOF. Let $\Omega = \mathbb{R}^n$. Let $r > 0$. Note that for any $\xi \in B_r(0)$, $x \mapsto v_k(\xi + x)$ is a subsolution of

$$\inf_{\eta \in B_r(0)} F(x + \eta, Du(x), u(x)) = 0 \quad \text{in } \Omega.$$

So, $x \mapsto \sup\{v_k(y) : k > \frac{1}{r},\ |y - x| < r\}$ is a subsolution of the above HJ equation. The stability under monotone convergence (Theorem 4) completes the proof. □

Theorem 6 (Perron's method)

Let f, g be, respectively, a sub and supersolution of (1). Assume $f \in \mathrm{LSC}(\Omega)$ and $g \in \mathrm{USC}(\Omega)$ and that $f \le g$ in Ω. Set

$$u(x) = \sup\{v(x) : v \in \mathcal{S}^-,\ f \le v \le g \ \text{ in } \Omega\} \ \text{ for } x \in \Omega,$$

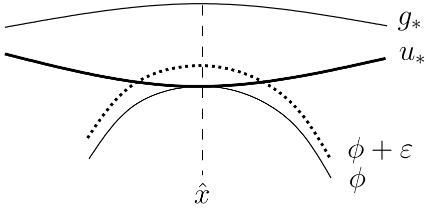
where $\mathcal{S}^- =$ the set of all subsolutions of (1). Then u is a solution of (1).

PROOF. Since, by definition, u is a pointwise sup of a family of subsolutions, it is a subsolution.

Let $\phi \in C^1$ and $\min(u_* - \phi) = (u_* - \phi)(\hat{x})$ for some $\hat{x} \in \Omega$. Assume that min = a strict min. Two cases:

Case 1: $\phi(\hat{x}) = g_*(\hat{x})$. Then, $\phi \leq u_* \leq g_*$ in Ω. ϕ touches g_* from below at $\hat{x}$. Since $g \in \mathcal{S}^+$, where $\mathcal{S}^+ =$ the set of all supersolultions of (1), we find that $F(\hat{x}, D\phi(\hat{x}), g_*(\hat{x})) \geq 0$ $(F(\hat{x}, D\phi(\hat{x}), u_*(\hat{x})) \geq 0)$.

Case 2: $\phi(\hat{x}) < g_*(\hat{x})$. Suppose by contradiction that $F(\hat{x}, D\phi(\hat{x}), \phi(\hat{x})) < 0$.



The function $\max\{u, \phi + \varepsilon\}$ $(0 < \varepsilon \ll 1)$ is against the maximality of u. □

Let H be a Hamiltonian satisfying the Lipschitz condition: for some constant $C > 0$,

$$|H(x,t,p) - H(x,t,q)| \le C|p-q|,$$
$$|H(x,t,p) - H(y,s,p)| \le C(|x-y| + |t-s|)(|p|+1).$$

Theorem 7

Let $H = H(x,p)$ satisfy the above Lipschitz condition as well as the boundedness: $|H(x,0)| \le C$. Let $\lambda > 0$. There exists a solution $u \in \mathrm{BC}(\mathbb{R}^n)$ of

$$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{R}^n. \tag{2}$$

PROOF. Set $f(x) = -C/\lambda$, $g(x) = C/\lambda$. Then f, g are, respectively, a sub and super solultion of (2). Set

$$u(x) = \sup\{v(x) : v \in \mathcal{S}^-,\ f \le v \le g \ \text{ in } \mathbb{R}^n\},$$

where $\mathcal{S}^-$ =the set of all subsolutions of (2). By Perron's method, u is a solution of (2).

By the comparison theorem, applied to a subsolution u^* and a supersolution u_*, we find that $u^* \le u_*$ in $\mathbb{R}^n$, from which $u \le u^* \le u_* \le u$ in $\mathbb{R}^n$. That is, $u = u^* = u_*$ and hence, $u \in C(\mathbb{R}^n)$. □

Theorem 8

Let H satisfy the above Lipschitz condition and the boundedness: $|H(x,t,0)| \le C$. Let $h \in \mathbf{BC}(\mathbb{R}^n)$. Then there exists a solution $u \in C(\mathbb{R}^n \times [0,\infty))$, bounded on $\mathbb{R}^n \times [0,T]$ for any $T > 0$, of

$$\text{(3)} \qquad \begin{cases} u_t + H(x,t,Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = h & \text{on } \mathbb{R}^n. \end{cases}$$

PROOF. We may assume that $|h(x)| \le C$. Set

$$g_0(x,t) = C(1+t) \quad \text{and} \quad f_0 = -g_0,$$

and note that f, g are, resp., a sub and super solutions of $u_t + H = 0$.

Want to have a sub and super solutions f, g such that $f(\cdot, 0) = g(\cdot, 0) = h$. Fix any $y \in \mathbb{R}^n$, $\varepsilon > 0$ and choose a constant $A(y, \varepsilon) > 0$ so that

$$|h(x) - h(y)| < \varepsilon + A(y, \varepsilon)|x - y| \ \ \forall x.$$

Note:

$$|H(x, t, p)| \leq |H(x, t, 0)| + C|p| \leq C(1 + |p|).$$

and choose a constant $B(y, \varepsilon) > 0$ so that if $|p| \leq A(y, \varepsilon)$,

$$|H(x, t, p)| \leq B(y, \varepsilon).$$

Set

$$g_{y,\varepsilon}(x, t) = h(y) + \varepsilon + A(y, \varepsilon)|x - y| + B(y, \varepsilon)t,$$
$$f_{y,\varepsilon}(x, t) = h(y) - (\varepsilon + A(y, \varepsilon)|x - y| + B(y, \varepsilon)t),$$

and note that $f_{y,\varepsilon}$, $g_{y,\varepsilon}$ are, resp., a sub and super solution of our HJ equation.

Moreover, we have

$$f_{y,\varepsilon}(x,t) \le h(x) \le g_{y,\varepsilon}(x,t) \quad \forall (x,t),$$
$$|f_{y,\varepsilon}(y,0) - h(y)| = |g_{y,\varepsilon}(y,0) - h(y)| = \varepsilon.$$

Finally, define $g, f : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ by

$$g(x,t) = g_0(x,t) \wedge \inf_{y,\varepsilon} g_{y,\varepsilon}(x,t),$$
$$f(x,t) = f_0(x,t) \vee \sup_{y,\varepsilon} f_{y,\varepsilon}(x,t).$$

Then,

$$g \in \mathcal{S}^+, \quad f \in \mathcal{S}^-, \quad g \in \text{USC}, \quad f \in \text{LSC},$$

f, g are bounded on $\mathbb{R}^n \times [0,T]$ $\forall T < \infty$,

$$f(x,t) \le h(x) \le g(x,t) \ \forall (x,t), \quad f(\cdot,0) = h = g(\cdot,0).$$

Perron's method yields a solution u such that $f \le u \le g$, which implies that $u^*(\cdot,0) = u_*(\cdot,0) = h$ on $\mathbb{R}^n$. The comparison theorem shows that $u^* = u_* = u$ and $u \in C$. □

Homogenization of Hamilton-Jacobi equations I

(Lions-Papanicolaou-Varadhan)

Consider the HJ equation

(1) $u_t + |Du|^2 - f(x/\varepsilon) = 0$ in $\mathbb{R}^n \times (0, \infty)$, with $\varepsilon > 0$,

together with initial condition

(2) $$u(x, 0) = h(x) \quad \text{for } x \in \mathbb{R}^n.$$

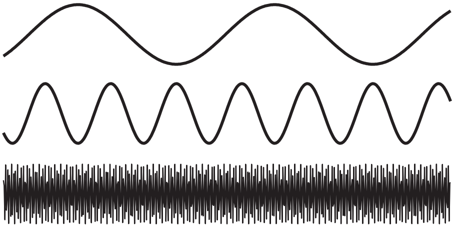
The Hamiltonian H is:

$$H(x, p) = |p|^2 - f(x),$$

where $f \in C(\mathbb{T}^n)$ is assumed, and our HJ equation reads

$$u_t + H(x/\varepsilon, D_x u) = 0.$$

The main question here is: If u_ε is a solution of the above HJ equation, what happens with u_ε as $\varepsilon \to 0+$.



$$f(x) = \sin 2\pi x,$$

$$f(5x) = \sin 5 \cdot 2\pi x,$$

$$f(100x) = \sin 100 \cdot 2\pi x$$

- Formal expansion:
  Suppose that we have an expansion

$$u_\varepsilon(x,t) = u_0(x,t) + \varepsilon u_1(x/\varepsilon, t) + \varepsilon^2 u_2(x/\varepsilon, t) + \cdots .$$

Insert this into the HJ equation, to get

$$0 = u_{0,t}(x,t) + \varepsilon u_{1,t}(x/\varepsilon, t) + O(\varepsilon^2)$$
$$+ H(x/\varepsilon, D_x u_0(x,t) + D_x u_1(x/\varepsilon, t) + O(\varepsilon)).$$

Because of a high oscillation when $\varepsilon \to 0+$, one may look at x/ε as if an independent variable y.

Then, in the limit $\varepsilon \to 0+$, the above asymptotic identity suggests that for some u_0, u_1,

$$u_\varepsilon(x,t) \to u_0(x,t) \quad \text{as } \varepsilon \to 0+,$$
$$u_{0,t} + H(y, D_x u_0(x,t) + D_y u_1(y,t)) = 0 \quad \text{for all } x, y, t.$$

If we have a solution u_0, u_1 of the above identity, we are in a good shape to conclude the above convergence. Thus, the question is how to find u_0, u_1 which satisfy

$$u_{0,t} + H(y, D_x u_0(x,t) + D_y u_1(y,t)) = 0 \quad \text{for all } x, y, t.$$

If we can write

$$\overline{H}(p) = H(y, p + D_y u_1(y,t)),$$

then the above equation can be stated as

$$u_{0,t} + \overline{H}(D_x u_0) = 0.$$

Here a big question is when we can write

$$\overline{H}(p) = H(y, p + D_y u_1(y,t)).$$

We consider this as a solvability problem: given $p \in \mathbb{R}^n$, find $(c, v) \in \mathbb{R} \times C(\mathbb{T}^n)$ such that

$$H(y, p + Dv(y)) = c \quad \text{in } \mathbb{T}^n. \tag{3}$$

(In fact, a crucial point is not the periodicity of v, but the sublinear growth of v.) Notice that the correspondence: $(c, v) \leftrightarrow (\overline{H}(p), u_1)$.

The problem of solving a solution (c, v) is called a *cell problem.* (Aslo, ergodic problem, additive eigenvalue problem, weak KAM problem)

Example 1

Consider the case $n = 1$ and $f(x) = -\cos(2\pi x)$. The case $p = 0$:

$$|v_x(x)|^2 = c - \cos(2\pi x).$$

For the solvability, RHS$\geq 0 \iff c \geq 1$.

When v is a solution of

$$H(y, p + Dv(y)) = c \quad \text{in } \mathbb{R}^n, \tag{3'}$$

then $w(y) = p \cdot y + v(y)$ is a solution of

$$H(y, Dw(y)) = c \quad \text{in } \mathbb{R}^n.$$

The sublinear growth of the solution v identifies the p term in the equation.

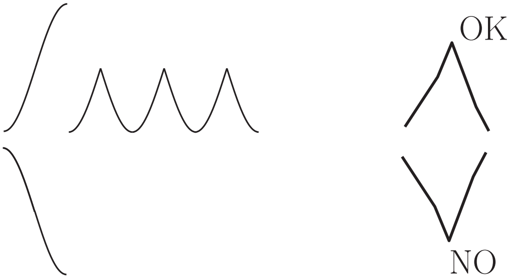
If $c > 1$, then RHS $\geq c - 1 > 0$, which implies NO periodic (viscosity) solution: any function is tested from below at its minimum point, if any, by constant functions.

Thus, $c = 1$. If $c = 1$, then

$$|v_x(x)| = \sqrt{1 - \cos(2\pi x)} = \sqrt{2}|\sin(\pi x)|.$$

Integrate, to get

$$v(x) = \text{constant} \pm \frac{\sqrt{2}}{\pi}\cos(\pi x) \quad \text{for } 0 \leq x \leq 1.$$



The periodic function

$$v(x) = -\frac{\sqrt{2}}{\pi}\cos(\pi x) \quad \text{for } -\frac{1}{2} \le x \le \frac{1}{2},$$

with period 1, is a viscosity solution for $p = 0$ and $c = 1$.

For general $p \in \mathbb{R}$, we have to solve

$$|p + v_x| = \sqrt{c - \cos(2\pi x)},$$

with $c \ge 1$, which reads

$$v_x = -p \pm \sqrt{c - \cos(2\pi x)}.$$

Let $c = 1$ and

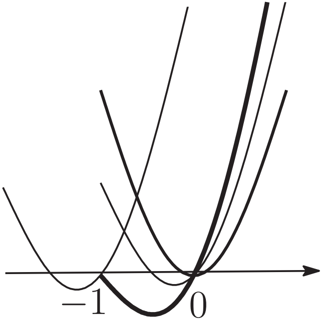
$$v(x) := -px + \frac{\sqrt{2}}{\pi}(1 - \cos(\pi x)).$$

Note that $v(0) = 0$ and solve

$$v(-1) = 0,$$

to find that

$$-p = \frac{2\sqrt{2}}{\pi}.$$



So, as far as $|p| \leq \frac{2\sqrt{2}}{\pi}$, the problem

$$|p + v_x|^2 = 1 - \cos(2\pi x)$$

has a periodic viscosity solultion. Moreover, if $|p| > \frac{2\sqrt{2}}{\pi}$,

$$|p + v_x|^2 = c - \cos(2\pi x)$$

has a periodic solution v only when $c > 1$.

We will know that if v is a (viscosity) solution of

$$|p + v_x| = \sqrt{2}|\sin \pi x|,$$

then v is Lipschitz continuous and satisfies the equation in the a.e. If it is periodic with period 1, then

$$\int_0^1 |p + v_x| dx \begin{cases} = \sqrt{2} \int_0^1 \sin \pi x \, dx = \frac{2\sqrt{2}}{\pi}, \\ \geq \left| \int_0^1 (p + v_x) dx \right| = |p|. \end{cases}$$

As a function of p, $c = \overline{H}(p)$ and, in the above case of f,

$$\overline{H}(p) \begin{cases} = 1 & \text{if } |p| \leq \frac{2\sqrt{2}}{\pi}, \\ > 1 & \text{otherwise }. \end{cases}$$

In homogenization theory, $\overline{H}$ is called the *effective Hamiltonian.*

Some properties of $\overline{H}$:

- $\overline{H}$ is a continuous function on $\mathbb{R}$.
- $\overline{H}$ is a convex function on $\mathbb{R}$.
- $\overline{H}$ is coercive on $\mathbb{R}$. That is, $\lim_{|p|\to\infty} \overline{H}(p) = \infty$.

Theorem 1

Assume that $h \in \mathbf{BUC}(\mathbb{R}^n)$. Then there exists a unique solution u_ε on $\mathbb{R}^n \times [0,\infty)$ of the Cauchy problem (1) – (2) such that $u_\varepsilon \in \mathbf{BUC}(\mathbb{R}^n \times [0,T])$ for every $T > 0$. Also, there exists a unique solution u on $\mathbb{R}^n \times [0,\infty)$ of

$$(4) \qquad \begin{cases} u_t + \overline{H}(D_x u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = h & \text{on } \mathbb{R}^n, \end{cases}$$

such that $u \in \mathbf{BUC}(\mathbb{R}^n \times [0,T))$ for every $T > 0$. Furthermore, as $\varepsilon \to 0+$,

$$u_\varepsilon(x,t) \to u(x,t) \quad \text{locally uniformly on } \mathbb{R}^n \times [0,\infty).$$

- The main steps in the proof of the convergence:
  - Show that $\{u_\varepsilon\}_{\varepsilon\in(0,1)}$ is unif-bounded and equi-continuous on $\mathbb{R}^n \times [0,T]\ \ \forall T>0$.
  - $v := \lim_{j\to\infty} u_{\varepsilon_j}$ for some $\varepsilon_j \to 0+$, where the convergence is locally uniform on $\mathbb{R}^n \times [0,\infty)$.
  - Show that $v = u$.
- Method of purterbed test functions (Evans).

To show the last step of the above list, we need to prove that v is a solution of $v_t + \overline{H}(D_x v) = 0$ in $\mathbb{R}^n \times (0,\infty)$.

Let $\psi \in C^1(\mathbb{R}^n \times (0,\infty))$ and assume that $v-\psi$ takes a strict maximum at $(\hat{x},\hat{t})$. Fix a compact neighborhood $K \subset \mathbb{R}^n \times (0,\infty)$ of $(\hat{x},\hat{t})$.

Classical argument: Let $(x_\varepsilon,t_\varepsilon)\in K$ be a maximum point of $u_\varepsilon - \psi$ on K. We have

$$\lim_{\varepsilon\to 0+}(x_\varepsilon,t_\varepsilon) = (\hat{x},\hat{t}).$$

For sufficiently small $\varepsilon>0$, we have $(x_\varepsilon,t_\varepsilon)\in \operatorname{int} K$ and

$$\psi_t(x_\varepsilon,t_\varepsilon) + H(x_\varepsilon/\varepsilon, D_x\psi(x_\varepsilon,t_\varepsilon)) \le 0.$$

This way, we can show that v is a subsolultion of $v_t + \min_y H(y, D_x v) = 0$ and a supersolution of $v_t + \max_y H(y, D_x v) = 0$. This is not enough to conclude that $v = u$.

The formal exapansion suggests that $v(x,t) + \varepsilon w(x/\varepsilon)$ should be a good approximation of u_ε.

Set $\hat{p} = D_x\psi(\hat{x}, \hat{t})$. Let $w \in C(\mathbb{T}^n)$ be a solution of

$$H(y, \hat{p} + D_y w(y)) = \overline{H}(\hat{p}) \quad \text{for } y \in \mathbb{T}^n.$$

Temporarily, we assume that $w \in C^1$ and consider the function

$$u_\varepsilon(x,t) - \psi(x,t) - \varepsilon w(x/\varepsilon).$$

Let $(x_\varepsilon, t_\varepsilon) \in K$ be a maximum point of this function. Then

$$\lim_{\varepsilon \to 0+} (x_\varepsilon, t_\varepsilon) = (\hat{x}, \hat{t}),$$

and if $\varepsilon > 0$ is small enough, $(x_\varepsilon, t_\varepsilon) \in \operatorname{int} K$ and

$$\psi_t(x_\varepsilon, t_\varepsilon) + H(x_\varepsilon/\varepsilon, D_x\psi(x_\varepsilon, t_\varepsilon) + Dw(x_\varepsilon/\varepsilon)) \le 0.$$

For some $\varepsilon_j \to 0+$, we may assume that for some $\hat{y} \in \mathbb{T}^n$,

$$\lim_{j\to\infty} x_{\varepsilon_j}/\varepsilon_j = \hat{y} \pmod{\mathbb{Z}^n}$$

Sending $\varepsilon_j \to 0+$ yields

$$\psi_t(\hat{x},\hat{t}) + H(\hat{y}, D_x\psi(\hat{x},\hat{t}) + Dw(\hat{y})) \leq 0,$$

while we had

$$H(y, D_x\psi(\hat{x},\hat{t}) + D_y w(y)) = \overline{H}(D_x\phi(\hat{x},\hat{t})) \quad \text{for } y \in \mathbb{T}^n.$$

Thus,

$$\psi_t(\hat{x},\hat{t}) + \overline{H}(D_x\psi(\hat{x},\hat{t})) \leq 0,$$

proving that v is a subsolution of $v_t + \overline{H} = 0$.

In general, we have only the Lipschitz regularity of w and we need to use the doubling variable argument.

Similarly, we conclude that v is a supersolution of $v_t + \overline{H} = 0$. Thus, $v = u$. □

Homogenization of Hamilton-Jacobi equations II

Consider the equation

(1) $$u_t + H(x, x/\varepsilon, D_x u) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

where

- $H \in C(\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^n)$.
- $H(x,y,p)$ is bounded and uniformly continuous on $\mathbb{R}^n \times \mathbb{T}^n \times B_R$ for every $R > 0$.
- H is coercive, i.e.,

$$\lim_{|p|\to\infty} H(x,y,p) = \infty \quad \text{uniformly in } (x,y).$$

The cell problem is: given $(x,p) \in \mathbb{R}^{2n}$, we solve $(c,w) \in \mathbb{R} \times C(\mathbb{T}^n)$ such that

(2) $$H(x, y, p + D_y w(y)) = c \quad \text{for } y \in \mathbb{T}^n.$$

Theorem 1

Under the above hypotheses on H, there exists a solution (c,w) for each $(x,p) \in \mathbb{R}^{2n}$. The constant c is unique and defines a function $\overline{H}(x,p)$. That is, $\overline{H}(x,p) = c$.

A standard proof goes this way: consider the discounted problem

$$(3) \qquad \lambda w + H(x, y, p + D_y w) = 0 \quad \text{in } \mathbb{T}^n, \quad \text{with } \lambda > 0,$$

and send $\lambda \to 0+$.

1) Choose $C > 0$ so large that $|H(x, y, p)| \leq C$ and observe that $\lambda^{-1}C$ (resp. $-\lambda^{-1}C$) is a super (resp. sub) solution of (3). Perron's method yields a solution w_λ of (3).

2) By comparison, $|w_\lambda| \leq \lambda^{-1}C$ (and hence, $\lambda|w_\lambda| \leq C$) on $\mathbb{T}^n$.

3) By the coercivity, choose $L > 0$ so that if $|q| > L$, then $H(x, y, p + q) > C$ for all (x, y). Since $H(x, y, p + D_y w_\lambda) \leq -\lambda w_\lambda \leq C$, we have $|Dw_\lambda| \leq L$. This implies that w_λ is Lipschitz continuous with Lipschitz bound L.

4) Fix $y_0 \in \mathbb{T}^n$. the family $\{w_\lambda - w_\lambda(y_0)\}_{\lambda>0}$ is unif-bounded and equi-Lipschitz. We may choose $\lambda_j \to 0+$ so that, as $\lambda_j \to 0+$,

$$\lambda_j w_{\lambda_j}(y_0) \to -c \ (\exists c \in \mathbb{R}),$$

$$w_{\lambda_j} - w_{\lambda_j}(y_0) \to w \ (\exists w \in \mathbf{Lip}(\mathbb{T}^n)).$$

To repeat, as $\lambda_j \to 0+$,

$$\lambda_j w_{\lambda_j}(y_0) \to -c \ (\exists c \in \mathbb{R}),$$
$$\overline{w}_j := w_{\lambda_j} - w_{\lambda_j}(y_0) \to w \ (\exists w \in \mathbf{Lip}(\mathbb{T}^n)).$$

Then:

$$\lambda_j \overline{w}_j + H(x, y, p + D_y \overline{w}_j) = -\lambda_j w_{\lambda_j}(y_0).$$

In the limit $k \to \infty$,

$$H(x, y, p + D_y w) = c \quad \text{for } y \in \mathbb{T}^n.$$

We have used the following regularity results.

Theorem 2

Let $\Omega \subset \mathbb{R}^n$ be open and convex. Let $F \in C(\Omega \times \mathbb{R}^n)$ satisfy the condition that $\exists R > 0$ such that

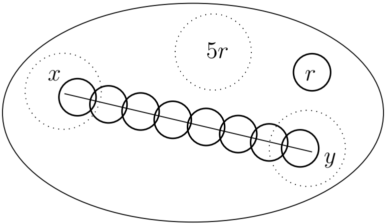
$$F(x,p) > 0 \quad \text{if } |p| > R.$$

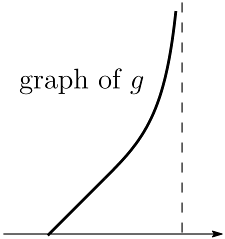
If $v \in \mathrm{USC}(\Omega)$ is a subsolution of $F(x, Du) = 0$ in Ω, then $|v(x) - v(y)| \leq R|x - y|$ for all $x, y \in \Omega$.

PROOF. Fix $z \in \Omega$ and $r > 0$ so that $B_{5r}(z) \subset \Omega$. We claim that

$$|v(x) - v(y)| \leq R|x - y| \;\; \forall x, y \in B_r(z).$$

This is enough to conclude the proof.



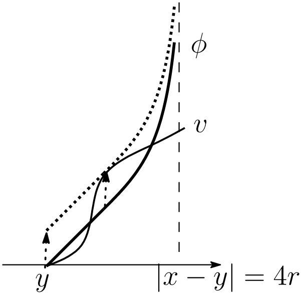


Let $g : [0, 4r) \to [0, \infty)$ be a smooth function such that $g(t) = t$ for $0 \le t \le 2r$, $g'(t) \ge 1$ for all $0 \le t < 4r$, and $\lim_{t \to 4r-} g(t) = \infty$.

For each fixed $y \in B_r(z)$ and $\varepsilon > 0$, consider the function $\phi : x \mapsto v(y) + (R + \varepsilon) g(|x - y|)$ on $B_{4r}(y) \subset B_{5r}(z)$.

If $v(x) \le \phi(x)$ on $B_{4r}(y)$, then $v(x) - v(y) \le (R + \varepsilon)|x - y|$ for all $x \in B_r(z) \subset B_{2r}(y)$.

Otherwise,



The slope of $\phi \geq R + \varepsilon$,

$F(x,p) > 0$ if $|p| > R$.

Hence,

$$F(x, D\phi(x)) > 0.$$

Theorem 3

Let $F \in C(\mathbb{R}^n \times \mathbb{R}^n)$ and $a < b$. Assume that $F \in \mathrm{BUC}(\mathbb{R}^n \times B_R)$ for any $R > 0$. Let $v, w \in \mathrm{B}(\mathbb{R}^n)$ be a subsolution of $F(x, Du) = a$ in $\mathbb{R}^n$ and a supersolution of $F(x, Du) = b$ in $\mathbb{R}^n$, respectively. Assume that either v or w is Lipschitz continuous in $\mathbb{R}^n$. Then, $v \leq w$ in $\mathbb{R}^n$.

PROOF. We consider only the case when $v \in \mathbf{Lip}$. Choose $\varepsilon > 0$ be such that $a + \varepsilon < b$. Choose $\delta > 0$ small enough so that $v_\delta(x) := v(x) - \delta\langle x\rangle$ is a subsolution of $F(x, Du) = a + \varepsilon$ in $\mathbb{R}^n$. This is possible since $v \in \mathbf{Lip}$ and $F \in \mathrm{UC}(\mathbb{R}^n \times B_R)$ for any $R > 0$.

We only need to prove that $v_\delta \le w_*$. By contradiction, we suppose that $\sup(v_\delta - w_*) > 0$. We fix $r > 0$ large enough so that

$$v_\delta - w_* < 0 \quad \text{on } \mathbb{R}^n \setminus B_r.$$

Consider the function

$$\Phi_k(x, y) = v_\delta(x) - w_*(y) - k|x - y|^2$$

on $\overline{B}_r \times \overline{B}_r$. Let (x_k, y_k) be a maximum point of Φ_k. Let $L > 0$ be a Lipschitz bound of the function v_δ and note that

$$\Phi_k(x_k, y_k) \ge \Phi_k(y_k, y_k),$$

which reads

$$k|x_k - y_k|^2 \le v_\delta(x_k) - v_\delta(y_k) \le L|x_k - y_k|.$$

This yields

$$k|x_k - y_k| \le L.$$

With this estimate in hand, we go as in the proof of the previous comparison theorems, to find for sufficient large k,

$$F(x_k, 2k(x_k - y_k)) \leq a + \varepsilon \quad \text{and} \quad F(y_k, 2k(x_k - y_k)) \geq b,$$

and, along a subsequence,

$$\lim(x_k, y_k) = (x_0, x_0) \quad \text{for some } x_0 \in B_r.$$

We may assume that, after taking a further subsequence,

$$\lim 2k(x_k - y_k) = p_0 \quad \text{for some } p_0 \in \mathbb{R}^n.$$

Consequently,

$$F(x_0, p_0) \leq a + \varepsilon < b \leq F(x_0, p_0).$$

This is a contradiction. □

Recall Theorem 1:

> **Theorem 1**
> Under the hypotheses above on H, there exists a solution (c, w), for each $(x, p) \in \mathbb{R}^{2n}$, of
>
> (2) $$H(x, y, p + D_y w(y)) = c \quad \text{for } y \in \mathbb{T}^n.$$
>
> The constant c is unique and defines a function $\overline{H}(x, p)$. That is, $\overline{H}(x, p) = c$.

PROOF OF THE UNIQUENESS. Let (c, w) and (d, v) be solutions of (2). If $c < d$, then, by Theorem 3 (the comparison theorem),

$$w + C \leq v \quad \text{in } \mathbb{T}^n,$$

where C is an arbitrary constant, which is a contradiction. Hence, we have $c \geq d$. By symmetry, we have $d \geq c$. □

Theorem 5

Under the above hypotheses on H, the effective Hamiltonian $\overline{H}$ has the properties:

- $\overline{H} \in \mathrm{BUC}(\mathbb{R}^n \times B_R)$ for every $R > 0$.
- $\overline{H}$ is coercive, i.e.,

$$\lim_{|p|\to\infty} \overline{H}(x,p) = \infty \quad \text{uniformly in } x \in \mathbb{R}^n.$$

1) We have

$$\overline{H}(x,p) = \min\{c \in \mathbb{R} : \exists z \in \mathrm{Lip}(\mathbb{T}^n) \text{ s.t.} \\ H(x,y,p+Dz) \le c \ \text{ in } \mathbb{T}^n\}.$$

Let $w \in \mathrm{Lip}(\mathbb{T}^n)$ be a solution of $H(x,y,p+Dw(y)) = \overline{H}(x,p)$ in $\mathbb{T}^n$. If $c \ge \overline{H}(x,p)$, then $H(x,y,p+Dw(y)) \le c$ (subsolution) in $\mathbb{T}^n$. If $z \in \mathrm{Lip}(\mathbb{T}^n)$ be a subsolution of $H(x,y,p+Dz(y)) \le c$ in $\mathbb{T}^n$, with $c < \overline{H}(x,p)$, then, by the comparison theorem, $z + C \le w$ in $\mathbb{T}^n$ for all $C \in \mathbb{R}$, which is impossible.

Thus, the formula above is valid.

2) Set

$$m_0 := \inf H > -\infty.$$

Then

$$\overline{H}(x,p) \geq m_0 \quad \text{for all } (x,p) \in \mathbb{R}^{2n}.$$

$(H(x,y,p+Dw(y)) = c$, with $c < m_0$, cannot have a solution w.)

Fix $R > 0$. Set

$$M_R = \sup_{x,y,|p| \leq R} H(x,y,p).$$

Note that $z(y) = 0$ satisfies

$$H(x,y,p+Dz(y)) \leq M_R, \quad \text{if } |p| \leq R$$

and that

$$\overline{H}(x,p) \leq M_R \quad \text{for all } x \in \mathbb{R}^n,\ p \in B_R.$$

Thus,

$$\overline{H} \quad \text{is bounded on } \mathbb{R}^n \times B_R, \ \ \forall R > 0.$$

3) Fix $R > 0$ and let $M_R > 0$ be as above. There is $L > 0$ such that

$$H(x, y, r) - M_R > 0 \quad \text{if } |r| > L.$$

Fix any $(x, p) \in \mathbb{R}^n \times B_R$. Let w be a solution of

$$H(x, y, p + Dw(y)) = \overline{H}(x, p) \quad \text{in } \mathbb{T}^n.$$

Since $H(x, y, p + Dw(y)) \leq M_R$ (subsolution), the function w is in $\mathrm{Lip}(\mathbb{T}^n)$, with Lipschitz constant $\leq L + |p| \leq L + R$.

4) Set $K = 2R + L + 1$ and note that $H \in \mathrm{UC}(\mathbb{R}^{2n} \times B_K)$. $\forall \varepsilon > 0$, $\exists \delta \in (0, 1)$ such that for all $(x', p') \in B_\delta(x, p)$,

$$H(x', y, p' + Dw(y)) \leq H(x, y, p + Dw(y)) + \varepsilon,$$

which assures

$$H(x', y, p' + Dw(y)) \leq \overline{H}(x, p) + \varepsilon \quad \text{for all } (x', p') \in B_\delta(x, p),$$

and

$$\overline{H}(x', p') \leq \overline{H}(x, p) + \varepsilon \quad \text{for all } (x', p') \in B_\delta(x, p).$$

Notice that δ can be chosen uniformly in (x, p, w) in the above. Thus, $\overline{H}$ is uniformly continuous on $\mathbb{R}^n \times B_R$, $\forall R > 0$.

5) Let w be a solution of

$$H(x, y, p + Dw(y)) = \overline{H}(p) \quad \text{in } \mathbb{T}^n.$$

w takes a maximum at some $y_0 \in \mathbb{T}^n$, and then

$$H(x, y_0, p) \leq \overline{H}(x, p).$$

Since H is coercive, this shows that $\overline{H}$ is coercive.

Theorem 6

Assume in addition that $p \mapsto H(x, y, p)$ is convex. Then $p \mapsto \overline{H}(x, p)$ is convex.

PROOF. To check this, let v and w be solutions of

$$H(x, y, p + Dv(y)) = \overline{H}(x, p) \quad \text{in } \mathbb{T}^n,$$

$$H(x, y, q + Dw(y)) = \overline{H}(x, q) \quad \text{in } \mathbb{T}^n.$$

Let $\theta \in (0,1)$. Assuming that $v, w \in C^1$, we observe that

$$\begin{aligned} & H\big(x, y, \theta(p + Dv(y)) + (1-\theta)(q + Dw(y))\big) \\ & \le \theta H(x, y, p + Dv(y)) + (1-\theta) H(x, y, q + Dw(y)) \\ & \le \theta \overline{H}(x, p) + (1-\theta)\overline{H}(x, q). \end{aligned}$$

In general, we deduce (a.e. subsolution or the doubling variable argument) that $\theta v + (1-\theta) w$ is a subsolution of

$$H(x, y, \theta p + (1-\theta) q + Du(y)) \le \theta \overline{H}(p) + (1-\theta)\overline{H}(q) \quad \text{in } \mathbb{T}^n,$$

which proves that

$$\overline{H}(x, \theta p + (1-\theta) q) \le \theta \overline{H}(x, p) + (1-\theta)\overline{H}(x, q). \qquad \square$$

Theorem 7

Assume

- $H \in \mathrm{BC}(\mathbb{R}^n \times B_R)$ for every $R > 0$;
- H is coercive, i.e.,

$$\lim_{|p|\to\infty} H(x,p) = \infty \quad \text{uniformly in } x;$$

- $h \in \mathrm{Lip} \cap \mathrm{B}(\mathbb{R}^n)$.

Then there is a solution $u \in \mathrm{Lip}(\mathbb{R}^n \times [0,\infty))$ of

$$\text{(4)} \qquad \begin{cases} u_t + H(x, D_x u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot, 0) = h & \text{on } \mathbb{R}^n. \end{cases}$$

REMARK. The Lipschitz constant of u is bounded by a constant which depends only on the "structural bounds" for H and the Lipschitz constant of h.

$$\sup_{\mathbb{R}^n \times B_R} |H|, \qquad \inf_{\mathbb{R}^n \times (\mathbb{R}^n \setminus B_R)} H, \quad \text{with } R > 0.$$

PROOF. Let $C_h > 0$ be a Lipschitz bound for h. Set

$$C = C_{h,H} := \sup_{|p| \leq C_h} |H(x,p)|.$$

Note that $f(x,t) = h(x) - Ct$ and $g(x,t) = h(x) + Ct$ are in $\mathcal{S}^-$ and $\mathcal{S}^+$, respectively.

Moreover, $f(x,t) \leq h(x) \leq g(x,t)$ and $f(x,0) = h(x) = g(x,0)$ for all (x,t). Perron's method yields a solution u such that $f \leq u_* \leq u \leq u^* \leq g$ on $\mathbb{R}^n \times (0,\infty)$. These inequalities imply

$$u(x,0) := \lim_{t \to 0+} u(x,t) = h(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Note:

$$u(x,t) = \sup\{v(x,t) : v \in \mathcal{S}^-, \, v \leq g \text{ on } \mathbb{R}^n \times (0,\infty)\},$$

$u \in \mathrm{USC}(\mathbb{R}^n \times [0,\infty))$, and

$$u(x,t) = \max\{v(x,t) : v \in \mathcal{S}^-, \, v \leq g \text{ on } \mathbb{R}^n \times (0,\infty)\}.$$

Fix any $\delta > 0$. Note

$$(x,t) \mapsto u(x,\delta+t) \in \mathcal{S}^-, \quad \le g(x,t+\delta) = g(x,t) + C\delta.$$

Hence,

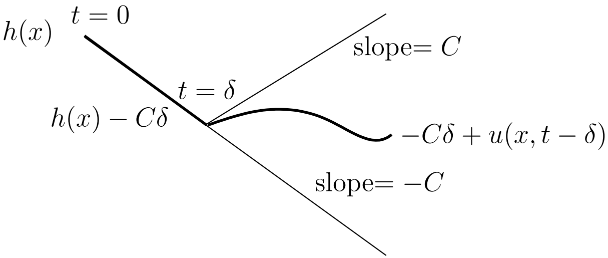
$$u(x,t) \ge u(x,t+\delta) - C\delta$$

and $u(x,\delta+t) \le u(x,t) + C\delta$.

Set

$$u^\delta(x,t) = \begin{cases} f(x,t) & \text{if } t \in [0,\delta], \\ -C\delta + u(x,t-\delta) & \text{if } t > \delta. \end{cases}$$

Observe: $u^\delta \in \mathcal{S}^-$ and $u^\delta \le g$.



Hence,

$$u(x,\delta+t) \geq u^\delta(x,\delta+t) = u(x,t) - C\delta,$$

and $t \mapsto u(x,t)$ is Lipschitz continuous with Lipschitz bound C. This implies that $|u_t| \leq C$, $u_t \geq |u_t| - 2|u_t| \geq |u_t| - 2C$, and

$$|u_t| + H(x, D_x u) - 2C \leq 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Since $F(x,t,p,q) := |q| + H(x,p) - 2C$ is coercive, u is Lipschitz continuous on $\mathbb{R}^n \times (0,\infty)$. □

Theorem 8

Let $0 < T < \infty$. Assume that $H \in \mathrm{BUC}(\mathbb{R}^n \times (0,T) \times B_R)$ for every $R > 0$. Consider

$$\text{(5)} \qquad u_t + H(x,t,D_x u) = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Let v, w be a sub and super-solution of (5). Assume that v, w are bounded, $v, -w \in \mathrm{USC}$, and $v(x,0) \leq w(x,0)$ for all $x \in \mathbb{R}^n$. Assume moreover either v or w is *Lipschitz continuous*. Then, $v \leq w$ on $\mathbb{R}^n \times (0,T)$.

REMARK. The Lipshictz regularity assumption above can be replaced by the existence of a Lipschitz continuous solution u such that $v(x,0) \le u(x,0) \le w(x,0)$.

REMARK. In the doubling variable argument, we consider the function

$$\Phi_k(x,t,y,s) = v(x,t) - w(y,s) - k[|x-y|^2 + (t-s)^2]$$

and its maximum point (x_k, t_k, y_k, s_k). If $v \in \mathbf{Lip}$, then

$$\Phi_k(x_k, t_k, y_k, s_k) \ge \Phi_k(y_k, s_k, y_k, s_k)$$

yields

$$\begin{aligned} k[|x_k - y_k|^2 + (t_k - s_k)^2] &\le v(x_k, t_k) - v(y_k, s_k) \\ &\le C(|x_k - y_k| + |t_k - s_k|), \end{aligned}$$

and

$$k[|x_k - y_k| + |t_k - s_k|] \le C'.$$

This is the *boundedness of the gradient* of our test functions, which allows us to take the limit as $k \to \infty$:

$$2(t_k - s_k) + H(x_k, t_k, 2k(x_k - y_k)) \le -\eta,$$
$$2(t_k - s_k) + H(y_k, s_k, 2k(x_k - y_k)) \ge 0.$$

Theorem 9

Assume that $h \in \mathbf{BUC}(\mathbb{R}^n)$. Then there exists a unique solution u_ε on $\mathbb{R}^n \times [0,\infty)$ of the Cauchy problem

$$\begin{cases} u_t + H(x, x/\varepsilon, D_x u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot, 0) = h \end{cases}$$

such that $u_\varepsilon \in \mathbf{BUC}(\mathbb{R}^n \times [0,T])$ for every $T > 0$. Also, there exists a unique solution u on $\mathbb{R}^n \times [0,\infty)$ of

$$\begin{cases} u_t + \overline{H}(x, D_x u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot, 0) = h & \text{on } \mathbb{R}^n, \end{cases}$$

such that $u \in \mathbf{BUC}(\mathbb{R}^n \times [0,T))$ for every $T > 0$. Furthermore, as $\varepsilon \to 0+$,

$$u_\varepsilon(x,t) \to u(x,t) \quad \text{locally uniformly on } \mathbb{R}^n \times [0,\infty).$$

Long-time behavior of solutions I

Example 1

Let $\lambda > 0$. Consider the HJ equation

$$(1) \qquad u_t + \lambda u + |D_x u|^2 - f(x) = 0 \quad \text{in } \mathbb{T}^n \times (0, \infty).$$

The Hamiltonian H is:

$$H(x, p, u) = \lambda u + |p|^2 - f(x),$$

where $f \in C(\mathbb{T}^n)$. If there is a solution $u_0 \in C(\mathbb{T}^n)$ of

$$(2) \qquad H(x, D_x u_0, u_0) = 0 \quad \text{in } \mathbb{T}^n,$$

then $u(x,t) = u_0(x)$ is a solution of (1).

Let $v \in C(\mathbb{T}^n \times [0, \infty))$ be another solution of (1). By comparison, we have

$$(3) \quad \|(u - v)(\cdot, t)\|_\infty \le \|(u - v)(\cdot, 0)\|_\infty e^{-\lambda t} \quad \text{for all } t > 0.$$

Indeed,

$$w(x,t) := v(x,t) + \|u(\cdot,0) - v(\cdot,0)\|_\infty e^{-\lambda t}$$

satisfies

$$w_t + \lambda w + |D_x w|^2 - f(x) = v_t + \lambda v + |D_v|^2 - f(x) = 0,$$
$$u(\cdot,0) \le w(\cdot,0),$$

and, by the comparison theorem, $u(x,t) \le w(x,t)$. Similarly, we have $v(x,t) \le u(x,t) + \|u(\cdot,0) - v(\cdot,0)\|_\infty e^{-\lambda t}$.

Theorem 1
Problem (2) has a unique solution $u_0 \in \mathbf{Lip}(\mathbb{T}^n)$. For any $h \in C(\mathbb{T}^n)$, the Cauchy problem for (1) with initial condition $u(\cdot, 0) = h$ has a unique solution $u \in C(\mathbb{T}^n \times [0, \infty))$. Moreover, as $t \to \infty$,

$$v(x, t) \to u_0(x) \quad \text{uniformly and exponentially on } \mathbb{T}^n.$$

- The conclusion of the above theorem holds true if H is replaced by a general continuous Hamiltonian H:
  - $u \mapsto H(x, p, u) - \lambda u$ is nondecreasing for some $\lambda > 0$.
  - For some $C > 0$ and for all $x, y \in \mathbb{T}^n, p \in \mathbb{R}^n, u \in \mathbb{R}$,

$$|H(x, p, u) - H(y, p, u)| \le C|x - y|(|p| + 1).$$

Example 2

(Barles-Souganidis) Consider the HJ equation

$$u_t + |u_x + 2\pi| - 2\pi = 0 \quad \text{in } \mathbb{T}^1 \times [0, \infty).$$

$n = 1$. The function $u(x,t) = \sin 2\pi(x - t)$ is a classical solution. The point is

$$|u_x + 2\pi| = |2\pi \cos 2\pi(x - t) + 2\pi| = 2\pi \cos 2\pi(x - t) + 2\pi.$$

$t \mapsto \sin 2\pi(x - t)$ is periodic with minimal period 1.

In this example, the Hamiltonian is given by

$$H(x,p) = H(p) = |p + 2\pi| - 2\pi.$$

Note that $p \mapsto H(x,p)$ is convex and coercive.

$$\lim_{|p| \to \infty} H(p) = \infty.$$

Example 3

(Namah-Roquejoffre) Consider

$$u_t + |D_x u|^2 - f(x) = 0 \quad \text{in } \mathbb{T}^n \times [0, \infty). \tag{4}$$

Assume that for some $x_0 \in \mathbb{T}^n$ and all $x \in \mathbb{T}^n$,

$$f(x) \geq f(x_0) = 0. \tag{5}$$

Set

$$v_0(x) = \sup\{v(x) : v \in \mathcal{S}^-, v(x_0) = 0\},$$

where $\mathcal{S}^-$ denotes the set of all subsolutions of $H(x, Du) := |Du|^2 - f(x) = 0$ in $\mathbb{T}^n$.

It follows that $0 \leq v_0(x) \leq o(|x - x_0|)$.
($|Dv_0(x)| \leq \sqrt{f(x)}$.) Moreover, the function v_0 is a solution of $H(x, Du) = 0$ in $\mathbb{T}^n$.

Let $u \in C(\mathbb{T}^n \times [0, \infty))$ be a solution of (4). Note that $H(x_0, p) \geq 0$ for all $p \in \mathbb{R}^n$. Hence, $u_t(x_0, t) \leq 0$ for all $t \in (0, \infty)$ and, therefore, $t \mapsto u(x_0, t)$ is nonincreasing. This monotonicity property is valid for any zero point $\in \mathbb{T}^n$ of f. That is, if we set $Z = f^{-1}(0) = \{x : f(x) = 0\}$, then $t \mapsto u(x, t)$ is nonincreasing for all $x \in Z$.

Select $C > 0$ so that $v_0 - C \leq u(\cdot, 0) \leq v_0 + C$ on $\mathbb{T}^n$. By the comparison theorem, $v_0 - C \leq u(x, t) \leq v_0(x) + C$ for all $(x, t) \in \mathbb{T}^n \times [0, \infty)$.

By Theorem 9 in the last lecture, u is uniformly continuous on $\mathbb{T}^n \times [0, \infty)$. Thus, the family $\{u(\cdot, t) : t \geq 0\}$ is unif-bounded and equi-continuous on $\mathbb{T}^n$.

The monotonicity on Z of u and the unif-boundedness and equi-continuity properties, together with AA theorem, assure that for some function $u_0 \in C(\mathbb{T}^n)$, as $t \to \infty$,

- $u(x,t) \to u_0(x)$ uniformly and monotonically for $x \in Z$,
- $u(x,t) \to u_0(x)$ uniformly for $x \in \mathbb{T}^n$ along a sequence of t.

At this point, it is not clear if u_0 is a solution of $H(x, Du) = 0$ in $\mathbb{T}^n$. Define

$$w^{\pm}(x,t) = \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \{u(x,t+s) : s \geq 0\} \text{ for all } (x,t) \in \mathbb{T}^n \times [0,\infty).$$

The function w^+ (resp., w^-) is a subsolution (resp., a supersolution) of $w_t + H(x, D_x w) = 0$ in $\mathbb{T}^n \times (0,\infty)$, they are bounded, uniformly continuous on $\mathbb{T}^n \times [0,\infty)$, $t \mapsto w^+(x,t)$ (resp., $t \mapsto w^-(x,t)$) is nonincreasing (resp., nondecreasing) for all $x \in M$, and $w^+(x,t) = u(x,t)$ (resp., $w^-(x,t) = u_0(x)$) on $Z \times [0,\infty)$. Thus, as $t \to \infty$, for some $w_0^{\pm} \in C(\mathbb{T}^n)$,

$$w^{\pm}(x,t) \to w_0^{\pm}(x) \text{ uniformly and monotonically on } \mathbb{T}^n.$$

It follows that $w_0^\pm = u_0$ on Z and that w_0^+ (resp., w_0^-) is a subsolultion (resp., supersolution) of $H(x, Du) = 0$ in $\mathbb{T}^n$. Also, by the definition of $w_0^\pm$, we have $w_0^+ \geq w_0^-$ on $\mathbb{T}^n$. Once we have shown that $w_0^+ = w_0^-$ on $\mathbb{T}^n$, we see easily that $u_0 = w_0^\pm$ on $\mathbb{T}^n$, which implies that as $t \to \infty$,

$$u(x, t) \to u_0 \quad \text{uniformly on } \mathbb{T}^n.$$

We claim that $w_0^+ = w_0^-$ on $\mathbb{T}^n$. It is enough to prove that

$$w_0^+ \leq w_0^- \quad \text{on } \mathbb{T}^n \setminus Z.$$

By adding a large constant to $w_0^\pm$, we may assume that both $w_0^\pm$ are positive functions. Let $\theta \in (0, 1)$ and set $w_\theta = \theta w_0^+$. Note that

$$H(x, Dw_\theta) = \theta^2 |Dw_0^+|^2 - f(x) = \theta^2 H(x, Dw_0^+) - (1 - \theta^2) f(x),$$

and that

$$w_\theta(x) < w_0^-(x) \quad \text{on } Z.$$

Let Z_δ be the closed δ-neighborhood of Z ($\delta > 0$) such that

$$w_\theta(x) < w_0^-(x) \quad \text{for all } x \in Z_\delta.$$

Set $U_\delta := \mathbb{T}^n \setminus Z_\delta$. There exists $\eta > 0$ such that

$$f(x) \geq \eta \quad \text{for all } x \in U_\delta.$$

Note that

$$(1-\theta^2)f(x) > (1-\theta^2)\eta \quad \text{on } U_\delta,$$

and hence, w_θ is a subsolution of

$$H(x, Du) \leq -(1-\theta^2)\eta \ \text{ in } U_\delta.$$

By the comparison principle, we have

$$w_\theta \leq w_0^- \quad \text{on } U_\delta \text{ (and } \text{ on } \mathbb{T}^n).$$

Theorem 2

Let u be a solution of (4). Assume (5) ($f \geq f(x_0) = 0$). Then, for some $u_0 \in C(\mathbb{T}^n)$, as $t \to \infty$,

$$u(x,t) \to u_0(x) \quad \text{uniformly on } \mathbb{T}^n.$$

One can replace $H(x,p) = |p|^2 - f(x)$ by a general continuous $H(x,p)$ which satisfies:

- $p \mapsto H(x,p)$ is convex for every $x \in \mathbb{T}^n$.
- $p \mapsto H(x,p)$ is coercive for every $x \in \mathbb{T}^n$.
- $\min_{p\in\mathbb{R}^n} H(x,p) = H(x,0) \ \ \forall x \in \mathbb{T}^n$, $\max_{x\in\mathbb{T}^n} H(x,0) = 0$.

Some convenient technical theorems are as follows.

Theorem 3

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $F = F(x,p,u)$ is a continuous convex (in p) Hamiltonian on $\Omega \times \mathbb{R}^n \times \mathbb{R}$. Let $u \in \mathrm{Lip}(\Omega)$. Then

$$u \in \mathcal{S}^-(F) \iff u \in \mathcal{S}^-_{\mathrm{ae}}(F).$$

• $\mathcal{S}^-$ =the set of all viscosity subsolutions, $\mathcal{S}^-_{\mathrm{ae}}$ =the set of all a.e. subsolutions ($F(x,Du(x),u(x)) \le 0$ a.e.).

PROOF. Local property! We may assume that Ω is bounded (and convex).

1) Assume that $u \in \mathcal{S}^-(F)$. Since $u \in \mathbf{Lip}$ and is differentiable a.e. in Ω. Fix any differentiability point x of u, and choose $\phi \in C^1(\Omega)$ such that ϕ tests u from above at x. Note that $D\phi(x) = Du(x)$. Then, since $u \in \mathcal{S}^-$,

$$0 \geq F(x, D\phi(x), u(x)) = F(x, Du(x), u(x)).$$

2) Assume now that $u \in \mathcal{S}^-_{\mathbf{ae}}(F)$. Since $u \in \mathbf{Lip}$, it is differentiable a.e. in Ω and the derivative Du is identified with the distributional derivative of u. Choose a constant $M > 0$ so that $|u(x)| + |Du(x)| \leq M$ a.e. We may assume that F is uniformly continuous on $\Omega \times B_{M+1} \times [-M-1, M+1]$ (if needed, replace Ω by a smaller one). For each $0 < \varepsilon \ll 1$, choose $\delta(\varepsilon) > 0$ so that

$$F(x, Du(y), u(x)) \leq F(y, Du(y), u(y)) + \varepsilon \quad \text{a.e. } y \in \Omega, \forall x \in B_{\delta(\varepsilon)}(y).$$

Mollifying the above with a standard kernel (and using the convexity), to get

$$F(x, u_\varepsilon(x), u(x)) \leq \varepsilon \quad \text{in } \Omega,$$

where u_ε is the mollified function of u. Now, u_ε is a classical (hence, viscosity) subsolution of $F(x, Du_\varepsilon(x), u(x)) \leq \varepsilon$. In the limit as $\varepsilon \to 0$, we see that $u \in \mathcal{S}^-(F)$. □

We write $\mathcal{S}^-_{\mathbf{BJ}}(F)$ for the set of all functions $u \in \mathbf{Lip}(\Omega)$ such that if $\phi \in C^1(\Omega)$ touches *from below* at $x \in \Omega$, then $F(x, D\phi(x), u(x)) \leq 0$. (Barron-Jensen)

Theorem 4

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $F = F(x, p, u)$ is a continuous convex (in p) Hamiltonian on $\Omega \times \mathbb{R}^n \times \mathbb{R}$. Let $u \in \mathbf{Lip}(\Omega)$. Then

$$u \in \mathcal{S}^-(F) \iff u \in \mathcal{S}^-_{\mathbf{BJ}}(F).$$

PROOF. We need to show that

$$\boldsymbol{u} \in \boldsymbol{\mathcal{S}}^{-}_{\mathbf{ae}}(\boldsymbol{F}) \iff \boldsymbol{u} \in \boldsymbol{\mathcal{S}}^{-}_{\mathbf{BJ}}(\boldsymbol{F}).$$

The previous proof applies to show this claim. □

A consequence of the above is:

Theorem 5

Let $\boldsymbol{\Omega} \subset \mathbb{R}^{\boldsymbol{n}}$ be an open set. Let $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u})$ be a continuous convex (in $\boldsymbol{p}$) Hamiltonian on $\boldsymbol{\Omega} \times \mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}$. Let $\boldsymbol{\mathcal{F}} \neq \emptyset$ be a locally unif-bounded, equi-Lipschitz continuous collection of subsolutions of $\boldsymbol{F} = \mathbf{0}$ in $\boldsymbol{\Omega}$. Then the function

$$\boldsymbol{u}(\boldsymbol{x}) := \inf\{\boldsymbol{v}(\boldsymbol{x}) : \boldsymbol{v} \in \boldsymbol{\mathcal{F}}\}$$

is in $\boldsymbol{\mathcal{S}}^{-}(\boldsymbol{F})$.

PROOF. The proof is parallel to that of the assertion that the pointwise $\sup$ of a family of subsolutions is a subsolution: replace "touching from above" and "$\sup$" by "touching from below" and "$\inf$", respectively, which is also parallel to that of the theorem saying that the pointwise $\inf$ of a family of supersolutions is a supersolution: replace $\geq$ by $\leq$. □

REMARK. Roughly speaking, if u is differentiable at y and it is a subsolution of $F = 0$, then

$$F(y, Du(y), u(y)) \leq 0.$$

Indeed, we may choose a continuous function ω on $[0, 1]$ such that $\omega(0) = 0$, $\omega(t) \geq 0$, and

$$u(x) - u(y) \leq p \cdot (x - y) + \omega(|x - y|)|x - y| \quad \text{if } x \in B_1(y),$$

where $p = Du(y)$. We may assume that ω is nondecreasing.

Note that

$$\omega(t)t \leq \int_t^{2t} \omega(r)dr \quad \text{for all } t \in [0,\ 1/2].$$

Setting

$$\psi(t) = \int_t^{2t} \omega(r)dr \quad \text{for all } t \in [0, 1/2],$$

and

$$\phi(x) = u(y) + p \cdot (x - y) + \psi(|x - y|) \quad \text{for all } x \in B_{1/2}(y),$$

we observe that $\phi \in C^1(B_{1/2}(y))$, $D\phi(y) = p$,

$$u(x) \leq \phi(x)\ \forall x \in B_{1/2}(y) \quad \text{and} \quad u(y) = \phi(y).$$

Extending ϕ smoothly outside $B_{1/3}(y)$ so that $u(x) \leq \phi(x)$ on the domain of definition of u. We now find that

$$0 \geq F(y, D\phi(y), u(y)) = F(y, Du(y), u(y)).$$

In the above discussion, the differentiability can be weakened as follows:

$$u(x) - u(y) \leq p \cdot (x - y) + o(|x - y|) \quad \text{as } x \to y$$

for some $p \in \mathbb{R}^n$. If this is the case and u is a subsolution of $F = 0$, then

$$F(y, p, u(y)) \leq 0.$$

The set of all $p \in \mathbb{R}^n$ for which the above asymptotic relation hold is called the *superdifferentials* of u at y and is denoted by $D^+u(y)$. By making the upside-down in the above discussion, we define $D^-u(y)$, called the *subdifferentials* of u at y.

Theorem 6

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ locally bounded. Let $F \in C(\Omega \times \mathbb{R}^n \times \mathbb{R})$. The function u is a (viscosity) subsolution (resp., supersolution) of $F(x, Du, u) = 0$ in Ω if and only if

$$F(x, p, u^*(x)) \leq 0 \quad \text{for all } p \in D^+u^*(x)$$

$$(\text{resp., } F(x, p, u_*(x)) \geq 0 \quad \text{for all } p \in D^-u_*(x)).$$

Long-time behavior of solutions II

Long-time behavior of solutions to a general HJE

(1) $$u_t + H(x, D_x u) = 0 \quad \text{in } \mathbb{T}^n \times (0, \infty).$$

Assumptions on H:

- $H \in C(\mathbb{T}^n \times \mathbb{R}^n)$.
- $p \mapsto H(x, p)$ is coercive for every (uniformly) x. i.e.,

$$\lim_{r \to \infty} \inf_{|p| \geq r} H(x, p) = \infty.$$

Recall the following theorem (the proof was done for bounded functions on $\mathbb{R}^n$).

Theorem 1

Let $h \in \mathbf{Lip}(\mathbb{T}^n)$. Under the above assumptions, there is a solution $u \in \mathbf{Lip}(\mathbb{T}^n \times [0, \infty))$ of

(2) $$\begin{cases} u_t + H(x, D_x u) = 0 & \text{in } \mathbb{T}^n \times (0, \infty), \\ u(\cdot, 0) = h & \text{on } \mathbb{T}^n. \end{cases}$$

Note also that the comparison principle holds for sub and super solutions of (1), which is crucial to establish the following theorem.

Theorem 2

Let $h \in C(\mathbb{T}^n)$. Under the above assumptions, there is a solution $u \in \mathrm{UC}(\mathbb{T}^n \times [0, \infty))$ of

$$(2) \qquad \begin{cases} u_t + H(x, D_x u) = 0 & \text{in } \mathbb{T}^n \times (0, \infty), \\ u(\cdot, 0) = h & \text{on } \mathbb{T}^n. \end{cases}$$

PROOF. Choose a sequence $h_k \in \mathbf{Lip}(\mathbb{T}^n) \to h$ in $C(\mathbb{T}^n)$ and let $u_k \in \mathbf{Lip}(\mathbb{T}^n \times [0, \infty)$ be the solution of the Cauchy problem (2) with h replaced by h_k. Choose a monotone sequence $\varepsilon_k \to 0+$ so that

$$\|h_j(x) - h_k\|_\infty \leq \varepsilon_k \quad \forall j > k.$$

By the comparison principle, if $j > k$, then

$$|u_j(x, t) - u_k(x, t)| \leq \varepsilon_k \quad \forall (x, t).$$

That is, for some $u \in \mathbf{UC}(\mathbb{T}^n \times [0,\infty))$,

$$\lim_k u_k(x,t) = u(x,t) \quad \text{uniformly on } \mathbb{T}^n \times [0,\infty).$$

The function u is a solution of (2). □

Limit problem:

$$H(x, Du) = c \quad \text{in } \mathbb{T}^n. \tag{3}$$

This ergodic problem has a solution $(c, u) \in \mathbb{R} \times \mathbf{Lip}(\mathbb{T}^n)$. The ergodic constant c is uniquely determined.

We follow the argument due to Barles-Souganidis. The argument has been modified (or simplified) by Barles-HI-Mitake. Another important approach is the one due to Davini-Siconolfi (after Fathi).

We add another requirement on H:

- There exist constants $\eta_0 > 0$ and $\theta_0 > 1$ and for each $(\eta, \theta) \in (0, \eta_0) \times (1, \theta_0)$ a constant $\psi = \psi(\eta, \theta) > 0$ such that for all $x, p, q \in \mathbb{R}^n$, if $H(x, p) \leq c$ and $H(x, q) \geq c + \eta$, then

$$H(x, p + \theta(q - p)) \geq c + \eta\theta + \psi.$$

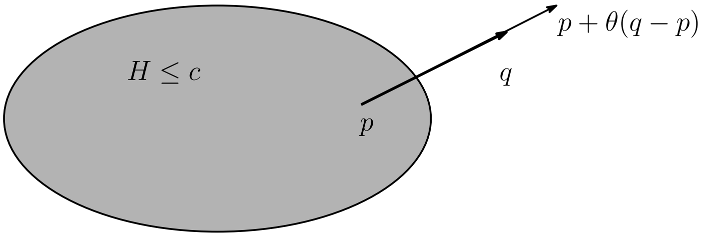
This is a kind of strict convexity of H. Indeed, if $p \mapsto H(x, p)$ is strictly convex, one can show that the above condition is satisfied.

Indeed, if H is strictly convex, since $q = \theta^{-1}(p + \theta(q - p)) + (1 - \theta^{-1})p$,

$$\begin{aligned} c + \eta \leq H(x, q) &< \theta^{-1} H(x, p + \theta(q - p)) + (1 - \theta^{-1}) H(x, p) \\ &< \theta^{-1} H(x, p + \theta(q - p)) + (1 - \theta^{-1}) c, \end{aligned}$$

i.e.,

$$H(x, p + \theta(q - p)) > c + \theta\eta.$$



Figure

Theorem 3

Let $h \in C(\mathbb{T}^n)$ and c be the ergodic constant. Let $u = u(x,t,h) \in \mathrm{UC}(\mathbb{T}^n \times [0,\infty))$ be the solution of the Cauchy problem (2). Then, for some $h_\infty \in \mathcal{S}(H-c) \cap \mathrm{Lip}(\mathbb{T}^n)$, as $t \to \infty$,

$$u(x,t,h) + ct \ \to \ h_\infty(x) \quad \text{uniformly in } \mathbb{T}^n.$$

Outline of proof. By the comparison principle,

$$\|u(\cdot,t,h) - u(\cdot,t,g)\|_\infty \leq \|h-g\|_\infty.$$

we may assume that $h \in \mathrm{Lip}(\mathbb{T}^n)$ and $u \in \mathrm{Lip}(\mathbb{T}^n \times [0,\infty))$.

Note that the function $v = u(x,t,h) + ct$ is a solution of $v_t + H - c = 0$. By rewriting H for $H - c$, we henceforth assume that $c = 0$.

Fix a $v_0 \in \mathcal{S}(H)$. By choosing $C > 0$ so that

$$v_0 - C \leq h \leq v_0 + C \quad \text{on } \mathbb{T}^n.$$

we have by the comparison principle,

$$|u(x,t,h) - v_0(x)| \leq C \ \ \forall (x,t).$$

Thus,

$$u(\cdot,\cdot,h) \in (\mathbf{Lip}\cap \mathbf{B})(\mathbb{T}^n \times [0,\infty)).$$

We assume by adding a constant to v_0 that

$$u(x,t) - v_0(x) \geq 0 \quad \forall (x,t).$$

Let θ, η, ψ be as in the above condition on H. Define

$$w(x,t) = \sup_{s\geq t}[u(x,t) - v_0(x) - \theta(u(x,s) - v_0(x) + \eta(s-t))]$$

Let $M > 0$ be a Lipschitz bound of u and v_0. Define

$$\omega(r) = \max\{|H(x,p) - H(x,q)| : p,q \in \overline{B}_R,\ |p-q| \leq r\},$$

where $R = (2\theta_0 + 1)M$.

Theorem 4

The function w is a subsolution of

$$\min\{w, w_t - \omega(|D_x w|) + \psi\} \leq 0 \quad \text{in } \mathbb{T}^n \times (0, \infty).$$

In particular, setting

$$m(t) = \max_x w(x, t),$$

we have

$$\min\{m, m_t + \psi\} \leq 0.$$

The last inequality implies that for a finite time $\tau > 0$,

$$m(t) \leq 0 \quad \forall t \geq \tau.$$

Then, for any $t \geq \tau$, $x \in \mathbb{T}^n$, $s \geq t$,

$$u(x, t) - v_0(x) \leq \theta(u(x, s) - v_0(x) + \eta(s - t)).$$

The constant $\tau = \tau_{\theta,\eta}$ depends on θ, η.

(AA theorem) $\exists t_j \to \infty$ such that for some $u_\infty \in \mathbf{Lip}(\mathbb{T}^n)$,

$$u(x, t_j, h) \;\to\; u_\infty(x) \quad \text{in } C(\mathbb{T}^n).$$

Then, we have

$$u(x,t+t_j,h) \ \to \ u(x,t,u_\infty) \ \ \forall (x,t).$$

$$\begin{pmatrix} \|u(\cdot,t,u(\cdot,t_j,h)) - u(\cdot,t,u_\infty)\|_\infty & \\ \qquad \le \|u(\cdot,t_j,h) - u_\infty\|_\infty \ \ \forall t \ge 0 & \text{by comparison.} \end{pmatrix}$$

Hence, for all $t \ge 0$, $s \ge t$, $x \in \mathbb{T}^n$,

$$u(x,t,u_\infty) - v_0(x) \le \theta(u(x,s,u_\infty) - v_0(x) + \eta(s-t)).$$

This holds for any $\theta \in (1,\theta_0)$ and $\eta > 0$. Thus,

$$u(x,t,u_\infty) - v_0(x) \le u(x,s,u_\infty) - v_0(x) \quad \text{if } s \ge t.$$

That is, $t \mapsto u(x,t,u_\infty)$ is nondecreasing. Monotone in t.

(AA theorem) $\exists h_\infty \in \mathbf{Lip}(\mathbb{T}^n)$ such that

$$h_\infty(x) = \lim_{t\to\infty} u(x,t,u_\infty) \quad \text{in } C(\mathbb{T}^n).$$

Since

$$\|u(\cdot,t+t_j,h) - u(\cdot,t,u_\infty)\|_\infty \le \|u(\cdot,t_j,h) - u_\infty\|_\infty \ \ \forall t \ge 0,$$

we have

$$h_\infty(x) = \lim_{t\to\infty} u(x,t,h) \quad \text{in } C(\mathbb{T}^n).$$

Since

$$\|u(\cdot, t+t_j, h) - h_\infty\|_\infty \to 0 \quad \text{as } j \to \infty,$$

we find that $\partial_t h_\infty + H(x, D_x h_\infty) = 0$ and $h_\infty \in \mathcal{S}(H)$. □

Outline of the proof of the VI:

$$\min\{w, w_t - \omega(|D_x w|) + \psi\} \le 0, \qquad \text{where}$$

$$w(x,t) := \sup_{s \ge t}[u(x,t) - v_0(x) - \theta(u(x,s) - v_0(x) + \eta(s-t))].$$

Fix any $(x,t) \in \mathbb{T}^n \times (0,\infty)$. If $w(x,t) \le 0$, we have VI at (x,t).

Assume that $w(x,t) > 0$. Suppose that $u \in C^1$ and $v_0 \in C^1$ and that for some $s > t$,

$$w(x,t) = u(x,t) - v_0(x) - \theta(u(x,s) - v_0(x) + \eta(s-t)),$$

and show that

$$w_t - \omega(|D_x w|) + \psi \le 0.$$

Set

$$p = Dv_0(x), \quad q = D_x u(x,s), \quad r = D_x u(x,t),$$
$$a = u_t(x,s), \qquad b = u_t(x,t).$$

We have

$$\begin{aligned} H(x,p) &\le 0. \\ a + H(x,q) &\ge 0, \\ b + H(x,r) &\le 0. \end{aligned}$$

The function

$$-w(x',t') + u(x',t') - v_0(x') - \theta(u(x',s') - v_0(x') + \eta(s'-t'))$$

≤ 0 and attains the maximum value 0 at (x,t,s), which yields

$$\begin{aligned} D_x w(x,t) &= r - p - \theta(q-p), \\ w_t(x,t) &= b + \theta\eta, \\ 0 &= -\theta(a+\eta). \end{aligned}$$

$a + H(x,q) \geq 0$ and $a + \eta = 0$ yield

$$H(x,q) \geq \eta.$$

This and $H(x,p) \leq 0$, the key assumption on H,

$$H(x, p + \theta(q-p)) \geq \theta\eta + \psi.$$

Since $r = D_x w(x,t) + p + \theta(q-p)$,

$$H(x,r) = H(x, D_x w(x,t) + p + \theta(q-p)).$$

Note:

$$|r| = |D_x u(x,t)| \leq M \leq R \,, \; |p+\theta(q-p)| \leq (1+2\theta)M \leq R.$$

Hence,

$$\begin{aligned} H(x,r) &\geq H(x, p+\theta(q-p)) - \omega(|D_x w(x,t)|) \\ &\geq -\omega(|D_x w(x,t)|) + \theta\eta + \psi. \end{aligned}$$

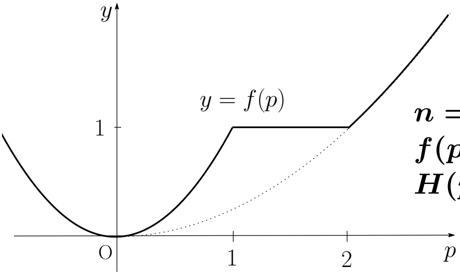
Now,

$$\begin{aligned} & w_t(x,t) = b + \theta\eta, \\ & 0 \geq b + H(x,r) \geq b - \omega(|D_x w|) + \theta\eta + \psi \end{aligned}$$

yield

$$0 \geq w_t - \omega(|D_x w|) + \psi. \qquad \square$$

Example 1 (Non-convex H)

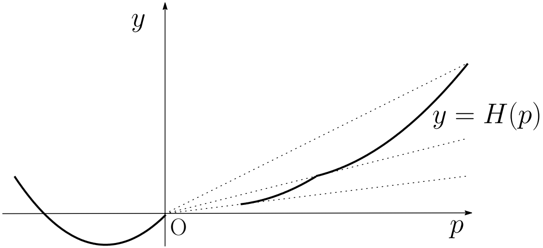


$n = 1$,
$f(p) = \max\{\frac{1}{2}p^2, \min\{p^2, 1\}\}$,
$H(p) = f(p+1) - 1$.

Note that constant functions are solutions of $H = 0$. Hence, $c(H) = 0$. Since H is "strictly convex" on $\{H > 0\} = \{f > 1\}$, our key condition is satisfied.

The key condition implies that $\{p : H(x,p) \leq c\}$ is convex.

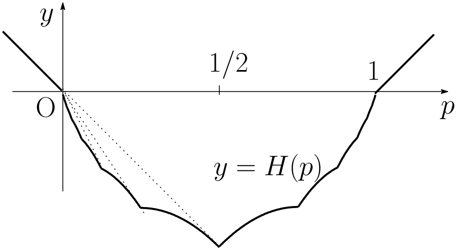
The key assumption requires a kind of "strict convexity" of H in a neighborhood of $\{p : H(x,p) \leq c\}$ in $\{p : H(x,p) > c\}$.



The following condition replaces the key condition:

- There exist constants $\eta_0 > 0$ and $\theta_0 > 1$ and for each $(\eta, \theta) \in (0, \eta_0) \times (1, \theta_0)$ a constant $\psi = \psi(\eta, \theta) > 0$ such that for all $x \in \mathbb{T}^n$, $p, q \in \mathbb{R}^n$, if $H(x, p) \le c$ and $H(x, q) \ge c - \eta$, then

$$H(x, p + \theta(q - p)) \ge c - \eta\theta + \psi.$$



Vanishing discount problem for Hamilton-Jacobi equations I

Let $\lambda > 0$. Consider the stationary problem

$$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{T}^n. \tag{1}$$

In view of optimal control theory, the constant λ is called a discoutn factor. Here we study the asymptotic behavior of the solution u_λ of (1) as $\lambda \to 0+$.

Assumptions on H:

- $H \in C(\mathbb{T}^n \times \mathbb{R}^n)$.
- H is coercive, i.e.,

$$\lim_{r\to\infty} \inf_{\mathbb{T}^n \times (\mathbb{R}^n \setminus B_r)} H(x, p) = \infty.$$

- H is convex, i.e., $p \mapsto H(x, p)$ is convex, $\forall x \in \mathbb{R}^n$.

Theorem 1

PDE (1) has a unique solution u_λ in the class $\mathbf{Lip}(\mathbb{T}^n)$. The comparison principle is valid for sub and super solutions in the class $\mathbf{B}(\mathbb{T}^n)$.

Remark. $\exists C > 0$ (independent of $\lambda > 0$) such that

$$\lambda |u_\lambda(x)| \leq C.$$

$\exists M > 0$ such that

$$|p| > M \implies -C + H(x,p) > 0.$$

Since u_λ is a subsolution of

$$-C + H(x, Du) \leq 0 \quad \text{in } \mathbb{T}^n,$$

M is a Lipschitz bound of u_λ.
M can be chosen independently of λ.

The above observations imply together with AA theorem that for a sequence $\lambda_k \to 0+$, u_{λ_k} "converge" to a function $u_0 \in C(\mathbb{T}^n)$ and for some constant c (the ergodic constant), u_0 is a solution of

$$H(x, Du) = c \quad \text{in } \mathbb{T}^n. \tag{2}$$

The main result is roughly stated as follows.

Claim 2

The whole family $\{u_\lambda\}_{\lambda>0}$ "converges" to a function u_0 in $C(\mathbb{T}^n)$.

(Davini-Fathi-Iturriaga-Zavidovique)

- *Mather measures* play an important role in the proof.

1) $\exists M > 0$ such that $\|Du_\lambda\|_\infty \le M$ for all $\lambda > 0$.

2) u_λ is the value function of the optimal control system:

$$\begin{cases} H(x,p) = \sup_\xi(\xi \cdot p - L(x,\xi)), \\ \dot{X}(t) = -\alpha(t) \quad X(0) = x, \\ J(x,\alpha) = \int_0^\infty e^{-\lambda t} L(X(t),\alpha(t))dt. \end{cases}$$

That is,

$$\begin{aligned} u_\lambda(x) &= \inf_{X(0)=x} \int_0^\infty e^{-\lambda t} L(X(t), -\dot{X}(t))dt \\ &= \inf_{Y(0)=x} \int_{-\infty}^0 e^{\lambda t} L(Y(t), \dot{Y}(t))dt. \end{aligned}$$

3) $\xi \mapsto L(x,\xi)$ has a superlinear growth:

$$L(x,\xi) \ge \xi \cdot \frac{A\xi}{|\xi|} - H(x, \tfrac{A\xi}{|\xi|}), \quad \forall A > 0, \xi \neq 0.$$

$\forall |p| \le M, \ \exists \rho > 0$ such that

$$H(x,p) = \max_{|\xi| \le \rho} \xi \cdot p - L(x,\xi).$$

Set

$$H_\rho(x,p) := \max_{|\xi|\le\rho} \xi\cdot p - L(x,\xi).$$

u_λ is a solution of

$$\lambda u + H_\rho(x,Du) = 0 \quad \text{in } \mathbb{T}^n,$$

and

$$u_\lambda(x) = \inf_{X(0)=x,\,|\dot X(t)|\le\rho} \int_0^\infty e^{-\lambda t} L(X(t),-\dot X(t))dt.$$

4) Set $K = K_\rho =: \mathbb{T}^n \times \overline{B}_\rho$. Let $\mathsf{M} = \mathsf{M}(\mathbb{T}^n\times\mathbb{R}^n)$ denote the set of all finite Borel measures μ on $\mathbb{T}^n\times\mathbb{R}^n$. Set

$$\mathsf{M}_\rho = \mathsf{M}_\rho(\mathbb{T}^n\times\mathbb{R}^n) = \{\mu\in\mathsf{M} : \operatorname{supp}\mu \subset K_\rho\},$$
$$\mathsf{M}_\rho^+ = \mathsf{M}_\rho^+(\mathbb{T}^n\times\mathbb{R}^n) = \{\mu\in\mathsf{M}_\rho : \mu\ge 0\}.$$

Set

$$\mathcal{C}_\rho(x) = \{X\in C([0,\infty),\mathbb{T}^n) : X\in \mathrm{AC}[0,T], \forall T>0,$$
$$X(0)=x,\ |\dot X(t)|\le\rho \text{ a.e. }\}.$$

Given $z \in \mathbb{T}^n$ and $X \in \mathcal{C}(z)$, consider the functional

$$C(K) \ni \phi \mapsto \int_0^\infty e^{-\lambda t} \phi(X(t), -\dot{X}(t)) dt \in \mathbb{R}.$$

Note:

$$\left| \int_0^\infty e^{-\lambda t} \phi(X(t), -\dot{X}(t)) dt \right| \le \|\phi\|_\infty \int_0^\infty e^{-\lambda t} dt = \lambda^{-1} \|\phi\|_\infty.$$

Each $z \in \mathbb{T}^n$ and $X \in \mathcal{C}(z)$ define a continuous linear functional on $C(K)$, an element of $C^*(K)$, and by Riesz' theorem, $\exists \mu \in \mathsf{M}_\rho$ such that

$$\lambda \int_0^\infty e^{-\lambda t} \phi(X(t), -\dot{X}(t)) dt = \int_K \phi(x, \xi) \mu(dx, d\xi).$$

If $\phi = 1$ (resp., $\phi \ge 0$), then

$$\lambda \int_0^\infty e^{-\lambda t} \phi(X(t), -\dot{X}(t)) dt = 1 \text{ (resp., } \ge 0).$$

Hence, $\mu \in \mathsf{M}_\rho^+$ and a probability measure.

Let $\mathsf{P}_\rho = \{\mu \in \mathsf{M}_\rho^+ : \mu(K) = 1\}$. If we write $\mu_{z,X}$ for the measure defined above, then

$$\lambda u_\lambda(z) = \inf_{X \in \mathcal{C}(z)} \int_K L(x, \xi) \mu_{z,X}(dx, d\xi).$$

P_ρ has a good stability property: the compactness in the weak-star convergence in $C^*(K)$ (the weak convergence in the sense of measures). The Banach-Alaoglu theorem. On the other hand, the implication of "convergence" of $\{X_k\}$ to the functionals

$$\int_0^\infty e^{-\lambda t} \phi(X_k(t), -\dot{X}_k(t)) dt$$

is not easy. What is the limit?

$$\mu_{z,X_k} \xrightarrow{\text{weak}^*} \mu = \mu_{z,X} \ (\exists X \in \mathcal{C}(z)?).$$

Want to replace $\{\mu_{z,X} : X \in \mathcal{C}(z)\}$ by a good $G \subset \mathsf{P}_\rho$ such that

$$\lambda u_\lambda(z) = \inf_{\mu \in G} \int_K L \mu(dx, d\xi).$$

$G = \mathsf{P}_\rho$ is too big.

5) Note that if $u_\lambda \in C^1(\mathbb{T}^n)$, then

$$\lambda u_\lambda(x) + \xi \cdot Du_\lambda(x) \le L(x,\xi) \ \ \forall (x,\xi) \in K.$$

Integrate both sides by $\mu = \mu_{z,X}$, to get

$$\int_K (\lambda u_\lambda(x) + \xi \cdot Du_\lambda(x))\mu(dx,d\xi) \le \int_K L(x,\xi)\mu(dx,d\xi).$$

Compute that

$$\begin{aligned}\int_K &(\lambda u_\lambda(x) + \xi \cdot Du_\lambda(x))\mu_{x,X}(dx,d\xi) \\ &= \lambda \int_0^\infty e^{-\lambda t}(\lambda u_\lambda(X(t)) - \dot{X}(t)\cdot Du_\lambda(X(t)))dt \\ &= \lambda \int_0^\infty \frac{d}{dt}\left(-e^{-\lambda t}u_\lambda(X(t))\right)dt = \lambda u_\lambda(z).\end{aligned}$$

Hence, for any $\mu = \mu_{z,X}$,

$$\int_K L(x,\xi)\mu(dx,d\xi) \ge \lambda u_\lambda(z).$$

Let $\mathsf{P_c}$ denote the set of all (Borel) probability measures with *compact support*. Note: $\mathsf{P}_\rho \subset \mathsf{P_c}$.

We introdue the condition on $\mu \in \mathsf{P_c}$ that $\forall \psi \in C^1(\mathbb{T}^n)$,

$$\lambda\psi(z) = \int_{\mathbb{T}^n\times\mathbb{R}^n} (\lambda\psi(x) + \xi\cdot D\psi(x))\mu(dx, d\xi). \tag{3}$$

In general, "$u_\lambda \in C^1(\mathbb{T}^n)$" does not hold, but the above condition always makes sense.

We call $\mu \in \mathsf{P_c}$ a *closed measure* for (z, λ) if (3) holds. We write $\mathfrak{C}(z,\lambda)$ for the set of all closed measures for (z,λ). Note that $\mathfrak{C}(z,\lambda)$ is irrelevant to our HJE. Since all $\mu_{z,X}$ are in $\mathfrak{C}(z,\lambda)$, we have

$$\lambda u_\lambda(z) \geq \inf_{\mu\in\mathfrak{C}(z,\lambda)} \int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,\xi)\mu(dx,d\xi).$$

Theorem 3

$$\lambda u_\lambda(z) = \min_{\mu\in\mathfrak{C}(z,\lambda)} \int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,\xi)\mu(dx,d\xi).$$

Proof. 1) A first step is: $\forall \mu \in \mathfrak{C}(z, \lambda)$,

$$\lambda u_\lambda(z) \le \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, \xi) \mu(dx, d\xi). \tag{4}$$

Since $u_\lambda \in \mathbf{Lip}(\mathbb{T}^n)$, it is a.e. differentiable and the pointwise derivative is identified with the distributional derivative. Let u_λ^ε and $(Du_\lambda)^\varepsilon$ be the mollified functions of u_λ and Du_λ, respectively, with the same millification kernel. We have $Du_\lambda^\varepsilon = (Du_\lambda)^\varepsilon$. H is uniformly continuous on $\mathbb{T}^n \times B_M$, and so

$$\lambda u_\lambda(y) + H(x, Du_\lambda(y)) \le \delta(\varepsilon) \text{ a.e. } \{(x, y) \in \mathbb{T}^{2n} : |x - y| < \varepsilon\},$$

where $\delta(\varepsilon) \to 0+$ $(\varepsilon \to 0+)$. By the convexity of H, we find

$$\lambda u_\lambda^\varepsilon(x) + H(x, Du_\lambda^\varepsilon(x)) \le \delta(\varepsilon) \text{ on } \mathbb{T}^n.$$

Integrate

$$\lambda u_\lambda^\varepsilon(x) + \xi \cdot Du_\lambda^\varepsilon(x) \le L(x, \xi) + \delta(\varepsilon),$$

by $\mu \in \mathfrak{C}(z, \lambda)$, to get

$$\lambda u_\lambda^\varepsilon(z) \le \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, \xi) \mu(dx, d\xi) + \delta(\varepsilon); \quad \text{hence, (4).}$$

Recall that

$$\lambda u_\lambda(z) \geq \inf_{\mu \in \mathfrak{C}(z,\lambda)} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x,\xi)\mu(dx,d\xi),$$

to conclued that

$$\lambda u_\lambda(z) = \inf_{\mu \in \mathfrak{C}(z,\lambda)} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x,\xi)\mu(dx,d\xi).$$

2) The next and last step is to replace $\inf$ by $\min$. Choose $\{X_k\} \subset \mathcal{C}(z)$ so that

$$\int_K L(x,\xi)\mu_{z,X_k}(dx,d\xi) \to u_\lambda(z).$$

By replacing by a subsequence, we may assume that $\mu_{z,X_k} \overset{\text{weak}^*}{\longrightarrow} \mu$ for some $\mu \in \mathsf{P}_\rho$.

3) "Lower semicontinuity + weak* convergence" imply:

$$\int_K L\,\mu(dx,d\xi) \leq \liminf_k \int_K L\,\mu_{z,X_k}(dx,d\xi)\ (= \lambda u_\lambda(z)).$$

4) Need to check that μ is a closed measure for (z,λ):
$\forall\psi \in C^1(\mathbb{T}^n)$, $\phi(x,\xi) := \lambda\psi(x) + \xi\cdot D\psi(x)$ is in $C^1(K)$.
Hence,

$$\lambda\psi(z) = \int_K \phi(x,\xi)\mu_{x,X_k}(dx,d\xi) \to \int_K \phi(x,\xi)\mu(dx,d\xi).$$

Thus, $\mu \in \mathfrak{C}(x,\lambda) \cap \mathsf{P}_\rho$ and

$$\lambda u_\lambda(z) = \int_{T^n\times\mathbb{R}^n} L\mu(dx,d\xi).$$

- We call a minimizer $\mu \in \mathfrak{C}(z,\lambda)$ as *generalized Mather measure* for (z,λ). We write $\mathfrak{M}(z,\lambda)$ for all minimizers $\mu \in \mathsf{P}_{\mathbf{c}}(z,\lambda)$. Also, called as a discounted Mather measure
- One can show that $\mathfrak{M}(z,\lambda) \subset \mathsf{P}_\rho$.

Another approach to the existence of Mather measures.

Assume that

$$L \in C(K).$$

For $\phi \in C(K)$, set

$$H_\phi(x,p) := \max_{|\xi|\le\rho} \xi\cdot p - \phi(x,\xi),$$
$$F_{\lambda,\phi}(x,p,u) := \lambda u + H_\phi(x,p).$$

Let Γ denote the set of all $(\psi,\phi) \in C(\mathbb{T}^n)\times C(K)$ such that $\psi \in \mathcal{S}^-(F_{\lambda,\phi})$. That is,

$$\lambda\psi(x) + \xi\cdot D\psi(x) \le \phi(x,\xi) \quad \text{for all } (x,\xi)\in K.$$

For fixed (z,λ), let

$$G(z,\lambda) = \{\phi - \lambda\psi(z) : (\psi,\phi)\in\Gamma\}.$$

Γ and $G(z,\lambda)$ are closed convex cones with vertex at the origin in $C(\mathbb{T}^n)\times C(K)$ and $C(K)$, respectively.

Let $G^*(z,\lambda)$ denote the dual cone, i.e.,

$$G^*(z,\lambda) := \{\nu \in C^*(K) : \langle \nu, g\rangle \geq 0 \ \forall g \in G(z,\lambda)\}.$$

We invoke the Hahn-Banach theorem:

1) $G(z,\lambda)$ has nonempty interior. Choose $(0,1) \in \Gamma$ so that $1 \in G(z,\lambda)$. For any $\phi \in C(K)$ such that $\|\phi\|_\infty \leq 1$, we have $(0, 1+\phi) \in \Gamma$ and $1+\phi \in G(z,\lambda)$.

2) $L - \lambda u_\lambda(z) \in \partial G(z,\lambda)$. Indeed, $L - \lambda u_\lambda(z) \in G(z,\lambda)$ and $L - \lambda u_\lambda(z) - \frac{1}{k} \notin G(z,\lambda)$ for all $k \in \mathbb{N}$.

3) HB theorem $\implies$ $\exists \nu \in C^*(K)$ such that, $\nu \neq 0$, and

$$\langle \nu, g - (L - \lambda u_\lambda(z))\rangle \geq 0 \ \ \forall g \in G(z,\lambda).$$

4) Select $g = t(L - \lambda u_\lambda(z))$, $t > 0$, in the above, to find

$$(t-1)\langle \nu, L - \lambda u_\lambda(z)\rangle \geq 0,$$

and

$$\langle \nu, L\rangle = \lambda u_\lambda(z)\langle \nu, 1\rangle.$$

5) Select $g = L - \lambda u_\lambda(z) + f$, with any $f \geq 0$, to find that

$$\langle \nu, f\rangle \geq 0, \text{ i.e., } \nu \in \mathsf{M}_\rho^+.$$

Set

$$\mu := \frac{\nu}{\nu(K)} \in \mathsf{P}_\rho.$$

6) Fix any $(\psi, \phi) \in \Gamma$ and note that $(\psi, \phi) + (L, u_\lambda) \in \Gamma$ and $\phi + L - \lambda(\psi + u_\lambda)(z) \in G(z, \lambda)$. Select $g = \phi + L - \lambda(\psi + u_\lambda)(z)$, to see

$$\langle \mu, \phi\rangle \geq \lambda\psi(z).$$

Let $\psi \in C^1(\mathbb{T}^n)$. Choose $\phi = \lambda\psi(x) + \xi \cdot D\psi(x)$, to find

$$\langle \mu, \lambda\psi + \xi \cdot D\psi(x)\rangle \geq \lambda\psi(z)$$

This is valid also for $-\psi$ in place of ψ. Hence,

$$\lambda\psi(z) = \langle \mu, \lambda\psi + \xi \cdot D\psi\rangle \;\; \forall\psi \in C^1(\mathbb{T}^n).$$

7) The conclusion:

$$\mu \in \mathfrak{C}(z, \lambda) \quad \text{and} \quad \lambda u_\lambda(z) = \langle \mu, L \rangle = \int_K L\mu.$$

EXERCISES. 1. Prove that Γ is a convex set.

2. Prove that if $a > 0$, then $L - \lambda u_\lambda(z) - a \notin G(z, \lambda)$.

Vanishing discount problem for Hamilton-Jacobi equations II

Our HJE is as follows:

$$\lambda u + H(x, Du) = 0 \quad \text{in } \mathbb{T}^n. \tag{1}$$

Assumptions on H:

- $H \in C(\mathbb{T}^n \times \mathbb{R}^n)$.
- H is coercive, i.e.,

$$\lim_{r\to\infty} \inf_{\mathbb{T}^n \times (\mathbb{R}^n \setminus B_r)} H(x, p) = \infty.$$

- H is convex, i.e., $p \mapsto H(x, p)$ is convex, $\forall x \in \mathbb{T}^n$.

Theorem 1

$$\lambda u_\lambda(z) = \min_{\mu \in \mathfrak{C}(z,\lambda)} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, \xi)\mu(dx, d\xi).$$

The $\min$ is attained at $\mu \in \mathsf{P}_\rho \cap \mathfrak{C}(z, \lambda)$, where, for $\mu \in \mathsf{P}_\rho$, $\operatorname{supp} \mu \subset K = \mathbb{T}^n \times \overline{B}_\rho$ and ρ does not depend of $\lambda > 0$.

The closedness of $\mu \in \mathfrak{C}(z, \lambda)$ is described as: $\forall \psi \in C^1(\mathbb{T}^n)$,

$$\lambda \psi(z) = \int_{\mathbb{T}^n \times \mathbb{R}^n} (\lambda \psi(x) + \xi \cdot D\psi(x)) \mu(dx, d\xi).$$

This condition is stable under the weak* convergence of sequences in P_ρ. For instance, if $\lambda_j \to 0+$ and $\mathsf{P}_\rho \cap \mathfrak{C}(z, \lambda_j) \ni \mu_j \xrightarrow{\text{weak}^*} \mu$, then

$$0 = \int_{\mathbb{T}^n \times \mathbb{R}^n} \xi \cdot D\psi(x) \mu(dx, d\xi) \quad \forall \psi \in C^1(\mathbb{T}^n). \tag{2}$$

We call $\mu \in \mathsf{P}_{\mathbf{c}}$ a *closed measure* (for $\lambda = 0$) if (2) holds. Let $\mathfrak{C}(0)$ denote the set of all closed measures $\mu \in \mathsf{P}_{\mathbf{c}}$.

Recall the ergodic problem:

$$H(x, Du) = c \quad \text{in } \mathbb{T}^n. \tag{3}$$

We know the following.

Theorem 2

Let c be the ergodic constant. Then

- $u_\lambda - \max_{\mathbb{T}^n} u_\lambda \to u_0$ in $C(\mathbb{T}^n)$ along a sequence $\lambda_j \to 0+$,
- $\lambda u_\lambda \to -c$ in $C(\mathbb{T}^n)$ as $\lambda \to 0+$,
- u_0 is a solution of (3).

We have a representation theorem for c.

Theorem 3

Let c be the ergodic constant. Then

$$-c = \min_{\mu \in \mathfrak{C}(0)} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, \xi) \mu(dx, d\xi).$$

PROOF. 1) Let $u_0 \in \mathbf{Lip}(\mathbb{T}^n)$ be a solution of $H = c$ in $\mathbb{T}^n$. We have $\|Du_0\|_\infty < \infty$. By approximation, $\exists u_0^\varepsilon \in C^1(\mathbb{T}^n)$, $\delta(\varepsilon) > 0$ such that

$$\begin{cases} -c + H(x, Du_0^\varepsilon(x)) \leq \delta(\varepsilon) \ \text{ in } \mathbb{T}^n, \\ u_0^\varepsilon \to u_0 \ \text{ in } C(\mathbb{T}^n) \ (\varepsilon \to 0+), \\ \delta(\varepsilon) \to 0+ \ (\varepsilon \to 0+). \end{cases}$$

In particular,

$$-c + \xi \cdot Du_0^\varepsilon(x) \leq L(x, \xi) + \delta(\varepsilon) \ \ \forall (x, \xi).$$

Integrating by $\mu \in \mathfrak{C}(0)$ and sending $\varepsilon \to 0+$ yield

$$-c \leq \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, \xi) \mu(dx, d\xi).$$

Thus,

$$-c \leq \inf_{\mu \in \mathfrak{C}(0)} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, \xi) \mu(dx, d\xi).$$

2) Existence of a minimizer: Fix $z \in \mathbb{T}^n$ and for each $\lambda > 0$ choose $\mu_\lambda \in \mathfrak{M}(z, \lambda) \cap \mathsf{P}_\rho$ so that

$$\lambda u_\lambda(z) = \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, \xi)\mu_\lambda(dx, d\xi).$$

Recall that

$$\lim_{\lambda \to 0+} \lambda u_\lambda(z) = -c.$$

We can choose $\lambda_j \to 0+$ so that

$$\mu_{\lambda_j} \xrightarrow{\text{weak}^*} \mu_0 \in \mathsf{P}_\rho.$$

As in the argument for a fixed $\lambda > 0$, we find that $\mu_0 \in \mathfrak{C}(0)$,

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} L\, \mu_0(dx, d\xi) \le \liminf_{j \to \infty} \int_{\mathbb{T}^n \times \mathbb{R}^n} L\, \mu_{\lambda_j}(dx, d\xi) = -c.$$

Hence, μ_0 is a minimizer:

$$-c = \int_{\mathbb{T}^n \times \mathbb{R}^n} L\, \mu_0(dx, d\xi). \qquad \square$$

- Any minimizer $\mu \in \mathfrak{C}(0)$ is called a *Mather measure.* Denoted by $\mathfrak{M}(0)$.

Our purpose here is:

Claim 4

The whole family $\{u_\lambda\}_{\lambda>0}$ "converges" to a function u_0.

Formal expansion:

$$\lambda u_\lambda \approx -c + \lambda u_0(x) + \lambda^2 u_1(x) + \cdots .$$

Then,

$$u_\lambda \approx -\lambda^{-1}c + u_0(x) + \lambda u_1(x) + \cdots ;$$
$$0 = \lambda u_\lambda + H(x, Du_\lambda) \approx -c + H(x, Du_0 + \cdots) + \cdots ,$$

and hence,

$$-c + H(x, Du_0) = 0.$$

$$0 \gtrapprox -c + \lambda u_0 + \cdots + \xi \cdot (Du_0 + \lambda Du_1 + \cdots) - L(x, \xi).$$

If $\mu_0 \in \mathfrak{M}(0)$, then

$$\int (-c - L)\mu_0 = 0, \quad \int \xi \cdot (Du_0 + \lambda Du_1 + \cdots)\mu_0 \approx 0.$$

Hence,

$$0 \gtrapprox \lambda \int u_0\mu_0, \quad \text{i.e.,} \int u_0\mu_0 \leq 0.$$

Theorem 5

The whole family $\{u_\lambda + \lambda^{-1}c\}_{\lambda>0}$ converges to a solution u_0 in $C(\mathbb{T}^n)$ of (3).

(Davini-Fathi-Iturriaga-Zavidovique=2016)

PROOF. 1) Note that $v_\lambda := u_\lambda + \lambda^{-1}c$ satisfies

$$\lambda v_\lambda + H(x, Dv_\lambda) = \lambda u_\lambda + c + H(x, Du_\lambda) = c \quad \text{in } \mathbb{T}^n.$$

If we set $H_c(x,p) = H(x,p) - c$, then v_λ is a solution of $\lambda v_\lambda + H_c = 0$ in $\mathbb{T}^n$. If u_0 is a solution of $H = c$ in $\mathbb{T}^n$, then it is also a solution of $H_c(x, Du_0) = 0$ in $\mathbb{T}^n$. Note that the Lagrangian corresponding to H_c is given by

$$L_c(x,\xi) := \sup_p \xi \cdot p - H_c(x,p) = L(x,\xi) + c.$$

Replacing (H, L) by (H_c, L_c), we may assume that $c = 0$. We need to show that the solutions u_λ of $\lambda u + H(x, Du) = 0$ in $\mathbb{T}^n$ converge to a solution u_0 of $H(x, Du) = 0$ in $\mathbb{T}^n$.

2) Let $v_0 \in \mathbf{Lip}(\mathbb{T}^n)$ be a solution of $H = 0$ in $\mathbb{T}^n$. Choose $C_0 > 0$ so that $\|v_0\|_\infty \le C_0$. Note that

$$\lambda(v_0 + C_0) + H(x, Du_0) \ge 0, \quad \lambda(v_0 - C_0) + H \le 0 \quad \text{in } \mathbb{T}^n.$$

By comparison,

$$v_0 + C_0 \ge u_\lambda \ge v_0 - C_0 \quad \text{in } \mathbb{T}^n.$$

Hence,

$$|u_\lambda(x)| \le 2C_0 \quad \text{in } \mathbb{T}^n,$$

and the family $\{u_\lambda\}$ is unif-bounded on $\mathbb{T}^n$. Thus, the family $\{u_\lambda\}$ is unif-bounded and equi-Lipschitz continuous on $\mathbb{T}^n$.

3) Let $\mathcal{V}$ denote the set of all limit points in $C(\mathbb{T}^n)$ of $\{u_\lambda\}_{\lambda>0}$ as $\lambda \to 0+$. We have $\mathcal{V} \ne \emptyset$. Since

$$\lambda u_\lambda \to 0 \ \text{ in } C(\mathbb{T}^n) \ (\lambda \to 0+),$$

we find that $v \in \mathcal{V}$ is a solution of $H = 0$ in $\mathbb{T}^n$.

We claim:

$$\int v(x)\mu(dx,d\xi)\le 0 \quad \forall (v,\mu)\in \mathcal{V}\times \mathfrak{M}(0).$$

Let $v\in\mathcal{V}$ and $\mu\in\mathfrak{M}(0)$. Choose a sequence $\lambda_j\to 0+$ such that u_{λ_j} converge to v in $C(\mathbb{T}^n)$. Note that u_λ is a solution of

$$\widetilde{H}(x,Du_\lambda)=0 \quad \text{in } \mathbb{T}^n, \quad (\text{the ergodic constant} = 0!)$$

where $\widetilde{H}(x,p)=\sup_\xi(\xi\cdot p-L(x,\xi)+\lambda u_\lambda(x))$, which implies that

$$0=\min_{\nu\in\mathfrak{C}(0)}\int (L(x,\xi)-\lambda u_\lambda(x))\nu(dx,d\xi).$$

Since $\mu\in\mathfrak{C}(0)$,

$$\begin{aligned}0&\le\int (L(x,\xi)-\lambda u_\lambda(x))\mu(dx,d\xi)\\&=-\lambda\int u_\lambda\mu(dx,d\xi).\end{aligned}$$

Sending $\lambda = \lambda_j \to 0+$, we find that

$$\int v(x)\mu(dx, d\xi) \leq 0.$$

Let $\mathcal{W}$ denote the set of all solutions w of $H = 0$ in $\mathbb{T}^n$ such that

$$\int w(x)\mu(dx, d\xi) \leq 0 \quad \forall \mu \in \mathfrak{M}(0).$$

We have shown that

$$\mathcal{V} \subset \mathcal{W}.$$

4) We claim that

$$w \le v \text{ on } \mathbb{T}^n \ \ \forall (w,v) \in \mathcal{W} \times \mathcal{V},$$

which assures that for all $v \in \mathcal{V}$,

$$v(x) = \max_{w \in \mathcal{W}} w(x) \ \ \forall x \in \mathbb{T}^n.$$

In particular, if we set $v(x) := \max_{w \in \mathcal{W}} w(x)$, then $\mathcal{V} = \{v\}$, and, as $\lambda \to 0+$,

$$u_\lambda \ \to \ v \quad \text{in } C(\mathbb{T}^n).$$

5) To show the above, fix any $w \in \mathcal{W}, v \in \mathcal{V}$. Choose $\lambda_j \to 0+$ so that

$$u_{\lambda_j} \to v \quad \text{in } C(\mathbb{T}^n) \ (j \to \infty).$$

Fix any $z \in \mathbb{T}^n$. Fix a $\mu_\lambda \in \mathfrak{M}(z, \lambda) \cap \mathsf{P}_\rho$ for each $\lambda > 0$. Note that

$$\lambda w + \widetilde{H}(x, Dw) = 0 \quad \text{in } \mathbb{T}^n,$$

where $\widetilde{H}(x,p) := \sup_\xi(\xi \cdot p - L(x,\xi) - \lambda w(x))$.

By the formula

$$\lambda w(z) = \min_{\mu \in \mathfrak{C}(z,\lambda)} \int (L(x,\xi) + \lambda w(x))\mu(dx, d\xi),$$

we have

$$\begin{aligned} \lambda w(z) &\leq \int (L(x,\xi) + \lambda w(x))\mu_\lambda \\ &= \lambda u_\lambda(z) + \lambda \int w(x)\mu_\lambda \\ &= \lambda u_\lambda(z) + \lambda \int w(x)\mu_\lambda. \end{aligned}$$

By passing to a subsequence, we may assume that for some $\mu_0 \in \mathfrak{M}(0)$,

$$\mu_\lambda \xrightarrow{\text{weak}^*} \mu_0 \ \ (\lambda = \lambda_j \to 0+).$$

In the limit as $\lambda = \lambda_j \to 0+$,

$$w(z) \leq v(z) + \int w(x)\mu_0(dx, d\xi) \leq v(z). \qquad \square$$

- We have shown

$$\lim_{\lambda \to 0^+} u_\lambda(x) = \max_{w \in \mathcal{W}} w(x).$$

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