METASTABILITY FOR PARABOLIC EQUATIONS WITH DRIFT: PART II. THE QUASILINEAR CASE

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Abstract. This is the second part of our series of papers on metastability results for parabolic equations with drift. The aim is to present a self contained study, using partial differential equations methods, of the metastability properties of quasi-linear parabolic equations with a drift and to obtain results similar to those in Freidlin and Koralov [6, 8].

Notation. We work in $\mathbb{R}^n$ and write $\mathbb{S}^n$ for the space of real $n\times n$ symmetric matrices. For any $\theta\in(0,1]$, $\mathbb{S}^n(\theta)$ denotes the subset of all $a\in\mathbb{S}^n$ satisfying $\theta I\le a\le\theta^{-1}I$, where I denotes the $n\times n$ identity matrix. If $a\in\mathbb{S}^n$, then $\operatorname{tr} a$ denotes its trace, and, for $a,b\in\mathbb{S}^n$, $a\le b$ if and only if $b-a$ is a nonnegative definite matrix. Given $p\in\mathbb{R}^n$, $p\otimes p:=\Sigma_{i,j=1}^n p_ip_j$. If U is a subset of R^k for some $k\in\mathbb{N}$, then $C(U;\mathbb{S}^n(\theta))$ is the set of $\mathbb{S}^n(\theta)$-valued continuous maps from U into $\mathbb{S}^n$. For $a\in\mathbb{S}^n$ and $p\in\mathbb{R}^n$, $ap\cdot p:=\Sigma_{i,j=1}^n a_{ij}p_jp_i$. If $r_1,\,r_2\in\mathbb{R}$, then $r_1\wedge r_2:=\min\{r_1,r_2\}$ and $r_1\vee r_2:=\max\{r_1,r_2\}$ and, for $r\in\mathbb{R}$, $r_+=r\vee 0$ and $r_-=(-r)\vee 0$. We use the convention $\inf\emptyset=\infty$ and $\sup\emptyset=-\infty$. The open ball in $\mathbb{R}^n$ with radius $R>0$ and center at $x\in\mathbb{R}^n$ is $B_R(x)$, and $B_R:=B_R(0)$. Given $\Omega\subset\mathbb{R}^n$ and $\delta>0$, we write $\Omega_\delta:=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)\ge\delta\}$, and, for $T>0$, $Q_T:=\Omega\times(0,T)$; if $T=\infty$, then we write Q instead of Q_∞. The parabolic boundary of Q_T is $\partial_{\rm p}Q_T:=(\overline{\Omega}\times\{0\})\cup(\partial\Omega\times(0,T))$. We denote by $\operatorname{Lip}(A,\mathbb{R}^k)$ the set of the R^k valued Lipschitz continuous functions defined in $A\subset\mathbb{R}^k$; when $k=1$, we often write $\operatorname{Lip}(A)$. We write $\operatorname{USC}(A)$ and $\operatorname{LSC}(A)$ for the set of upper- and lower-lower semicontinuous functions defined on A, and, when A is open, $C^{2,1}(A)$ is the space of functions which are continuously differentiable twice with respect to space and once with respect to time. Given a bounded family of functions $f_\delta:A\to\mathbb{R}$, $\limsup^\star f_\delta(x):=\lim_{r\to0}\sup\{f_\delta(x+y)\,:\,x+y\in A,|y|+\delta\le r\}$ and $\liminf_\star f_\delta(x):=\lim_{r\to0}\inf\{f_\delta(x+y):x+y\in A,|y|+\delta\le r\}$. If A is a closed subset of $\mathbb{R}^n$ and $f:A\to\mathbb{R}$, $\arg\min(f|A):=\{x\in A\,:\,f(x)=\min_{y\in A}f(y)\}$. We use C to denote constants, which may change from line to line. If we want to display the dependence of a constant C on a parameter a, we write $C=C(a)$. Finally, for $a,b\in\mathbb{R}$, $a\approx b$ means that a and b are close to each other in a controlled way.

1. Introduction

This is the second part of our series of papers on metastability results for parabolic equations with drift. The aim is to present a self contained study, using partial differential

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equations methods, of the metastability properties of quasi-linear parabolic equations with a drift and to obtain results similar to those in Freidlin and Koralov [6, 8].

More precisely we are interested in the asymptotic behavior, as $\varepsilon \to 0$ and $t \to \infty$, of the solution $u^\varepsilon = u^\varepsilon(x,t)$ of the initial-boundary value problem

$$u_t^\varepsilon = \varepsilon \operatorname{tr}[a(x,u^\varepsilon)D^2u^\varepsilon] + b(x)\cdot Du^\varepsilon \quad \text{in} \quad Q, \tag{1.1}$$

and

$$u^\varepsilon = g \quad \text{on } \partial_{\mathrm{p}}Q, \tag{1.2}$$

where

$$\Omega \text{ is a bounded } C^1\text{-domain with outward normal vector } \nu \tag{1.3}$$

and

$$g \in C(\partial_{\mathrm{p}}Q). \tag{1.4}$$

Throughout the paper we assume that, for some $\theta_0 \in (0,\,1]$,

$$a \in C(\overline{\Omega}\times\mathbb{R};\mathbb{S}^n(\theta_0)), \tag{1.5}$$

and

$$b \in \mathrm{Lip}(\mathbb{R}^n;\mathbb{R}^n) \quad \text{with} \quad b(0)=0. \tag{1.6}$$

is such that

$$\begin{aligned}&\text{the origin is a (unique) globally asymptotically stable point of}\\ &\text{the dynamical system } \dot X = b(X) \text{ generated by } b.\end{aligned} \tag{1.7}$$

This last assumption is further quantified by the additional requirements that b points inward at the boundary points of Ω, that is,

$$b\cdot\nu < 0 \quad \text{on} \quad \partial\Omega, \tag{1.8}$$

and there exist $b_0 > 0$ and $r_0 > 0$ such that such that $\overline{B}_{r_0} \subset \Omega$, and

$$b(x)\cdot x \le -b_0|x|^2 \quad \text{for all} \quad x \in B_{r_0}. \tag{1.9}$$

For later use we summarize all the above assumptions in the list

$$(1.3),\ (1.4),\ (1.5),\ (1.6),\ (1.7),\ (1.8) \text{ and } (1.9). \tag{1.10}$$

As mentioned above the goal is to present a self-contained proof, which are stated below as Theorem 1. Our arguments are based entirely on a partial differential (pde for short) methods and the main tools are the comparison principle and the construction of two kinds of barrier functions for parabolic equations. The later was a main subject of our previous paper [11].

We work with either classical or viscosity solutions depending on the context and most of the times we say solution without making a distinction.

An important tool is the quasi-potential V^c associated, for each $c \in \mathbb{R}$, with $(a(\cdot,c),b)$, which is characterized by the property

$$V^c \text{ is the maximal subsolution of } H^c(x,Du)=0 \text{ in } \Omega \text{ and } u(0)=0,$$

where the Hamiltonian $H^c \in C(\overline{\Omega}\times\mathbb{R}^n)$ is given by

$$H^c(x,p) := a(x,c)p\cdot p + b(x)\cdot p.$$

Next we introduce some terminology and recall the notation and hypotheses in [6, 8].

Given $g \in C(\partial_{\mathrm{p}} Q)$, we set

$$g_{\min} := \min_{\overline{\Omega}} g, \quad g_{\max} := \max_{\overline{\Omega}} g, \quad g_1 := \min_{\partial\Omega} g, \quad g_2 := \max_{\partial\Omega} g,$$

and note that $[g_1,\, g_2] \subset [g_{\min},\, g_{\max}]$.

Consider the map $M : [g_{\min},\, g_{\max}] \to \mathbb{R}$ given by

$$M(c) := \min_{\partial\Omega} V^c. \tag{1.11}$$

The continuity of a and the stability properties of viscosity solutions yield that the functions $[g_{\min},\, g_{\max}] \ni c \to M(c)$ and $\overline{\Omega} \times [g_{\min},\, g_{\max}] \ni (x, c) \to V^c(x) \in \mathbb{R}$ are continuous. Indeed the continuity of the latter is an easy consequence of the fact that V^c is the unique (viscosity) solution $u \in \mathrm{Lip}(\overline{\Omega})$ of the state-constraints problem for the Hamilton-Jacobi equation $H(x, Du) = 0$ in Ω, with the additional condition that $u(0) = 0$. (See Lemma C.1 in Appendix C for the uniqueness of this state-constraints problem, and also Soner [14], Fleming and Soner [5] and Ishii [10] for some related aspects.)

Following [6, 8], we assume that

(1.12) there exist finitely many $c^1, ..., c^k \in [g_{\min},\, g_{\max}]$ such that, if $c \in [g_{\min},\, g_{\max}] \setminus \{c^1, ..., c^k\}$, then the minimum in (1.11) is achieved at a single point,

$$c_0 := g(0) \notin \{c^1, ..., c^k\}, \tag{1.13}$$

and

(1.14) for any $i \in \{1, ..., k\}$, the minimum in (1.11), with $c = c^i$, is achieved exactly at two points in $\partial\Omega$.

For $c \in [g_{\min},\, g_{\max}] \setminus \{c^1, ..., c^k\}$, $x_*(c)$ denotes the unique minimum point in (1.11). It is easily seen by (1.12) and the joint continuity of $V^c(x)$ in x and c, that $x_* : [g_{\min},\, g_{\max}] \setminus \{c^1, ..., c^k\} \to \partial\Omega$ is continuous.

Moreover, (1.14) and again the continuity of $V^c(x)$ in (x, c) imply that x_* has left and right limits at the points c^i of discontinuity.

For $i \in \{1, ..., k\}$, we set

$$x_1(c^i) := \lim_{c \to c^i,\, c < c^i} x_*(c) \;\text{ if }\; c^i \neq g_{\min} \;\text{ and }\; x_2(c^i) := \lim_{c \to c^i,\, c > c^i} x_*(c) \;\text{ if }\; c^i \neq g_{\max}.$$

If $c^i = g_{\min}$ (resp. $c^i = g_{\max}$), $x_1(c^i)$ (resp. $x_2(c^i)$) is the minimum point in (1.11) with $c = c^i$ different from $x_2(c^i)$ (resp. $x_1(c^i)$).

We assume that

$$\text{for any } i \in \{1, ..., k\}, \text{ if } g_{\min} < c^i < g_{\max}, \text{ then } \lim_{c \to c^i, c < c^i} x_*(c) \neq \lim_{c \to c^i, c > c^i} x_*(c), \tag{1.15}$$

which implies that

$$x_1(c^i) \neq x_2(c^i) \;\text{ for all }\; i \in \{1, ..., k\}.$$

Let $G_1(c^i) := g(x_1(c^i))$ and $G_2(c^i) := g(x_2(c^i))$ and consider the piecewise continuous function $G : [g_{\min},\, g_{\max}] \to [g_1, g_2]$ defined by

$$\begin{cases} G(c) := g(x_*(c)) & \text{for } c \in [g_{\min},\, g_{\max}] \setminus \{c^1, ..., c^k\}, \\ G(c^i) := G_1(c^i) & \text{for } i \in \{1, ..., k\}. \end{cases}$$

We define c_1 as follows: if $G(c_0) \geq c_0$, then $c_1 := \inf\{c \in [c_0, \infty) : G(c) \leq c\}$, and, if $G(c_0) \leq c_0$, then $c_1 := \sup\{c \in (-\infty, c_0] : G(c) \geq c\}$, and note that , since $G([g_{\min}, g_{\max}]) \subset [g_1, g_2]$, we always have $c_1 \in [g_1, g_2]$.

Next we suppose that the graph of G crosses the diagonal from the left to the right at c_1, that is

$$\begin{cases} \text{for all } \delta_0 > 0, \text{ there exists } \delta \in (0, \delta_0] \text{ such that} \\ \qquad \text{if } c_1 > g_{\min}, \text{ then } G(c_1 - \delta) > c_1 - \delta, \\ \qquad \text{if } c_1 < g_{\max}, \text{ then } G(c_1 + \delta) < c_1 + \delta. \end{cases} \tag{1.16}$$

We define the function $c : (0, \infty) \to [g_{\min}, g_{\max}]$ as follows: For each $\lambda \in (0, \infty)$,

$$c(\lambda) := \begin{cases} c_0 \ \text{ if either } \lambda < M(c_0) \ \text{ or } \ c_1 = c_0, \\ \min(c_1, \inf\{c \in [c_0, c_1] : M(c) = \lambda\}) \ \text{ if } \ \lambda \geq M(c_0) \ \text{ and } \ c_1 > c_0, \\ \max(c_1, \sup\{c \in [c_1, c_0] : M(c) = \lambda\}) \ \text{ if } \ \lambda \geq M(c_0) \ \text{ and } \ c_1 < c_0. \end{cases} \tag{1.17}$$

For later use we summarize the above assumptions in the list

$$(1.12),\ (1.13),\ (1.14),\ (1.15) \text{ and } (1.16). \tag{1.18}$$

The definition of $c(\lambda)$ is cumbersome. For clarity and to compare with the linear problem, we discuss what happens when $a(x, c)$ is independent of c. In this case the quasi-potential V and its minimum value $M = \min_{\partial\Omega} V$ do not depend on c. Assumption (1.12) then states that V takes its minimum value M over $\partial\Omega$ at a unique point x^*. The function G is a constant given by $G(c) = g(x^*)$ and we have $c_0 = g(0)$ and $c_1 = g(x^*)$. It is easily checked that, if $g(0) = g(x^*)$, then $c(\lambda) = g(0) = g(x^*)$ for all $\lambda > 0$, and, if either $g(0) < g(x^*)$ or $g(0) > g(x^*)$,

$$c(\lambda) = \begin{cases} c_0 & \text{if } \ \lambda \leq M, \\ c_1 & \text{if } \ \lambda > M. \end{cases}$$

Note that, if either $g(0) < g(x^*)$ or $g(0) > g(x^*)$, $c(\lambda)$ is discontinuous at $\lambda = M$.

The main result, which is similar to [6, Theorem 3.1; 8], is:

Theorem 1. *Assume* (1.10) *and* (1.18) *and let $\lambda > 0$ be a point of continuity of c. If, for $\varepsilon \in (0, 1)$, $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ is a solution of* (1.1) *and* (1.2)*, then, for each $\delta > 0$ so that $\Omega_\delta \neq \emptyset$,*

$$\lim_{\varepsilon \to 0} u^\varepsilon(\cdot, \exp(\lambda/\varepsilon)) = c(\lambda) \quad \textit{uniformly in} \quad \Omega_\delta.$$

In view of the previous discussion, when $a(x, c)$ is independent of c, that is for linear equations, Theorem 1 is a slightly less general version of [11, Theorem 1].

As in [6, 8], to prove Theorem 1 we need to show the following proposition, which was proved in [8] using several large deviation results from [9].

Theorem 2 (Lemma 3.11 of [8])**.** *Assume* (1.10) *and* (1.18) *and let $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a solution of* (1.1) *and* (1.2) *with $\varepsilon \in (0, 1)$. Suppose there exist constants $a_1, a_2, \mu_k, \lambda_k, \beta_1, \beta_2$ and $\varepsilon_k \to 0$ as $k \to \infty$ such that, for all $k \in \mathbb{N}$,*

$$0 < a_1 \leq \mu_k < \lambda_k \leq a_2, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \textit{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

If $g_{\min} < \beta_1 < \beta_2 < g_{\max}$, then neither of the following is possible:

(A) *There exists* $\delta > 0$ *such that* $\lambda_k < M(\beta_2) - \delta$,
(B) *There exists* $\delta > 0$ *such that* $G(c) < \beta_2 - \delta$ *for all* $c \in [\beta_2 - \delta,\, \beta_2 + \delta]$.

If $g_{\min} < \beta_2 < \beta_1 < g_{\max}$, *then neither of the following is possible:*

(A′) *There exists* $\delta > 0$ *such that* $\lambda_k < M(\beta_2) - \delta$,
(B′) *There exist* $\delta > 0$ *such that* $G(c) > \beta_2 + \delta$ *for all* $c \in [\beta_2 - \delta,\, \beta_2 + \delta]$.

We discuss next some of the new ideas that are needed to prove the main theorem.

Recall that we are interested in the asymptotic behavior, as $(\varepsilon, t) \to (0, \infty)$, of the solution u^ε of (1.1) and (1.2) in a logarithmic time scale, that is, in the behavior, as $\varepsilon \to 0$, of $u^\varepsilon(x, \exp(\lambda/\varepsilon))$ for any fixed $\lambda > 0$. It turns out that this is a consequence of what we call "uniform asymptotic constancy" which yields that $u^\varepsilon(\cdot, t)$ behaves similarly to $u^\varepsilon(0, t)$ in the space $C(\Omega)$ equipped with the locally uniform convergence topology,

The uniform asymptotic constancy (see Theorem 8 below) is a crucial observation that goes beyond [11]. Roughly it says that, if u^ε is a bounded solution of (2.1), then, as $\varepsilon \to 0$, for any compact $K \subset \Omega$ and $\delta > 0$,

$$u^\varepsilon(x,t) \approx u^\varepsilon(0,t) \quad \text{uniformly for } \ (x,t) \in K \times [\mathrm{e}^{\delta/\varepsilon},\, \infty).$$

With the asymptotic constancy at hand the main theorem, Theorem 1, is an easy consequence of Theorem 2.

The proof of Theorem 2 is based on the comparison (or maximum) principle and, thus, on the construction of barriers, that is sub- and super-solutions of (1.1). We have already built such functions in our previous work [11], where the matrix $a(x, c)$ is independent of c. Here (see Theorem 11) we modify the construction of one class of barrier functions in order to make the comparison argument straightforward.

The building block of the barrier functions in [11] and here is viscosity solutions of $H_\alpha(x, Du) = 0$ with some additional normalization conditions, where $\alpha \in C(\overline{\Omega}; \mathbb{S}^n(\theta_0))$ is is selected as explained below and $H_\alpha(x,p) := \alpha(x)p\cdot p + \underline{\ }(x)\cdot p$. If V_α is the quasi-potential associated with (α, b), then $V_\alpha > 0$ in $\overline{\Omega} \setminus \{0\}$ and $M_\alpha := \min_{\partial\Omega} V_\alpha > 0$.

The barriers $w^\varepsilon : \overline{Q} \to \mathbb{R}$ are supersolutions of (1.1) of the form

$$w^\varepsilon(x,t) := \exp\left(\frac{v(x) - m}{\varepsilon}\right) + d_\varepsilon t,$$

where m and d_ε are positive constants such that $0 < m < M_\alpha$ and $d_\varepsilon = \exp(-\lambda_\varepsilon/\varepsilon)$ for some $\lambda_\varepsilon \approx m$, and v is an appropriately chosen smooth approximation of V_α. The choice of m yields that, for ε sufficiently small, w^ε is compatible with the Dirichlet data g on $\partial\Omega \times [0, \infty)$.

In view of the fact that a priori we have little knowledge of the uniform in ε regularity of solutions of (1.1), given such a solution u^ε, we treat $a(x, u^\varepsilon(x,t))$ as an arbitrary element $a^\varepsilon = a(x, u^\varepsilon(x,t))$ of $C(\overline{Q}; \mathbb{S}^n(\theta_0))$.

To motivate the choice of α in the barrier function given the a^ε above we compute in Q

$$\begin{aligned} &w^\varepsilon_t - \operatorname{tr}[a^\varepsilon(x,t)D^2 w^\varepsilon] - b\cdot Dw^\varepsilon \\ &\quad = d_\varepsilon - \varepsilon^{-1}\exp\left(\frac{v(x)-m}{\varepsilon}\right)\left(H_\varepsilon(x,t,Dv) - \varepsilon\operatorname{tr}[a^\varepsilon D^2 v]\right) \end{aligned}$$

with $H_\varepsilon(x,t,p) := a^\varepsilon(x,t)p\cdot p + b(x)\cdot p$.

If α satisfies $a^\varepsilon(x,t) \le \alpha(x)$ for any $(x,t) \in Q$, then

$$w_t^\varepsilon - \operatorname{tr}[a^\varepsilon(x,t)D^2 w^\varepsilon] - b \cdot Dw^\varepsilon \ge d_\varepsilon - \varepsilon^{-1} \exp\left(\frac{v(x)-m}{\varepsilon}\right)(H_\alpha(x,Dv) - O(\varepsilon)) \ge 0,$$

with the last the inequality holding, if ε is sufficiently small, because of the choice of v and d_ε –the details are given in Theorem 10.

A very important fact in our analysis (see Theorem 9 below for the precise statement) is that the local uniform topology of $C(\Omega)$ is strong enough to imply that, if $\alpha(x) \approx a(x,c)$ in $C(\Omega)$, then $M_\alpha \approx M(c)$ and $\arg\min(V_\alpha \,|\, \partial\Omega) \approx \arg\min(V^c \,|\, \partial\Omega)$.

To describe the idea which is in the core of the proof of, for example, the claims (A) and (A′) of Theorem 2, we consider the very special case that, for $\varepsilon > 0$ sufficiently small and some constants c, $\gamma > 0$ and $0 < \delta < \mu < \lambda$,

$$|u^\varepsilon(0,t) - c| < \gamma \quad \text{for all} \quad t \in [\exp(\delta/\varepsilon), \exp(\lambda/\varepsilon)],$$

$$u^\varepsilon(0, \exp(\delta/\varepsilon)) = c \quad \text{and} \quad u^\varepsilon(0, \exp(\mu/\varepsilon)) > c + \eta \quad \text{for some } \eta \in (0,\gamma).$$

We then choose $\alpha \in C(\overline{\Omega}; \mathbb{S}^n(\theta_0))$ so that $a^\varepsilon(x,t) \le \alpha(x)$ for all $(x,t) \in \Omega \times [t_\varepsilon, T_\varepsilon]$, where $t_\varepsilon := \exp(\delta/\varepsilon)$ and $T_\varepsilon := \exp(\lambda/\varepsilon)$. Using the barrier w^ε as in the linear case (see [11, Theorem 1 (i)]), we conclude that, as $\varepsilon \to 0$, for any $\rho < M_\alpha$, $u^\varepsilon(0,t) \to c$ for all $t \in [t_\varepsilon, T_\varepsilon \wedge \exp(\rho/\varepsilon)]$, which implies that $\mu \ge M_\alpha$. Furthermore, according to the previous arguments, α can be chosen, so that, as $\gamma \to 0$, $M_\alpha \to M^c$.

Organization of the paper. The rest of the paper is organized as follows. In Section 2 we study the asymptotic constancy, that is the effect of the drift term in parabolic equations like (1.1). In Section 3 we introduce Hamilton-Jacobi equations related to (1.1), which have quadratic nonlinearity, and study the continuity properties of the associated quasi-potentials. Section 4 is devoted to the construction of two kind of barrier functions, or sub- and super-solutions, which are used to study the asymptotic behavior of solutions of linear parabolic equations, that is, equations like (1.1) with $a \in C(\overline{Q}; \mathbb{S}^n(\theta_0))$. The proofs of Theorem 2 and Theorem 1 are given in Sections 5 and 6 respectively. Some basic properties of viscosity solutions are explained in the Appendices A, B and C.

2. The Asymptotic constancy

We consider the linear pde

$$u_t^\varepsilon = \varepsilon \operatorname{tr}[a^\varepsilon(x,t)D^2 u^\varepsilon] + b(x) \cdot Du^\varepsilon \quad \text{in} \quad Q. \tag{2.1}$$

We assume, in addition to (1.6) and (1.9), that

$$a^\varepsilon \in C(\overline{Q}, \mathbb{S}^n(\theta_0)). \tag{2.2}$$

The goal here is to show that the effect of the drift term in (2.1) is to propagate, as $\varepsilon \to 0$, the values of the solutions u^ε at $x = 0$ to Ω. We call this phenomenon the asymptotic constancy.

It turns out that the asymptotic constancy does not depend on any properties of a^ε other than (2.2). It is, therefore, technically more convenient to study, in some instances, instead of (2.1), the problem

$$v_t = \varepsilon P^+(D^2 v) + b(x) \cdot Dv \quad \text{in} \quad Q, \tag{2.3}$$

where P^+ is the Pucci operator associated with $\mathbb{S}^n(\theta_0)$ defined by

$$P^+(X) = \sup\{\operatorname{tr}[AX] : A \in \mathbb{S}^n(\theta_0)\}, \tag{2.4}$$

which is, obviously, uniformly elliptic with constants θ_0 and θ_0^{-1}, that is, for all matrices $X, Y \in \mathbb{S}^n$ such that $X \le Y$,

$$\theta_0 \operatorname{tr}(Y - X) \le P^+(Y) - P^+(X) \le \theta_0^{-1} \operatorname{tr}(Y - X). \tag{2.5}$$

Some useful barrier functions. We fix an auxiliary function $h \in C^2([0, \infty))$ with the properties

$$0 \le h \le 1, \ h = 0 \ \text{ in } \ [0, 1/2], \ h = 1 \ \text{ in } \ [1, \infty) \text{ and } h' \ge 0, \tag{2.6}$$

set

$$k := b_0/2 \quad \text{and} \quad R_0 := 2\sqrt{2n}/\sqrt{b_0\theta_0},$$

choose $R \in [R_0, \infty), r \in (0, r_0]$, where r_0 is as in (1.9), and $\varepsilon_0 \in (0, 1)$ so that

$$\sqrt{\varepsilon}_0 R < r, \tag{2.7}$$

and, for $\varepsilon \in (0, \varepsilon_0]$, let

$$\tau = \tau(\varepsilon) := \frac{1}{k} \log\left(\frac{r}{R\sqrt{\varepsilon}}\right). \tag{2.8}$$

With all these choices at hand we introduce the functions $p^\varepsilon, q^\varepsilon : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ defined by

$$p^\varepsilon(x, t) := h((R\sqrt{\varepsilon})^{-1}|x| \, \mathrm{e}^{-kt}) \tag{2.9}$$

and

$$q^\varepsilon(x, t) := p^\varepsilon(x, t) + \frac{\|h''\|_{L^\infty}}{R^2\theta_0} \int_0^t \mathrm{e}^{-2kt} \ dt; \tag{2.10}$$

observe that, since h vanishes identically in a neighborhood of the positive time axis $l := \{0\} \times (0, \infty)$, p^ε and q^ε are smooth in $\mathbb{R}^n \times (0, \infty)$.

We note that p^ε appears in the proof of [6, Lemma 3.6; 8]. The difference is that these references consider equations like (2.1), while here we study (2.3).

The following lemma summarizes the properties of q^ε. Its proof is based on long explicit but also straightforward calculations. The reader may want to skip the details on first reading.

Lemma 1. *Assume* (1.6), (1.9) *and* (2.5). *With the above choices of k, R, ε_0, ε and τ, the function q^ε given by* (2.10) *is a supersolution to* (2.3) *in $B_{r_0} \times (0, \infty)$. Moreover,*

$$\begin{cases} q^\varepsilon(\cdot, 0) \ge 0 \ \ in \ \ B_r, \quad q^\varepsilon(\cdot, 0) \ge 1 \ \ in \ B_r \setminus B_{\sqrt{\varepsilon}R}, \\ q^\varepsilon \ge 1 \ \ in \ \partial B_r \times [0, \tau] \ \ and \ \ q^\varepsilon(\cdot, \tau) \le \frac{\|h''\|_{L^\infty}}{b_0\theta_0 R^2} \ \ on \ \ B_{r/2}. \end{cases}$$

Proof. First note that

$$p^\varepsilon(x, 0) = 1 \quad \text{if} \ \ |x| \ge R\sqrt{\varepsilon} \quad \text{and} \quad p^\varepsilon(x, t) = 0 \quad \text{if} \ \ |x| \le \frac{1}{2} R\sqrt{\varepsilon} \, \mathrm{e}^{kt}.$$

For $(x,t) \in B_{r_0} \times (0,\infty)$ we write $\rho = \frac{1}{R\sqrt{\varepsilon}}$, $r_{x,t} = (R\sqrt{\varepsilon})^{-1}|x|\,\mathrm{e}^{-kt}$ and $\bar{x} := x/|x|$ (since, in view of the above, p^ε vanishes in a neighborhood of the origin we do not have to be concerned about $x=0$), and find

$$\begin{aligned} p_t^\varepsilon(x,t) &= -kh'(r_{x,t})|x|\rho\,\mathrm{e}^{-kt}, \qquad Dp^\varepsilon(x,t) = h'(r_{x,t})\rho\bar{x}\,\mathrm{e}^{-kt}, \\ D^2p^\varepsilon(x,t) &= h'(r_{x,t})\rho\,\mathrm{e}^{-kt}\frac{1}{|x|}(I-\bar{x}\otimes\bar{x}) + h''(r_{x,t})\rho^2\,\mathrm{e}^{-2kt}\,\bar{x}\otimes\bar{x}. \end{aligned}$$

Moreover, for any $a \in \mathbb{S}^n(\theta_0)$ and all $(x,t) \in \overline{Q}$ with $x \neq 0$, we have

$$|\operatorname{tr}[a(I-\bar{x}\otimes\bar{x})]| \le \theta_0^{-1}(n-1) < \theta_0^{-1}n \quad \text{ and } \quad |\operatorname{tr}[a\bar{x}\otimes\bar{x}]| \le \theta_0^{-1},$$

and, therefore,

$$\begin{aligned} &p_t^\varepsilon - \varepsilon\operatorname{tr}[aD^2p^\varepsilon] - b(x)\cdot Dp^\varepsilon \\ &\quad = h'(r_{x,t})\rho|x|\,\mathrm{e}^{-kt}\left\{-k - |x|^{-1}b(x)\cdot\bar{x} - \frac{\varepsilon}{|x|^2}\operatorname{tr}[a(I-\bar{x}\otimes\bar{x})]\right\} \\ &\quad\quad - \varepsilon h''(r_{x,t})\rho^2\,\mathrm{e}^{-2kt}\operatorname{tr}[a\bar{x}\otimes\bar{x}] \\ &\quad \ge h'(r_{x,t})\rho|x|\,\mathrm{e}^{-kt}\left\{-k + b_0 - \frac{n\varepsilon}{\theta_0|x|^2}\right\} - \varepsilon\|h''\|_{L^\infty}\rho^2\,\mathrm{e}^{-2kt}\,\theta_0^{-1}. \end{aligned}$$

Observe that

$$\frac{1}{2} \le |x|\rho\,\mathrm{e}^{-kt} \le 1 \quad \text{ if and only if } \quad \frac{1}{2}R\sqrt{\varepsilon}\,\mathrm{e}^{kt} \le |x| \le R\sqrt{\varepsilon}\,\mathrm{e}^{kt}, \tag{2.11}$$

and

$$h'(|x|\rho\,\mathrm{e}^{-kt})\frac{1}{|x|^2} \le h'(|x|\rho\,\mathrm{e}^{-kt})\frac{4\,\mathrm{e}^{-2kt}}{R^2\varepsilon} \le h'(|x|\rho\,\mathrm{e}^{-kt})\frac{4}{R^2\varepsilon}.$$

Using the observations above and (1.9) and recalling the choices of the constants and that $a \in \mathbb{S}^n(\theta_0)$ is arbitrary, we get

$$\begin{aligned} &p_t^\varepsilon - \varepsilon P^+(D^2p^\varepsilon) - b(x)\cdot Dp^\varepsilon \\ &\quad \ge h'(r_{x,t})\rho|x|\,\mathrm{e}^{-kt}\left\{-k + b_0 - \frac{4n}{\theta_0R^2}\right\} - \|h''\|_{L^\infty}\frac{\mathrm{e}^{-2kt}}{\theta_0R^2} \ge -\|h''\|_{L^\infty}\frac{\mathrm{e}^{-2kt}}{\theta_0R^2}. \end{aligned}$$

Thus, noting that, for all $t > 0$,

$$p_t^\varepsilon(0,t) - \varepsilon P^+(D^2p^\varepsilon(0,t)) - b(0)\cdot Dp^\varepsilon(0,t) = 0$$

we conclude that

$$p_t^\varepsilon - \varepsilon P^+(D^2p^\varepsilon) - b(x)\cdot Dp^\varepsilon \ge -\|h''\|_{L^\infty}\frac{\mathrm{e}^{-2kt}}{\theta_0R^2} \quad \text{in} \quad B_{r_0}\times(0,\infty),$$

and, hence, q^ε is a supersolution of (2.3) in $B_{r_0}\times(0,\infty)$.

Finally, we observe that, if $0 \le t \le \tau$ and $x \in \partial B_r$, then

$$\frac{|x|\,\mathrm{e}^{-kt}}{\sqrt{\varepsilon}R} \ge \frac{r\,\mathrm{e}^{-k\tau}}{\sqrt{\varepsilon}R} = 1 \quad \text{ and } \quad q^\varepsilon(x,t) \ge p^\varepsilon(x,t) = 1,$$

and, if $x \in B_{r/2}$, then

$$\frac{|x|\,\mathrm{e}^{-k\tau}}{\sqrt{\varepsilon}R} \le \frac{r\,\mathrm{e}^{-k\tau}}{2\sqrt{\varepsilon}R} = \frac{1}{2} \quad \text{ and } \quad q^\varepsilon \le p^\varepsilon + \frac{\|h''\|_{L^\infty}}{b_0\theta_0R^2} = \frac{\|h''\|_{L^\infty}}{b_0\theta_0R^2}.$$

Moreover,

$$q^\varepsilon(x,0) = p^\varepsilon(x,0) = h(|x|/(\sqrt{\varepsilon}R)) \geq \begin{cases} 0 & \text{for all } \ x \in B_r, \\ 1 & \text{for all } \ x \in B_r \setminus B_{\sqrt{\varepsilon}R}. \end{cases} \qquad \square$$

An application of the Harnack inequality. We use a consequence of the Harnack inequality to obtain an a priori bound for the oscillations of the u^ε's, which are uniform in ε and t up to ∞.

If $u^\varepsilon \in C^{2,1}(Q)$ is a solution of (2.1), then

$$v^\varepsilon(y,t) := u^\varepsilon(\sqrt{\varepsilon}y, t) \quad \text{ for } \ (y,t) \in B_{r_0/\sqrt{\varepsilon}} \times [0,\infty),$$

satisfies

$$v^\varepsilon_t = \mathrm{tr}[a^\varepsilon(\sqrt{\varepsilon}y,t)D^2 v^\varepsilon] + \frac{b(\sqrt{\varepsilon}y)}{\sqrt{\varepsilon}} \cdot Dv^\varepsilon \ \text{ in } \ B_{r_0/\sqrt{\varepsilon}} \times (0,\infty). \tag{2.12}$$

It also follows from (1.6) that there exist $L_b > 0$ such that

$$|b(x)| \leq \mathcal{L}_b|x| \quad \text{ for all } \ x \in B_{r_0},$$

and, hence,

$$\frac{|b(\sqrt{\varepsilon}y)|}{\sqrt{\varepsilon}} \leq L_b|y| \quad \text{ for all } \ y \in B_{r_0/\sqrt{\varepsilon}}. \tag{2.13}$$

Next we recall the following consequence of the Harnack inequality from Krylov [12, Theorem 4.2.1].

Theorem 3. *Assume* (2.2) *and* (2.13)*, fix* $R \in (0,2]$*,* $(z,\tau) \in \mathbb{R}^n \times (0,\infty)$ *such that* $B_R(z) \subset B_{r_0/\sqrt{\varepsilon}}$ *and* $\tau > 2R^2$*, and let* $w \in C^{2,1}(B_R(z) \times (\tau - 2R^2, \tau))$ *be a nonnegative solution of* (2.12) *in* $B_R(z) \times (\tau - 2R^2, \tau)$*. There exists a constant* $C = C(R, \theta_0, L_b, n) > 1$ *such that*

$$w(z, \tau - R^2) \leq C \inf_{y \in B_{R/2}(z)} w(y,\tau).$$

We use now Theorem 3 to obtain the following improvement of oscillation-type result for solutions to (2.1).

Corollary 4. *Assume* (2.2) *and* (2.13) *and, for* $\varepsilon \in (0,1)$*, let* $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ *be a solution of* (2.1) *in* Q*. Fix* $m \in \mathbb{N}$ *and* $T > 0$ *and assume that* $(m+2)\sqrt{\varepsilon} \leq r_0$*,* $T > 4(m+1)$ *and*

$$\begin{cases} u^\varepsilon(0,t) \leq 0 & \textit{for all } \ t \in (0,T), \\ u^\varepsilon(x,t) \leq 1 & \textit{for all } \ (x,t) \in B_{(m+2)\sqrt{\varepsilon}} \times (0,T). \end{cases} \tag{2.14}$$

There exists a constant $\eta = \eta(m, \theta_0, L_b, n) \in (0,1)$ *such that*

$$u^\varepsilon \leq \eta \ \textit{ in } \ B_{m\sqrt{\varepsilon}} \times (4(m+1), T).$$

Proof. Noting that the function $v^\varepsilon(y,t) = u^\varepsilon(\sqrt{\varepsilon}y,t)$ is defined on $B_{m+2} \times (0,T)$, we set

$$w(x,t) = 1 - v^\varepsilon(x,t) \quad \text{ for } \ (x,t) \in B_{m+2} \times (0,T).$$

Observe that w is a solution of (2.12) in $B_{m+2} \times (0,T)$ and, by (2.14), that w is a nonnegative function on $B_{m+2} \times (0,T)$ and satisfies

$$w(0,t) \geq 1 \quad \text{ for all } \ t \in (0,T).$$

Let $(x,t) \in B_m \times (4(m+1), T)$ and choose a finite sequence $B_1(x_1), ..., B_1(x_m) \subset B_m$ of balls so that $x_1 = 0$, $x \in B_1(x_m)$ and, if $1 \le i < m$, then $B_1(x_{i+1}) \cap B_1(x_i) \neq \emptyset$. We apply Theorem 3, with $R = 2$, to get, for some $C = C(\theta_0, L_b, n) > 1$,

$$w(0, t-4m) \le C \inf_{y \in B_1(x_1)} w(y, t - 4(m-1)).$$

Hence, if $m = 1$, we have

$$w(0, t-4m) \le C^m w(x,t).$$

If $m > 1$, repeating the argument above we obtain

$$\begin{aligned} w(0,t-4m) &\le C w(x_2, t-4(m-1)) \le C^2 \inf_{y \in B_1(x_2)} w(y, t-4(m-2)) \\ &\le \cdots \le C^m \inf_{y \in B_1(x_m)} w(y,t) \le C^m w(x,t). \end{aligned}$$

Thus, we have $w(0,t-4m) \le C^m w(x,t)$, and, since $w(0,t-4m) \ge 1$ by (2.14), we get

$$1 \le C^m (1 - v^\varepsilon(x,t)),$$

which yields

$$v^\varepsilon(x,t) \le 1 - \frac{1}{C^m},$$

and, hence, with $\eta = 1 - 1/C^m$,

$$u^\varepsilon(x,t) \le \eta \quad \text{for all } (x,t) \in B_{m\sqrt{\varepsilon}} \times (4(m+1), T). \qquad \square$$

The asymptotic constancy. Let Π be a relatively open, possibly empty, subset of $\partial\Omega$, set $\Omega^\Pi := \Omega \cup \Pi$, and, for any $\delta > 0$,

$$\Omega_\delta := \{x \in \overline{\Omega} \,:\, \operatorname{dist}(x, \partial\Omega) > \delta\} \quad \text{and} \quad \Omega_\delta^\Pi := \{x \in \overline{\Omega} \,:\, \operatorname{dist}(x, \partial\Omega \setminus \Pi) > \delta\}.$$

The next result is the first indication of what we call asymptotic constancy, which is a straightforward generalization of [11, Theorem 14]. Roughly it says that, for ε small, if a solution of (2.1) is bounded and small (say negative) in a small cylinder around the positive time axis l and a portion of the parabolic boundary, then it is small (of order $\delta > 0$) in a large part of Q after some uniform time depending on δ.

Theorem 5. *Assume* (1.3)*,* (1.6)*,* (1.7)*,* (1.9) *and* (2.2) *and fix* $\delta \in (0, r_0)$*. There exist* $T_\delta > 0$ *and* $\varepsilon_0 \in (0,1)$*, which depend only on* δ*,* θ_0*,* b *and* Ω*, such that, if, for* $\varepsilon \in (0, \varepsilon_0)$*,* $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ *is a solution of* (2.1) *and satisfies, for some* $T(\varepsilon) \in (T_\delta, \infty]$*,*

$$u^\varepsilon \le 1 \;\; in \;\; \Omega \times [0, T(\varepsilon)) \;\; and \;\; u^\varepsilon \le 0 \;\; in \;\; (B_\delta \cup \Pi) \times [0, T(\varepsilon)),$$

then

$$u^\varepsilon(x,t) \le \delta \quad for\ all \;\; (x,t) \in \Omega_\delta^\Pi \times [T_\delta, T(\varepsilon)).$$

For the proof of Theorem 5 it is necessary to first describe some preliminary facts that are consequence of the asymptotic stability property of the vector field b.

We fix $\delta > 0$ and set

$$\tau(x) := \sup\{t \ge 0 \,:\, X(t,x) \notin B_\delta\} \quad \text{for} \;\; x \in \overline{\Omega},$$

where $X(t) = X(t,x)$ denotes the solution of

$$\dot{X}(t;x) = b(X(t;x)) \quad \text{and} \quad X(0;x) = x.$$

Since Ω is bounded and the origin is a globally asymptotically stable point of b, it is immediate that, if

$$T_\delta := \sup_{x\in\overline{\Omega}} \tau(x), \tag{2.15}$$

then

$$0 < T_\delta < \infty \ \text{ and } X(t,x) \in B_\delta \text{ for all } (x,t) \in \overline{\Omega} \times t \geq T_\delta. \tag{2.16}$$

We consider the transport problem

$$\begin{cases} U_t \leq b \cdot DU & \text{in } \Omega \times (0, T_\delta], \\ \min\{U_t - b\cdot DU, U\} \leq 0 & \text{on } \Pi \times (0, T_\delta], \\ U \leq 0 & \text{in } B_\delta \times \{0\}. \end{cases} \tag{2.17}$$

The first inequality in (2.17) should be understood in the viscosity subsolution sense and while the second is a viscosity interpretation of the Dirichlet condition, $U \leq 0$, on Π (see [10]).

Lemma 2. *Assume* (1.3), (1.6), (1.7) *and* (1.8). *If* $U \in \mathrm{USC}(\overline{\Omega} \times [0, T_\delta])$ *is a subsolution of* (2.17), *then* $U(x, T_\delta) \leq 0$ *for all* $x \in \Omega^\Pi$.

Proof. Fix any $x \in \Omega^\Pi$ and, for $t \in [0, T_\delta]$, set

$$u(t) = U(X(T_\delta - t, x), t).$$

It is a standard observation (see Lemma A.1 in Appendix A) that $u \in \mathrm{USC}([0, T_\delta])$ is a subsolution, if $x \in \Omega$, of

$$u' \leq 0 \quad \text{in } \ (0, T_\delta], \tag{2.18}$$

and, if $x \in \Pi$, of

$$\begin{cases} u' \leq 0 & \text{in } \ (0, T_\delta), \\ u' \leq 0 \ \text{ or } \ u \leq 0 & \text{on } \ \{T_\delta\}. \end{cases} \tag{2.19}$$

Suppose that $\max_{[0, T_\delta]} u > 0$. Since $X(T_\delta, x) \in B_\delta$ and $u(0) = U(X(T_\delta, x), 0) \leq 0$, there must exist $\alpha > 0$ and $\tau \in (0, T_\delta]$ such that the function $[0, T_\delta] \ni t \to u(t) - \alpha t$ attains its maximum on $[0, T_\delta]$ at τ. In view of (2.18), if $x \in \Omega$, then $\alpha \leq 0$, which is a contradiction. If $x \in \Pi$, then either $\alpha \leq 0$ or $\tau = T_\delta$ and $u(T_\delta) \leq 0$, which is again a contradiction. Thus, we conclude that $u \leq 0$ on $[0, T_\delta]$. In particular, $u(T_\delta) \leq 0$, which shows that $U(x, T_\delta) \leq 0$ for all $x \in \Omega^\Pi$. □

We proceed with the proof of Theorem 5.

Proof of Theorem 5. Let $T_\delta > 0$ be the number defined by (2.15). For any $\varepsilon \in (0, 1)$, let $\mathcal{V}_\varepsilon$ denote the set of all (viscosity) subsolutions $v \in \mathrm{USC}(\overline{\Omega} \times [0, T_\delta])$ of (2.3) such that

$$v \leq 1 \ \text{ on } \ \overline{\Omega} \times [0, T_\delta] \ \text{ and } \ v \leq 0 \ \text{ on } \ (B_\delta \cup \Pi) \times [0, T_\delta], \tag{2.20}$$

and note that $\mathcal{V}_\varepsilon$, which is clearly nonempty, depends only on δ, T_δ, θ_0, b and Ω.

It turns out that $\mathcal{V}_\varepsilon$ has a maximum element. Indeed, for $(x,t) \in \overline{\Omega} \times [0, T_\delta]$, set

$$v^\varepsilon(x,t) := \sup\{v(x,t) : v \in \mathcal{V}_\varepsilon\}$$

and consider its upper semicontinuous envelope

$$\bar v^\varepsilon(x,t) := \lim_{r\to 0} \sup\{v^\varepsilon(y,s) : (y,s) \in \overline{\Omega} \times [0,\, T_\delta], |(y,s)-(x,t)| < r\}.$$

Standard arguments from the theory of viscosity solutions yield that $\bar v^\varepsilon \in \mathcal{V}_\varepsilon$ and, since $0 \in \mathcal{V}_\varepsilon$, $\bar v^\varepsilon \geq 0$ on $\overline{\Omega} \times [0,\, T_\delta]$.

Let $U \in \mathrm{USC}(\overline{\Omega} \times [0,\, T_\delta])$ be the half-relaxed upper limit of $\bar v^\varepsilon$, that is, for $(x,t) \in \overline{\Omega} \times [0,\, T_\delta]$,

$$U(x,t) := \limsup_{\varepsilon\to 0}{}^* \bar v^\varepsilon(x,t);$$

we refer to Crandall, Ishii and Lions [3] for more discussion about the half relaxed upper and lower limits.

It follows from Lemma 2 that

$$U(x,T_\delta) \leq 0 \quad \text{ for all } \ x \in \Omega^\Pi,$$

and, hence, in view of the uniformity encoded in the definition of U, there exists a constant $\varepsilon_0 \in (0,\, 1)$, depending only on δ, θ_0, b and Ω, such that

$$v^\varepsilon(x,T_\delta) \leq \delta \quad \text{ for all } \ x \in \Omega^\Pi_\delta \ \text{ and } \ \varepsilon \in (0,\, \varepsilon_0).$$

Finally, since, for each ε, the function

$$\overline{\Omega} \times [0,\, T_\delta] \ni (x,t) \mapsto u^\varepsilon(x, s+t),$$

with $0 \leq s < T(\varepsilon) - T_\delta$, belongs to $\mathcal{V}_\varepsilon$, it follows that, if $s \in [0,\, T(\varepsilon) - T_\delta)$, then

$$u^\varepsilon(x, s+T_\delta) \leq v^\varepsilon(x,T_\delta) \leq \delta \quad \text{ for all } \ x \in \Omega^\Pi_\delta \ \text{ and } \ \varepsilon \in (0, \varepsilon_0],$$

and, thus,

$$u^\varepsilon(x,t) \leq \delta \quad \text{ for all } (x,t) \in \Omega^\Pi_\delta \times [T_\delta,\, T(\varepsilon)) \ \text{ and } \ \varepsilon \in (0,\, \varepsilon_0]. \qquad \square$$

Next we use Corollary 4 and the previous theorem to obtain a refinement. Here we assume an upper bound, say 1, only in a cylindrical neighborhood of the positive time axis l and show that, if, in addition, the solutions are small, say less than 0 on the half line l, then they are small, say less than δ, after a time, of order $|\log\varepsilon|$, in a small cylindrical neighborhood of l. We remark that a time period of order $|\log\varepsilon|$ is "very short" in the logarithmic scale of time, that is, as $\varepsilon \to 0$, if $\exp(\lambda_\varepsilon/\varepsilon) = O(|\log\varepsilon|)$, then $\lambda_\varepsilon \to 0$.

Theorem 6. *Assume* (1.3)*,* (1.6)*,* (1.7)*,* (1.9) *and* (2.2)*. For any $\delta > 0$, there exist $\varepsilon_0 \in (0,\, 1)$ and a family $\{\tau(\varepsilon)\}_{0<\varepsilon\leq\varepsilon_0} \subset (0,\, \infty)$, both depending on r_0, θ_0, b, δ and n, and $\gamma \in (0,\, 1)$, such that, if, for $\varepsilon \in (0,\, \varepsilon_0]$, u^ε is a solution of* (2.1) *with the property that, for some $T(\varepsilon) \in (\tau(\varepsilon),\, \infty]$,*

$$u^\varepsilon \leq 1 \ \text{ in } B_{r_0} \times (0, T(\varepsilon)) \ \text{ and } \ u^\varepsilon(0,t) \leq 0 \ \text{ for all } \ t \in (0,\, T(\varepsilon)), \tag{2.21}$$

then

$$u^\varepsilon \leq \delta \quad \text{in} \quad B_{\gamma r_0} \times (\tau(\varepsilon),\, T(\varepsilon)).$$

Moreover, there exists a constant $C > 0$, which depends on r_0, θ_0, b, δ and n, such that

$$\tau(\varepsilon) \leq C(|\log\varepsilon| + 1) \quad \text{ for all } \ \varepsilon \in (0,\, \varepsilon_0].$$

Although it appears similar, Theorem 6 is actually very different from [11, Theorem 13]. Indeed the second condition in (2.21) on the solutions is required only at the origin, while in [11, Theorem 13] it is assumed on a neighborhood of the origin. This refinement, which is important for the proof of Theorem 2, depends technically on the barrier functions q^ε in Lemma 1 and the Harnack inequality (Theorem 3).

Proof of Theorem 6. To simplify the argument, we assume that $T(\varepsilon) = \infty$ since the general case can be treated similarly.

Fix $\delta > 0$, choose $h \in C^2([0, \infty))$ satisfying (2.6) and $m = m(\theta_0, n, \|h''\|_{L^\infty}) \in \mathbb{N}$ such that

$$\frac{\|h''\|_{L^\infty}}{b_0\theta_0 m^2} \leq \frac{1}{2} \quad \text{and} \quad m \geq \frac{2\sqrt{2n}}{\sqrt{b_0\theta_0}},$$

let $\eta = \eta(\theta_0, L_b, n) \in (0, 1)$ be the constant in Corollary 4, set $\tau_0 = 4(m + 1)$ and fix $\varepsilon_1 = \varepsilon_1(r_0, m) \in (0, 1)$ so that

$$(m + 2)\sqrt{\varepsilon_1} \leq r_0.$$

Then, for any $\varepsilon \in (0, \varepsilon_1]$, Corollary 4 gives

$$u^\varepsilon(x, t) \leq \eta \quad \text{for all} \;\; (x, t) \in B_{m\sqrt{\varepsilon}} \times (\tau_0, \infty).$$

Define

$$v^\varepsilon := (1 - \eta)^{-1} (u^\varepsilon - \eta) \;\; \text{in} \;\; \Omega \times [0, \infty),$$

and note that v^ε is a solution of (2.1), and, moreover,

$$v^\varepsilon \leq 1 \;\; \text{in} \;\; B_{r_0} \times (0, \infty) \quad \text{and} \;\; v^\varepsilon \leq 0 \;\; \text{on} \;\; \overline{B}_{m\sqrt{\varepsilon}} \times [\tau_0, \infty).$$

Let q^ε be given by (2.10) with R and r replaced by m and r_0 respectively. It follows from Lemma 1 and the comparison principle that, for any fixed $s \geq \tau_0$,

$$v^\varepsilon(\cdot, s + \cdot) \leq q^\varepsilon \;\; \text{in} \;\; B_{r_0} \times [0, \tau_1],$$

where $\tau_1 = \tau_1(\varepsilon) > 0$ is given by

$$\frac{\theta_0\tau_1}{2} = \log\left(\frac{r_0}{m\sqrt{\varepsilon}}\right).$$

Hence,

$$v^\varepsilon(\cdot, \cdot + \tau_1) \leq \frac{\|h''\|_{L^\infty}}{b_0\theta_0 m^2} \leq \frac{1}{2} \;\; \text{in} \;\; B_{r_0/2} \times [\tau_0, \infty),$$

which, with $T_1(\varepsilon) := \tau_0 + \tau_1(\varepsilon)$, can be rewritten as

$$u^\varepsilon \leq \eta + \frac{1 - \eta}{2} = \frac{1}{2}(1 + \eta) \;\; \text{in} \;\; B_{r_0/2} \times [T_1(\varepsilon), \infty). \tag{2.22}$$

Next, for $j = 2, , 3...$, we choose $\varepsilon_j \in (0, \varepsilon_{j-1})$ so that

$$(m + 2)\sqrt{\varepsilon_j} \leq \frac{r_0}{2^{j-1}},$$

and, for any $\varepsilon \in (0, \varepsilon_j)$, select $\tau_j = \tau_j(\varepsilon) > \tau_{j-1}(\varepsilon)$ so that

$$\frac{\theta_0\tau_j(\varepsilon)}{2} = \log\left(\frac{r_0}{2^{j-1}m\sqrt{\varepsilon}}\right),$$

and set, for $\varepsilon \in (0, \varepsilon_j)$,

$$T_j(\varepsilon) := T_{j-1}(\varepsilon) + \tau_0 + \tau_j(\varepsilon) = j\tau_0 + \sum_{i=1}^{j} \tau_i(\varepsilon).$$

We prove by induction that

$$u^\varepsilon \le \left(\frac{1+\eta}{2}\right)^j \quad \text{in} \quad B_{r_0/2^j} \times [T_j(\varepsilon), \infty). \tag{2.23}$$

Since (2.22) yields that (2.23) holds for $j = 1$, we assume that (2.23) is valid for some $j \in \mathbb{N}$, set

$$w^\varepsilon := \left(\frac{2}{1+\eta}\right)^j u^\varepsilon(\cdot, \cdot + T_j(\varepsilon)) \quad \text{in} \quad Q,$$

observe that w^ε is a solution of (2.1), with $a^\varepsilon(\cdot,\cdot)$ replaced by $a^\varepsilon(\cdot, \cdot + T_j(\varepsilon))$ and satisfies $w^\varepsilon(0,t) \le 0$ for all $t \in [0, \infty)$ and $w^\varepsilon \le 1$ in $B_{r_0/2^j} \times [0, \infty)$.

Using Lemma 1 and Corollary 4 as before, with the same m and τ_0, but with u^ε, r_0 and τ_1 replaced by w^ε, $r_0/2^j$ and τ_{j+1} respectively, we obtain

$$w^\varepsilon \le \frac{1+\eta}{2} \quad \text{in} \quad B_{r_0/2^{j+1}} \times (\tau_0 + \tau_{j+1}(\varepsilon), \infty),$$

which, after been rewritten as

$$u^\varepsilon \le \left(\frac{1+\eta}{2}\right)^{j+1} \quad \text{in } B_{r_0/2^{j+1}} \times [T_{j+1}(\varepsilon), \infty),$$

yields the claim.

Finally, selecting $j \in \mathbb{N}$ so that

$$\left(\frac{1+\eta}{2}\right)^j \le \delta,$$

setting $\varepsilon_0 = \varepsilon_j$, $\gamma = 2^{-j}$ and $\tau(\varepsilon) = T_j(\varepsilon)$, and observing that, as $\varepsilon \to 0+$, $\tau(\varepsilon) = O(|\log \varepsilon|)$ we complete the proof. □

We have by now completed all the technical steps needed for the next theorem, which is a nontrivial refinement of Theorem 5. It asserts that bounded solutions to (2.1), which are small on the positive time axis l and a part of the parabolic boundary, are actually small in almost the whole domain after some time of order $|\log \varepsilon|$. This is the mathematical statement of what we called asymptotic constancy.

Theorem 7. *Assume* (1.3), (1.6), (1.7), (1.9) *and* (2.2) *and let* $\{T(\varepsilon)\}_{\varepsilon \in (0,1)}$ *be a collection of positive numbers. For each* $\delta > 0$ *and* $C_0 > 0$*, there exist constants* $\varepsilon_0 \in (0, 1)$ *and* $C > 0$ *such that, if, for* $\varepsilon \in (0, \varepsilon_0]$, $u^\varepsilon \in C^{2,1}(Q)$ *is a solution of* (2.1) *satisfying*

$$u^\varepsilon \le C_0 \quad \textit{in} \quad \Omega \times [0, T(\varepsilon)) \quad \textit{and} \quad u^\varepsilon \le 0 \quad \textit{in } (\{0\} \cup \Pi) \times [0, T(\varepsilon)),$$

then

$$u^\varepsilon(x,t) \le \delta \qquad \textit{for all } (x,t) \textit{ in } \Omega_\delta^\Pi \times (C|\log \varepsilon|, T(\varepsilon)).$$

Proof. Theorem 6 yields constants ε_1, $\gamma \in (0, 1)$ and $C_1 > 0$ such that, for all $0 < \varepsilon \le \varepsilon_1$,

$$u^\varepsilon \le \frac{\delta}{2} \quad \text{in} \quad B_{\gamma r_0} \times [C_1 |\log \varepsilon|, T(\varepsilon)).$$

Theorem 5 applied to $v^\varepsilon(x,t) := C_0^{-1}(u^\varepsilon(x, t + C_1|\log \varepsilon|) - \delta)$ instead u^ε implies the existence of T_δ and $\varepsilon_0 \in (0, \varepsilon_1)$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$v^\varepsilon \le \frac{\delta}{2C_0} \quad \text{in} \quad \Omega_\delta^\Pi \times [T_\delta, T(\varepsilon) - C_1|\log \varepsilon|), \tag{2.24}$$

which says that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$u^\varepsilon \le \delta \ \text{ in } \ \Omega_\delta^\Pi \times [T_\delta + C_1|\log\varepsilon|,\, T(\varepsilon)),$$

and the proof is complete. □

Next we use the last result to control the difference between values of $u^\varepsilon(\cdot, t)$ and $u^\varepsilon(0, t)$.

Theorem 8. *Assume* (1.3)*,* (1.6)*,* (1.7)*,* (1.9) *and* (2.2)*. For each $\delta > 0$ and $C_0 > 0$ there exist constants $\varepsilon_0 \in (0, 1)$ and $C > 0$ such that, if, for $\varepsilon \in (0, \varepsilon_0]$, u^ε is a solution of* (2.1) *satisfying*

$$|u^\varepsilon| \le C_0 \ \text{ in } \ \Omega \times [0, \infty),$$

then

$$|u^\varepsilon(x,t) - u^\varepsilon(0,t)| \le \delta \ \text{ for all } \ (x,t) \ \text{ in } \Omega_\delta \times [C|\log\varepsilon|, \infty).$$

Proof. We double the variables and define the function $v^\varepsilon : \Omega \times \Omega \times [0, \infty) \to \mathbb{R}$ by

$$v^\varepsilon(x,y,t) := u^\varepsilon(x,t) - u^\varepsilon(y,t).$$

It is standard that v^ε solves in $\Omega \times \Omega \times (0, \infty)$ the doubled equation

$$\begin{aligned} v_t^\varepsilon &= \operatorname{tr}[a^\varepsilon(x,t)D_x^2 v^\varepsilon] + \operatorname{tr}[a^\varepsilon(y,t)D_y^2 v^\varepsilon] + b(x)\cdot D_x v^\varepsilon + b(y)\cdot D_y v^\varepsilon \\ &= \operatorname{tr}[A^\varepsilon(x,y,t)D^2 v^\varepsilon] + B(x,y)\cdot Dv^\varepsilon, \end{aligned}$$

where

$$B(x,y) := (b(x), b(y)) \quad \text{ and } \quad A^\varepsilon(x,y,t) := \begin{pmatrix} a^\varepsilon(x,t) & 0 \\ 0 & a^\varepsilon(y,t) \end{pmatrix}.$$

The conclusion follows if we apply Theorem 7, with $\Pi = \emptyset$, to $\pm v^\varepsilon$, since $v^\varepsilon(0,0,t) = 0$ for all $t \ge 0$ and $|v^\varepsilon| \le 2C_0$ in $\Omega \times \Omega \times [0, \infty)$.

The only issue is that the boundary of $\Omega \times \Omega$ does not have the C^1- regularity required for the theorem.

To overcome this difficulty, we only need to approximate $\Omega \times \Omega$ by smaller C^1-domains. That is, for fixed $\delta > 0$, we choose a C^1-domain $W \subset \mathbb{R}^{2n}$ so that

$$\Omega_\delta \times \Omega_\delta \subset W_{\delta/2} \subset W \subset \Omega \times \Omega,$$

where $W_{\delta/2} := \{(x,y) \in W \,:\, \operatorname{dist}((x,y), \partial W) < \delta/2\}$, and

$$B(x,y)\cdot N(x,y) < 0 \quad \text{ for all } (x,y) \in \partial W,$$

where $N(x,y)$ denotes the outward unit normal vector at $(x,y) \in \partial W$. □

3. Quasi-potentials

We establish here an important continuity property under perturbations for the minimum and the arg min map of the quasi-potentials we introduced earlier in the introduction.

We begin with some notation and the introduction of several auxiliary quantities needed to define the perturbations. To this end, we fix $c_0 \in [g_{\min}, g_{\max}]$, define $H_0 \in C(\overline{\Omega} \times \mathbb{R}^n)$ by

$$H_0(x,p) = a(x,c_0)p\cdot p + b(x)\cdot p,$$

choose some $\delta_0 > 0$, and, for $\delta \in (0, \delta_0)$,

$$\theta(\delta) := \max\{|(a(x,c) - a(x,c_0))\xi\cdot\xi| \,:\, x \in \overline{\Omega},\, \xi \in \mathbb{R}^n,\, |\xi| \le 1,\, c \in [c_0 - \delta,\, c_0 + \delta]\}.$$

The continuity of $a(x,c)$ (recall (1.5)) yields $\lim_{\delta\to 0}\theta(\delta)=0$, and, hence, selecting $\delta_0>0$ sufficiently small, we assume henceforth that

$$\theta(\delta)\le \theta_0/2 \quad \text{for all } \delta\in(0,\,\delta_0).$$

We define $a_\delta^\pm\in C(\overline{\Omega},\,\mathbb{S}^n)$ and $H_\delta^\pm\in C(\overline{\Omega}\times\mathbb{R}^n)$, respectively, by

$$a_\delta^\pm(x):=a(x,c_0)\pm\theta(\delta)I \ \text{ and } \ H_\delta^\pm(x,p):=a_\delta^\pm(x)p\cdot p+b(x)\cdot p,$$

and note that, for all $(x,c)\in\overline{\Omega}\times[c_0-\delta,\,c_0+\delta]$,

$$(\theta_0/2)I\le a_\delta^-(x)\le a(x,c)\le a_\delta^+(x)\le(\theta_0^{-1}+\theta_0/2)I.$$

We choose $\chi_\delta\in C(\mathbb{R}^n;[0,1])$ such that

$$\chi_\delta=1 \quad\text{in}\quad x\in\Omega_\delta \qquad\text{and}\qquad \chi_\delta=0 \quad\text{in}\quad \mathbb{R}^n\setminus\Omega_{\delta/2},$$

and define $\mathcal{H}_\delta^\pm\in C(\overline{\Omega}\times\mathbb{R}^n)$ by

$$\begin{aligned}\mathcal{H}_\delta^+(x,p)&=\chi_\delta(x)H_\delta^+(x,p)+(1-\chi_\delta(x))(\theta_0^{-1}|p|^2+b(x)\cdot p),\\ \mathcal{H}_\delta^-(x,p)&=\chi_\delta(x)H_\delta^-(x,p)+(1-\chi_\delta(x))(\theta_0|p|^2+b(x)\cdot p),\end{aligned}$$

and note that, for all $(x,c)\in\Omega_{\delta/2}\times[c_0-\delta,\,c_0+\delta]\cup(\Omega\setminus\Omega_{\delta/2})\times\mathbb{R}$ and $p\in\mathbb{R}^n$,

$$\mathcal{H}_\delta^-(x,p)\le a(x,c)p\cdot p+b(x)\cdot p\le\mathcal{H}_\delta^+(x,p).$$

We also have

$$\mathcal{H}_\delta^\pm(x,p)=H_\delta^\pm(x,p) \ \text{ for all } (x,p)\in\Omega_\delta\times\mathbb{R}^n,$$

while, for all $(x,p)\in(\overline{\Omega}\setminus\Omega_{\delta/2})\times\mathbb{R}^n$,

$$\mathcal{H}_\delta^+(x,p)=\theta_0^{-1}|p|^2+b(x)\cdot p \quad\text{and}\quad \mathcal{H}_\delta^-(x,p)=\theta_0|p|^2+b(x)\cdot p.$$

If we set

$$\alpha_\delta^+(x)=\chi_\delta(x)a_\delta^+(x)+(1-\chi_\delta(x))\theta_0^{-1}I \qquad\text{and}\qquad \alpha_\delta^-(x)=\chi_\delta(x)a_\delta^-(x)+(1-\chi_\delta(x))\theta_0 I,$$

then, for all $(x,p)\in\overline{\Omega}\times\mathbb{R}^n$,

$$\mathcal{H}_\delta^\pm(x,p)=\alpha_\delta^\pm(x)p\cdot p+b(x)\cdot p.$$

Let V_0 and $V_\delta^\pm$ be respectively the maximal subsolutions of

$$\begin{cases}H_0(x,Du)=0 \ \text{ in } \ \Omega,\\ u(0)=0,\end{cases} \tag{3.1}$$

and

$$\begin{cases}\mathcal{H}_\delta^\pm(x,Du)=0 \quad \text{in } \Omega,\\ u(0)=0.\end{cases} \tag{3.2}$$

We note by [11, Corollary 5] that $V_\delta^\pm(x)>0$ and $V_0(x)>0$ for all $x\in\overline{\Omega}\setminus\{0\}$. Since $\mathcal{H}_\delta^-\le H_0\le\mathcal{H}_\delta^+$ on $\Omega\times\mathbb{R}^n$, it is clear that

$$V_\delta^+\le V_0\le V_\delta^- \quad\text{on}\quad \overline{\Omega}. \tag{3.3}$$

We set

$$M_0:=\min_{\partial\Omega}V_0,\quad \Gamma_0:=\arg\min(V_0|\partial\Omega),\quad M_\delta^\pm:=\min_{\partial\Omega}V_\delta^\pm,\quad \Gamma_\delta^\pm:=\arg\min(V_\delta^\pm|\partial\Omega),$$

and note that

$$M_\delta^+\le M_0\le M_\delta^-.$$

We establish the following result about the continuity of $M_\delta^\pm$ and $\Gamma_\delta^\pm$ with respect to δ.

Theorem 9. *Assume* (1.3), (1.5), (1.6), (1.7) *and* (1.8). *Then*

$$\lim_{\delta\to0+} M_\delta^+ = \lim_{\delta\to0+} M_\delta^- = M_0 \tag{3.4}$$

and

$$\limsup_{\delta\to0+} \Gamma_\delta^+ \cup \limsup_{\delta\to0+} \Gamma_\delta^- \subset \Gamma_0. \tag{3.5}$$

The set limit in (3.5) is understood in the sense of Kuratowski, that is, for a given $\{\Gamma_\delta\}_{\delta\in(0,\,\delta_0)} \subset \mathbb{R}^n$,

$$\limsup_{\delta\to0+} \Gamma_\delta := \bigcap_{r\in(0,\,\delta_0)} \overline{\bigcup_{\delta\in(0,\,r)} \Gamma_\delta} = \{x\in\mathbb{R}^n \,:\, x = \lim_{k\to\infty} x_k,\ x_k\in\Gamma_{\delta_k},\ \lim_{k\to\infty}\delta_k = 0\}.$$

Now we prove Theorem 9.

Proof of Theorem 9. The uniform in x and δ coercivity of the Hamiltonians $\mathcal{H}_\delta^\pm$, that is the fact that $\mathcal{H}_\delta^\pm(x,p)\to\infty$ as $|p|\to\infty$ uniformly in x and δ, yields that the families $\{V_\delta^\pm\}_{\delta\in(0,\delta_0)}$ are equi-Lipschitz continuous on $\overline{\Omega}$, and, since $V_\delta^\pm(0)=0$, relatively compact in $C(\overline{\Omega})$.

To prove (3.4) and (3.5), it is enough to show that, if $\{\delta_j\}_{j\in\mathbb{N}} \subset (0,\,\delta_0)$ is such that both $\{V_{\delta_j}^\pm\}_{j\in\mathbb{N}}$ converge in $C(\overline{\Omega})$ to some $V_0^\pm \in C(\overline{\Omega})$, that is

$$V_0^\pm = \lim_{j\to\infty} V_{\delta_j}^\pm \quad \text{uniformly on } \ \overline{\Omega},$$

then

$$M_0 = \min_{\partial\Omega} V_0^+ = \min_{\partial\Omega} V_0^-. \tag{3.6}$$

and

$$\arg\min(V_0|\partial\Omega) = \arg\min(V_0^+|\partial\Omega) = \arg\min(V_0^-|\partial\Omega). \tag{3.7}$$

For notational convenience, we set

$$M_0^\pm := \min_{\partial\Omega} V_0^\pm \quad \text{and} \quad \Gamma_0^\pm = \arg\min(V_0^\pm|\partial\Omega).$$

It is well-known (see Lemma B.1 in the Appendix) that the $V_\delta^\pm$'s satisfy in the viscosity sense

$$\mathcal{H}_\delta^\pm(x, DV_\delta^\pm) \ge 0 \ \text{ on } \overline{\Omega} \quad \text{and} \quad \mathcal{H}_\delta^\pm(x, DV_\delta^\pm) \le 0 \ \text{ in } \Omega,$$

that is, the $V_\delta^\pm$'s are solutions of the state-constraints problems

$$\mathcal{H}_\delta^\pm(x, DV_\delta^\pm) = 0 \quad \text{in } \Omega.$$

By the stability of viscosity properties, the $V_0^\pm$'s satisfy

$$H_0(x, DV_0^\pm(x)) \le 0 \ \text{ in } \ \Omega \quad \text{and} \quad H_{\theta_0}^+(x, DV_0^+(x)) \ge 0 \quad \text{on } \overline{\Omega},$$

where

$$H_{\theta_0}^+(x,p) := \begin{cases} H_0(x,p) & \text{for } (x,p)\in\Omega\times\mathbb{R}^n, \\ \theta_0^{-1}|p|^2 + b(x)\cdot p & \text{for } (x,p)\in\partial\Omega\times\mathbb{R}^n. \end{cases}$$

Here we used that

$$\limsup_{\delta\to 0}{}^{*}\,\mathcal{H}_\delta^{\pm}(x,p)=\liminf_{\delta\to 0}{}_{*}\,\mathcal{H}_\delta^{\pm}(x,p)=H_0(x,p)\quad\text{for all } (x,p)\in\Omega\times\mathbb{R}^n,$$

and

$$\limsup_{\delta\to 0}{}^{*}\,\mathcal{H}_\delta^{+}(x,p)=H_{\theta_0}^{+}(x,p)\quad\text{for all } (x,p)\in\overline{\Omega}\times\mathbb{R}^n.$$

The maximality of V_0 implies that $V_0^- \le V_0$ on Ω and, since, in view of (3.3), $V_0 \le V_0^-$ in $\overline{\Omega}$, we have $V_0^- = V_0$, which, obviously gives

$$M_0 = M_0^- \quad\text{and}\quad \Gamma_0^- = \Gamma_0. \tag{3.8}$$

The argument for M_0^+ and Γ_0^+ is slightly more complicated.

Since (3.3) yields $V_0^+ \le V_0$, it is immediate that

$$M_0^+ \le M_0.$$

Next we show that

$$\min\{V_0,\, M_0\} \le V_0^+ \ \text{ in }\ \overline{\Omega}, \tag{3.9}$$

which, together the previous inequality, give

$$M_0^+ = M_0 \quad\text{and}\quad \Gamma_0 \subset \Gamma_0^+. \tag{3.10}$$

We proceed with the proof of (3.9). Fix $l \in (0,\, M_0)$, choose $\gamma_1 \in (0,\, \delta_0)$ so that

$$V_0 > l \ \text{ on }\ \overline{\Omega}\setminus\Omega_{\gamma_1},$$

fix $\mu \in (0,\, 1)$ sufficiently close to 1 so that

$$\mu V_0 > l \ \text{ on } \overline{\Omega}\setminus\Omega_{\gamma_1},$$

and choose $\gamma_2 \in (0,\, \gamma_1)$ so that

$$\mu(a(x,c_0)+\theta(\delta)I)\le a(x,c_0)\quad \text{for all }\ x\in\overline{\Omega}\ \text{ and }\ \delta\in(0,\,\gamma_2).$$

Observe that, if $u_\mu(x) := \mu V_0(x)$, then, for all $\delta \in (0,\, \gamma_2)$,

$$u_\mu > l \ \text{ in }\ \overline{\Omega}\setminus\Omega_\delta,$$

and, for all $\delta \in (0,\, \gamma_2)$, in the viscosity sense,

$$\begin{aligned} H_\delta^+(x,Du_\mu) &= \mu(\mu(a(x,c_0)+\theta(\delta))|DV_0|^2 + b(x)\cdot DV_0) \\ &\le \mu(a(x,c_0)|DV_0|^2 + b\cdot DV_0) \le \mu H_0(x,DV_0)\le 0 \quad \text{in }\ \Omega. \end{aligned}$$

Now set $u_\mu^l := \min\{u_\mu,\, l\}$ and note that the convexity of $H_\delta^+(x,p)$ in p yields that, if $\delta \in (0,\, \gamma_2)$, then

$$\mathcal{H}_\delta^+(x,Du_\mu^l) = H_\delta^+(x,Du_\mu^l)\le 0 \quad \text{in }\ \Omega_\delta.$$

Also, if $\delta \in (0,\, \gamma_2)$, then, since $u_\mu^l(x) = l$ in an open neighborhood $N_\delta \subset \Omega$ of $\Omega\setminus\Omega_\delta$,

$$\mathcal{H}_\delta(x, Du_\mu^l(x)) = 0 \quad \text{in }\ N_\delta.$$

Thus we deduce that, for any $\delta \in (0, \gamma_2)$, u_μ^l is a subsolution of $\mathcal{H}_\delta^+(x, Du_\mu^l) \le 0$ in Ω, and, hence, $u_\mu^l \le V_\delta^+$ in $\overline{\Omega}$ by the maximality of V_δ^+. Sending $\delta \to 0$, along the sequence $\{\delta_j\}$, $\mu \to 1$ and $l \to M_0$ in this order, we conclude that (3.9) holds.

Next we show that $\Gamma_0^+ \subset \Gamma_0$. Let $z \in \Gamma_0^+ \setminus \Gamma_0$ and observe that, since $V_0(z) > M_0$, there is an open, relatively to $\overline{\Omega}$, neighborhood $N_z \subset \overline{\Omega}$, such that $V_0 > M_0$ in N_z, while (3.9) gives $V_0^+ \geq M_0$ in N_z.

Let $\rho \in C^1(\mathbb{R}^n)$ be a defining function of Ω, that is, $\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}$ and $|D\rho| \neq 0$ on $\partial\Omega$, and, in particular, $D\rho/|D\rho| = \nu$ on $\partial\Omega$.

For any $\varepsilon > 0$, $x \mapsto V_0^+(x) - \varepsilon\rho(x)$ achieves a minimum at z over N_z. Since $H_{\theta_0}^+(x, DV_0^+) \geq 0$ on $\overline{\Omega}$, we have

$$0 \leq H_{\theta_0}^+(z, \varepsilon D\rho(z)) = \varepsilon(\varepsilon\theta_0^{-1}|D\rho(z)|^2 + b(z) \cdot D\rho(z)),$$

which is a contradiction, in view of the fact that the right hand side is negative if ε is sufficiently small.

It follows that $\Gamma_0^+ \setminus \Gamma_0 = \emptyset$, that is, $\Gamma_0^+ \subset \Gamma_0$, which, together with (3.10), proves the claim. □

4. Barrier functions

We adapt and modify here the main argument of building barrier functions of [11] to obtain information on the behavior of the solutions u^ε of (2.1) along the positive time axis l, that is on $u^\varepsilon(0,t)$, for a sufficiently long time interval $[0, T(\varepsilon))$, under the assumption that the matrices $a^\varepsilon \in C(\overline{Q}_{T(\varepsilon)})$ are bounded by $\alpha \in C(\overline{Q}_{T(\varepsilon)})$ from above or from below.

Recall that, for any $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$, $H_\alpha \in C(\overline{\Omega} \times \mathbb{R}^n)$ be the Hamiltonian given by $H_\alpha(x,p) = \alpha(x)p \cdot p + b(x) \cdot p$, $V_\alpha \in \mathrm{Lip}(\overline{\Omega})$ is the quasi-potential corresponding to (α, b), and $M_\alpha = \min_{\partial\Omega} V_\alpha$, and set

$$\Sigma_\alpha := \{x \in \overline{\Omega} : V_\alpha(x) \leq M_\alpha\} \ \Gamma_\alpha := \Sigma_\alpha \cap \partial\Omega,$$

and, any $m > 0$,

$$\Sigma_\alpha^m := \{x \in \overline{\Omega} : V_\alpha(x) \leq m\}.$$

We consider the again (2.1) for a family of $a^\varepsilon \in C(\overline{Q}, \mathbb{S}^n(\theta_0))$ with $\varepsilon \in (0, 1)$.

We state two results one for an upper and one for the lower bound. The upper bound is valid up to λ smaller than M_α in the logarithmic time scale, and the lower bound is valid up to ∞, provided u^ε, on the boundary portion $\Gamma_\alpha \times [0, T(\varepsilon))$, is larger than a lower bound.

We begin with the former, which corresponds to [11, Theorem 1 (i)] in its nature. The latter is related to [11, Theorem 1(ii)].

Theorem 10. *Assume* (1.2) *and* (1.10) *and fix* $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$, $T(\varepsilon) \in (0, \infty]$ *and* $m \in (0, M_\alpha)$. *If, for* $a^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}; \mathbb{S}^n(\theta_0))$, *where* $\varepsilon \in (0, 1)$, *such that*

$$a^\varepsilon(x,t) \leq \alpha(x) \quad \text{in} \ \ (x,t) \in Q_{T(\varepsilon)},$$

$u^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}) \cap C^{2,1}(Q_{T(\varepsilon)})$ *is a subsolution of* (2.1) *in* $Q_{T(\varepsilon)}$ *such that*

$$u^\varepsilon(x,0) \leq 0 \ \ \text{for all} \ \ x \in \Sigma_\alpha^m \ \ \text{and} \ \ \sup_{Q_{T(\varepsilon)}} u^\varepsilon < \infty,$$

then, for any $\delta > 0$, *there exists* $\varepsilon_0 \in (0, 1)$ *such that, if* $\varepsilon \in (0, \varepsilon_0)$, *then*

$$u^\varepsilon(0,t) \leq \delta \quad \text{for all} \ \ t \in [0, \exp((m-\delta)/\varepsilon) \wedge T(\varepsilon)].$$

The lower bound is stated next.

Theorem 11. *Assume* (1.2), (1.10), *fix* $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$, $T(\varepsilon) \in (0, \infty]$ *and* $m > M_\alpha$. *If, for* $a^\varepsilon \in C(\bar{Q}_{T(\varepsilon)}; \mathbb{S}^n(\theta_0))$, *where* $\varepsilon \in (0, 1)$, *such that*

$$a^\varepsilon(x,t) \geq \alpha(x) \quad \text{in } (x,t) \in Q_{T(\varepsilon)},$$

$u^\varepsilon \in C(\overline{Q}_{T(\varepsilon)}) \cap C^{2,1}(Q_{T(\varepsilon)})$ *is a solution of* (2.1) *and* (1.2) *such that*

$$\begin{cases} u^\varepsilon(x,0) \geq 0 & \text{for all } x \in \Sigma_\alpha^m, \\ u^\varepsilon(x,t) \geq 0 & \text{for all } (x,t) \in (\Sigma_\alpha^m \cap \partial\Omega) \times (0, T(\varepsilon)), \end{cases}$$

and

$$\inf_{Q_{T(\varepsilon)}} u^\varepsilon > -\infty,$$

then, for any $\delta > 0$, *there exists* $\varepsilon_0 \in (0, 1)$ *such that, if* $\varepsilon \in (0, \varepsilon_0)$, *then*

$$u^\varepsilon(0,t) \geq -\delta \quad \text{for all } t \in [0, T(\varepsilon)].$$

The proofs of Theorem 10 and Theorem 11 use the next two lemmata. We state them without proof for which we refer to [11].

Lemma 3. *Assume* (1.10) *and fix* $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$. *For any* $r \in (0, r_0)$, *there exist* $v_r \in C^2(\overline{\Omega})$ *and* $\eta \in (0,1)$ *such that*

$$\begin{cases} H_\alpha(x, Dv_r) \leq -\eta & \text{in } \Omega \setminus B_r, \\ H_\alpha(x, Dv_r) \leq 1 & \text{in } B_r, \\ \|v_r - V_\alpha\|_{L^\infty(\Omega)} < r. \end{cases} \tag{4.1}$$

Lemma 4. *Assume* (1.10) *and fix* $\alpha \in C(\overline{\Omega}, \mathbb{S}^n(\theta_0))$. *For each* $m > M_\alpha$, *there exists* $w_m \in \mathrm{Lip}(\overline{\Omega})$ *and* $\eta > 0$ *such that*

$$0 < \min_{\overline{\Omega}} w_m \leq \max_{\overline{\Omega}} w_m < m, \tag{4.2}$$

and, in the viscosity supersolution sense,

$$H_\alpha(x, -Dw_m) \geq \eta \quad \text{in } \Omega \quad \text{and} \quad D^2 w_m(x) \leq \eta^{-1} I \text{ in } \Omega. \tag{4.3}$$

We continue with the proof of Theorem 10 which parallels that of [11, Theorem 8].

Proof of Theorem 10. For $r \in (0, r_0)$ to be fixed below, let $v = v_r \in C^2(\bar{\Omega})$ (for notational simplicity we omit the subscript r in what follows) and $\eta > 0$ be given by Lemma 3, set, for $x \in \overline{\Omega}$,

$$w^\varepsilon(x) := \exp\left(\frac{v(x) - m + 2r}{\varepsilon}\right),$$

compute, for any $(x,t) \in Q$,

$$\begin{aligned} &\mathrm{tr}[a^\varepsilon(x,t) D^2 w^\varepsilon] + b(x) \cdot Dw^\varepsilon \\ &\quad = \frac{w^\varepsilon}{\varepsilon}\left(a^\varepsilon(x,t) Dv \cdot Dv + b \cdot Dv + \varepsilon\, \mathrm{tr}[a^\varepsilon(x,t) D^2 w]\right) \\ &\quad \leq \frac{w^\varepsilon}{\varepsilon}\left(\alpha(x) Dv \cdot Dv + b(x) \cdot Dv + \varepsilon\, \mathrm{tr}[a^\varepsilon(x,t) D^2 w]\right) \\ &\quad \leq \frac{w^\varepsilon}{\varepsilon}\left(H_\alpha(x, Dv) + \varepsilon\, \mathrm{tr}[a^\varepsilon(x,t) D^2 w]\right). \end{aligned}$$

and choose $\varepsilon_0 \in (0, 1)$ so that, for all $\varepsilon \in (0, 1)$,

$$\varepsilon_0 \left(\operatorname{tr} a^\varepsilon(x,t) D^2 v\right)_+ \le \min\{\eta, r, 1\};$$

note that ε_0 can be chosen so as to depend on a^ε only through θ_0.

We assume henceforth that $\varepsilon \in (0, \varepsilon_0)$ and observe that, from the computation above, we get

$$\operatorname{tr}[a^\varepsilon(x,t) D^2 w^\varepsilon] + b(x) \cdot Dw^\varepsilon \le \begin{cases} 0 & \text{for all } (x,t) \in \Omega \setminus B_r \times (0, \infty), \\ \frac{2}{\varepsilon} w^\varepsilon & \text{for all } (x,t) \in B_r \times (0, \infty). \end{cases} \tag{4.4}$$

Let $C_0 > 0$ be a Lipschitz bound of b, and note that, if $H_\alpha(x,p) \le 0$, then $|p| \le C_0 \theta_0^{-1}$, which implies that $V_\alpha(x) \le C_0 |x|^2/(2\theta_0) \le C_0 r^2/(2\theta_0)$ for all $x \in B_r$. We may thus assume by replacing, if needed, $r > 0$ by a smaller number that $V_\alpha \le r$ in B_r. Accordingly we have

$$v - m + 2r \le V_\alpha - m + 3r \le -m + 4r \quad \text{in} \quad B_r,$$

and

$$w^\varepsilon \le \exp\left(\frac{-m + 4r}{\varepsilon}\right) \quad \text{in} \quad B_r. \tag{4.5}$$

Observe also that

$$v - m + 2r > V_\alpha - m + r \ge r \quad \text{in} \quad \overline{\Omega} \setminus \Sigma_\alpha^m,$$

and

$$w^\varepsilon > \exp\left(\frac{r}{\varepsilon}\right) \quad \text{in} \quad \overline{\Omega} \setminus \Sigma_\alpha^m. \tag{4.6}$$

Next set $d_\varepsilon = \frac{2}{\varepsilon} \exp(\frac{-m+4r}{\varepsilon})$ and

$$z^\varepsilon(x,t) = w^\varepsilon(x) + d_\varepsilon t \quad \text{for} \quad (x,t) \in \overline{\Omega} \times [0, \infty).$$

It is immediate from (4.4) and (4.5) that

$$z_t^\varepsilon \ge \varepsilon \operatorname{tr}[a^\varepsilon D^2 z^\varepsilon] + b \cdot Dz^\varepsilon \quad \text{in} \quad Q. \tag{4.7}$$

We choose $C_1 > 0$ so that, for all $\varepsilon \in (0, 1)$,

$$u^\varepsilon \le C_1 \quad \text{on} \quad \overline{Q},$$

and by replacing, if necessary, $\varepsilon_0 > 0$ by a smaller number we may assume that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$C_1 < \exp\left(\frac{r}{\varepsilon}\right).$$

It follows from (4.6) that

$$z^\varepsilon \ge w^\varepsilon \ge \exp\left(\frac{r}{\varepsilon}\right) > C_1 \ge u^\varepsilon \quad \text{on } (\overline{\Omega} \setminus \Sigma_\alpha^m) \times [0, \infty);$$

note that, since $m < M_\alpha$, we have $\partial\Omega \subset \overline{\Omega} \setminus \Sigma_\alpha^m$.

On the other hand, for any $x \in \Sigma_\alpha^m$, we have

$$z^\varepsilon(x, 0) = w^\varepsilon(x) > 0 \ge u^\varepsilon(x, 0),$$

and, hence,

$$u^\varepsilon \le z^\varepsilon \quad \text{on } \partial_{\mathrm{p}} Q.$$

We find from the above, (4.7) and the comparison principle that

$$u^\varepsilon \le z^\varepsilon \quad \text{on} \quad \overline{Q},$$

and, in particular, for any $t \in [0, \exp((m-5r)/\varepsilon)]$,

$$u^\varepsilon(0,t) \le z^\varepsilon(0,t) \le w^\varepsilon(0) + \frac{2}{\varepsilon}\exp\left(\frac{-r}{\varepsilon}\right) \le \exp\left(\frac{-m+3r}{\varepsilon}\right) + \frac{2}{\varepsilon}\exp\left(\frac{-r}{\varepsilon}\right).$$

It is now clear that, for a given $\delta > 0$, we may choose $r > 0$ and $\varepsilon_0 \in (0, 1)$ so that if $\varepsilon \in (0, \varepsilon_0)$, and, then

$$u^\varepsilon(0,t) \le \exp\left(\frac{-m+3r}{\varepsilon}\right) + \frac{2}{\varepsilon}\exp\left(\frac{-r}{\varepsilon}\right) < \delta \quad \text{for all } \ t \in [0, \exp((m-\delta)/\varepsilon)]. \qquad \square$$

We continue with

Proof of Theorem 11. We fix $r \in (0, r_0)$ small enough so that, as in the previous proof, $V_\alpha(x) \le r$ for all $x \in B_r$ and $m - 5r > M_\alpha$. In view of Lemma 3 and Lemma 4, we may choose $v \in C^2(\overline{\Omega})$, $w \in \mathrm{Lip}(\overline{\Omega})$ and $\eta > 0$ so that, in addition to (4.1), $0 < \min_{\overline{\Omega}} w < \max_{\overline{\Omega}} w < m - 5r$, and, in the viscosity supersolution sense,

$$H_\alpha(x, -Dw) \ge \eta \quad \text{and} \quad D^2 w \le \eta^{-1} I \quad \text{in } \ \Omega.$$

Setting $u = -w$, $\rho^- = \min_{\overline{\Omega}} w$ and $\rho^+ = \max_{\overline{\Omega}} w$, we get that $\rho^+ < m - 5r$, $0 > -\rho^- \ge u(x) \ge -\rho^+$ for all $x \in \overline{\Omega}$ and, in the viscosity subsolution sense,

$$H_\alpha(x, Du) \ge \eta \quad \text{and} \quad D^2 u \ge -\eta^{-1} I \quad \text{in } \ \Omega.$$

For $\varepsilon \in (0, 1)$, we set

$$z^\varepsilon = -\exp\left(\frac{v-m+2r}{\varepsilon}\right) + \exp\left(\frac{u}{\varepsilon}\right) - \exp\left(\frac{-\rho^-}{\varepsilon}\right),$$

and find that, in the viscosity subsolution sense,

$$\begin{aligned}\operatorname{tr}[a^\varepsilon D^2 z^\varepsilon] + b\cdot Dz^\varepsilon \ge &-\frac{1}{\varepsilon}\exp\left(\frac{v-m+2r}{\varepsilon}\right)\big(H_\alpha(x,Dv) + \varepsilon\operatorname{tr}[a^\varepsilon D^2 v]\big)\\ &+\frac{1}{\varepsilon}\exp\left(\frac{u}{\varepsilon}\right)\big(H_\alpha(x,Du) + \varepsilon\operatorname{tr}[a^\varepsilon D^2 u]\big) \quad \text{in } \ Q.\end{aligned}$$

Let $\varepsilon_0 \in (0, 1)$ be a constant to be specified later and assume henceforth that $\varepsilon \in (0, \varepsilon_0)$. Observing that in the viscosity subsolution sense,

$$\operatorname{tr}[a^\varepsilon D^2 u] \ge -\eta^{-1}\operatorname{tr} a^\varepsilon \ge -n(\theta_0\eta)^{-1} \quad \text{in } \ Q,$$

and

$$\operatorname{tr}[a^\varepsilon D^2 v] \ge -\|D^2 v\|_{L^\infty(\Omega)}\operatorname{tr} a^\varepsilon \le -n\theta_0^{-1}\|D^2 v\|_{L^\infty(\Omega)} \quad \text{in } \ Q,$$

and setting, for $x \in \overline{\Omega}$,

$$\begin{aligned}f(x) = &-\frac{1}{\varepsilon}\exp\left(\frac{v(x)-m+2r}{\varepsilon}\right)\big(H_\alpha(x,Dv(x)) + \varepsilon n\theta_0^{-1}\|D^2 v\|_{L^\infty(\Omega)}\big)\\ &+\frac{1}{\varepsilon}\exp\left(\frac{u(x)}{\varepsilon}\right)\big(\eta - \varepsilon n(\eta\theta_0)^{-1}\big),\end{aligned}$$

we obtain, in the viscosity subsolution sense,

$$\operatorname{tr}[a^\varepsilon D^2 z^\varepsilon] + b\cdot Dz^\varepsilon \ge f(x) \quad \text{in } \ Q. \tag{4.8}$$

Choosing $\varepsilon_0 \in (0, 1)$ so that

$$\varepsilon_0 n\theta_0^{-1}\|D^2 v\|_{L^\infty(\Omega)} \le \min\{\eta, 1\} \quad \text{and} \quad \varepsilon_0 n(\eta\theta_0)^{-1} \le \frac{\eta}{2},$$

we get

$$\eta - \varepsilon n(\eta\theta_0)^{-1} \geq \frac{\eta}{2} \quad \text{and} \quad H_\alpha(x, Dv) + \varepsilon n\theta_0^{-1}\|D^2 v\|_{L^\infty(\Omega)} \leq \begin{cases} 0 & \text{for all } \ x \in \Omega \setminus B_r, \\ 2 & \text{for all } \ x \in B_r, \end{cases}$$

and, accordingly,

$$f \geq \begin{cases} 0 & \text{in } \ \Omega \setminus B_r, \\ -\dfrac{2}{\varepsilon} \exp\left(\dfrac{-m+4r}{\varepsilon}\right) + \dfrac{\eta}{2\varepsilon} \exp\left(\dfrac{-\rho^+}{\varepsilon}\right) & \text{in } \ B_r. \end{cases}$$

Since $\rho^+ < m - 5r$, we have

$$\begin{aligned} -2\exp\left(\frac{-m+4r}{\varepsilon}\right) + \frac{\eta}{2}\exp\left(\frac{-\rho^+}{\varepsilon}\right) &\geq -2\exp\left(\frac{-\rho^+ - r}{\varepsilon}\right) + \frac{\eta}{2}\exp\left(\frac{-\rho^+}{\varepsilon}\right) \\ &= \exp\left(\frac{-\rho^+}{\varepsilon}\right)\left(-2\exp\left(\frac{-r}{\varepsilon}\right) + \frac{\eta}{2}\right). \end{aligned}$$

We may assume by replacing $\varepsilon_0 \in (0, 1)$ by a smaller number that

$$2\exp\left(\frac{-r}{\varepsilon_0}\right) \leq \frac{\eta}{2},$$

and, therefore,

$$-2\exp\left(\frac{-m+4r}{\varepsilon}\right) + \frac{\eta}{2}\exp\left(\frac{-\rho^+}{\varepsilon}\right) \geq 0,$$

which ensures that $f \geq 0$ in Ω, and, hence, z^ε, as a function of $(x,t) \in Q$, is a subsolution of (2.1).

Next observe that

$$z^\varepsilon < 0 \quad \text{on} \quad \overline{\Omega},$$

and, if $V_\alpha(x) > m$,

$$z^\varepsilon(x) \leq -\exp\left(\frac{V_\alpha(x) - m + r}{\varepsilon}\right) \leq -\exp\left(\frac{r}{\varepsilon}\right).$$

Fix a constant $C_1 > 0$ so that, for $\varepsilon \in (0, 1)$, $u^\varepsilon \geq -C_1$ on $\overline{Q}$, and, assume henceforth that $\varepsilon_0 \in (0, 1)$ is small enough so that

$$\exp\left(\frac{r}{\varepsilon_0}\right) \geq C_1.$$

Consequently, we have

$$z^\varepsilon(x) \leq \begin{cases} -\exp\left(\frac{r}{\varepsilon}\right) \leq -C_1 \leq u^\varepsilon(x,t) & \text{for all } \ (x,t) \in (\overline{\Omega} \setminus \Sigma_\alpha^m) \times [0, \infty), \\ 0 \leq u^\varepsilon(x, 0) & \text{for all } \ x \in \Sigma_\alpha^m, \\ 0 \leq u^\varepsilon(x,t) & \text{for all } \ (x,t) \in (\Sigma_\alpha^m \cap \partial\Omega) \times (0, \infty), \end{cases}$$

that is

$$z^\varepsilon(x) \leq u^\varepsilon(x,t) \ \text{ for all } (x,t) \in \partial_{\mathrm{p}} Q,$$

and, hence, by the comparison principle, we get

$$z^\varepsilon(x) \leq u^\varepsilon(x,t) \quad \text{for all } \ (x,t) \leq \overline{Q}.$$

Finally, we note that

$$\begin{aligned} z^\varepsilon(0) &= -\exp\left(\frac{v(0)-m+2r}{\varepsilon}\right) - \exp\left(\frac{-\rho^-}{\varepsilon}\right) \\ &\geq -\exp\left(\frac{-m+4r}{\varepsilon}\right) - \exp\left(\frac{-\rho^-}{\varepsilon}\right) \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned}$$

which completes the proof. □

5. The proof of Theorem 2

We begin with the proof of the assertions of Theorem 2 concerning (A) and (A′), which can be restated as follows.

Theorem 12. *Assume* (1.5), (1.2), (1.10) *and* (1.18) *and, for* $\varepsilon \in (0, 1)$, *let* $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ *be a solution of* (1.1). *Assume furthermore that the collection* $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ *is uniformly bounded on* Q *and suppose that there exist sequences* $\mu_k < \lambda_k$ *and* $\varepsilon_k \in (0, 1)$, *and constants* $0 < a_1 < a_2$ *and* $\beta_1, \beta_2 \in \mathbb{R}$ *such that* $\lim_{k\to\infty} \varepsilon_k = 0$, *and, for all* $k \in \mathbb{N}$,

$$0 < a_1 \leq \mu_k < \lambda_k \leq a_2, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \textit{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

If either $\beta_1 < \beta_2$ *or* $\beta_2 < \beta_1$, *then*

$$\limsup_{k\to\infty} \lambda_k \geq M(\beta_2).$$

Proof. Since the arguments are similar here we treat only the case $\beta_1 < \beta_2$.

We argue by contradiction and suppose that

$$\limsup_{k\to\infty} \lambda_k < M(\beta_2). \tag{5.1}$$

Let $\delta > 0$ be a constant to be fixed later, define α_δ^+ and $\mathcal{H}_\delta^+$ as in Section 3, with c_0 replaced by β_2, and, as in Section 3, let V_δ^+ be the maximal subsolution of

$$\mathcal{H}_\delta^+(x, Du) = 0 \quad \text{in } \ \Omega, \quad u(0) = 0,$$

and set $M_\delta^+ = \min_{\partial\Omega} V_\delta^+$.

Since Theorem 9 yields

$$\lim_{\delta\to 0+} M_\delta^+ = M(\beta_2),$$

in view of (5.1), we may choose $\delta > 0$ so that

$$\limsup_{k\to\infty} \lambda_k + \delta < M_\delta^+.$$

We fix $m \in \mathbb{R}$ so that

$$\limsup_{k\to\infty} \lambda_k + \delta < m < M_\delta^+,$$

and, by passing to a subsequence if necessary, we may assume that

$$\lambda_k \leq m - \delta \quad \text{for all } \ k \in \mathbb{N}.$$

Set

$$\Sigma = \{x \in \overline{\Omega} \, : \, V_\delta^+(x) \leq m\},$$

and note that Σ is a compact subset of Ω.

In view of the continuity of the map $t \mapsto u^\varepsilon(0,t)$, reselecting, if needed, β_1, μ_k and λ_k, we may assume that, all $t \in [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)]$ and $k \in \mathbb{N}$,

$$\beta_2 - \frac{\delta}{2} < \beta_1 \le u^{\varepsilon_k}(0,t) \le \beta_2. \tag{5.2}$$

Now we choose $\gamma \in (0,\, \delta/2)$ small enough, so that

$$\Sigma \subset \Omega_\gamma \quad \text{and} \quad \beta_2 - \beta_1 > 2\gamma. \tag{5.3}$$

Theorem 8 gives $\varepsilon_0 \in (0,\, 1)$ such that, if $\varepsilon \in (0,\, \varepsilon_0)$,

$$|u^\varepsilon(x,t) - u^\varepsilon(0,t)| < \gamma \quad \text{for all } (x,t) \in \Omega_\gamma \times [\exp(a_1/\varepsilon)\,,\, \infty). \tag{5.4}$$

We assume that $\varepsilon_k < \varepsilon_0$ for all $k \in \mathbb{N}$, and combine (5.4) and (5.2), to find

$$|u^\varepsilon(x,t) - \beta_2| \le \delta \quad \text{for all } (x,t) \in \Omega_\gamma \times [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)], \tag{5.5}$$

and

$$u^{\varepsilon_k}(x, \exp(\mu_k/\varepsilon_k)) \le \beta_1 + \gamma \quad \text{for all } x \in \Omega_\gamma \text{ and } k \in \mathbb{N}.$$

Since (5.5) implies that

$$a(x, u^{\varepsilon_k}(x,t)) \le \alpha_\delta(x) \quad \text{for all } (x,t) \in \Omega \times [\exp(\mu_k/\varepsilon_k), \exp(\lambda_k/\varepsilon_k)],\ k \in \mathbb{N},$$

setting

$$\begin{cases} v^k(x,t) = u^{\varepsilon_k}(x, t + \exp(\mu_k/\varepsilon_k)) - \beta_1 - \gamma, \\ a^k(x,t) = a(x, u^{\varepsilon_k}(x, t + \exp(\mu_k/\varepsilon_k))), \end{cases}$$

we see that

$$v_t^k = \varepsilon_k \operatorname{tr}[a^k(x,t) D^2 v^k] + b(x) \cdot Dv^k \quad \text{for all } (x,t) \in Q.$$

Furthermore, we have

$$v^k(x,0) \le 0 \quad \text{for all } x \in \Omega_\gamma,$$

which ensures that

$$v^k(x,0) \le 0 \quad \text{for all } x \in \Sigma.$$

An application of Theorem 10, with ε_k, v^k and γ in place of ε, u^ε and δ, respectively, guarantees that, for sufficiently large k, we have

$$v^k(0,t) \le \gamma \quad \text{for all } t \in [0,\, \exp(\lambda_k/\varepsilon_k) - \exp(\mu_k/\varepsilon_k)],$$

which, in particular, yields

$$v^k(0, \exp(\lambda_k/\varepsilon_k) - \exp(\mu_k/\varepsilon_k)) \le \gamma.$$

This shows that

$$u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) \le \beta_1 + 2\gamma < \beta_2,$$

which is a contradiction. □

Next, we prove the assertions of Theorem 2 concerning (B) and (B′), which we state as follows.

Theorem 13. *Assume* (1.5), (1.2), (1.10) *and* (1.18) *and, for* $\varepsilon \in (0,\, 1)$, *let* $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ *be a solution of* (1.1) *and* (1.2). *Assume further that there exist sequences* $\mu_k < \lambda_k$ *and* $\varepsilon_k \in (0,\, 1)$ *and constants* $0 < a_1 < a_2$ *and* $\beta_1, \beta_2 \in [g_{\min}, g_{\max}]$ *such that* $\lim_{k\to\infty} \varepsilon_k = 0$, *and, for all* $k \in \mathbb{N}$,

$$0 < a_1 \le \mu_k < \lambda_k \le a_2, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \text{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

If $g_{\min} < \beta_1 < \beta_2 < g_{\max}$, *then*

$$\lim_{r\to 0+} \sup_{|c-\beta_2|\le r} G(c) \ge \beta_2,$$

and if $g_{\min} < \beta_2 < \beta_1 < g_{\max}$, *then*

$$\lim_{r\to 0+} \inf_{|c-\beta_2|\le r} G(c) \le \beta_2.$$

Proof. Since the arguments are similar, here we only consider the case where $g_{\min} < \beta_2 < \beta_1 < g_{\max}$ holds.

We suppose that

$$\lim_{r\to 0+} \inf_{|c-\beta_2|\le r} G(c) > \beta_2, \tag{5.6}$$

and obtain a contradiction.

For a small constant $\delta > 0$ to be chosen later, define α_δ^- and $\mathcal{H}_\delta^-$ as in Section 3, with c_0 replaced by β_2, let V_δ^- be the quasi-potential corresponding to (α_δ^-, b), that is the maximal subsolution of

$$\mathcal{H}_\delta^-(x, Du) = 0 \quad \text{in} \ \ \Omega \quad \text{and} \quad u(0) = 0.$$

and V^{β_2} the quasi-potential corresponding to the pair $(a(\cdot, \beta_2), b)$, set

$$M_\delta^- = \min_{\partial\Omega} V_\delta^-, \quad \Gamma_\delta^- = \arg\min(V_\delta^- | \partial\Omega) \quad \text{and} \quad \Gamma^{\beta_2} = \arg\min(V^{\beta_2} | \partial\Omega),$$

and observe that, in view of assumptions (1.12) and (1.15),

$$\lim_{r\to 0+} \inf_{|c-\beta_2|\le r} G(c) = \min_{\Gamma^{\beta_2}} g.$$

Hence, by (5.6), we get

$$\min_{\Gamma^{\beta_2}} g > \beta_2.$$

Furthermore, in view of (3.5), we may choose $\delta > 0$ so that

$$\min_{\Gamma_\delta^-} g > \beta_2 + \delta. \tag{5.7}$$

Finally replacing, if necessary, β_1, μ_k and λ_k we may assume

$$\beta_1 \ge u^\varepsilon(0, t) \ge \beta_2 \quad \text{for all} \ \ t \in [\exp(\mu_k/\varepsilon_k), \, \exp(\lambda_k/\varepsilon_k)], \ k \in \mathbb{N},$$

and

$$\beta_1 < \beta_2 + \delta/2. \tag{5.8}$$

Since the maximum principle gives $g_{\min} \le u^\varepsilon \le g_{\max}$ in $\overline{Q}$, we find that Theorem 8 yields $\varepsilon_0 \in (0, 1)$ such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$|u^\varepsilon(x, t) - u^\varepsilon(0, t)| < \delta/2 \quad \text{for all} \ \ (x, t) \in \Omega_{\delta/2} \times [\exp(a_1/\varepsilon), \, \infty). \tag{5.9}$$

Consequently, if $k \in \mathbb{N}$ is sufficiently large, then $\varepsilon_k < \varepsilon_0$ and

$$|u^{\varepsilon_k}(x, t) - \beta_2| < \delta \quad \text{for all} \ \ (x, t) \in \Omega_{\delta/2} \times [\exp(\mu_k/\varepsilon_k), \, \exp(\lambda_k/\varepsilon_k)]. \tag{5.10}$$

Henceforth, passing if necessary to a subsequence, we assume that (5.10) holds for all $k \in \mathbb{N}$ and, thus

$$\alpha_\delta^-(x) \le a(x, u^{\varepsilon_k}(x, t)) \quad \text{for all} \ \ (x, t) \in \overline{\Omega} \times [\exp(\mu_k/\varepsilon_k), \, \exp(\lambda_k/\varepsilon_k)], \ k \in \mathbb{N}. \tag{5.11}$$

We set $\Pi = \{x \in \partial\Omega \,:\, g(x) > \beta_2+\delta\}$ and note by (5.7) that Π is an open neighborhood, relative to $\partial\Omega$, of Γ_δ^- and

$$\{x \in \overline{\Omega} \,:\, V_\delta^-(x) \le M_\delta^-\} = \{x \in \Omega \,:\, V_\delta^-(x) \le M_\delta^-\} \cup \Gamma_\delta^- \subset \Omega^\Pi,$$

and deduce that, for $\gamma > 0$ sufficiently small,

$$\{x \in \overline{\Omega} \,:\, V_\delta^-(x) \le M_\delta^- + \gamma\} \subset \Omega_\gamma^\Pi. \tag{5.12}$$

We fix $\gamma > 0$ so that (5.12) and $5\gamma < \beta_1 - \beta_2$ hold, set

$$\Sigma = \{x \in \overline{\Omega} \,:\, V_\delta^-(x) \le M_\delta^- + \gamma\},$$

and select a sequence $\{\nu_k\}_{k\in\mathbb{N}}$ so that

$$\begin{cases} \mu_k < \nu_k < \lambda_k, \qquad u^{\varepsilon_k}(0, \exp(\nu_k/\varepsilon_k)) = \beta_1 - 3\gamma \quad \text{for all } \ k \in \mathbb{N}, \\ \beta_1 \ge u^{\varepsilon_k}(0,t) \ge \beta_1 - 3\gamma \quad \text{for all } \ t \in [\exp(\mu_k/\varepsilon_k),\, \exp(\nu_k/\varepsilon_k)], \ k \in \mathbb{N}. \end{cases} \tag{5.13}$$

Furthermore, observe that

$$\Sigma \subset \Omega_\gamma^\Pi,$$

and, in view of (5.7) and (5.8),

$$g(x) > \beta_1 + \delta/2 > \beta_1 \quad \text{for all } \ x \in \Pi. \tag{5.14}$$

Similarly to (5.9), using Theorem 8, we may assume that, for some $r \in (0,\, r_0)$,

$$|u^{\varepsilon_k}(x,t) - u^{\varepsilon_k}(0,t)| < \gamma \ \text{ for all } \ (x,t) \in B_r \times [\exp(a_1/\varepsilon_k), \infty) \text{ and } k \in \mathbb{N}.$$

We set

$$v^k(x,t) = u^{\varepsilon_k}(x, t + \exp(\mu_k/\varepsilon_k)) - \beta_1 + \gamma \quad \text{for } \ (x,t) \in \overline{Q}, \ k \in \mathbb{N},$$

and note that

$$v^k(x,0) \ge 0 \quad \text{for all } \ x \in B_r.$$

We apply Theorem 10, with ε, u^ε and α replaced respectively by ε_k, $-v^k$ and $\theta_0^{-1}I$, to deduce that, for sufficiently large $k \in \mathbb{N}$ and for some $\rho > 0$,

$$-v^k(0,t) \le \gamma \quad \text{for all } \ t \in [0,\, \exp(\rho/\varepsilon_k)],$$

that is,

$$u^{\varepsilon_k}(0,t) \ge \beta_1 - 2\gamma \quad \text{for all } \ t \in [\exp(\mu_k/\varepsilon_k),\, \exp(\mu_k/\varepsilon_k) + \exp(\rho/\varepsilon_k)],$$

In view of the choice of ν_k, this implies that for sufficiently large $k \in \mathbb{N}$,

$$\exp(\nu_k/\varepsilon_k) > \exp(\mu_k/\varepsilon_k) + \exp(\rho/\varepsilon_k). \tag{5.15}$$

Next we set

$$w^k(x,t) = u^{\varepsilon_k}(x, t + \exp(\mu_k/\varepsilon_k)) - \beta_1 + 3\gamma \quad \text{for } \ (x,t) \in \overline{Q}, \ k \in \mathbb{N},$$

and note that, in view of (5.13) and (5.14),

$$\begin{cases} w^k(0,t) \ge 0 & \text{for all } \ t \in [0,\, \exp(\nu_k/\varepsilon_k) - \exp(\mu_k/\varepsilon_k)] \\ w^k(x,t) = g(x) - \beta_1 + 3\gamma \ge 0 & \text{for all } \ (x,t) \in \Pi \times [0,\, \infty). \end{cases}$$

Recalling (5.15), we apply Theorem 7, with ε and u^ε replaced by ε_k and $-w_k$, to get, for sufficiently large k,

$$-w^k(x, \exp(\nu_k/\varepsilon) - \exp(\mu_k/\varepsilon_k)) \le \gamma \quad \text{for all } \ x \in \Omega_\gamma^\Pi,$$

which reads

$$u^{\varepsilon_k}(x, \exp(\nu_k/\varepsilon)) \geq \beta_1 - 4\gamma \quad \text{for all } \ x \in \Omega_\gamma^\Pi.$$

Finally, we set

$$z^k(x,t) = u^{\varepsilon_k}(x, t + \exp(\nu_k/\varepsilon_k)) - \beta_1 + 4\gamma \quad \text{for } \ (x,t) \in \overline{Q},$$

observe that

$$\begin{cases} z^k(x,0) \geq 0 \quad \text{for all } \ x \in \Sigma, \\ z^k(x,t) = g(x) - \beta_1 + 4\gamma \geq 0 \quad \text{for all } \ (x,t) \in \Pi \times [0,\, \infty), \end{cases}$$

and invoke Theorem 11, to conclude that for sufficiently large $k \in \mathbb{N}$,

$$z^k(0, \exp(\lambda_k/\varepsilon_k) - \exp(\nu_k/\varepsilon_k)) \geq -\gamma,$$

and, hence,

$$u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) \geq \beta_1 - 5\gamma > \beta_2,$$

which is a contradiction. □

6. Proof of the main theorem

The proof of Theorem 1 is an easy consequence of Theorem 2 as shown in [6,8]. For the reader's convenience, we reproduce it here. We begin with an introductory lemma.

Lemma 5. *Assume* (1.5), (1.10) *and* (1.4) *and let* $u^\varepsilon \in C(\overline{Q}) \cap C^{2,1}(Q)$ *be a solution of* (1.1) *and* (1.2). *For any* $\delta > 0$ *there exist* $\lambda_0 > 0$ *and* $\varepsilon_0 \in (0,\, 1)$ *such that*

$$|u^\varepsilon(0,t) - g(0)| \leq \delta \quad \textit{for all } \ t \in [0,\, \exp(\lambda_0/\varepsilon)] \quad \textit{and} \quad \varepsilon \in (0,\, \varepsilon_0). \tag{6.1}$$

Proof. Let $V \in \mathrm{Lip}(\overline{\Omega})$ be the quasi-potential associated with $(\theta_0^{-1} I, b)$. We choose $m > 0$ small enough so that $m < \min_{\partial\Omega} V$ and

$$\{x \in \Omega \,:\, V(x) \leq m\} \subset \{x \in \Omega \,:\, |g(x) - g(0)| \leq \delta/2\}.$$

Applying Theorem 10, with $a^\varepsilon(x,t) = a(x, u^\varepsilon(x,t))$ and $\alpha(x) = \theta_0^{-1} I$ and u^ε replaced by $\pm(u^\varepsilon - g(0)) - \delta/2$, we get that, for each $\gamma > 0$, there is $\varepsilon_0 \in (0,\, 1)$ such that

$$\pm(u^\varepsilon(0,t) - g(0)) - \delta/2 \leq \gamma \quad \text{for all } \ t \in [0,\, \exp((m-\gamma)/\varepsilon)] \quad \text{and} \quad \varepsilon \in (0,\, \varepsilon_0).$$

We fix $\gamma > 0$ small enough so that $\gamma < \min\{\delta/2,\, m\}$, and we get (6.1) with $\lambda_0 = m - \gamma$. □

Proof of Theorem 1. In view of Theorem 8, we only need to show that

$$\lim_{\varepsilon \to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) = c(\lambda). \tag{6.2}$$

The comparison principle yields that

$$g_{\min} \leq u^\varepsilon \leq g_{\max} \text{ on } \ \overline{Q}.$$

We fix $\lambda > 0$ and we consider first the case $\lambda < M(c_0)$, which implies that $c(\lambda) = c_0$, and prove that

$$\limsup_{\varepsilon \to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \leq c(\lambda) = c_0. \tag{6.3}$$

We argue by contradiction and suppose that

$$\limsup_{\varepsilon \to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) > c_0. \tag{6.4}$$

Using the continuity of the function M, we choose $\beta_1, \beta_2 \in \mathbb{R}$ so that

$$c_0 < \beta_1 < \beta_2 < \limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \quad \text{and} \quad M(\beta_2) > \lambda,$$

and note that, in view of Lemma 5, there are constants $\lambda_0 \in (0, \lambda)$ and $\varepsilon_0 \in (0, 1)$ such that

$$u^\varepsilon(0, \exp(\lambda_0/\varepsilon)) \le \beta_1 \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{6.5}$$

On the other hand, (6.4) yields a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \varepsilon_0)$ such that $\varepsilon_k \to 0$ and

$$u^{\varepsilon_k}(0, \exp(\lambda/\varepsilon_k)) \ge \beta_2 \quad \text{for all } \ k \in \mathbb{N},$$

while, (6.5) gives

$$u^{\varepsilon_k}(0, \exp(\lambda_0/\varepsilon_k)) \le \beta_1 \quad \text{for all } k \in \mathbb{N}.$$

The continuity of $t \mapsto u^{\varepsilon_k}(0, t)$ implies that, for each $k \in \mathbb{N}$, there exist $\mu_k, \lambda_k \in [\lambda_0, \lambda]$ such that $\lambda_0 \le \mu_k < \lambda_k \le \lambda$ and

$$u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \text{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

Hence we have

$$g_{\min} \le c_0 < \beta_1 < \beta_2 < \limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \le g_{\max} \ \text{ and } \lambda_k \le \lambda < M(\beta_2) \text{ for all} k \in \mathbb{N},$$

which contradicts the assertion of Theorem 2 concerning condition (A).

A similar argument shows that

$$\liminf_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \ge c(\lambda),$$

and, thus, we have (6.2) in the case where $\lambda < M(c_0)$.

Next we consider the case where $\lambda \ge M(c_0)$ and $c_1 = c_0$ and recall that, by definition, $c(\lambda) = c_0$. We first suppose that

$$\limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) > c_0.$$

We use (1.16) and the piecewise continuity of G to select $\beta_2 \in \mathbb{R}$ so that G is continuous at β_2, $c_0 < \beta_2 < \limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon))$ and $G(\beta_2) < \beta_2$, and, hence, for $\delta > 0$ small enough, we have $G(c) < \beta_2 - \delta$ for all $c \in [\beta_2 - \delta, \beta_2 + \delta]$.

Choosing, for instance, $\beta_1 = (c_0 + \beta_2)/2$, so that $c_0 < \beta_1 < \beta_2$, and, using Lemma 5 as in the previous case, we may choose sequences $\varepsilon_k \to 0$, and $\{\mu_k\}$, $\{\lambda_k\}$ such that, for some $\lambda_0 > 0$ and for all $k \in \mathbb{N}$,

$$\lambda_0 \le \mu_k < \lambda_k \le \lambda, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1 \quad \text{and} \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

This is a situation that condition (B) of Theorem 2 holds, which is a contradiction. Thus, we conclude that

$$\limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \le c_0.$$

A similar argument shows

$$\liminf_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \ge c_0,$$

and, hence, we have (6.2) when $\lambda \ge M(c_0)$ and $c_1 = c_0$.

Now we consider the case where $\lambda \ge M(c_0)$ and $c_1 > c_0$. The definition of c_1 implies that $G(c) > c$ for all $c \in [c_0, c_1)$ and $\min\{G(c) - c \,:\, c \in (c_1, c_2)\} \le 0$ for all $c_2 > c_1$. Moreover,

$c(\lambda) \in [c_0,\, c_1]$, $M(c) \neq \lambda$ for all $c \in [c_0,\, c(\lambda))$, and, if $c(\lambda) < c_1$, then $M(c(\lambda)) = \lambda$. Since M is continuous and $\lambda \geq M(c_0)$, it follows that $\lambda > M(c)$ for all $c \in [c_0,\, c(\lambda))$.

Suppose that

$$\limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) > c(\lambda).$$

We assume first that $c(\lambda) = c_1$, which implies that $c_1 < g_{\max}$. Then (1.16) yields $\beta_2 \in \mathbb{R}$ so that G is continuous at β_2, $G(\beta_2) < \beta_2$ and $c_1 < \beta_2 < \limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon))$. Fixing $\beta_1 \in (c_1,\, \beta_2)$, we argue, as in the previous case, with c_1 in place of c_0 and find sequences $\varepsilon_k \to 0$, $\{\mu_k\}$ and $\{\lambda_k\}$, and constants $\lambda_0 > 0$ and $\delta > 0$ such tha for all $k \in \mathbb{N}$,

$$\lambda_0 \leq \mu_k < \lambda_k \leq \lambda, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k) = \beta_1, \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2,$$

and

$$G(c) < \beta_2 - \delta \quad \text{ for all } \ c \in [\beta_2 - \delta,\, \beta_2 + \delta],$$

which together contradict Theorem 2.

Assume next that $c(\lambda) < c_1$. As noted above, we have $M(c(\lambda)) = \lambda$ and $M(c) < \lambda$ for all $c \in [c_0,\, c(\lambda))$, and, in particular,

$$M(c) \leq \lambda \quad \text{for all } \ c \in [c_0,\, c(\lambda)]. \tag{6.6}$$

Since the function c is continuous at λ, we may choose $\eta > 0$ so that $c(r) < c_1$ for all $r \in [\lambda,\, \lambda+\eta]$. For any $r \in (\lambda,\, \lambda+\eta]$, noting that $r > M(c_0)$, we find by the definition of $c(r)$ that $M(c(r)) = r$, which together with (6.6) implies that $c(r) > c(\lambda)$. We choose $\gamma \in (0,\, \eta)$ small enough so that $c(\lambda + \gamma) < \limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon))$. If we set $\beta_2 = c(\lambda + \gamma)$ and fix $\beta_1 \in (c(\lambda), \beta_2)$, then we have $c(\lambda) < \beta_1 < \beta_2 < \limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon))$.

As before, we choose sequences $\varepsilon_k \to 0$, $\{\mu_k\}$ and $\{\lambda_k\}$ such that, for some $\lambda_0 > 0$ and for all $k \in \mathbb{N}$,

$$\lambda_0 \leq \mu_k < \lambda_k \leq \lambda, \quad u^{\varepsilon_k}(0, \exp(\mu_k/\varepsilon_k)) = \beta_1, \quad u^{\varepsilon_k}(0, \exp(\lambda_k/\varepsilon_k)) = \beta_2.$$

Furthermore, noting that $M(\beta_2) = M(c(\lambda + \gamma)) = \lambda + \gamma > \lambda$, we may choose $\delta > 0$ so that $\lambda_k < M(\beta_2) - \delta$ for all $k \in \mathbb{N}$. This contradicts Theorem 2.

Thus, in the case when $\lambda \geq M(c_0)$ and $c_1 > c_0$, we have

$$\limsup_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \leq c(\lambda),$$

and, by similar considerations, we find

$$\liminf_{\varepsilon\to 0} u^\varepsilon(0, \exp(\lambda/\varepsilon)) \geq c(\lambda),$$

and we conclude that (6.2) holds when $\lambda \geq M(c_0)$ and $c_1 > c_0$.

A similar argument proves that (6.2) holds when $\lambda \geq M(c_0)$ and $c_1 < c_0$, and the proof is complete. □

Appendix A. A subsolution property

For $T > 0$ and a (relatively) open subset Π of $\partial\Omega$, we consider the problem

$$\begin{cases} U_t \leq b(x) \cdot DU \quad \text{ in } \quad \Omega \times (0,\, T], \\ \min\{U_t - b(x) \cdot DU,\, U\} \leq 0 \quad \text{on} \quad \Pi \times (0,\, T]. \end{cases} \tag{A.1}$$

Lemma A.1. *Let $U \in \mathrm{USC}(\overline{Q}_T)$ be a subsolution of* (A.1), *fix $z \in \varOmega^{\varPi}$ and set*

$$u(t) = U(X(T-t, z), t) \quad \text{for } t \in [0, T].$$

Then $u \in \mathrm{USC}([0, T])$ and, if $z \in \varOmega$, it is a subsolution of

$$u' \le 0 \quad \text{in} \ \ (0, T] \tag{A.2}$$

and, if $z \in \varPi$, it is a subsolution of

$$\begin{cases} u' \le 0 & \text{in} \ \ (0, T), \\ \min\{u', u\} \le 0 & \text{on} \ \ \{T\}. \end{cases} \tag{A.3}$$

We note that observations like the lemma above concerning the restriction of viscosity solutions to lower dimensional manifolds go back to Crandall and Lions [4, Proposition I.13].

Proof. Let $\phi \in C^1((0, T])$ and assume that $u - \phi$ has a strict maximum at $\hat{t} \in (0, T]$.

For $\alpha > 0$ consider the function $\Phi : \overline{Q}_T \to \mathbb{R}$ given by

$$\Phi(x,t) := U(x,t) - \phi(t) - \alpha|x - X(T-t, z)|^2,$$

let $(x_\alpha, t_\alpha) \in \overline{Q}_T$ be a maximum point of Φ, set $\hat{x} = X(T - \hat{t}, z)$, and observe that, as $\alpha \to \infty$, $(x_\alpha, t_\alpha) \to (\hat{x}, \hat{t})$, $\alpha|x_\alpha - X(T - t_\alpha, z)|^2 \to 0$ and $U(x_\alpha, t_\alpha) \to U(\hat{x}, \hat{t})$.

Then, for α sufficiently large, we may assume that $(x_\alpha, t_\alpha) \in \varOmega \times (0, T]$ if either $z \in \varOmega$ or $\hat{t} < T$, and $(x_\alpha, t_\alpha) \in \varOmega^{\varPi} \times (0, T]$ if $z \in \varPi$.

If $(x_\alpha, t_\alpha) \in \Omega \times (0, T]$, (A.1) yields

$$\phi'(t_\alpha) - 2\alpha(X(T - t_\alpha, z) - x_\alpha) \cdot \dot{X}(T - t_\alpha, z) \le 2\alpha b(x_\alpha) \cdot (x_\alpha - X(T - t_\alpha, z)),$$

and then

$$\begin{aligned} \phi'(t_\alpha) &\le 2\alpha(x_\alpha - X(T - t_\alpha, z)) \cdot (b(x_\alpha) - b(X(T - t_\alpha, z))) \\ &\le 2\|Db\|_{L^\infty(\varOmega)} \alpha |x_\alpha - X(T - t_\alpha, z))|^2. \end{aligned}$$

Similarly, if $(x_\alpha, t_\alpha) \in \varPi \times (0, T]$, then we get

$$\phi'(t_\alpha) \le 2\|Db\|_{L^\infty(\varOmega)} \alpha |x_\alpha - X(T - t_\alpha, z))|^2 \quad \text{or} \quad U(x_\alpha, t_\alpha) \le 0.$$

Sending $\alpha \to \infty$ yields

$$\phi'(\hat{t}) \le 0 \quad \text{if either} \ \ z \in \varOmega \ \ \text{or} \ \ \hat{t} < T,$$

and

$$\phi'(\hat{t}) \le 0 \quad \text{or} \quad u(\hat{t}) \le 0 \quad \text{if} \ \ z \in \varPi \ \ \text{and} \ \ \hat{t} = T.$$

□

Appendix B. The supersolution property up to the boundary

For $H(x,p) = \alpha(x)p \cdot p + b(x) \cdot p$ and $\alpha \in C(\overline{\varOmega}, \mathbb{S}^n(\theta_0))$ consider the

$$\begin{cases} H(x, Du) = 0 & \text{in} \ \ \varOmega, \\ u(0) = 0. \end{cases} \tag{B.1}$$

Lemma B.1. *The maximal subsolution $V \in \mathrm{Lip}(\overline{\varOmega})$ of* (B.1) *with $V(0) = 0$ satisfies, in the viscosity sense,*

$$H(x, DV) \ge 0 \quad \text{on} \ \ \overline{\varOmega}.$$

Note that the importance of the lemma above is that the viscosity inequality holds up to the boundary.

Proof. Let $\phi \in C^1(\overline{\Omega})$ and assume that $V - \phi$ has a strict minimum at $\hat{x} \in \overline{\Omega}$ and $V(\hat{x}) = \phi(\hat{x})$.

To prove the assertion of the lemma, we argue by contradiction and suppose that $H(\hat{x}, D\phi(\hat{x})) < 0$.

Indeed, if $\hat{x} = 0$, then

$$H(\hat{x}, D\phi(\hat{x})) = \alpha(0)D\phi(0) \cdot D\phi(0) \geq 0,$$

and, henceforth, we may assume that $\hat{x} \neq 0$.

We may choose constants $r > 0$ and $\varepsilon > 0$ so that $0 \notin B_r(\hat{x})$ and

$$H(x, D\phi) \leq 0 \quad \text{for all} \;\; x \in \overline{\Omega} \cap B_r(\hat{x}), \tag{B.2}$$

$$\varepsilon + \phi(x) < V(x) \quad \text{for all } \hat{x} \in \overline{\Omega} \cap \partial B_r(\hat{x}). \tag{B.3}$$

It follows from (B.2) that, in the viscosity sense,

$$H(x, D\phi) \leq 0 \quad \text{in} \;\; \Omega \cap B_r(\hat{x}).$$

Set

$$W(x) = \max\{V(x), \varepsilon + \phi(x)\} \quad \text{for} \;\; x \in \overline{\Omega},$$

and observe that $\Omega = N \cup M$, where $N = \Omega \cap B_r(\hat{x})$, $M = \{x \in \Omega \,:\, V(x) > \varepsilon + \phi(x)\}$ (note that N, M are both open subsets of Ω),

$$H(x, DW) \leq 0 \quad \text{in} \;\; N \quad \text{in the viscosity sense},$$

$W = V$ in M and $\hat{x} \in M$. Hence, W is a subsolution of (B.1), such that $W(\hat{x}) > V(\hat{x})$, which contradicts the maximality of V. □

Appendix C. A comparison theorem

We follow the arguments of [10, Corollary 2.2 & Remark 2.4] to give a proof of following lemma.

Lemma C.1. *Let $a_0 \in C(\mathbb{R}^n, \mathbb{S}^n(\theta_0))$ and $H(x,p) = a_0(x)p \cdot p + b(x) \cdot p$. If $v \in \mathrm{Lip}(\overline{\Omega})$ and $w \in \mathrm{LSC}(\overline{\Omega})$ are respectively a subsolution and a supersolution of the state-constraints problem*

$$H(x, Du) = 0 \quad \textit{in} \;\; \Omega,$$

that is, v and w satisfy, respectively,

$$H(x, Dv) \leq 0 \quad \textit{in } \Omega \quad \textit{and} \quad H(x, Dw) \geq 0 \quad \textit{on} \;\; \overline{\Omega},$$

and $v(0) \leq w(0)$, then $u \leq v$ on $\overline{\Omega}$.

Note that the viscosity property of v and w at the origin is indeed not required in the lemma above. That is, it is enough to assume that v and w are a subsolution of

$$H(x, Dv) \leq 0 \quad \text{in} \;\; \Omega \setminus \{0\},$$

and a supersolution of

$$H(x, Dw) \geq 0 \quad \text{on} \;\; \overline{\Omega} \setminus \{0\}.$$

Proof. Fix $\varepsilon > 0$ and choose $r \in (0, r_0)$ sufficiently small so that

$$\max_{\partial B_r} v \le \min_{\partial B_r} w + \varepsilon,$$

set $\Omega(r) := \Omega \setminus \overline{B}_r$, define $h \in C(\partial\Omega(r))$ and $v_\varepsilon \in \mathrm{Lip}(\overline{\Omega})$ by

$$v_\varepsilon = v + \varepsilon \quad \text{and} \quad h(x) = \begin{cases} \min_{\partial B_r} w & \text{if } x \in \partial B_r, \\ \max_{\partial\Omega} v & \text{if } x \in \partial\Omega, \end{cases}$$

and observe that v_ε and w are, respectively, a subsolution and a supersolution of the Dirichlet problem in the viscosity sense (see [10]):

$$\begin{cases} H(x, Du) = 0 & \text{in} \quad \Omega(r), \\ u = h \quad \text{or} \quad H(x, Du) = 0 & \text{on} \quad \partial\Omega(r). \end{cases}$$

It follows from [11, Corollary 4] that there exists $\psi \in \mathrm{Lip}(\overline{\Omega}(r))$ which is a subsolution of $H(x, D\psi) \le -\eta$ in $\Omega(r)$ for some $\eta > 0$ and note, that we may assume by adding, if necessary a constant, that $\psi \le v$ on $\Omega(r)$.

Define $v^\varepsilon \in \mathrm{Lip}(\overline{\Omega}(r))$ by $v^\varepsilon(x) = (1 - \varepsilon)v(x) + \varepsilon\psi(x)$ and note that v^ε is a subsolution of

$$\begin{cases} H(x, Du) \le -\varepsilon\eta & \text{in} \quad \Omega(r), \\ u \le h \quad \text{or} \quad H(x, Du) \le -\varepsilon\eta & \text{on} \quad \partial\Omega(r). \end{cases}$$

It is clear that the domain $\Omega(r)$ satisfies the uniform interior cone condition and, hence, we apply [10, Corollary 2.2 & Remark 2.4] to v^ε and w_ε, to conclude that $v^\varepsilon \le w_\varepsilon$ in $\overline{\Omega}(r)$, from which, after sending $\varepsilon \to 0$, we get $v \le w$ on $\overline{\Omega}$. □

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