TWO REMARKS ON PERIODIC SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

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We show firstly the equivalence between existence of a periodic solution of the Hamilton-Jacobi equation $u_t + H(x, Du) = f(t)$ in $\Omega \times \mathbb{R}$, where Ω is a bounded domain of $\mathbb{R}^n$, with the Dirichlet boundary condition $u = g(x,t)$ and that of a subsolution of the stationary problem $H(x, Dv) = \langle f \rangle$ under the assumptions that the function $(f(t), g(x,t))$ is periodic in t and H is coercive. Here $\langle f \rangle$ denotes the average of f over the period. This proposition is a variant of a recent result for $\Omega = \mathbb{R}^n$ due to Bostan–Namah, and we give a different and simpler approach to such an equivalence. Secondly, we establish that any periodic solution $u(x,t)$ of the problem, $u_t + H(x, Du) = 0$ in $\Omega \times \mathbb{R}$ and $u = g$ on $\partial\Omega \times \mathbb{R}$, is constant in t on the Aubry set for H. Here H is assumed to be convex, coercive and strictly convex in a sense.

Keywords: Hamilton-Jacobi equations; periodic solutions; Aubry sets.

1. Introduction

In this paper we consider the Hamilton-Jacobi equation with the Dirichlet boundary condition:

$$\begin{cases} u_t(x,t) + H(x, u(x,t), Du(x,t)) = f(t) \quad \text{for } (x,t) \in \Omega \times \mathbb{R}, \\ u(x,t) = g(x,t) \quad \text{for } (x,t) \in \partial\Omega \times \mathbb{R}. \end{cases} \tag{D}$$

Here Ω is a bounded domain of $\mathbb{R}^n$, $u = u(x,t)$ is a function on $\overline{\Omega} \times \mathbb{R}$ which represents the unknown function, H is a given function on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ which is the so-called Hamiltonian, f is a given periodic function with period $T > 0$, $g = g(x, t)$ is a given function on $\partial\Omega \times \mathbb{R}$, the functions u, H, f and g are scalar functions and u_t and Du denote the derivatives $\partial u/\partial t$ and $(\partial u/\partial x_1, ..., \partial u/\partial x_n)$, respectively. In this note we will be concerned only with viscosity solutions of Hamilton-Jacobi equations and thus we call them just solutions. The boundary condition, $u = g$ on $\partial\Omega\times\mathbb{R}$, is also understood in the viscosity sense. We refer to [1,2,6] for overviews on viscosity solutions theory.

We assume throughout the following (A1)–(A5):

(A1) $H \in \mathrm{BUC}(\overline{\Omega} \times [-R, R] \times B(0, R))$ for all $R > 0$.
(A2) H is monotone. That is, for each $(x, p) \in \overline{\Omega} \times \mathbb{R}^n$, the function $H(x, \cdot, p)$ is non-decreasing on $\mathbb{R}$.
(A3) H is coercive. That is, for every $r \in \mathbb{R}$,

$$\lim_{|p|\to\infty} H(x, r, p) = \infty \quad \text{uniformly for } x \in \overline{\Omega}.$$

(A4) Ω is a bounded, open, connected subset of $\mathbb{R}^n$ with C^0 boundary.
(A5) $g \in \mathrm{BUC}(\partial\Omega \times \mathbb{R})$.

Here and in what follows $\mathrm{BUC}(X)$ denotes the space of bounded, uniformly continuous functions on metric space X. In (A4), it is assumed that Ω has C^0 boundary. We mean by this C^0 regularity that for each $z \in \partial\Omega$ there are a neighborhood V of z and a C^1 diffeomorphism Φ of V to $B(0, r)$, with $r > 0$, such that $\Phi(z) = 0$ and $\Phi(\Omega \cap V) = B(0, r) \cap \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > h(x')\}$ for some $h \in C(\mathbb{R}^{n-1})$. The role of (A4) in this note is to guarantee (see [19]) that if $u \in C(\Omega)$ satisfies $|Du(x)| \le C$ in Ω in the viscosity sense for some constant $C > 0$, then u is uniformly continuous on Ω, so that it can be extended uniquely to $\overline{\Omega}$ as a continuous function.

In this note we often deal with soloutions $u(x, t)$ of (D) which is periodic in t. Given a set X and a function $w = w(x, t)$ defined in $X \times \mathbb{R}$, we call w *periodic* with period T if it is periodic in t with period T for all $x \in X$.

In the next section we assume that the function $g(x, t)$ is also periodic with period T and establish the equivalence between existence of a periodic solution $u \in C(\overline{\Omega} \times \mathbb{R})$ of (D) with period T and that of a subsolution of

$$H(x, v, Dv) = \langle f\rangle \quad \text{in } \Omega. \tag{S}$$

The result here gives a variant of [5,Theorem 4.1], due to Bostan–Namah, where Ω is replaced by $\mathbb{R}^n$. Our proof is somewhat simpler than the one in [5]. Our result covers also the equivalence between existence of a bounded solution of (D) and that of a subsolution of (S) without the assumption that g is periodic.

We refer for instance to [23, 22, 17] for some existence results of periodic solutions of Hamilton-Jacobi equations.

In the final section, Section 3, we restrict ourselves to the case where $f = 0$, $H = H(x,p)$ and $H(x,p)$ is convex in p. We establish a theorem on representation of bounded solutions u of (D) and then show under some additional assumptions that if g is periodic with period T, then any solution $u(x,t)$ of (D) is constant in t on the Aubry set $\mathcal{A}$. Actually, we show this constancy result under an assumption more general than the periodicity of g. We give the definition of the Aubry set $\mathcal{A}$ in Section 3, following [11–13,19,20]. The constancy in t of periodic solutions $u(x,t)$ of (D) on the Aubry set indicates a new characteristic of the Aubry set for problem (D) in the periodic setting in t, and indeed has an important role in the dynamical approach to the asymptotic behavior for large t of solutions of the Cauchy–Dirichlet problem for Hamilton-Jacobi equation $u_t + H(x, Du) = 0$ with the Dirichlet condition $u = g$. See [22] for the asymptotic behavior of solutions of the Cauchy-Dirichlet problem with periodic Dirichlet data in t. We refer to [8–13] for Aubry sets and weak KAM theory.

2. Existence of periodic solutions

Throughout this section we assume that $f \in C(\mathbb{R})$ is a periodic function with period $T > 0$. The following theorem is our main result in this section.

Theorem 2.1. (i) *Problem* (D) *has a bounded solution* $u \in C(\overline{\Omega} \times \mathbb{R})$ *if and only if* (S) *has a subsolution* $v \in C(\overline{\Omega})$. (ii) *Assume that* g *is periodic with period* T. *Then* (D) *has a periodic solution* $u \in C(\overline{\Omega} \times \mathbb{R})$ *with period* T *if and only if* (S) *has a subsolution* $v \in C(\overline{\Omega})$.

We set

$$F(t) = \int_0^t (f(s) - \langle f \rangle)\mathrm{d}s \quad \text{for } t \in \mathbb{R}.$$

Note that F is a C^1 periodic function on $\mathbb{R}$ with period T. Also, we set $G(x,t) = g(x,t) - F(t)$ for $(x,t) \in \partial\Omega \times \mathbb{R}$. We consider the problem

$$\begin{cases} w_t(x,t) + H(x,\, w(x,t) + F(t),\, Dw(x,t)) = \langle f \rangle & \text{in } \Omega \times \mathbb{R}, \\ w(x,t) = G(x,t) & \text{on } \partial\Omega \times \mathbb{R}. \end{cases} \tag{D$'$}$$

Lemma 2.1. (i) *Problem* (D) *has a bounded solution* $u \in C(\overline{\Omega} \times \mathbb{R})$ *if and only if* (D′) *has a bounded solution* $w \in C(\overline{\Omega} \times \mathbb{R})$. (ii) *Problem* (D) *has a periodic solution* $u \in C(\overline{\Omega} \times \mathbb{R})$ *with period* T *if and only if* (D′) *has a periodic solution* $w \in C(\overline{\Omega} \times \mathbb{R})$ *with period* T.

Proof. Observe that if u is a solution of (D), then $w(x,t) := u(x,t) - F(t)$ is a solution of (D′). On the other hand, if w is a solution of (D′), then $u(x,t) := w(x,t) + F(t)$ is a solution of (D). Note that u is bounded on $\overline{\Omega} \times \mathbb{R}$ if and only if so is the function $w(x,t) := u(x,t) - F(t)$. Note also that u is periodic with period T if and only if so is the function w. □

It is a useful and classical observation on solutions of (D) or (D′), which is a consequence of the coerciveness of the Hamiltonian H, that if u is an upper semi-continuous subsolution of (D) or (D′) on $\overline{\Omega} \times (a,\, b)$, then $u(x,t) \le g(x,t)$ on $\partial\Omega \times (a,\, b)$.

Proof of Theorem 2.1. First of all, assume that (D) has a bounded continuous solution. By Lemma 2.1, there is a bounded solution $w \in C(\overline{\Omega} \times \mathbb{R})$ of (D′). Let $M > 0$ be a constant such that $|w(x,t)| \vee |F(t)| \le M$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. Set

$$v(x) = \sup_{t \in \mathbb{R}} w(x,t) \quad \text{for } x \in \Omega,$$

and observe by the stability of viscosity property that $u := v^*$ is a subsolution of

$$H(x, u - M, Du) = \langle f \rangle \quad \text{in } \Omega,$$

where v^* denotes the upper semi-continuous envelope of v. Since H is coercive and Ω has C^0 boundary, we find that $v^* \in C(\overline{\Omega})$ and $v^* - M$ is a subsolution of (S).

Next, we suppose that (S) has a subsolution $v \in C(\overline{\Omega})$ and show in view of Lemma 2.1 that (D′) has a bounded solution $w \in C(\overline{\Omega} \times \mathbb{R})$. In view of the monotonicity of H, we see that the function $v + C$, with any negative constant, is a subsolution of (S). By adding a negative constant to v if necessary, we may assume that $v(x) \le G(x,t)$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$. It is obvious that $v - M$ is a subsolution of (D′). In view of the coerciveness of H, we may choose a supersolution $\psi \in C(\overline{\Omega})$ of (S) so that $\psi(x) \ge G(x,t)$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$. Note that $\psi + M$ is a supersolution of (D′). We define $w : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} w(x,t) = \sup\{\phi(x,t) :\ & \phi \text{ is a subsolution of (D}'\text{)}, \\ & \phi(x,t) \le \psi(x) + M \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}\}. \end{aligned}$$

It is clear that $v(x) \le w(x,t) \le \psi(x) + M$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. In view of the Perron method we see that the upper semi-continuous envelope w^* (respectively., the lower semi-continuous envelope w_*) of w is a subsolution (respectively., supersolution) of (D′). In particular, since $w^*(x,t) \le \psi(x) + M$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, we find that $w^* \le w$ and hence $w^* = w$ on $\overline{\Omega} \times \mathbb{R}$. Also, it is clear by the definition of w that if g is periodic with period T, then the function w is periodic with period T.

It remains only to show that $w \in C(\overline{\Omega} \times \mathbb{R})$. Let ω be a modulus such that

$$|G(x,t) - G(x,s)| \vee |F(t) - F(s)| \le \omega(|t-s|) \quad \text{for all } t,s \in \mathbb{R},\ x \in \partial\Omega.$$

For any $h \in \mathbb{R}$ we consider the function $w^h(x,t) := w(x,t+h) - \omega(|h|)$. Observe that w^h is a subsolution of (D′) and satisfies $w^h(x,t) \le \psi(x) + M$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. Therefore, by the definition of w, we have $w^h(x,t) \le w(x,t)$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. That is, we have $w(x,t+h) \le w(x,t) + \omega(|h|)$ for all $(x,t,h) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$. Hence we get

$$|w(x,t) - w(x,s)| \le \omega(|t-s|) \ \text{ for all } t,s \in \mathbb{R},\ x \in \overline{\Omega}. \tag{1}$$

We suppose for the moment that $|G(x,t) - G(x,s)| \le L|t-s|$ for all $(x,t,s) \in \partial\Omega \times \mathbb{R}^2$ and for some $L > 0$. Then, since $F \in C^1(\mathbb{R})$, we may assume by replacing L by a larger number if necessary that

$$|G(x,t) - G(x,s)| \vee |F(t) - F(s)| \le L|t-s| \quad \text{for all } t,s \in \mathbb{R},\ x \in \partial\Omega.$$

This combined with (1) ensures that $|w(x,t) - w(x,s)| \le L|t-s|$ for all $x \in \overline{\Omega}$ and $t,s \in \mathbb{R}$, from which we infer that

$$H(x,\ w(x,t) - M,\ Dw(x,t)) \le L + \langle f \rangle \ \text{ in } \Omega \times \mathbb{R}.$$

The coerciveness of H and the C^0 boundary regularity of Ω guarantee that $|w(x,t) - w(y,t)| \le \omega_0(|x-y|)$ for all $(x,y,t) \in \overline{\Omega}^2 \times \mathbb{R}$ and for some modulus ω_0. Thus, w is uniformly continuous on $\overline{\Omega} \times \mathbb{R}$.

Now, we treat the general situation where $g \in \mathrm{BUC}(\partial\Omega \times \mathbb{R})$. We approximate G by a sequence of functions $G_k(x,t)$, $k = 1,2,...$, such that $G(x,t) - 1/k \le G_k(x,t) \le G(x,t)$ and $|G_k(x,t) - G_k(x,s)| \le L_k|t-s|$ for all $(x,t,s) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}$ and for some $L_k > 0$. Consider the problem

$$\begin{cases} z_t + H(x,\ z + F,\ Dz) = \langle f \rangle & \text{in } \Omega \times \mathbb{R} \\ z(x,t) = G_k(x,t) & \text{for } (x,t) \in \partial\Omega \times \mathbb{R}. \end{cases} \tag{D'_k}$$

We define $w_k : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$w_k(x,t) = \sup\{\phi(x,t) : \phi \text{ is a subsolution of } (\mathrm{D}'_k),\\ \phi(x,t) \le \psi(x) + M \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}\}.$$

We have already observed that $w_k \in C(\overline{\Omega} \times \mathbb{R})$ since the family of functions $G_k(x, \cdot)$, with $x \in \overline{\Omega}$, are equi-Lipschitz continuous on $\mathbb{R}$ and that w_k is a solution of (D'_k). Since $G_k \le G$, by the definition of w, we see that $w_k \le w$ in $\overline{\Omega} \times \mathbb{R}$. Similarly, we see that $w - 1/k \le w_k$ in $\overline{\Omega} \times \mathbb{R}$. Thus we see that w is a uniform limit of a sequence of functions in $\mathrm{BUC}(\overline{\Omega} \times \mathbb{R})$. Hence, we have $w \in \mathrm{BUC}(\overline{\Omega} \times \mathbb{R})$. □

Remark 2.1. (i) The above proof shows that if there is a subsolution of (S), then there is a solution $u \in \mathrm{BUC}(\overline{\Omega} \times \mathbb{R})$ of (D). Moreover the solution constructed in the above proof is the maximal solution of (D) in the sense that it is the pointwise maximum of all subsolutions of (D). (ii) The periodicity of f can be replaced by its almost periodicity in Theorem 2.1. In the case of almost periodic f, we have to modify the definition of $\langle f \rangle$ and to replace it by

$$\langle f \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t)\mathrm{d}t.$$

3. Constancy on Aubry sets

In this section we always assume that $H(x,r,p)$ does not depend on r and $f = 0$. We write $H(x,p)$ for $H(x,r,p)$. Our problems (D) and (S) thus read

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \Omega \times \mathbb{R}, \\ u = g & \text{on } \partial\Omega \times \mathbb{R} \end{cases} \tag{D}$$

and

$$H(x, Dv) = 0 \quad \text{in } \Omega. \tag{S}$$

We investigate here properties of bounded solutions of (D). Thus, in view of Theorem 2.1 (i), we make the following assumption:

(A6) There is a subsolution of (S).

In addition to (A1)–(A6), we assume the following throughout this section:

(A7) H is convex. That is, the function $p \mapsto H(x,p)$ is convex for any $x \in \overline{\Omega}$.

(A8) Either of the following $(A8)_+$ or $(A8)_-$ holds:

$(A8)_+$ There exists a modulus ω satisfying $\omega(r) > 0$ for all $r > 0$ such that for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$ such that $H(x,p) = 0$ and for all $\xi \in D_2^- H(x,p)$, $q \in \mathbb{R}^n$,

$$H(x,p+q) \geq H(x,p) + \xi \cdot q + \omega((\xi \cdot q)_+)$$

where $D_2^- H(x,p)$ stands for the subdifferential of H with respect to the second variable p and $r_+ := \max\{0, r\}$ for $r \in \mathbb{R}$.

$(A8)_-$ There exists a modulus ω satisfying $\omega(r) > 0$ for all $r > 0$ such that for all $(x,p) \in \overline{\Omega} \times \mathbb{R}^n$ such that $H(x,p) = 0$ and for all $\xi \in D_2^- H(x,p)$, $q \in \mathbb{R}^n$,

$$H(x,p+q) \geq H(x,p) + \xi \cdot q + \omega((\xi \cdot q)_-)$$

where $r_- := \max\{0, -r\}$ for $r \in \mathbb{R}$.

We remark that condition (A8) appears in the study of the asymptotic behavior of solutions of the Cauchy problem for Hamilton-Jacobi equations. For this, see [14–16] and also [3]. Condition (A8) is a sort of strict convexity requirement on H at the level of $H = 0$.

Let L denote the Lagrangian of H, that is,

$$L(x,\xi) = \sup_{p \in \mathbb{R}^n} (\xi \cdot p - H(x,p)) \quad \text{for } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Define the functions $e : \overline{\Omega} \times \overline{\Omega} \times (0,\infty) \to \mathbb{R} \cup \{\infty\}$, $d : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ and $b : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} e(x,y,t) &= \inf \Big\{ \int_0^t L[\gamma] \ : \ \gamma \in \mathrm{AC}([0,t], \overline{\Omega}),\ \gamma(t) = x,\ \gamma(0) = y \Big\}, \\ d(x,y) &= \inf_{t>0} e(x,y,t), \\ b(x,t) &= \inf \left\{ e(x,y,\tau) + g(y, t-\tau) : \tau > 0,\ y \in \partial\Omega \right\}. \end{aligned}$$

Here and henceforth $\mathrm{AC}([a,b], \overline{\Omega})$ denotes the space of absolutely continuous functions on $[a,\, b]$ with values in $\overline{\Omega}$. Also, we use the abbreviated notation $\int_a^b L[\gamma]$ to denote the integral $\int_a^b L(\gamma(s),\, \dot{\gamma}(s))\mathrm{d}s$. We extend the domain of definition of e to $\overline{\Omega}^2 \times [0,\infty)$ by setting $e(x,x,0) = 0$ and $e(x,y,0) = \infty$ if $x \neq y$.

Recall that

$$d(x,y) = \sup\{v(x) - v(y) \ : \ v \in C(\overline{\Omega}) \text{ is a subsolution of (S)}\}.$$

See [7,12,13] for similar results on the n-dimensional torus. We note by (A6), the coerciveness (A3) and this formula that $d \in C(\overline{\Omega}^2)$. The validity of this formula for d can be seen as follows. Let $w(x,y)$ denote the right hand side of the above formula. It follows from [20, Proposition 5.1] that $v(x) - v(y) \leq d(x,y)$ for any $x, y \in \overline{\Omega}$ if v is a subsolution of (S). Hence, we have $w(x,y) \leq d(x,y)$. On the other hand, since $\sup_{\overline{\Omega}\times B(0,\delta)} L < \infty$ for some $\delta > 0$ ([18, Proposition 2.1]) and Ω has C^0 boundary, it is not hard to check (see the proof of Lemma 3.2 below for a related argument) that d is bounded above on $\overline{\Omega}^2$. Then, by using [18, Theorem A.1], one sees that $d(\cdot, y)$ is a subsolution of (S), which implies that $d(x,y) \leq w(x,y)$. A standard remark here is that the function w (and hence d) is uniformly continuous on $\overline{\Omega}^2$ because the family of subsolutions of (S) is equi-continuous due to the coerciveness of H and the C^0 boundary regularity of Ω.

We now consider the state-constraint problem for $H(x, Du) = 0$ in $\overline{\Omega}$. That is, we consider the problem of finding solutions u of two inequalities:

$$\begin{cases} H(x, Du(x)) \leq 0 & \text{in } \Omega, \\ H(x, Du(x)) \geq 0 & \text{on } \overline{\Omega}. \end{cases} \tag{SC}$$

We now introduce the (projected) *Aubry set* $\mathcal{A}$ for H, associated with (SC), by setting

$$\mathcal{A} = \{y \in \overline{\Omega} \,:\, d(\cdot, y) \text{ is a solution of (SC)}\}.$$

We refer to [19,20] for related observations and to [11–13] for general properties of Aubry sets. In particular, it is known (see [20, Proposition 6.4]) that $\mathcal{A} \neq \emptyset$ if and only if (SC) has a solution and that $\mathcal{A}$ is a compact set. Also it is known that, under hypothesis (A6), $\mathcal{A} \neq \emptyset$ if and only if (SC) has a supersolution. Also, the following characterization is a classical and crucial observation regarding Aubry sets. Let $\tau > 0$. A point $y \in \overline{\Omega}$ is in $\mathcal{A}$ if and only if

$$\inf\Big\{ \int_0^t L[\gamma] : t \geq \tau,\, \gamma \in \mathrm{AC}([0,\, t], \overline{\Omega}),\, \gamma(0) = \gamma(t) = y \Big\} = 0. \tag{2}$$

See [18, Proposition A.3] or [12, Theorem 4.3] for a proof of this characterization.

The function e is a "fundamental solution" of the state-constraint problem for $u_t + H(x, Du) = 0$ in $\overline{\Omega} \times (0, \infty)$. Indeed, for any $f \in C(\overline{\Omega})$, the solution u of the Cauchy problem of

$$\begin{cases} u_t + H(x, Du) \le 0 & \text{in } \ \Omega \times (0, \infty), \\ u_t + H(x, Du) \ge 0 & \text{in } \ \overline{\Omega} \times (0, \infty), \\ u(\cdot, 0) = f, \end{cases}$$

can be written (see [20, Eq. (5.1)]) as

$$u(x, t) = \inf\{f(y) + e(x, y, t) \ : \ y \in \Omega\}.$$

Lemma 3.1. (i) *The function e is bounded below on $\overline{\Omega}^2 \times [0, \infty)$.* (ii) *$e$ is a lower semi-continuous function on $\overline{\Omega}^2 \times [0, \infty)$.* (iii) *For each $y \in \overline{\Omega}$ the function $u := e(\cdot, y, \cdot)$ is a solution of the state-constraint problem for $u_t + H(x, Du) = 0$ in $\overline{\Omega} \times (0, \infty)$ in the sense of Barron-Jensen* [4]. *That is, for any $(x, t) \in \overline{\Omega} \times (0, \infty)$ and $\phi \in C^1(\overline{\Omega} \times (0, \infty))$, if $u - \phi$ attains a minimum at (x, t), then*

$$\begin{cases} \phi_t(x, t) + H(x, D\phi(x, t)) = 0 & \text{if } x \in \Omega, \\ \phi_t(x, t) + H(x, D\phi(x, t)) \ge 0 & \text{if } x \in \partial\Omega. \end{cases}$$

Remark 3.1. In the following presentation, we will not use the above assertion (iii).

Proof. In this proof we set $Q = \overline{\Omega}^2 \times (0, \infty)$. By the definition of d, we have $d(x, y) \le e(x, y, t)$ for all $(x, y, t) \in Q$, with $t > 0$. Clearly, we have $d(x, y) \le e(x, y, 0)$ for all $x, y \in \overline{\Omega}$. Thus, e is bounded below on $\overline{Q}$.

To see that e is lower semi-continuous on Q, we fix any $(x, y, t) \in Q$ and assume that there is a sequence $\{(x_k, y_k, t_k)\}_{k\in\mathbb{N}} \subset Q$ such that $\lim_{k\to\infty}(x_k, y_k, t_k) = (x, y, t)$ and $\lim_{k\to\infty} e(x_k, y_k, t_k) = e_0$ for some $e_0 \in \mathbb{R}$. We choose a sequence of curves $\gamma_k \in \mathrm{AC}([0,\, t_k], \overline{\Omega})$ such that for all $k \in \mathbb{N}$, $\gamma_k(t_k) = x_k$, $\gamma_k(0) = y_k$ and

$$e(x_k, y_k, t_k) + \frac{1}{k} > \int_0^{t_k} L[\gamma_k].$$

By using [18, Lemmas 6.3, 6.4], we deduce that there is a curve $\gamma \in \mathrm{AC}([0, t], \overline{\Omega})$ such that $\gamma(t) = x$, $\gamma(0) = y$ and $\int_0^t L[\gamma] \le e_0$. Hence, we get $e(x, y, t) \le e_0$, which shows the lower semi-continuity of e at (x, y, t).

We remark here that the variational problem

$$e(x, y, t) = \inf\{\int_0^t L[\gamma] : \gamma \in \mathrm{AC}([0, t], \overline{\Omega}), \gamma(t) = x, \gamma(0) = y\}$$

has a minimizer for every $(x, y, t) \in Q$.

Next, we show that e is lower semi-continuous at points in $\overline{\Omega}^2 \times \{0\}$. Fix any $R > 0$ and a constant $C_R > 0$ so that $H(x,p) \le C_R$ for all $(x,p) \in \overline{\Omega} \times B(0,R)$. Also, fix any $y \in \overline{\Omega}$. The function $w(x,t) = R|x-y| - C_R t$ of (x,t) on $\overline{\Omega} \times [0,\infty)$ is a subsolution of $w_t + H(x,Dw) = 0$ in $\Omega \times (0,\infty)$. Due to [20, Proposition 5.1], we obtain

$$w(\gamma(t),t) \le w(\gamma(0),0) + \int_0^t L[\gamma]$$

for any $t > 0$ and $\gamma \in \mathrm{AC}([0,t],\overline{\Omega})$. From this we get

$$e(x,y,t) \ge R|x-y| - C_R t \quad \text{for all } (x,y,t) \in Q.$$

Thus, for any $(x_0,y_0) \in \overline{\Omega}^2$, we obtain

$$\liminf_{(x,y)\to(x_0,y_0),\, t\to 0+} e(x,y,t) \ge R|x_0 - y_0|.$$

As $R > 0$ is arbitrary, we see that

$$\liminf_{(x,y)\to(x_0,y_0),\, t\to 0+} e(x,y,t) \ge e(x_0,y_0,0).$$

This completes the proof of the lower semi-continuity of e on $\overline{Q}$.

Now, we fix $y \in \overline{\Omega}$ and set $u(x,t) := e(x,y,t)$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. Let $\phi \in C^1(\overline{\Omega} \times (0,\infty))$ and assume that $u - \phi$ attains a strict minimum at $(\bar{x},\bar{t})$. We choose a minimizer $\gamma \in \mathrm{AC}([0,\bar{t}],\overline{\Omega})$ for $e(\bar{x},y,\bar{t})$, i.e., the curve γ has the properties: $\gamma(\bar{t}) = \bar{x}$, $\gamma(0) = y$ and

$$e(\bar{x},y,\bar{t}) = \int_0^{\bar{t}} L[\gamma].$$

We need to show that

$$\phi_t(\bar{x},\bar{t}) + H(\bar{x},D\phi(\bar{x},\bar{t})) \ge 0 \tag{3}$$

and also

$$\phi_t(\bar{x},\bar{t}) + H(\bar{x},D\phi(\bar{x},\bar{t})) \le 0 \ \text{ if } \bar{x} \in \Omega. \tag{4}$$

We suppose the contrary of (3), i.e, $\phi_t(\bar{x},\bar{t}) + H(\bar{x},D\phi(\bar{x},\bar{t})) < 0$. We choose an $\varepsilon \in (0,\bar{t})$ so that $\phi_t(x,t) + H(x,D\phi(x,t)) \le 0$ in $B(\bar{x},\varepsilon) \times [\bar{t}-\varepsilon,\bar{t}]$. We select $\tau \in [\bar{t}-\varepsilon,\,\bar{t})$ so that $\gamma(s) \in B(\bar{x},\varepsilon) \cap \overline{\Omega}$ for all $s \in [\tau,\,\bar{t}]$ and either $\tau = \bar{t} - \varepsilon$ or $\gamma(\tau) \in \partial B(\bar{x},\varepsilon) \setminus \overline{\Omega}$. We may assume by adding a constant to

ϕ that $u(\bar{x},\bar{t}) = \phi(\bar{x},\bar{t})$. Then we have $u > \phi$ on $(\partial B(\bar{x},\varepsilon)\cap\overline{\Omega})\times[\bar{t}-\varepsilon,\,\bar{t}]\cup B(\bar{x},\varepsilon)\times\{\bar{t}-\varepsilon\}$. We observe that

$$\phi(\gamma(\bar{t}),\bar{t}) - \phi(\gamma(\tau),\tau) = \int_\tau^{\bar{t}} (\phi_t(\gamma(s),s) + D\phi(\gamma(s),s)\cdot\dot{\gamma}(s))\,\mathrm{d}\,s$$
$$\le \int_\tau^{\bar{t}} (\phi_t(\gamma(s),s) + H(\gamma(s),D\phi(\gamma(s),s)) + L(\gamma(s),\dot{\gamma}(s)))\,\mathrm{d}\,s \le \int_\tau^{\bar{t}} L[\gamma]$$

and therefore

$$u(\bar{x},\bar{t}) = \phi(\bar{x},\bar{t}) < u(\gamma(\tau),\tau) + \int_\tau^{\bar{t}} L[\gamma] = \int_0^{\bar{t}} L[\gamma] = e(\bar{x},y,\bar{t}) = u(\bar{x},\bar{t}).$$

This is a contradiction, which shows that (3) is valid.

We next prove inequality (4). We assume that $\bar{x}\in\Omega$. We suppose, contrary to (4), that $\phi_t(\bar{x},\bar{t}) + H(\bar{x},D\phi(\bar{x},\bar{t})) > 0$. We choose an $r > 0$ so that $B(\bar{x},\,r)\subset\Omega$ and $\phi_t(x,t) + H(x,D\phi(x,t)) \ge 0$ in $B(\bar{x},r)\times[\bar{t},\,\bar{t}+r]$. As before, we assume that $u = \phi$ at $(\bar{x},\bar{t})$. Note that $u(x,t) > \phi(x,t)$ if $(x,t)\ne(\bar{x},\bar{t})$. As in the proof of [18, Theorem A.1], we find a curve $\eta\in\mathrm{AC}([\bar{t},\,\tau],\Omega)$, with $\bar{t} < \tau \le \bar{t}+r$, such that for a.e. $s\in(\bar{t},\,\tau)$,

$$D\phi(\eta(s),s)\cdot\dot{\eta}(s) = L(\eta(s),\dot{\eta}(s)) + H(\eta(s),D\phi(\eta(s),s)).$$

By replacing τ by a smaller number ($> \bar{t}$) if necessary, we may assume that $\eta(s)\in B(\bar{x},r)$ for all $s\in[\bar{t},\tau]$. We now compute that

$$u(\eta(\tau),\tau) > \phi(\eta(\tau),\tau)$$
$$= \phi(\bar{x},\bar{t}) + \int_{\bar{t}}^{\tau} (\phi_t(\eta(s),s) + H(\eta(s),D\phi(\eta(s),s)) + L(\eta(s),\dot{\eta}(s)))\,\mathrm{d}\,s$$
$$= u(\bar{x},\bar{t}) + \int_{\bar{t}}^{\tau} L[\eta] = \int_0^{\bar{t}} L[\gamma] + \int_{\bar{t}}^{\tau} L[\eta] \ge u(\eta(\tau),\tau).$$

This is a contradiction, from which we conclude that (4) is valid. □

Lemma 3.2. (i) *There is a constant $C_0 > 0$ and for each $(z,\tau)\in\overline{\Omega}\times(0,\infty)$ a neighborhood V of z, ralative to $\overline{\Omega}$, such that*

$$e(x,y,\tau+t) \le C_0(\tau+t) \quad \textit{for all } x,y\in V,\ t\ge 0.$$

(ii) *There are constants $\tau_1 > 0$ and $C_1 > 0$ such that $e(x,y,\tau_1)\le C_1$ for all $(x,y)\in\overline{\Omega}^2$.*

Proof. As noted before, there are constants $\delta > 0$ and $C > 0$ such that

$$L(x,\xi) \le C \quad \text{for all } (x,\xi)\in\overline{\Omega}\times B(0,\delta). \tag{5}$$

For any $(x,t) \in \overline{\Omega} \times (0,\infty)$, if we set $\gamma(s) := x$, then

$$e(x,x,t) \leq \int_0^t L[\gamma] \leq Ct. \tag{6}$$

We note that for any $x,y,z \in \overline{\Omega}$ and $t,s \geq 0$,

$$e(x,y,t+s) \leq e(x,z,t) + e(z,y,s). \tag{7}$$

We show that assertion (i), with $C_0 = C$, holds. To see this, we fix $\tau > 0$ and $z \in \overline{\Omega}$. In view of (6) and (7), we need only to prove that there is a constant $\rho > 0$ such that

$$e(x,y,\tau) \leq C\tau \quad \text{for all } x,y \in B(z,\rho) \cap \overline{\Omega}. \tag{8}$$

According to [20, Lemma 4.2], there exists $\zeta \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ such that $x + \varepsilon\zeta(x) \in \Omega$ for all $(x,\varepsilon) \in \overline{\Omega} \times (0,\,1]$. Choose a constant $M > 0$ so that $\max_{\overline{\Omega}} |\zeta| \leq M$ and $\delta\tau \leq 3M$. Set $t = \tau/3$ and $\eta(x) = (\delta/M)\zeta(x)$ for $x \in \overline{\Omega}$. Note that $(\delta t/M) \leq 1$ and $\max_{\overline{\Omega}} |\eta| \leq \delta$. In particular, we have $x + s\eta(x) \in \Omega$ for all $(x,s) \in \overline{\Omega} \times (0,t]$. Select $r > 0$ so that $B(z + t\eta(z), r) \subset \Omega$ and also $2r \leq \delta t$. Next, in view of the continuity of η, we choose $\rho > 0$ so that $x + t\eta(x) \in B(z + t\eta(z),\, r)$ for all $x \in B(z,\rho) \cap \overline{\Omega}$.

Fix any $x,y \in B(z,\rho) \cap \overline{\Omega}$. We define the curve $\gamma \in \mathrm{AC}([0,\,\tau], \overline{\Omega})$ by concatenating three line segments $[y,\,q]$, $[q,\,p]$ and $[p,\,x]$, where $p := x + t\eta(x)$ and $q := y + t\eta(y)$, as follows:

$$\gamma(s) = \begin{cases} y + s\eta(y) & \text{for } 0 \leq s < t, \\ q + \frac{(s-t)}{t}(p-q) & \text{for } t \leq s < 2t, \\ x + (3t-s)\eta(x) & \text{for } 2t \leq s \leq 3t. \end{cases}$$

It is clear that $\gamma \in \mathrm{AC}([0,\,\tau], \overline{\Omega})$, $\gamma(0) = y$ and $\gamma(\tau) = x$. Noting that $|p - q|/t \leq 2r/t \leq \delta$, we see that $|\dot{\gamma}(s)| \leq \delta$ for a.e. $s \in (0,\,\tau)$ and consequently

$$e(x,y,\tau) \leq \int_0^\tau L[\gamma] \leq C\tau,$$

which completes the proof of (i).

Next we show that assertion (ii) is valid. Let $C_0 > 0$ be the constant from assertion (i). According to assertion(i), for each $z \in \overline{\Omega}$, we may choose an open neighborhood V_z of z such that

$$e(x,y,1) \leq C_0(1+t) \quad \text{for all } x,y \in V_z \cap \overline{\Omega},\ t \geq 0. \tag{9}$$

By the compactness of $\overline{\Omega}$, we may choose a finite points $z_1,...,z_k \in \overline{\Omega}$ such that $\{V_{z_j}\}_{j=1}^k$ covers $\overline{\Omega}$. We fix any $x,y \in \overline{\Omega}$. We use the connectedness of $\overline{\Omega}$, to see that there is a sub-family $\{W_1,...,W_m\}$ of $\{V_{z_j}\}$, with $m \leq k$, such that $x \in W_1$, $y \in W_m$ and $W_j \cap W_{j+1} \cap \overline{\Omega} \neq \emptyset$ for all $j = 1, ..., m-1$. We select a sequence $\{x_j\}_{j=1}^{m-1}$ so that $x_j \in W_j \cap W_{j+1} \cap \overline{\Omega}$ for all $j = 1, ..., m-1$. We observe by (9) and (7) that for any $t \geq 0$,

$$\begin{aligned} e(x, y, m+t) &\leq e(x, x_1, 1) + e(x_1, x_2, 1) + \cdots + e(x_{m-1}, y, 1+t) \\ &\leq C_0(m+t). \end{aligned}$$

Therefore, in general, we have

$$e(x, y, k) \leq C_0 k \quad \text{for all } (x, y) \in \overline{\Omega}^2.$$

Thus, assertion (ii) is valid with $\tau_1 = k$ and $C_1 = C_0 k$. □

Lemma 3.3. *The function b is bounded, uniformly continuous on $\overline{\Omega} \times \mathbb{R}$ and it is a solution of* (D). *Moreover we have for all $(x, t) \in \overline{\Omega} \times \mathbb{R}$,*

$$b(x, t) = \max\{v(x, t) : v \textit{ is a subsolution of } (\mathrm{D})\}. \tag{10}$$

Proof. By Lemma 3.1 (i), the function e is bounded below. Therefore, we see that b is bounded below on $\overline{\Omega} \times \mathbb{R}$. Indeed, we have $b(x, t) \geq \inf_{\overline{\Omega}^2} d + \inf_{\partial\Omega \times \mathbb{R}} g$ for all $(x, t) \in \overline{\Omega} \times \mathbb{R}$.

Next, by Lemma 3.2 (ii), there are $\tau_1 > 0$ and $C_1 > 0$ such that $e(x, y, \tau_1) \leq C_1$ for all $x, y \in \overline{\Omega}$. Note that

$$\begin{aligned} b(x, t) &\leq \inf\{e(x, y, \tau_1) + g(y, t - \tau_1) : y \in \partial\Omega\} \\ &\leq C_1 + \sup_{\partial\Omega \times \mathbb{R}} g \quad \text{for all } (x, t) \in \overline{\Omega} \times \mathbb{R}. \end{aligned}$$

Thus, the function b is bounded on $\overline{\Omega} \times \mathbb{R}$.

Next, we show that b is the maximal solution of (D), i.e, we show that (10) holds. We write $u(x, t)$ for the right hand side of (10). According to the proof of Theorem 2.1 (i) and Remark 2.1 (i), u is a solution of (D) and $u \in \mathrm{BUC}(\overline{\Omega})$. We regularize u by sup-convolutions in t as follows. Let $\varepsilon > 0$ and set

$$u^\varepsilon(x, t) = \sup_{s \in \mathbb{R}} \left(u(x, s) - \frac{|t - s|^2}{2\varepsilon} \right) \quad \text{for } (x, t) \in \overline{\Omega} \times \mathbb{R}.$$

As is well-known, the function u^ε is a subsolution of $u_t + H(x, Du) = 0$ in $\Omega \times \mathbb{R}$ and has the distributional first derivatives in $L^\infty(\Omega \times \mathbb{R})$. Moreover $|u^\varepsilon(x, t) - u(x, t)| \leq \omega(\varepsilon)$ for all $(x, t) \in \overline{\Omega} \times \mathbb{R}$, where ω is a modulus. In particular, we have $u^\varepsilon(x, t) \leq g(x, t) + \omega(\varepsilon)$ for all $(x, t) \in \partial\Omega \times \mathbb{R}$. Let $(x, t) \in \overline{\Omega} \times \mathbb{R}$ and $\tau > 0$, and fix any $\gamma \in \mathrm{AC}([0, \tau], \overline{\Omega})$ such that $\gamma(\tau) = x$ and $\gamma(0) \in \partial\Omega$. We apply [20, Proposition 5.1], to get

$$u^\varepsilon(x, t) \leq u^\varepsilon(\gamma(0), t - \tau) + \int_0^\tau L[\gamma] \leq g(\gamma(0), t - \tau) + \omega(\varepsilon) + \int_0^\tau L[\gamma],$$

from which we deduce that $u^\varepsilon(x,t) \le b(x,t) + \omega(\varepsilon)$. Moreover, since $\varepsilon > 0$ is arbitrary, we find that $u \le b$ on $\overline{\Omega} \times \mathbb{R}$.

Now, let b^* denote the upper semi-continuous envelope of b. If b^* is a subsolution of (D), then we have $b^* \le u$ on $\overline{\Omega} \times \mathbb{R}$ by the definition of u, which implies that $u = b$. Thus we only need to show that b^* is a subsolution of (D). But, it is a classical observation (see [18, Theorem A.1]) that b^* is a subsolution of $v_t + H(x, Dv) = 0$ in $\Omega \times \mathbb{R}$. Hence, it is enough to show that $b^* \le g$ on $\partial\Omega \times \mathbb{R}$. We fix any $(y, s) \in \partial\Omega \times \mathbb{R}$ and $\varepsilon > 0$. Let $C_0 > 0$ be the constant from Lemma 3.2 (i). We choose $\delta > 0$ so that $\max_{r \in [s-2\delta,\, s]} g(y, r) < g(y, s) + \varepsilon$ and $C_0\delta < \varepsilon$. By Lemma 3.2 (i), there is a neighborhood V of y, relative to $\overline{\Omega}$, such that

$$e(x, y, t) \le C_0 t \quad \text{for all } (x, t) \in V \times [\delta, \infty).$$

For any $(x, t) \in V \times [s - \delta,\, s + \delta]$, we obtain

$$b(x, t) \le e(x, y, \delta) + g(y, t - \delta) < C_0\delta + g(y, s) + \varepsilon \le g(y, s) + 2\varepsilon,$$

which ensures that $b^*(y, s) \le g(y, s)$. Hence, $b^* \le g$ on $\partial\Omega \times \mathbb{R}$, and b^* is a subsolution of (D). □

The following two theorems are the main results in this section.

Theorem 3.1. *Let u be a bounded solution of* (D) *on $\overline{\Omega} \times \mathbb{R}$. Set*

$$u_-(x) = \liminf_{t \to -\infty} u(x, t) \quad \textit{for } x \in \overline{\Omega},$$
$$u_0(x) = \inf\{u_-(y) + d(x, y) : \ y \in \mathcal{A}\} \quad \textit{for } x \in \overline{\Omega}.$$

Then

$$u(x, t) = u_0(x) \wedge b(x, t) \quad \textit{for all } (x, t) \in \overline{\Omega} \times \mathbb{R}. \tag{11}$$

In the above theorem, if $\mathcal{A} = \emptyset$, then $u_0(x) = \infty$ for all $x \in \overline{\Omega}$ and hence (11) asserts that $u = b$ on $\overline{\Omega} \times \mathbb{R}$.

It is standard observations that $u_0(x) = u_-(x)$ for all $x \in \mathcal{A}$ and $u_-(x) \le u_0(x)$ for all $x \in \overline{\Omega}$ and that u_- is a solution of $H(x, Du(x)) = 0$ in Ω and u_0 is a solution of (SC). Here the convexity of H is essential to conclude that u_- is a subsolution of $H(x,\, Du) = 0$ in Ω.

Theorem 3.2. *Let u and u_0 be as in Theorem* 3.1*. Assume that*

$$\liminf_{t \to -\infty} g(x, t) = \inf_{t \in \mathbb{R}} g(x, t) \quad \textit{for all } x \in \partial\Omega. \tag{12}$$

Then

$$u(x, t) = u_0(x) \quad \textit{for all } (x, t) \in \mathcal{A} \times \mathbb{R}.$$

A consequence of the above theorem is that if $g(x,t)$ is almost periodic in t for every $x \in \overline{\Omega}$, then condition (12) is satisfied and hence $u(x,t)$ is constant in t for any $x \in \mathcal{A}$. In particular, if g is periodic, then any bounded solution $u(x,t)$ of (D) is constant in t on the Aubry set $\mathcal{A}$.

A general observation on (D) is that the value of any solution $u \in C(\overline{\Omega} \times \mathbb{R})$ of (D) at $(x,t) \in \overline{\Omega} \times \mathbb{R}$ is represented as

$$\begin{aligned} u(x,t) = \inf\{e(x,y,t-s) + u(y,s) \,:\, y \in \overline{\Omega}\} \\ \wedge \inf\{e(x,y,t-\tau) + g(y,\tau) \,:\, y \in \partial\Omega,\ s < \tau < t\}, \end{aligned} \tag{13}$$

where $s \in (-\infty,\, t)$ is an arbitrarily fixed number. For a proof of this formula we refer to [21, Theorems 4.1, 4.3].

Let $v \in C(\overline{\Omega})$ be a solution of (SC). A curve $\gamma \in C((-\infty,\, 0], \overline{\Omega})$ is said to be *extremal* for v if, for any $-\infty < s < t \le 0$, γ is absolutely continuous on $[s,\, t]$ and satisfies

$$\int_s^t L[\gamma] = v(\gamma(t)) - v(\gamma(s)).$$

Let $\alpha(\gamma)$ denote the *alpha-limit set* of a curve $\gamma \in C((-\infty,\, 0], \overline{\Omega})$. That is,

$$\begin{aligned} \alpha(\gamma) &:= \{y \in \overline{\Omega} : \text{there exists a sequence } t_j \to \infty \text{ such that } \gamma(-t_j) \to y\} \\ &= \bigcap_{t \in \mathbb{R}} \overline{\gamma((-\infty, t])}. \end{aligned}$$

It is easily checked by recalling (2) that if γ is an extremal curve for some solution of (SC), then $\alpha(\gamma) \subset \mathcal{A}$.

Lemma 3.4. *Assume that* $\mathcal{A} \neq \emptyset$*. Let* u_0 *be the function from Theorem* 3.1*. Let* γ *be an extremal curve for* u_0*. Let* $\varepsilon > 0$*. Then there are a constant* $\tau_0 > 0$ *and a neighborhood* W *of* $\alpha(\gamma)$*, relative to* $\overline{\Omega}$*, and for each* $x, y \in W$ *a curve* $\eta \in \mathrm{AC}([-\tau,\, 0], \overline{\Omega})$*, with* $0 < \tau \le \tau_0$*, such that* $\eta(0) = x$*,* $\eta(-\tau) = y$ *and*

$$\int_{-\tau}^0 L[\eta] < \varepsilon + u_0(x) - u_0(y).$$

Proof. By Lemma 3.2 (i) and the compactness of $\overline{\Omega}$, we may choose constants $r > 0$, $\tau > 0$ and, for each $z \in \overline{\Omega}$ and $x, y \in B(z,\, r) \cap \overline{\Omega}$, a curve $\xi \in \mathrm{AC}([-\tau,\, 0], \overline{\Omega})$ such that $\xi(0) = x$, $\xi(-\tau) = y$ and

$$\int_{-\tau}^0 L[\xi] < \varepsilon.$$

Here, since $u_0 \in C(\overline{\Omega})$, we may assume by replacing $r > 0$ by a smaller positive number if necessary that $|u_0(z_1) - u_0(z_2)| \le \varepsilon$ if $z_1, z_2 \in \overline{\Omega}$ and $|z_1 - z_2| \le 2r$. Accordingly, we have

$$\int_{-\tau}^{0} L[\xi] < 2\varepsilon + u_0(x) - u_0(y).$$

Now, we set $K = \alpha(\gamma) \times \alpha(\gamma)$. Note that K is a compact subset of $\overline{\Omega}^2$. Let $(p, q) \in K$ and consider the neighborhood $V := (B(p, r) \cap \overline{\Omega}) \times (B(q, r) \cap \overline{\Omega}) \subset \overline{\Omega}^2$ of $(p,\, q)$. Fix any $x, y \in V$. Since $p, q \in \alpha(\gamma)$, we may choose numbers $0 < t_p < t_q < \infty$ so that $\gamma(-t_p) \in B(p, r)$ and $\gamma(-t_q) \in B(q, r)$. By the previous observation, there are curves $\xi_1 \in \mathrm{AC}([-\tau,\, 0], \overline{\Omega})$ and $\xi_2 \in \mathrm{AC}([-\tau,\, 0], \overline{\Omega})$ such that $\xi_1(0) = x$, $\xi_1(-\tau) = \gamma(-t_p)$, $\xi_2(0) = \gamma(-t_q)$ and $\xi_2(-\tau) = y$ and such that

$$\int_{-\tau}^{0} L[\xi_1] < 2\varepsilon + u_0(x) - u_0(\gamma(-t_p)),$$

and

$$\int_{-\tau}^{0} L[\xi_2] < 2\varepsilon + u_0(\gamma(-t_q)) - u_0(y).$$

Next, we concatenate three curves ξ_1, γ and ξ_2, to define the curve η. That is, we define the curve $\eta \in \mathrm{AC}([-t_{pq},\, 0], \overline{\Omega})$, with $t_{pq} = 2\tau + t_q - t_p$, by setting

$$\eta(s) = \begin{cases} \xi_1(s) & \text{for } -\tau < s \le 0, \\ \gamma(s + \tau - t_p) & \text{for } -\tau - t_q + t_p < s \le -\tau, \\ \xi_2(s + \tau + t_q - t_p) & \text{for } -t_{pq} \le s \le -\tau - t_q + t_p. \end{cases}$$

The curve η has the properties: $\eta(0) = x$, $\eta(-t_{pq}) = y$ and

$$\begin{aligned} \int_{-t_{pq}}^{0} L[\eta] &= \int_{-\tau}^{0} L[\xi_2] + \int_{-t_q}^{-t_p} L[\gamma] + \int_{-\tau}^{0} L[\xi_1] \\ &< 4\varepsilon + u_0(x) - u_0(y). \end{aligned}$$

For each $(p, q) \in K$ we fix t_p and t_q as above. Due to the compactness of K, we may find a finite sequence $\{(p_i, q_i)\}_{i=1}^{m} \subset K$ such that the family $\{B(p_i,\, r/2) \times B(q_i,\, r/2)\}_{i=1}^{m}$ covers K. We choose a constant $\delta > 0$ so that

$$\big(\alpha(\gamma) + B(0, \delta)\big) \times \big(\alpha(\gamma) + B(0, \delta)\big) \subset \bigcup_{i=1}^{m} B(p_i,\, r) \times B(q_i,\, r),$$

and set $W = \big(\alpha(\gamma) + B(0,\, \delta)\big) \cap \overline{\Omega}$. Clearly, W is a neighborhood of $\alpha(\gamma)$ relative to $\overline{\Omega}$. Also we set $\tau_0 = 2\tau + \max_{1 \le i \le m}(t_{q_i} - t_{p_i})$. It now follows that for each $(x, y) \in W \times W$ there is a curve $\eta \in \mathrm{AC}([-t,\, 0], \overline{\Omega})$, with $0 < t \le \tau_0$, such that $\eta(0) = x$, $\eta(-t) = y$ and

$$\int_{-t}^{0} L[\eta] < 4\varepsilon + u_0(x) - u_0(y),$$

which was to be proven. □

Lemma 3.5. *Let $\tau \in \mathbb{R}$ and set $I = (-\infty,\, \tau)$. Let $u, v \in \mathrm{BUC}(\overline{\Omega} \times I)$ be a subsolution and a supersolution of* (D) *in $\overline{\Omega} \times I$, respectively. Assume that $u \le v$ on $\mathcal{A} \times I$. Then $u \le v$ on $\overline{\Omega} \times I$.*

Remark 3.2. It follows from the above lemma that if $\mathcal{A} = \emptyset$ and if $u, v \in \mathrm{BUC}(\overline{\Omega} \times I)$ are a subsolution and a supersolution of (D) in $\overline{\Omega} \times I$, respectively, then $u \le v$ on $\overline{\Omega} \times \mathbb{R}$. In particular, if $\mathcal{A} = \emptyset$, then problem (D) has a unqiue solution in $\mathrm{BUC}(\overline{\Omega} \times \mathbb{R})$. For existence of such a solution of (D), see Remark 2.1.

Proof. Fix any $\varepsilon \in (0,\, 1)$, and set $u^{\varepsilon}(x, t) = u(x, t) - \varepsilon$ for $(x, t) \in \overline{\Omega} \times I$. There is a compact neighborhood K_{ε} of $\mathcal{A}$, relative to $\overline{\Omega}$, such that $u^{\varepsilon} \le v$ on $K_{\varepsilon} \times I$. (Needless to say, we take $K_{\varepsilon} = \emptyset$ if $\mathcal{A} = \emptyset$.) As a basic property of the Aubry set, there is a function $\psi \in C(\overline{\Omega})$ and, for each compact neighborhood K of $\mathcal{A}$, a constant $\delta_K > 0$ such that $H(x, D\psi(x)) \le -\delta_K$ in $\Omega \setminus K$ in the viscosity sense. For this property, see the proofs of [20, Theorem 3.3] and [12, Proposition 6.1]. We write δ_{ε} for $\delta_{K_{\varepsilon}}$. We may assume, by adding a constant to ψ if necessary, that $\psi(x) + 1 \le \inf_{\overline{\Omega} \times I} u$ for all $x \in \overline{\Omega}$, so that $\psi(x) \le u^{\varepsilon}(x, t)$ for all $(x, t) \in \overline{\Omega} \times I$. Accordingly, the function $w(x, t) := \psi(x)$ is a solution of

$$\begin{cases} w_t + H(x, Dw(x, t)) \le -\delta_{\varepsilon} & \text{in } (\Omega \setminus K_{\varepsilon}) \times I, \\ w(x, t) \le g(x, t) & \text{on } (\partial\Omega \setminus K_{\varepsilon}) \times I. \end{cases}$$

We may assume, by translation if necessary, that $\tau \le 0$. Fix any $\lambda \in (0,\, 1)$ and choose a constant $\nu_0 > 0$ so that $\lambda\nu_0 \le (1 - \lambda)\delta_{\varepsilon}$. For any $\nu \in (0,\, \nu_0)$ we define the functions $u^{\varepsilon\nu}$ and z on $\overline{\Omega} \times I$ by setting

$$\begin{aligned} u^{\varepsilon\nu}(x, t) &= u^{\varepsilon}(x, t) + \nu t, \\ z(x, t) &= \lambda u^{\varepsilon\nu}(x, t) + (1 - \lambda)\psi(x). \end{aligned}$$

It is easily seen that $u^{\varepsilon\nu}$ and z are, respectively, a solution of

$$\begin{cases} u^{\varepsilon\nu}_t + H(x, Du^{\varepsilon\nu}(x, t)) \le \nu & \text{in } (\Omega \setminus K_{\varepsilon}) \times I, \\ u^{\varepsilon\nu}(x, t) \le g(x, t) & \text{on } (\partial\Omega \setminus K_{\varepsilon}) \times I \end{cases}$$

and a solution of

$$\begin{cases} z_t + H(x, Dz(x,t)) \le \lambda\nu - (1-\lambda)\delta_\varepsilon & \text{in } (\Omega \setminus K_\varepsilon) \times I, \\ z(x,t) \le g(x,t) \quad \text{on } (\partial\Omega \setminus K_\varepsilon) \times I. \end{cases}$$

Note here that $\lambda\nu - (1-\lambda)\delta_\varepsilon \le 0$ and also that $\lim_{t\to-\infty} z(x,t) = -\infty$ uniformly for $x \in \overline{\Omega}$. Also, observe that

$$z(x,t) \le \max\{u^{\varepsilon\nu}(x,t), \psi(x)\} \le u^\varepsilon(x,t) \quad \text{for all } (x,t) \in \overline{\Omega} \times I.$$

Now, we choose a constant $t_\nu < \tau$ so that $z(x,t) \le v(x,t)$ for all $\overline{\Omega} \times (-\infty,\, t_\nu]$. We apply a comparison theorem to z and v on the set $(\overline{\Omega} \setminus K_\varepsilon) \times [t_\nu,\, \tau)$, to find that $z \le v$ on $(\overline{\Omega} \setminus K_\varepsilon) \times [t_\nu, \tau)$. Hence we have $z \le v$ on $\overline{\Omega} \times I$. That is, we have

$$\lambda(u(x,t) - \varepsilon + \nu t) + (1-\lambda)\psi(x) \le v(x,t) \quad \text{for all } (x,t) \in \overline{\Omega} \times I.$$

Sending $\nu \to 0$ and then $\varepsilon \to 0$, $\lambda \to 1$, we find that $u \le v$ on $\overline{\Omega} \times I$. □

Lemma 3.6. *Assume that $\mathcal{A} \ne \emptyset$. Let u and u_0 be as in Theorem* 3.1. *Then*

$$u(x,t) \le u_0(x) \quad \textit{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Proof. Fix any $(x,t) \in \overline{\Omega} \times \mathbb{R}$. There is an extremal curve γ for u_0 such that $\gamma(0) = x$. See [20, Theorem 6.1] for existence of such a curve.

Fix any $\varepsilon > 0$. Let $\tau_0 > 0$ and W be those from Lemma 3.4. Fix a point $y \in \alpha(\gamma)$ and a sequence $t_j \to \infty$ such that $\lim_{j\to\infty} u(y, t - t_j) = u_-(y)$. Note here that $y \in \mathcal{A}$ and hence $u_-(y) = u_0(y)$. By passing to a subsequence if necessary, we may assume that $\gamma(-t_j) \in W$, $\gamma(-t_j + \tau_0) \in W$ and $t_j > 2\tau_0$ for all $j \in \mathbb{N}$.

We now assume that $(A8)_+$ is satisfied. According to Lemma 3.4, for each $j \in \mathbb{N}$ there is a curve $\eta_j \in \mathrm{AC}([-\tau_j,\, 0], \overline{\Omega})$, with $0 < \tau_j \le \tau_0$, such that $\eta_j(0) = \gamma(-t_j)$, $\eta_j(-\tau_j) = y$ and

$$\int_{-\tau_j}^{0} L[\eta_j] < \varepsilon + u_0(\gamma(-t_j)) - u_0(y).$$

For each $j \in \mathbb{N}$ we fix $\delta_j > 0$ so that $t_j = (1+\delta_j)(t_j - \tau_j)$. That is, we set $\delta_j = \tau_j/(t_j - \tau_j)$. Define $\gamma_j \in \mathrm{AC}([-t_j + \tau_j,\, 0], \overline{\Omega})$ and $\xi_j \in \mathrm{AC}([-t_j,\, 0], \overline{\Omega})$, respectively, by $\gamma_j(s) = \gamma((1+\delta_j)s)$ and

$$\xi_j(s) = \begin{cases} \gamma_j(s) & \text{if } -t_j + \tau_j \le s \le 0, \\ \eta_j(s + t_j - \tau_j) & \text{if } -t_j \le s < -t_j + \tau_j. \end{cases}$$

Noting that $\xi_j(0) = x$, $\xi(-t_j + \tau_j) = \gamma(-t_j)$ and $\xi_j(-t_j) = y$, we observe that if j is large enough, then

$$\begin{aligned}\int_{-t_j}^{0} L[\xi_j] &= \int_{-t_j+\tau_j}^{0} L[\gamma_j] + \int_{-\tau_j}^{0} L[\eta_j] \\ &< u_0(x) - u_0(\gamma(-t_j)) + \tau_j\omega_1\Big(\frac{\tau_j}{t_j - \tau_j}\Big) + \varepsilon + u_0(\gamma(-t_j)) - u_0(y) \\ &= u_0(x) - u_0(y) + \tau_j\omega_1\Big(\frac{\tau_j}{t_j - \tau_j}\Big) + \varepsilon.\end{aligned}$$

Here we have used the fact (see for instance [16, Lemma 4.4] and the proof of [16, Theorem 4.3]) that for some modulus ω_1, if j is large enough, then

$$\int_{-t_j+\tau_j}^{0} L[\gamma_j] \le u_0(x) - u_0(\gamma(-t_j)) + \tau_j\omega_1\Big(\frac{\tau_j}{t_j - \tau_j}\Big).$$

We combine the above with

$$u(x,t) \le \int_{-t_j}^{0} L[\xi_j] + u(\xi_j(-t_j), t - t_j),$$

to get

$$u(x,t) < u_0(x) - u_0(y) + \tau_j\omega_1\Big(\frac{\tau_j}{t_j - \tau_j}\Big) + u(y, t - t_j) + \varepsilon.$$

Sending $j \to \infty$, we see that $u(x,t) < u_0(x) + \varepsilon$. Hence, we have $u(x,t) \le u_0(x)$.

We next assume that (A8)$_-$ is satisfied. Thanks to Lemma 3.4, for each $j \in \mathbb{N}$ there is a curve $\eta_j \in \mathrm{AC}([-\tau_j, 0], \overline{\Omega})$, with $0 < \tau_j \le \tau_0$, such that $\eta_j(0) = \gamma(-t_j + \tau_0)$, $\eta_j(-\tau_j) = y$ and

$$\int_{-\tau_j}^{0} L[\eta_j] < \varepsilon + u_0(\gamma(-t_j + \tau_0)) - u_0(y).$$

For each $j \in \mathbb{N}$ we set $\delta_j = (\tau_0 - \tau_j)/(t_j - \tau_j)$, so that $t_j - \tau_0 = (1 - \delta_j)(t_j - \tau_j)$ and $\delta_j \in (0, 1)$. Define $\gamma_j \in \mathrm{AC}([-t_j + \tau_j, 0], \overline{\Omega})$ and $\xi_j \in \mathrm{AC}([-t_j, 0], \overline{\Omega})$, respectively, by $\gamma_j(s) = \gamma((1 - \delta_j)s)$ and

$$\xi_j(s) = \begin{cases} \gamma_j(s) & \text{if } -t_j + \tau_j \le s \le 0, \\ \eta_j(s + t_j - \tau_j) & \text{if } -t_j \le s < -t_j + \tau_j. \end{cases}$$

We note that $\xi_j(0) = x$, $\xi_j(-t_j + \tau_j) = \gamma(-t_j + \tau_0)$ and $\xi_j(-t_j) = y$, and observe as before that for some modulus ω_1, if j is large enough, then

$$\begin{aligned}\int_{-t_j}^{0} L[\xi_j] &= \int_{-t_j+\tau_j}^{0} L[\gamma_j] + \int_{-\tau_j}^{0} L[\eta_j] \\ &< u_0(x) - u_0(\gamma(-t_j+\tau_0)) + \frac{\tau_0-\tau_j}{t_j-\tau_j}(t_j-\tau_0)\omega_1\Big(\frac{\tau_0-\tau_j}{t_j-\tau_j}\Big) \\ &\quad +\varepsilon + u_0(\gamma(-t_j+\tau_0)) - u_0(y) \\ &\le u_0(x) - u_0(y) + \tau_0\omega_1\Big(\frac{\tau_0}{t_j-\tau_j}\Big) + \varepsilon.\end{aligned}$$

Thus we get

$$\begin{aligned}u(x,t) &\le \int_{-t_j}^{0} L[\xi_j] + u(\xi_j(-t_j), t-t_j) \\ &< u_0(x) - u_0(y) + \tau_0\omega_1\Big(\frac{\tau_0}{t_j-\tau_j}\Big) + u(y, t-t_j) + \varepsilon.\end{aligned}$$

Sending $j \to \infty$, we conclude that $u(x,t) \le u_0(x)$. □

Proof of Theorem 3.1. Assume first that $\mathcal{A} = \emptyset$. Then b is the unique solution of (D) and $u_0(x) \equiv \infty$. Hence, we have $u(x,t) = b(x,t) \wedge u_0(x)$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

Next we assume that $\mathcal{A} \ne \emptyset$. Recall that $u_-(x) = u_0(x)$ for all $x \in \mathcal{A}$, that u_0 is a solution of (SC) and that b is a solution of (D). In particular, $b(x,t) \le g(x,t)$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$. It is now easy to check that the function $v(x,t) := u_0(x) \wedge b(x,t)$ is a solution of (D). Furthermore, we find from (10) or (13) that

$$u(x,t) \le b(x,t) \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{14}$$

According to Lemma 3.6, we have $u(x,t) \le u_0(x)$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. Hence, we see that $\lim_{t\to-\infty} u(x,t) = u_0(x)$ for all $x \in \mathcal{A}$. Since $u \in \mathrm{BUC}(\overline{\Omega} \times \mathbb{R})$, by the Ascoli–Arzela theorem, we infer that the above convergence is uniform for $x \in \mathcal{A}$. We now fix any $\varepsilon > 0$ and choose a $\tau \in \mathbb{R}$ so that $|u(x,t) - u_0(x)| \le \varepsilon$ for all $(x,t) \in \mathcal{A} \times (-\infty, \tau]$. By (14), we see that $|u(x,t) - v(x,t)| \le \varepsilon$ for all $(x,t) \in \mathcal{A} \times (-\infty, \tau]$. We apply Lemma 3.5, to observe that $|u(x,t) - v(x,t)| \le \varepsilon$ for all $\overline{\Omega} \times (-\infty, \tau]$. Moreover, we apply a comparison theorem for the initial-boundary value problem for (D) in $\overline{\Omega} \times (\tau, \infty)$, with initial data $u(\cdot,\tau)$ and $v(\cdot,\tau) \pm \varepsilon$, to conclude that $|u(x,t) - v(x,t)| \le \varepsilon$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. Finally, noting that $\varepsilon > 0$ is arbitrary, we complete the proof. □

Proof of Theorem 3.2. We set

$$g_-(x) := \liminf_{t\to-\infty} g(x,t) = \inf_{t\in\mathbb{R}} g(x,t) \quad \text{for } x \in \partial\Omega,$$

and note that

$$\inf_{t\in\mathbb{R}} b(x,t) = \inf\{d(x,y) + g_-(y) : y \in \partial\Omega\} \quad \text{for all } x \in \overline{\Omega}. \tag{15}$$

Indeed, we see immediately that

$$\begin{aligned}\inf_{t\in\mathbb{R}} b(x,t) &\geq \inf\{e(x,y,\tau) + g_-(y) : y \in \partial\Omega,\ \tau > 0\} \\ &= \inf\{d(x,y) + g_-(y) : y \in \partial\Omega\} \quad \text{for all } x \in \overline{\Omega}.\end{aligned}$$

On the other hand, for any $\varepsilon > 0$, $y \in \partial\Omega$ and $x \in \overline{\Omega}$, there are $\tau \in \mathbb{R}$ and $\sigma > 0$ such that $g_-(y) > -\varepsilon + g(y,\tau)$ and $d(x,y) > -\varepsilon + e(x,y,\sigma)$. Then,

$$\begin{aligned}d(x,y) + g_-(y) &> -2\varepsilon + g(y,\tau) + e(x,y,\sigma) \\ &\geq -2\varepsilon + b(x,\tau+\sigma) \geq -2\varepsilon + \inf_{t\in\mathbb{R}} b(x,t).\end{aligned}$$

Thus, (15) holds.

Next we show that

$$\liminf_{t\to-\infty} b(x,t) = \inf\{d(x,y) + g_-(y) : y \in \partial\Omega\} \quad \text{for all } x \in \overline{\Omega}. \tag{16}$$

In view of (15) we need only to show that

$$\liminf_{t\to-\infty} b(x,t) \leq \inf\{d(x,y) + g_-(y) : y \in \partial\Omega\} \quad \text{for all } x \in \overline{\Omega}. \tag{17}$$

For any $(x,y,\varepsilon) \in \overline{\Omega}\times\partial\Omega\times(0,\infty)$, there are a $\tau > 0$ and a sequence $\{t_j\} \subset \mathbb{R}$ diverging to $-\infty$ such that $d(x,y) > -\varepsilon + e(x,y,\tau)$ and $g_-(y) > -\varepsilon + g(y,t_j)$ for all j. Adding these two yields

$$d(x,y) + g_-(y) > -2\varepsilon + e(x,y,\tau) + g(y,t_j) \geq -2\varepsilon + b(x,\tau+t_j)$$

for all j, which guarantees that (17) is valid.

Now, by Theorem 3.1 and (16) we see that

$$u_-(x) \leq \liminf_{t\to-\infty} b(x,t) = \inf_{t\in\mathbb{R}} b(x,t) \quad \text{for all } x \in \overline{\Omega}.$$

Consequently, we find by Theorem 3.1 again that for any $x \in \mathcal{A}$,

$$u(x,t) = u_0(x) \wedge b(x,t) = u_0(x),$$

which was to be shown. □

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