Asymptotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator

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Abstract

We study the long time behavior of solutions of the Cauchy problem for semilinear parabolic equations with the Ornstein-Uhlenbeck operator in $\mathbb{R}^N$. The long time behavior in the main results is stated with help of the corresponding to ergodic problem, which complements, in the case of unbounded domains, the recent developments on long time behaviors of solutions of (viscous) Hamilton-Jacobi equations due to Namah, Roquejoffre, Fathi, Barles, and Souganidis. We refer to [N, NR, R, F, BS1, BS2] for these developments. We also establish existence and uniqueness results for solutions of the Cauchy problem and ergodic problem for semilinear parabolic equations with the Ornstein-Uhlenbeck operator.

Key Words: long time behavior, viscous Hamilton-Jacobi equations, maximum principle, viscosity solutions, Ornstein-Uhlenbeck operator.
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1 Introduction

We study the long time behavior of solutions of the Cauchy problem for the parabolic PDE with the Ornstein-Uhlenbeck operator

(1.1) $u_t(x,t) - \Delta u(x,t) + \alpha x \cdot Du(x,t) + H(Du(x,t)) = f(x)$ in $\mathbb{R}^N \times (0,\infty)$,
(1.2) $u(\cdot,0) = u_0$ on $\mathbb{R}^N$,

where $u : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is the unknown function, α is a given positive constant, Δ denotes the Laplace operator in $\mathbb{R}^N$, Du denotes the gradient of u, and H, f, and u_0 are given functions on $\mathbb{R}^N$. We refer the operator $\Delta - \alpha x \cdot D$ on $C^2(\mathbb{R}^N)$ as the *Ornstein-Uhlenbeck operator*.

As we will see, the "stationary states" of solutions u of (1.1) and (1.2) or "asymptotic solutions of (1.1) and (1.2)" are described by the following PDE

(1.3) $$c - \Delta v(x) + \alpha x \cdot Dv(x) + H(Dv(x)) = f(x) \quad \text{in } \mathbb{R}^N,$$

where the unknown is the pair of a constant $c \in \mathbb{R}$ and a function $v : \mathbb{R}^N \to \mathbb{R}$.

These PDE arise typically as the dynamic programming equations for stochastic optimal control or stochastic differential games of the systems described by controlled Ornstein-Uhlenbeck processes. In this regard, PDE (1.3) corresponds to ergodic control or ergodic differential games in which players try to optimize the long time average of costs or gains. In this view point we often call (1.1) a viscous Hamilton-Jacobi equation, H a Hamiltonian, and problem (1.3) an ergodic problem.

There has been a great interest on the behavior of solutions to nonlinear parabolic PDE by many authors. This investigation was strongly influenced by the recent developments on Hamilton-Jacobi equations and viscous Hamilton-Jacobi equations due to Namah, Roquejoffre, Fathi, Barles, Souganidis. In these developments they studied the asymptotic behavior of solutions of PDE on compact manifolds, for instance, torus $\mathbb{T}^N$. We refer to [N, NR, R, F, BS1, BS2] for these developments.

A typical result in this line is stated as follows: if u is a (unique) viscosity solution of Hamilton-Jacobi equation

(1.4) $$u_t(x,t) + H(x, Du(x,t)) = 0 \quad \text{in } \mathbb{T}^N,$$

satisfying the initial condition

$$u(\cdot,0) = u_0 \quad \text{on } \mathbb{T}^N,$$

where $u_0 \in C(\mathbb{R}^N)$, $H \in C^2(\mathbb{T}^N \times \mathbb{R}^N)$, and $H(x,\cdot)$ is uniformly convex on $\mathbb{R}^N$ for all $x \in \mathbb{T}^N$, then there is a constant $c \in \mathbb{R}$ and a viscosity solution $v \in C(\mathbb{T}^N)$ of

$$c + H(x, Dv(x)) = 0 \quad \text{in } \mathbb{T}^N,$$

such that $u(\cdot,t)-(ct+v)\to 0$ in $C(\mathbb{T}^N)$ as $t\to\infty$. Here c does not depend on u_0, but v depends on u_0. Here the function $ct+v$ is a solution of (1.1) having a special form which is frequently called a traveling front.

Our main interest here is if a similar long time behavior (i.e., the convergence to traveling fronts) of solutions of PDE on unbounded domains is valid or not. Our results in this paper answer affirmatively to this question. Indeed, we will show the convergence to traveling fronts of solutions of viscous Hamilton-Jacobi equation (1.1) as $t\to\infty$.

The paper is organized as follows. We establish existence and uniqueness results for the Cauchy problem (1.1) and (1.2) in Section 2. In Section 3 we show local Hölder continuity of solutions of (1.1) and (1.2) in the spatial variables x uniform in the time variable t. In Section 4 we establish an existence result for ergodic problem (1.3) as well as estimates on local Hölder continuities of the solutions. In Section 5, we establish a long time behavior of solutions of the Cauchy problem (1.1) and (1.2). One of basic assumptions in Sections 2–5 is that Hamiltonian H is Lipschitz continuous on $\mathbb{R}^N$. In Section 6 we remove this restriction by assuming stronger requirement on f and u_0, i.e., the Lipschitz continuity of f and u_0.

We collect here the notation we use in this paper.

Notation: we write

$$Q=\mathbb{R}^N\times(0,\infty),\quad Q_T=\mathbb{R}^N\times(0,T),\quad \text{and}\quad R_T=\mathbb{R}^N\times[0,T).$$

Let Ω be a subset of $\mathbb{R}^m$. $\mathrm{Lip}(\Omega)$ denotes the space of Lipschitz continuous functions on Ω. For $g\in L^\infty(\Omega)$ we write $\|g\|_\infty=\|g\|_{L^\infty(\Omega)}$. For $g:\Omega\to\mathbb{R}$, g^* and g_* denote, respectively, the upper semicontinuous and lower semicontinuous envelopes of g, i.e.,

$$\begin{aligned} g^*(x) &= \lim_{r\searrow 0}\sup\{g(y)\mid y\in\Omega,\ |y-x|\le r\} \qquad \text{for } x\in\overline{\Omega},\\ g_* &= -(-g)^*. \end{aligned}$$

For $\gamma\in(0,1]$ and $k=0,1,2,...$, $C^{k+\gamma}(\Omega)$ denotes the space of those functions $u\in C^k(\Omega)$ whose k-th derivatives are Hölder continuous with exponent γ on compact subsets of Ω. When $0<\gamma<1$, we write $C^\gamma(\Omega)$ for $C^{0+\gamma}(\Omega)$ as well. For $\Omega\subset\overline{Q}$, $C^{2+\gamma,1+\frac{\gamma}{2}}(\Omega)$ denotes the space of those functions $u(x,t)$ whose second derivatives in x and first derivative in t are both Hölder continuous with exponent γ in x and with exponent $\frac{\gamma}{2}$ in t on compact subsets of Ω and whose first derivatives in x is Hölder continuous with exponent $\frac{1+\gamma}{2}$ in t on compact subsets of Ω. Similarly, $C^{\gamma,\frac{\gamma}{2}}(\Omega)$ denotes the space of those functions $u(x,t)$ which are Hölder continuous with exponent γ in x and with exponent $\frac{\gamma}{2}$ in t on compact subsets of Ω. For $x,y\in\mathbb{R}^N$, $x\cdot y$ and $\langle x\rangle_\delta$, where $\delta>0$, denote, respectively, the Euclidean inner product of x, y and $(|x|^2+\delta^2)^{1/2}$. $\mathcal{S}^N$ denotes the set of $N\times N$ real symmetric matrices. For $a\in\mathbb{R}^N$ and $r>0$, $B(a,r)$ denotes the closed ball $\{x\in\mathbb{R}^N \mid |x-a|\le r\}$. We call a function $\omega:[0,\infty)\to[0,\infty)$ a modulus if it is upper semicontinuous and nondecreasing on $[0,\infty)$ and satisfies $\omega(0)=0$.

2 Comparison and existence theorems for the Cauchy problem

In this section we study the Cauchy problem for the semilinear parabolic PDE

$$u_t-\Delta u+\alpha x\cdot Du+H(Du)=f(x) \quad \text{in } Q, \tag{2.1}$$

with initial data

$$u(\cdot,0)=u_0 \quad \text{on } \mathbb{R}^N. \tag{2.2}$$

Here $\alpha>0$ is a given constant, H, f, and u_0 are given continuous functions on $\mathbb{R}^N$, and u is the unknown function on $\overline{Q}$.

For $\mu>0$ we define the function $\phi_\mu\in C^\infty(\mathbb{R}^N)$ by

$$\phi_\mu(x)=e^{\mu|x|^2} \quad \text{for } x\in\mathbb{R}^N. \tag{2.3}$$

Let $T\in(0,\infty)$. We introduce the spaces $\mathcal{E}_\mu^+(\Omega)$, $\mathcal{E}_\mu^-(\Omega)$, $\mathcal{E}_\mu(\Omega)$, with $\mu>0$, of functions on $\Omega=\mathbb{R}^N$, R_T, or $\overline{Q}$ as follows:

$$\begin{aligned}
\mathcal{E}_\mu^+(\mathbb{R}^N) &= \Big\{v:\mathbb{R}^N\to\mathbb{R} \ \Big|\ \limsup_{|x|\to\infty}\frac{v(x)}{\phi_\mu(x)}\le 0\Big\}\\
\mathcal{E}_\mu^+(R_T) &= \Big\{v:R_T\to\mathbb{R} \ \Big|\ \limsup_{|x|\to\infty}\sup_{0\le t<T}\frac{v(x,t)}{\phi_\mu(x)}\le 0\Big\},\\
\mathcal{E}_\mu^+(\overline{Q}) &= \Big\{v:\overline{Q}\to\mathbb{R} \ \Big|\ \limsup_{|x|\to\infty}\sup_{0\le t<T}\frac{v(x,t)}{\phi_\mu(x)}\le 0 \ \text{ for } T>0\Big\},
\end{aligned}$$

and for $\Omega=\mathbb{R}^N$, R_T, or $\overline{Q}$,

$$\mathcal{E}_\mu^-(\Omega):=-\mathcal{E}_\mu^+(\Omega), \quad \mathcal{E}_\mu(\Omega):=\mathcal{E}_\mu^+(\Omega)\cap\mathcal{E}_\mu^-(\Omega).$$

Throughout this section μ denotes a fixed constant satisfying

$$0<\mu<\frac{\alpha}{2}, \tag{2.4}$$

and we assume that

$$H \in \mathrm{Lip}(\mathbb{R}^N). \tag{2.5}$$

We write simply ϕ and L_H for ϕ_μ and $\|DH\|_\infty$, respectively, in this section.

We are now ready to state a comparison result for viscosity solutions of (2.1) and (2.2).

Theorem 2.1. *Let $0 < T \le \infty$ and let $u \in \mathrm{USC}(R_T)$ and $v \in \mathrm{LSC}(R_T)$ be a viscosity subsolution and a viscosity supersolution of* (2.1)*, respectively. Assume that*

$$u \in \mathcal{E}_\mu^+(R_T) \quad \text{and} \quad v \in \mathcal{E}_\mu^-(R_T) \tag{2.6}$$

and that $u(x,0) \le v(x,0)$ for all $x \in \mathbb{R}^N$. Then $u \le v$ on R_T.

Proof. 1. We may assume that $T < \infty$. Indeed, once we know that the assertion is valid for all $0 < T < \infty$, then we conclude immediately that the assertion for $T = \infty$ is valid as well.

We compute that for $x \in \mathbb{R}^N$,

$$D\phi(x) = 2\mu\phi(x)x, \qquad \Delta\phi(x) = 2\mu(2\mu|x|^2 + N)\phi(x),$$

$$-\Delta\phi(x) + \alpha x \cdot D\phi(x) - L_H|D\phi(x)| = 2\mu\phi(x)[(\alpha - 2\mu)|x|^2 - N - L_H|x|],$$

to deduce that there is a constant $B > 0$, depending only on $\mu,\ \alpha,\ N$, and L_H, such that

$$-\Delta\phi(x) + \alpha x \cdot D\phi(x) - L_H|D\phi(x)| \ge \phi(x) - B \quad \text{for } x \in \mathbb{R}^N. \tag{2.7}$$

2. Choose a constant $B > 0$ so that (2.7) holds. Fix any $\epsilon > 0$, and define the functions $u_\epsilon \in \mathrm{USC}(R_T)$ and $v_\epsilon \in \mathrm{LSC}(R_T)$ by

$$\begin{aligned} u_\epsilon(x,t) &= u(x,t) - \epsilon\phi(x) - \epsilon Bt, \\ v_\epsilon(x,t) &= v(x,t) + \epsilon\phi(x) + \epsilon Bt. \end{aligned}$$

Observe that $u_\epsilon, -v_\epsilon \in \mathrm{USC}(R_T)$ and that u_ϵ and v_ϵ are, respectively, a viscosity subsolution and a viscosity supersolution of (2.1) in Q_T. Indeed, using (2.7), we may compute formally that for $U = u_\epsilon$,

$$\begin{aligned} &U_t - \Delta U + \alpha x \cdot DU + H(DU) \\ &\quad \le \ u_t - \Delta u + \alpha x \cdot Du + H(Du) - \epsilon B - \epsilon(-\Delta\phi + \alpha x \cdot D\phi - L_H|D\phi|) \\ &\quad \le \ f(x) - \epsilon B + \epsilon B = f(x), \end{aligned}$$

which can be easily justified by using standard arguments in viscosity solutions theory. Similarly we can easily verify that v_ϵ is a viscosity supersolution of (2.1) in Q_T.

3. We observe by (2.6) that

$$\lim_{|x|\to\infty} \sup_{0\le t<T} u_\epsilon(x,t) = -\infty,$$
$$\lim_{|x|\to\infty} \inf_{0\le t<T} v_\epsilon(x,t) = \infty.$$

Hence, there is a constant $R_\epsilon > 0$ such that

$$u_\epsilon(x,t) \le v_\epsilon(x,t) \quad \text{for } (x,t) \in (\mathbb{R}^N \setminus \mathrm{int}B(0,R_\epsilon)) \times [0,T).$$

We apply a standard comparison theorem for viscosity sub- and supersolutions (e.g., Theorem 8.2 of [CIL]) on $B(0,R_\epsilon)\times[0,T)$, to find that $u_\epsilon \le v_\epsilon$ on $B(0,R_\epsilon)\times[0,T)$ (and hence on R_T). Sending $\epsilon \to 0$ allows us to conclude that $u \le v$ on R_T. □

The next theorem is one of main results in this section and establishes the existence and uniqueness of a solution of (2.1) and (2.2).

Theorem 2.2. *Assume that*

$$u_0, f \in \mathcal{E}_\mu(\mathbb{R}^N) \cap C(\mathbb{R}^N), \tag{2.8}$$
$$f \in C^{0+\gamma}(\mathbb{R}^N) \tag{2.9}$$

for some constant $\gamma \in (0,\,1]$. *Then there is a unique solution* $u \in \mathcal{E}_\mu(\overline{Q}) \cap C(\overline{Q}) \cap C^{2,1}(Q)$ *of* (2.1) and (2.2).

For the proof of the above theorem, we use the following lemma.

Lemma 2.3. *Let* $v \in \mathcal{E}_\mu(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. *Then for each* $\epsilon > 0$ *there is a constant* $K \equiv K(\epsilon) > 0$ *such that*

$$|v(x) - v(y)| \le \epsilon(\phi_\mu(x) + \phi_\mu(y)) + K|x-y| \qquad \text{for } x, y \in \mathbb{R}^N.$$

Proof. Fix any $\epsilon > 0$. Observe that

$$\lim_{|x|\to\infty} (v(x) - \epsilon\phi(x)) = -\infty,$$

and hence $v - \epsilon\phi$ is bounded above on $\mathbb{R}^N$. Similarly, the function $v + \epsilon\phi$ is bounded below on $\mathbb{R}^N$. Choose a constant $M > 0$ so that

$$v(x) - \epsilon\phi(x) \le M \quad \text{and} \quad v(x) + \epsilon\phi(x) \ge -M \quad \text{for } x \in \mathbb{R}^N.$$

Next, choose a constant $R > 0$ so that

$$v(x) - \epsilon\phi(x) \le -M \;\text{ and }\; v(x) + \epsilon\phi(x) \ge M \;\text{ for } x \in \mathbb{R}^N \setminus B(0,R).$$

Since v is uniformly continuous on $B(0,R)$, there is a constant $K \equiv K(\epsilon) > 0$ such that

$$v(x) - v(y) \leq \epsilon + K|x-y| \quad \text{for } x, y \in B(0,R).$$

It follows that for all $x, y \in B(0,R)$,

$$v(x) - v(y) \leq \epsilon(\phi(x) + \phi(y)) + K|x-y|.$$

Let $x, y \in \mathbb{R}^N$. If $x \in \mathbb{R}^N \setminus B(0,R)$, then

$$v(x) - \epsilon\phi(x) \leq -M \leq v(y) + \epsilon\phi(y),$$

and hence

$$v(x) - v(y) \leq \epsilon(\phi(x) + \phi(y)) + K|x-y|.$$

Similarly, if $y \in \mathbb{R}^N \setminus B(0,R)$, then

$$v(x) - v(y) \leq \epsilon(\phi(x) + \phi(y)) + K|x-y|.$$

All together we have for all $x, y \in \mathbb{R}^N$,

$$v(x) - v(y) \leq \epsilon(\phi(x) + \phi(y)) + K|x-y|,$$

which completes the proof. □

Proof of Theorem 2.2. 1. Uniqueness of solutions of (2.1) and (2.2) follows from Theorem 2.1. For the proof of the existence of a solution, we use the Perron method. For the Perron method in viscosity solutions theory, we refer for instance to Theorem 4.1 of [CIL].

By virtue of Lemma 2.3, for any $\epsilon \in (0,1)$ there is a constant $K(\epsilon) > 0$ such that for all $x, y \in \mathbb{R}^N$,

$$|u_0(x) - u_0(y)| \leq \epsilon(\phi(x) + \phi(y)) + K(\epsilon)|x-y|.$$

Fix such a collection $\{K(\epsilon) \mid \epsilon \in (0,1)\}$ of positive numbers.

Let $y \in \mathbb{R}^N$, $\epsilon, \delta \in (0,1)$, and $A > 0$, and set

$$g(x,t) = u_0(y) + \epsilon(\phi(x) + \phi(y)) + K(\epsilon)\langle x-y\rangle_\delta + At \quad \text{for } (x,t) \in \overline{Q}.$$

We compute that for $(x,t) \in Q$,

$$\begin{aligned} &g_t(x,t) - \Delta g(x,t) + \alpha x \cdot Dg(x,t) + H(Dg(x,t)) \\ &\quad \geq \; A + \epsilon(-\Delta\phi(x) + \alpha x \cdot D\phi(x) - L_H|D\phi(x)|) \end{aligned}$$

$$
\begin{aligned}
&+K(\epsilon)\left[-\Big(\frac{N}{\langle x-y\rangle_\delta}-\frac{|x-y|^2}{\langle x-y\rangle_\delta^3}\Big)+\frac{\alpha x\cdot(x-y)}{\langle x-y\rangle_\delta}\right]+H\left(K(\epsilon)\frac{x-y}{\langle x-y\rangle_\delta}\right)\\
\geq\ & A+\epsilon(-\Delta\phi(x)+\alpha x\cdot D\phi(x)-L_H|D\phi(x)|)\\
&+K(\epsilon)\left[-\frac{N}{\delta}+\alpha\frac{\langle x-y\rangle_\delta^2-\delta^2-|y||x-y|}{\langle x-y\rangle_\delta}\right]-|H(0)|-L_HK(\epsilon)\\
\geq\ & A+\epsilon\Big(-\Delta\phi(x)+\alpha x\cdot D\phi(x)-L_H|D\phi(x)|\Big)\\
&-K(\epsilon)\left[\frac{N}{\delta}+\alpha(\delta+|y|)+L_H\right]-|H(0)|.
\end{aligned}
$$

We choose a constant $B>0$ so that (2.7) holds. Since $f\in\mathcal{E}_\mu(\mathbb{R}^N)$, for each $\epsilon\in(0,1)$ we may choose a constant $C(\epsilon)>0$ so that

$$
|f(x)|\leq\epsilon\phi(x)+C(\epsilon)\quad\text{for } x\in\mathbb{R}^N. \tag{2.10}
$$

Also, for each $\epsilon\in(0,1)$, we choose a constant $\delta(\epsilon)\in(0,1)$ so that $K(\epsilon)\delta(\epsilon)\leq\epsilon$.

For each $y\in\mathbb{R}^N$ and $\epsilon\in(0,1)$ we set

$$
A(y,\epsilon)=K(\epsilon)\left(\frac{N}{\delta(\epsilon)}+\alpha(1+|y|)+L_H\right)+|H(0)|+\epsilon B+C(\epsilon),
$$

and define the functions $\psi^+\in C^\infty(\overline{Q})$, parametrized by y,ϵ, by

$$
\psi^+(x,t;y,\epsilon)=u_0(y)+\epsilon(\phi(x)+\phi(y))+K(\epsilon)\langle x-y\rangle_{\delta(\epsilon)}+A(y,\epsilon)t.
$$

Observe that, for any $y\in\mathbb{R}^N$ and $\epsilon\in(0,1)$, the function $h(x,t):=\psi^+(x,t;y,\epsilon)$ satisfies

$$
\begin{aligned}
&h_t(x,t)-\Delta h(x,t)+\alpha x\cdot Dh(x,t)+H(Dh(x,t))\\
&\quad\geq\ A(y,\epsilon)+\epsilon(\phi(x)-B)-K(\epsilon)\left(\frac{N}{\delta(\epsilon)}+\alpha(\delta(\epsilon)+|y|)+L_H\right)-|H(0)|\\
&\quad\geq\ \epsilon\phi(x)+C(\epsilon)\\
&\quad\geq\ f(x)\quad\text{for } (x,t)\in Q.
\end{aligned}
$$

That is, the functions $\psi^+(\cdot;y,\epsilon)$ are classical (and hence viscosity) supersolutions of (2.1). Observe also that

$$
\psi^+(x,t;y,\epsilon)\geq u_0(y)+\epsilon(\phi(x)+\phi(y))+K(\epsilon)\langle x-y\rangle_{\delta(\epsilon)}\geq u_0(x)\quad\text{for } (x,t)\in\overline{Q},
$$

and

$$
\psi^+(x,0;x,\epsilon)=u_0(x)+2\epsilon\phi(x)+K(\epsilon)\delta(\epsilon)\leq u_0(x)+2\epsilon\phi(x)+\epsilon\quad\text{for } x\in\mathbb{R}^N.
$$

2. We define the function $U \in \mathrm{USC}(\overline{Q})$ by

$$U(x,t) = \inf\{\psi^+(x,t;y,\epsilon) \mid y \in \mathbb{R}^N,\ \epsilon \in (0,1)\}.$$

Then U_*, the lower semicontinuous envelope of U, is a viscosity supersolution of (2.1) (see, e.g., Lemma 4.2 of [CIL] for this) and

$$u_0(x) \le U_*(x,t) \le U(x,t) \quad \text{for } (x,t) \in \overline{Q}.$$

Also, noting that

$$U(x,0) \le \psi^+(x,0;x,\epsilon) < u_0(x) + 2\epsilon\phi(x) + \epsilon \quad \text{for } x \in \mathbb{R}^N \text{ and } \epsilon \in (0,1),$$

we see that

$$U(x,0) = U_*(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}^N.$$

3. Similarly we define the function $V \in \mathrm{LSC}(\overline{Q})$ by

$$V(x,t) = \sup\{\psi^-(x,t;y,\epsilon) \mid y \in \mathbb{R}^N,\ \epsilon \in (0,1)\},$$

where

$$\psi^-(x,t;y,\epsilon) := u_0(y) - \epsilon(\phi(x) + \phi(y)) - K(\epsilon)\langle x-y\rangle_{\delta(\epsilon)} - A(y,\epsilon)t,$$

and then observe as before that V^* is a viscosity subsolution of (2.1) and that for $(x,t) \in \overline{Q}$,

$$u_0(x) \ge V^*(x,t) \ge V(x,t) \quad \text{and} \quad V(x,0) = V^*(x,0) = u_0(x).$$

4. Define $u : \overline{Q} \to \mathbb{R}$ by

$$u(x,t) = \sup\{\, v(x,t) \mid V \le v \le U \text{ on } \overline{Q},\ v \text{ is a viscosity subsolution of (2.1)}\}.$$

Then u^* and u_* are a viscosity subsolution and a viscosity supersolution of (2.1), respectively, and

$$V \le u \le U \quad \text{on } \overline{Q}.$$

In particular, we see that as $t \searrow 0$,

$$u(x,t) \to u_0(x),\ u^*(x,t) \to u_0(x),\ u_*(x,t) \to u_0(x) \quad \text{locally uniformly on } \mathbb{R}^N.$$

5. Next we show that $u \in \mathcal{E}_\mu(\overline{Q})$. Indeed, we have

$$\begin{aligned}
&|u(x,t) - u_0(0)| \\
&\quad \le \max\{U(x,t) - u_0(0), u_0(0) - V(x,t)\} \\
&\quad \le \epsilon(\phi(x) + \phi(0)) + K(\epsilon)\langle x\rangle_{\delta(\epsilon)} + A(0,\epsilon)t \quad \text{for } (x,t) \in \overline{Q},\ \epsilon \in (0,1),
\end{aligned}$$

which guarantees that $u \in \mathcal{E}_\mu(\overline{Q})$.

Now, we apply Theorem 2.1 to u^* and u_*, to find that

$$u^* \le u_* \quad \text{in } \overline{Q},$$

that is, $u \in C(\overline{Q})$.

6. Schauder theory for parabolic PDE (see the appendix or [LSU]) assures that for each $R > 0$ there is a classical solution $v \in C(\overline{\Omega_R}) \cap C^{2+\gamma,1+\frac{\gamma}{2}}(\Omega_R)$, where $\Omega_R := \operatorname{int} B(0,R) \times (0,R)$, of

$$\begin{cases} v_t - \Delta v + \alpha x \cdot Dv + H(Dv) = f(x) & \text{in } \Omega_R, \\ v = u & \text{on } \partial_p \Omega_R, \end{cases} \tag{2.11}$$

where $\partial_p \Omega_R := (\partial B(0,R) \times (0,R)) \cup (B(0,R) \times \{0\})$. This and standard comparison results for viscosity solutions yield that $u = v$ on Ω_R, which show that $u \in C^{2,1}(\Omega_R)$. Since $R > 0$ is arbitrary, we find that $u \in C^{2,1}(Q)$. The proof is now complete. □

Remark 2.4. In the above proof, since $V \le u \le U$ on $\overline{Q}$, we get

$$\begin{aligned} |u(x,t) - u_0(x)| &\le \max\{U(x,t) - u_0(x), u_0(x) - V(x,t)\} \\ &\le \inf_{0<\epsilon<1} (2\epsilon\phi(x) + \epsilon + A(x,\epsilon)t) \quad \text{for } (x,t) \in Q. \end{aligned}$$

Setting

$$\begin{aligned} A_R(\epsilon) &= \max\{A(x,\epsilon) \mid x \in B(0,R)\} \\ &\equiv K(\epsilon)\Big(\frac{N}{\delta(\epsilon)} + \alpha(R+1) + L_H\Big) + |H(0)| + \epsilon B + C(\epsilon) \quad \text{for } R > 0, \end{aligned}$$

and

$$\rho_R(t) = \inf_{0<\epsilon<1} \Big[\epsilon(1 + 2e^{\mu R^2}) + A_R(\epsilon)t\Big] \quad \text{for } t \ge 0 \text{ and } R > 0, \tag{2.12}$$

we have

$$|u(x,t) - u_0(x)| \le \rho_R(t) \quad \text{for } (x,t) \in B(0,R) \times [0,\infty) \text{ and } R > 0. \tag{2.13}$$

Note that $\rho_R \in \mathrm{USC}([0,\infty))$ and $\rho_R(0) = 0$ for all $R > 0$ and that we may select $\delta(\epsilon)$ for $\epsilon \in (0,1)$ by the formula

$$\delta(\epsilon) = \epsilon \min\Big\{1, \frac{1}{K(\epsilon)}\Big\}. \qquad \square \tag{2.14}$$

3 Estimates of uniform continuities

In the following we investigate estimates on local uniform continuities of solutions of (2.1).

In this section, as in the previous section, let μ be a fixed constant satisfying (2.4) and assume the Lipschitz continuity (2.5) of H. We write ϕ and L_H for ϕ_μ and $\|DH\|_\infty$, respectively.

We introduce the operators $\Xi : C^2(\mathbb{R}^{2N}) \to C(\mathbb{R}^{2N})$ by

$$\Xi\, g(x,y) = \Delta_x g(x,y) + \Delta_y g(x,y) + 2\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i \partial y_i} g(x,y).$$

Note that $-\Xi$ is a degenerate elliptic operator on $\mathbb{R}^{2N}$, i.e., if $g \in C^2(\mathbb{R}^{2N})$ attains a maximum at $(0,0)$, then

$$-\Xi\, g(0,0) = -\sum_{i=1}^{N} \frac{d^2}{dt^2} g(te_i, te_i)\Big|_{t=0} \geq 0,$$

where e_i denotes the unit vector with unity as its i-th entry for $i = 1, 2, ..., N$. Note also that for $u, v \in C^2(\mathbb{R}^N)$, if we set $g(x,y) := u(x) - v(y)$, then

$$\Xi\, g(x,y) = \Delta u(x) - \Delta v(y) \ \text{ for } x, y \in \mathbb{R}^N,$$

and that for $\varphi \in C^2(\mathbb{R}^N)$, if we set $g(x,y) := \varphi(x-y)$, then

$$\Xi\, g(x,y) = 0 \ \text{ for } x, y \in \mathbb{R}^N.$$

Let $\varphi, \psi_1, \psi_2 \in C^2(\mathbb{R}^N)$. Setting

$$g(x,y) = \varphi(x-y)(\psi_1(x) + \psi_2(y)),$$

we compute that

$$\begin{aligned}
Dg(x,y) &= (\psi_1(x) + \psi_2(y))(D\varphi(x-y), -D\varphi(x-y)) \\
&\quad + \varphi(x-y)(D\psi_1(x), D\psi_2(y)), \\
D^2 g(x,y) &= (\psi_1(x) + \psi_2(y)) \begin{pmatrix} D^2\varphi(x-y) & -D^2\varphi(x-y) \\ -D^2\varphi(x-y) & D^2\varphi(x-y) \end{pmatrix} \\
&\quad + \begin{pmatrix} D\varphi(x-y) \otimes D\psi_1(x) & -D\varphi(x-y) \otimes D\psi_1(x) \\ D\varphi(x-y) \otimes D\psi_2(y) & -D\varphi(x-y) \otimes D\psi_2(y) \end{pmatrix}
\end{aligned}$$

$$+\begin{pmatrix} D\psi_1(x)\otimes D\varphi(x-y) & D\psi_2(x)\otimes D\varphi(x-y) \\ -D\psi_1(x)\otimes D\varphi(x-y) & -D\psi_2(x)\otimes D\varphi(x-y) \end{pmatrix}$$

$$+\varphi(x-y)\begin{pmatrix} D^2\psi_1(x) & 0 \\ 0 & D^2\psi_2(y) \end{pmatrix},$$

$$\Xi\, g(x,y) = \varphi(x-y)(\Delta\psi_1(x)+\Delta\psi_2(y)). \tag{3.1}$$

Theorem 3.1. *Let* $u \in C(R_T)\cap C^{2,1}(Q_T)$ *be a solution of* (2.1) *and* (2.2). *Let* $\gamma \in (0,1]$ *and assume that*

$$u \in \mathcal{E}_\mu(R_T), \tag{3.2}$$

$$|f(x)-f(y)| \le C_f|x-y|^\gamma(\phi_\mu(x)+\phi_\mu(y)) \quad \text{for } x,y\in\mathbb{R}^N \tag{3.3}$$

$$|u_0(x)-u_0(y)| \le C_0|x-y|^\gamma(\phi_\mu(x)+\phi_\mu(y)) \quad \text{for } x,y\in\mathbb{R}^N. \tag{3.4}$$

Then there exists a constant $C_1>0$, *depending only on* α, γ, μ, C_0, C_f, $\|DH\|_\infty$, *and* N, *such that*

$$|u(x,t)-u(y,t)| \le C_1|x-y|^\gamma(\phi_\mu(x)+\phi_\mu(y)) \quad \text{for } x,y\in\mathbb{R}^N,\ t\in[0,T). \tag{3.5}$$

Proof. 1. We use the notation: $P_T = \mathbb{R}^{2N}\times(0,T)$ and $S_T=\mathbb{R}^{2N}\times[0,T)$. We define the function $w\in C(S_T)\cap C^{2,1}(P_T)$ by

$$w(x,y,t) = u(x,t)-u(y,t),$$

and observe that for all $(x,y,t)\in P_T$,

$$\begin{aligned} &w_t - \Xi\, w + \alpha x\cdot D_x w + \alpha y\cdot D_y w - L_H|D_x w + D_y w| \\ &\quad\le\ f(x)-f(y) \le C_f|x-y|^\gamma(\phi(x)+\phi(y)). \end{aligned} \tag{3.6}$$

2. Let $\delta>0$, $\epsilon>0$, $A>0$, and $C>0$ be constants to be fixed later on. Define the function $\zeta\in C(S_T)$ by

$$\zeta(x,y,t) = C(|x-y|^\gamma+\delta)(\phi(x)+\phi(y)+A)+\frac{\epsilon}{T-t}.$$

Using (3.1), we compute that for $(x,y,t)\in P_T$, if $x\neq y$, then

$$\begin{aligned} \zeta_t &= \frac{\epsilon}{(T-t)^2} \ge \frac{\epsilon}{T^2}, \\ \Xi\,\zeta &= C(|x-y|^\gamma+\delta)(\Delta\phi(x)+\Delta\phi(y)), \\ \alpha x\cdot D_x\zeta+\alpha y\cdot D_y\zeta &= \alpha\gamma C|x-y|^\gamma(\phi(x)+\phi(y)+A) \\ &\qquad + C(|x-y|^\gamma+\delta)(\alpha x\cdot D\phi(x)+\alpha y\cdot D\phi(y)), \\ L_H|D_x\zeta+D_y\zeta| &\le CL_H(|x-y|^\gamma+\delta)\left(|D\phi(x)|+|D\phi(y)|\right), \end{aligned}$$

and

$$
\begin{aligned}
&\zeta_t - \Xi\,\zeta + \alpha x\cdot D_x\zeta + \alpha y\cdot D_y\zeta - L_H|D_x\zeta + D_y\zeta| \\
\ge\ & \frac{\epsilon}{T^2} + C(|x-y|^\gamma + \delta)[(-\Delta\phi(x) + \alpha x\cdot D\phi(x) - L_H|D\phi(x)|) \\
& +(-\Delta\phi(y) + \alpha y\cdot D\phi(y) - L_H|D\phi(y)|)] + \alpha\gamma C|x-y|^\gamma(\phi(x)+\phi(y)+A).
\end{aligned}
$$

We choose a constant $B \equiv B(\alpha,\mu,L_H,N) > 0$ so that (2.7) holds. Then we have

$$
\begin{aligned}
&\zeta_t - \Xi\,\zeta + \alpha x\cdot D_x\zeta + \alpha y\cdot D_y\zeta - L_H|D_x\zeta + D_y\zeta| \\
\ge\ & \frac{\epsilon}{T^2} - 2\delta B + C|x-y|^\gamma(\phi(x)+\phi(y)+\alpha\gamma A - 2B) \ \text{ if } x \ne y.
\end{aligned}
$$

We fix constants A and C as

$$
A = \frac{2B}{\alpha\gamma}, \quad C = \max\{C_0, C_f\},
$$

so that

$$
\begin{aligned}
w(x,y,0) &\le \zeta(x,y,0) \ \text{ for } x,y \in \mathbb{R}^N, \\
\zeta_t - \Xi\,\zeta + \alpha x\cdot D_x\zeta + \alpha y\cdot D_y\zeta & - L_H|D_x\zeta + D_y\zeta| \\
&\ge \frac{\epsilon}{T^2} - 2\delta B + C_f|x-y|^\gamma(\phi(x)+\phi(y)) \ \text{ if } x \ne y.
\end{aligned}
$$

3. We want to show that

$$
w(x,y,t) \le C|x-y|^\gamma(\phi(x)+\phi(y)+A) \ \text{ for } (x,y,t) \in S_T. \tag{3.7}
$$

Note that (3.7) yields (3.5), with $C_1 = C(1+A)$, since $\phi(x) \ge 1$. To show (3.7), we argue by contradiction and thus suppose that

$$
\sup\{w(x,y,t) - C|x-y|^\gamma(\phi(x)+\phi(y)+A) \mid (x,y,t) \in S_T\} > 0.
$$

Then we can choose a constant $\epsilon > 0$ so that

$$
\sup\{w(x,y,t) - C(|x-y|^\gamma + \epsilon)(\phi(x)+\phi(y)+A) - \frac{\epsilon}{T-t} \mid (x,y,t) \in S_T\} > 0.
$$

We fix

$$
\delta = \epsilon \min\left\{1, \frac{1}{3BT^2}\right\}.
$$

Note that $0 < \delta \le \epsilon$ and

$$
\frac{\epsilon}{T^2} > 2\delta B,
$$

and consequently,

$$\sup_{S_T}(w-\zeta) > 0,$$

and

$$\begin{aligned} \zeta_t - \Xi\,\zeta + \alpha x\cdot D_x\zeta + \alpha y\cdot D_y\zeta - L_H|D_x\zeta + D_y\zeta| \\ > \ C_f|x-y|^\gamma(\phi(x)+\phi(y)) \ \text{ if } x\neq y. \end{aligned} \tag{3.8}$$

Since $u\in\mathcal{E}_\mu(R_T)$, we have

$$\lim_{|x|+|y|\to\infty}\sup_{0\le t<T}(w-\zeta)(x,y,t) = -\infty.$$

Also, for each $R>0$ we have

$$\lim_{t\nearrow T}\sup_{(x,y)\in B(0,R)\times B(0,R)}(w-\zeta)(x,y,t) = -\infty.$$

Thus the function $w-\zeta$ attains a positive maximum at a point $(\bar x,\bar y,\bar t)\in S_T$. Since $w(x,y,0)\le\zeta(x,y,0)$ for all $x,y\in\mathbb{R}^N$, we find that $\bar t>0$. Since $w(x,x,t)-\zeta(x,x,t) = -\zeta(x,x,t)<0$ for all $(x,t)\in Q_T$, we see that $\bar x\neq\bar y$. By the maximum principle, we have

$$(w-\zeta)_t(\bar x,\bar y,\bar t)=0, \quad D(w-\zeta)(\bar x,\bar y,\bar t)=0, \quad D^2(w-\zeta)(\bar x,\bar y,\bar t)\le 0.$$

These together with (3.6) yields

$$\begin{aligned} \zeta_t - \Xi\,\zeta + \alpha x\cdot D_x\zeta + \alpha y\cdot D_y\zeta - L_H|D_x\zeta + D_y\zeta| \\ \le \ C_f|x-y|^\gamma(\phi(x)+\phi(y)) \quad \text{at } (x,y,t)=(\bar x,\bar y,\bar t). \end{aligned} \tag{3.9}$$

This contradicts (3.8), which proves (3.7) and hence (3.5). □

Theorem 3.2. *In addition to the hypotheses of Theorem* 3.1, *assume that for each* $\epsilon\in(0,1)$ *there is a constant* $M(\epsilon)>0$ *such that*

$$|u(x,t)|\le\epsilon\phi_\mu(x)+M(\epsilon) \ \text{ for } (x,t)\in R_T. \tag{3.10}$$

Then for each $R>0$ *there is a modulus* ρ_R *such that*

$$|u(x,t)-u(x,s)|\le\rho_R(|t-s|) \ \text{ for } x\in B(0,R) \text{ and } t,s\in[0,\infty). \tag{3.11}$$

Proof. 1. By Theorem 3.1, we have

$$|u(x,t)-u(y,t)| \le C_1|x-y|^\gamma(\phi(x)+\phi(y)) \quad \text{for } x,y \in \mathbb{R}^N \text{ and } t \in [0,\infty)$$

for some constant $C_1 > 0$.

2. We show that for each $\epsilon \in (0,1)$ there is a constant $K(\epsilon) > 0$ such that for all $x, y \in \mathbb{R}^N$ and $t \ge 0$,

$$|u(x,t)-u(y,t)| \le \epsilon(\phi(x)+\phi(y)) + K(\epsilon)|x-y|. \tag{3.12}$$

For this fix any $\epsilon \in (0,1)$, $x, y \in \mathbb{R}^N$, and $t \ge 0$. If $C_1|x-y|^\gamma \le \epsilon$, then we have (3.12) for any $K(\epsilon) > 0$. If $C_1|x-y|^\gamma \ge \epsilon$, then we have

$$1 \le \left(\frac{C_1}{\epsilon}\right)^{\frac{1}{\gamma}} |x-y|,$$

and hence, by (3.10),

$$\begin{aligned} |u(x,t)-u(y,t)| &\le \epsilon(\phi(x)+\phi(y)) + M(\epsilon) \\ &\le \epsilon(\phi(x)+\phi(y)) + M(\epsilon)\left(\frac{C_1}{\epsilon}\right)^{\frac{1}{\gamma}} |x-y|. \end{aligned}$$

Thus we find that (3.12) holds with

$$K(\epsilon) = M(\epsilon)\left(\frac{C_1}{\epsilon}\right)^{\frac{1}{\gamma}}.$$

3. Now fix $s \ge 0$. Recalling Remark 2.4 and applying estimate (2.13), with $u(x,t+s)$ and $u(x,s)$, respectively, in place of $u(x,t)$ and $u_0(x)$, we find a modulus ρ_R for each $R > 0$ such that

$$|u(x,t+s)-u(x,t)| \le \rho_R(t) \quad \text{for } (x,t) \in B(0,R)\times[0,\infty). \tag{3.13}$$

Indeed, the function $\rho_R : [0,\infty) \to [0,\infty)$ defined by

$$\rho_R(t) = \inf_{0<\epsilon<1}[\epsilon(1+2e^{\mu R^2}) + A_R(\epsilon)t]$$

has the required properties, where

$$A_R(\epsilon) = K(\epsilon)\left(\frac{N}{\delta(\epsilon)} + \alpha(R+1) + L_H\right) + |H(0)| + \epsilon B + C(\epsilon)$$

with function $\delta : (0,1) \to (0,1)$ given by

$$\delta(\epsilon) = \epsilon \min\{1, \frac{1}{K(\epsilon)}\},$$

and with constant B and function $C : (0,1) \to (0,\infty)$ selected so that (2.7) and (2.10) hold. From (3.13) we conclude immediately that (3.11) holds. □

4 Ergodic problem

In this section we study the ergodic control problem

$$(4.1) \qquad c - \Delta v(x) + \alpha x \cdot Dv(x) + H(Dv(x)) = f(x) \quad \text{in } \mathbb{R}^N,$$

where the unknown is a pair of a constant c and a function $v \in C^2(\mathbb{R}^N)$. This is the equation which will describe the traveling fronts of solutions of (2.1) as $t \to \infty$.

Throughout this section we assume (2.5), that is, the Lipschitz continuity of H and that

$$(4.2) \qquad f \in \mathcal{E}_\mu(\mathbb{R}^N) \cap C^{0+\gamma}(\mathbb{R}^N),$$

for some constants $0 < \mu < \alpha/2$ and $\gamma \in (0,\, 1]$, where $\mathcal{E}_\mu(\mathbb{R}^N)$ denotes the space introduced in Section 2. Furthermore let L_H and ϕ denote the Lipschitz constant of H, i.e., $L_H = \|DH\|_\infty$, and ϕ_μ, respectively.

In order to solve (4.1), we first consider the approximate problem

$$(4.3) \qquad \lambda v^\lambda(x) - \Delta v^\lambda(x) + \alpha x \cdot Dv^\lambda(x) + H(Dv^\lambda(x)) = f(x) \quad \text{in } \mathbb{R}^N,$$

where $\lambda \in (0, 1)$ is a given constant to be sent zero. If H is a convex function, (4.3) may be regarded as the dynamic programming equation of an optimal control problem, where λ represents the discount factor. In this view point, especially when we call (4.1) an ergodic problem, we call (4.3) a discounted problem.

Theorem 4.1. *Let $u \in \mathrm{USC}(\mathbb{R}^N)$ and $v \in \mathrm{LSC}(\mathbb{R}^N)$ be a viscosity subsolution and a viscosity supersolution of* (4.3)*, respectively. Assume that*

$$(4.4) \qquad u \in \mathcal{E}_\mu^+(\mathbb{R}^N) \quad \text{and} \quad v \in \mathcal{E}_\mu^-(\mathbb{R}^N).$$

Then $u \le v$ in $\mathbb{R}^N$. $\square$

Proof. Recall (2.7) and fix a constant $B > 0$ such that

$$-\Delta\phi + \alpha x \cdot D\phi - L_H|D\phi| \ge \phi - B \quad \text{in } \mathbb{R}^N.$$

Fix such a constant B and, for $\epsilon \in (0, 1)$, define the functions u_ϵ, v_ϵ on $\mathbb{R}^N$, respectively, by

$$\begin{aligned} u_\epsilon(x) &= u(x) - \epsilon(\phi(x) + \lambda^{-1}B), \\ v_\epsilon(x) &= v(x) + \epsilon(\phi(x) + \lambda^{-1}B). \end{aligned}$$

Observe that $u_\epsilon, -v_\epsilon \in \mathrm{USC}(\mathbb{R}^N)$ and that u_ϵ and v_ϵ are a viscosity subsolution and a viscosity supersolution of (4.3), respectively. We examine these just for u_ϵ by calculating formally

$$\begin{aligned}&\lambda u_\epsilon - \Delta u_\epsilon + \alpha x \cdot Du_\epsilon + H(Du_\epsilon)\\ &\quad\le \lambda u - \Delta u + \alpha x \cdot Du + H(Du) - \epsilon(B - \Delta\phi + \alpha x \cdot D\phi - L_H|D\phi|)\\ &\quad\le f(x) \quad \text{for } x \in \mathbb{R}^N.\end{aligned}$$

By (4.4) we have

$$\lim_{|x|\to\infty} u_\epsilon(x) = -\infty \quad \text{and} \quad \lim_{|x|\to\infty} v_\epsilon(x) = \infty,$$

and hence, we find a constant $R \equiv R(\epsilon) > 0$ such that $u_\epsilon \le v_\epsilon$ on $\mathbb{R}^N \setminus \operatorname{int} B(0,R)$. We apply a standard comparison result to u_ϵ and v_ϵ on $B(0,R)$, to conclude that $u_\epsilon \le v_\epsilon$ on $B(0,R)$. Therefore we have $u_\epsilon \le v_\epsilon$ in $\mathbb{R}^N$. Since $\epsilon > 0$ is arbitrary, we conclude that $u \le v$ in $\mathbb{R}^N$. □

Theorem 4.2. *There is a unique solution $v^\lambda \in C^2(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$ of (4.3). Moreover there is a constant $C > 0$, independent of λ, such that*

$$\lambda |v^\lambda(0)| \le C. \tag{4.5}$$

Proof. We will apply the Perron method to find a viscosity solution of (4.3). In view of (2.7), we fix a constant $B > 0$ so that

$$-\Delta\phi(x) + \alpha x \cdot D\phi(x) - L_H|D\phi(x)| \ge \phi(x) - B \quad \text{for all } x \in \mathbb{R}^N.$$

Since $f \in \mathcal{E}_\mu(\mathbb{R}^N)$, there is a function $M : (0,1] \to (0,\infty)$ such that

$$|f(x)| \le \epsilon\phi(x) + M(\epsilon) \quad \text{for } x \in \mathbb{R}^N \text{ and } \epsilon \in (0,1].$$

For $\epsilon \in (0,1]$, we set

$$g(x) = \epsilon\phi(x) + A \quad \text{for } x \in \mathbb{R}^N,$$

where $A := \lambda^{-1}(M(\epsilon) + \epsilon B + |H(0)|)$, and calculate that

$$\begin{aligned}&\lambda g(x) - \Delta g(x) + \alpha x \cdot Dg(x) + H(Dg(x))\\ &\quad\ge \lambda A + \epsilon(\phi(x) - B) - |H(0)| = \epsilon\phi(x) + M(\epsilon) \ge f(x) \quad \text{for } x \in \mathbb{R}^N.\end{aligned}$$

We define the functions $U \in \mathrm{USC}(\mathbb{R}^N)$ and $V \in \mathrm{LSC}(\mathbb{R}^N)$ by

$$\begin{aligned} U(x) &= \inf_{0<\epsilon\le 1}[\epsilon\phi(x) + \lambda^{-1}(M(\epsilon) + \epsilon B + |H(0)|)], \\ V(x) &= -U(x). \end{aligned}$$

It is easy to check that U_* and V^* are a viscosity supersolution and a viscosity subsolution of (4.3), respectively.

Next define $v : \mathbb{R}^N \to \mathbb{R}$ by

$$\begin{aligned} v(x) = \sup\{w(x) \ \mid \ & V(x) \le w(x) \le U(x) \ \text{ for } x \in \mathbb{R}^N, \\ & w \text{ is a viscosity subsolution of (4.3)}\}. \end{aligned}$$

Then, in view of the Perron method, v^* and v_* are a viscosity subsolution and a viscosity supersolution of (4.3), respectively. Note that, since $V \le v \le U$ in $\mathbb{R}^N$,

$$|v(x)| \le \epsilon\phi(x) + \lambda^{-1}(M(\epsilon) + \epsilon B + |H(0)|), \tag{4.6}$$

for all $\epsilon \in (0,1]$ and $x \in \mathbb{R}^N$. Consequently, $v^*, v_* \in \mathcal{E}_\mu(\mathbb{R}^N)$. Now we apply Theorem 4.1, to find that $v^* \le v_*$ in $\mathbb{R}^N$. That is, $v \in C(\mathbb{R}^N)$. By virtue of Schauder theory (see the appendix or [LSU]), we find that $v \in C^2(\mathbb{R}^N)$.

Uniqueness of solutions of (4.3) is a direct consequence of Theorem 4.1. Finally, (4.5), with $C = 1 + M(1) + B + |H(0)|$, follows from (4.6). □

Theorem 4.3. *Assume that* (3.3) *holds for some constant* $C_f > 0$. *Let* $v^\lambda \in C^2(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$ *be the unique solution of* (4.3). *Then there is a constant* $K > 0$, *independent of* $\lambda \in (0,1)$, *such that*

$$|v^\lambda(x) - v^\lambda(y)| \le K|x-y|^\gamma(\phi(x) + \phi(y)) \quad \text{for } x, y \in \mathbb{R}^N,\ \lambda \in (0,1). \tag{4.7}$$

Proof. The following proof parallels that of Theorem 3.1. Set $w(x,y) := v^\lambda(x) - v^\lambda(y)$ for $x, y \in \mathbb{R}^N$. Note that for $x, y \in \mathbb{R}^N$,

$$\lambda w - \Xi\, w + \alpha x \cdot D_x w + \alpha y \cdot D_y w - L_H |D_x w + D_y w| \le f(x) - f(y),$$

where, as in Section 3, Ξ denotes the operator: $C^2(\mathbb{R}^N) \to C(\mathbb{R}^N)$ given by

$$\Xi\, g(x,y) = \Delta_x g(x,y) + 2\sum_{i=1}^N \frac{\partial^2}{\partial x_i \partial y_i} g(x,y) + \Delta_y g(x,y).$$

Recall that when we set $g(x,y) = \varphi(x-y)(\psi_1(x)+\psi_2(y))$, where $\varphi \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$ is an open set, and $\psi_1, \psi_2 \in C^2(\mathbb{R}^N)$, we have for $x, y \in \mathbb{R}^N$, if $x - y \in \Omega$,

$$\begin{aligned}\lambda g - \Xi\, g + \alpha x \cdot D_x g + \alpha y \cdot D_y g - L_H|D_x g + D_y g| \\ \geq\ & \lambda g + \varphi(x-y)(-\Delta\psi_1 + \alpha x \cdot D\psi_1 - L_H|D\psi_1|) \\ & +\varphi(x-y)(-\Delta\psi_2 + \alpha x \cdot D\psi_2 - L_H|D\psi_2|) \\ & +\alpha(x-y)\cdot D\varphi(x-y)(\psi_1+\psi_2).\end{aligned}$$

Let $B > 0$ be a constant such that

$$-\Delta\phi(x) + \alpha x \cdot D\phi(x) - L_H|D\phi(x)| \geq \phi(x) - B \quad \text{ for all } x \in \mathbb{R}^N.$$

Let $\delta > 0$ be a constant to be sent to zero and $A > 0$ a constant to be fixed later. Define $g \in C(\mathbb{R}^{2N})$ by

$$g(x,y) = C_f[\delta + (|x-y|^\gamma + \delta^2)(\phi(x) + \phi(y) + A)].$$

Then, for $x, y \in \mathbb{R}^N$, if $x \neq y$, we have

$$\begin{aligned}\lambda g - \Xi\, g + \alpha x \cdot D_x g + \alpha y \cdot D_y g - L_H|D_x g + D_y g| \\ \geq\ & \lambda g + C_f(|x-y|^\gamma + \delta^2)(\phi(x) + \phi(y) - 2B) \\ & \quad +\alpha\gamma C_f|x-y|^\gamma(\phi(x)+\phi(y)+A) \\ \geq\ & C_f(\lambda\delta + (|x-y|^\gamma + \delta^2)(\phi(x)+\phi(y)-2B) + \alpha\gamma A|x-y|^\gamma) \\ >\ & C_f[\delta(\lambda - 2\delta B) + |x-y|^\gamma(\phi(x)+\phi(y)) + (\alpha\gamma A - 2B)|x-y|^\gamma].\end{aligned}$$

Now, we assume that $\delta \leq \lambda/(2B)$ and fix

$$A = \frac{2B}{\alpha\gamma},$$

so that $\alpha\gamma A = 2B$ and $\lambda \geq 2\delta B$. Observe that for $x, y \in \mathbb{R}^N$, if $x \neq y$, then

$$\lambda g - \Xi\, g + \alpha x \cdot D_x g + \alpha y \cdot D_y g - L_H|D_x g + D_y g| > C_f|x-y|^\gamma(\phi(x)+\phi(y)). \tag{4.8}$$

We argue by contradiction, in order to prove that $w \leq g$ on $\mathbb{R}^{2N}$. Thus we suppose that

$$\sup_{\mathbb{R}^{2N}}(w-g) > 0.$$

Noting that

$$\lim_{|x|+|y|\to\infty}(w-g)(x,y) = -\infty,$$

we find an open bounded set $G \subset \mathbb{R}^{2N}$ such that $w \le g$ on $\mathbb{R}^{2N} \setminus G$. Let $(\hat{x}, \hat{y}) \in \overline{G}$ be a maximum point of the function $w - g$ over $\overline{G}$. Then we have $(w-g)(\hat{x}, \hat{y}) > 0$, which assures that $(\hat{x}, \hat{y}) \in G$ and $\hat{x} \ne \hat{y}$. Since

$$\begin{aligned}\lambda w - \Xi\, w + \alpha x \cdot D_x w + \alpha y \cdot D_y w - L_H |D_x w + D_y w| \\ \le C_f |x-y|^\gamma (\phi(x) + \phi(y)) \ \text{ for } x, y \in \mathbb{R}^N,\end{aligned}$$

by the maximum principle we get

$$\begin{aligned}\lambda g - \Xi\, g + \alpha x \cdot D_x g + \alpha y \cdot D_y g - L_H |D_x g + D_y g| \\ \le C_f |x-y|^\gamma (\phi(x) + \phi(y)) \ \text{ at } (x, y) = (\hat{x}, \hat{y}),\end{aligned}$$

which contradicts (4.8). This shows that $w \le g$ on $\mathbb{R}^{2N}$ and moreover that

$$v^\lambda(x) - v^\lambda(y) \le C_f |x-y|^\gamma \left(\phi(x) + \phi(y) + \frac{2B}{\alpha\gamma} \right) \quad \text{for all } x, y \in \mathbb{R}^N,$$

which guarantees (4.7), with $K = C_f(1 + 2B/(\alpha\gamma))$. □

We are ready to prove the following assertion.

Theorem 4.4. *Assume that* (3.3) *holds for some constant* $C_f > 0$. *Then there is a solution* $(c, v) \in \mathbb{R} \times C^2(\mathbb{R}^N)$ *of* (4.1) *such that*

$$v \in \bigcap_{\nu > \mu} \mathcal{E}_\nu(\mathbb{R}^N). \tag{4.9}$$

Proof. Let $v^\lambda \in C^2(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$, with $\lambda \in (0, 1)$, be the unique solution of (4.3). In view of Theorems 4.2 and 4.3 there is a constant $C > 0$ independent of $\lambda \in (0, 1)$ such that

$$\begin{aligned}\lambda |v^\lambda(0)| &\le C, \\ |v^\lambda(x) - v^\lambda(y)| &\le C|x-y|^\gamma (\phi(x) + \phi(y)) \ \text{ for } x, y \in \mathbb{R}^N.\end{aligned} \tag{4.10}$$

Define $w^\lambda,\ z^\lambda \in C^2(\mathbb{R}^N)$, with $\lambda \in (0, 1)$, by $w^\lambda(x) := v^\lambda(x) - v^\lambda(0)$ and $z^\lambda(x) := \lambda v^\lambda(x)$, respectively. Then we have for all $x, y \in \mathbb{R}^N$,

$$\begin{aligned}|z^\lambda(0)| &\le C, \\ |z^\lambda(x) - z^\lambda(0)| &\le \lambda C |x|^\gamma (\phi(x) + 1), && (4.11) \\ |w^\lambda(x)| &\le C|x|^\gamma (\phi(x) + 1), && (4.12) \\ |w^\lambda(x) - w^\lambda(y)| &\le C|x-y|^\gamma (\phi(x) + \phi(y)).\end{aligned}$$

Accordingly, $\{w^\lambda\}_{\lambda\in(0,1)}$ is a uniformly bounded and equi-continuous family on any balls of $\mathbb{R}^N$. Thus we can choose a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\,1)$ such that, as $j \to \infty$,

$$\lambda_j \to 0, \qquad z^{\lambda_j}(0) \to c,$$
$$w^{\lambda_j} \to v \quad \text{in } C(\mathbb{R}^N)$$

for some $c \in \mathbb{R}^N$ and $v \in C(\mathbb{R}^N)$. By (4.11) we have, as $j \to \infty$,

$$z^{\lambda_j}(x) \to c \quad \text{uniformly on balls of } \mathbb{R}^N.$$

By the stability of viscosity solutions (see, e.g., Lemma 6.1 of [CIL]), we find that v satisfies (4.1) in the viscosity sense. By using Schauder estimates (see the appendix or [LSU]), we deduce that $v \in C^2(\mathbb{R}^N)$. Let $\nu > \mu$. Since

$$\lim_{|x|\to\infty} \frac{|x|^\gamma \phi_\mu(x)}{\phi_\nu(x)} = 0,$$

we see from (4.12) that $v \in \mathcal{E}_\nu(\mathbb{R}^N)$. The proof is now complete. □

Theorem 4.5. *Let $(c, v),\ (d, w) \in \mathbb{R}\times C^2(\mathbb{R}^N)$ be solutions of ergodic problem (4.1) such that $v, w \in \mathcal{E}_\nu(\mathbb{R}^N)$ for some $\nu < \alpha/2$. Then $c = d$ and there is a constant $C \in \mathbb{R}$ such that $v - w = C$ in $\mathbb{R}^N$.*

As a consequence of Theorems 4.4 and 4.5, we have the following structure theorem on the solutions of ergodic problem (4.1).

Corollary 4.6. *Under the hypotheses of Theorem 4.4, let $(c, v) \in \mathbb{R}\times C^2(\mathbb{R}^N)$ be a solution of (4.1) such that $v \in \mathcal{E}_\nu(\mathbb{R}^N)$ for some constant $\nu < \alpha/2$. Then the set of solutions in $\mathcal{E}_\nu(\mathbb{R}^N)$ of (4.1) is given by*

$$\{(c, v + C) \in \mathbb{R}\times C^2(\mathbb{R}^N) \mid C \in \mathbb{R}\}.$$

Proof of Theorem 4.5. 1. To show that $c = d$, we argue by contradiction. Thus we suppose that $c \neq d$. We may assume that $c > d$. Define $u \in C^2(\mathbb{R}^N)$ by $u(x) = v(x) - w(x)$ and note that

$$-\Delta u + \alpha x\cdot Du - L_H|Du| \le d - c \quad \text{in } \mathbb{R}^N. \tag{4.13}$$

By adding a constant to u we may assume that $\sup_{\mathbb{R}^N} u > 0$.

Recall that there is a constant $B_\nu > 0$ such that

$$-\Delta\phi_\nu(x) + \alpha x\cdot D\phi_\nu(x) - L_H|D\phi_\nu(x)| \ge \phi_\nu(x) - B_\nu \ \text{ for } x \in \mathbb{R}^N. \tag{4.14}$$

Choose a constant $\epsilon > 0$ small enough so that $\epsilon B_\nu < c - d$ and set $U(x) = \epsilon\phi_\nu(x)$ for $x \in \mathbb{R}^N$. By the growth condition on v and w, we have

$$\lim_{|x|\to\infty} (u - U)(x) = -\infty.$$

Hence there is an open bounded set $\Omega \subset \mathbb{R}^N$ such that $u \leq U$ in $\mathbb{R}^N \setminus \Omega$. By (4.13) and (4.14), setting $\eta = u - U$, we have

$$-\Delta\eta(x) + \alpha x \cdot D\eta(x) - L_H|D\eta(x)| \leq d - c - \epsilon(\phi_\nu(x) - B_\nu) < 0 \quad \text{for } x \in \mathbb{R}^N.$$

By the maximum principle applied to η on the domain Ω, we have

$$\eta \leq 0 \quad \text{in } \Omega,$$

which shows that $u \leq U$ in $\mathbb{R}^N$. If we let $\epsilon \searrow 0$, we find that $u \leq 0$ in $\mathbb{R}^N$, which is a contradiction. Thus we see that $c = d$.

2. Next we show that for some constant $C \in \mathbb{R}$,

$$v - w = C \quad \text{in } \mathbb{R}^N. \tag{4.15}$$

In view of (4.14), there is a constant $R > 0$ such that

$$-\Delta\phi_\nu(x) + \alpha x \cdot D\phi_\nu(x) - L_H|D\phi_\nu(x)| > 0 \quad \text{for } x \in \mathbb{R}^N \setminus \operatorname{int} B(0, R).$$

We write $u = v - w$ as before and show that

$$\sup_{\mathbb{R}^N} u = \max_{B(0,R)} u. \tag{4.16}$$

For $\epsilon > 0$ we set $\eta(x) = u(x) - \epsilon\phi_\nu(x)$ and observe that

$$-\Delta\eta(x) + \alpha x \cdot D\eta(x) - L_H|D\eta(x)| < 0 \quad \text{for } x \in \mathbb{R}^N \setminus \operatorname{int} B(0R).$$

By the maximum principle, this shows that η cannot attain a maximum over $\mathbb{R}^N \setminus \operatorname{int} B(0, R)$ at any point in its interior. Noting that

$$\lim_{|x|\to\infty} \eta(x) = -\infty,$$

we see that (4.16) holds.

3. Now, to show (4.15), we utilize the strong maximum principle (see [PW] for a general reference). By (4.16) we have

$$\sup_{\mathbb{R}^N} u = \max_{B(0,r)} u \quad \text{for any } r > R.$$

We apply the strong maximum principle to u on $B(0, r)$, with arbitrary $r > R$, to conclude that u is a constant function on $B(0, r)$, with $r > R$, which clearly guarantees that $u(x) = u(0)$ for all $x \in \mathbb{R}^N$. This completes the proof. □

Remark 4.7. The above proof shows that if $v \in \mathcal{E}_\nu^+(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ and $w \in \mathcal{E}_\nu^-(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ are a subsolution and a supersolution of (4.1) with a common constant c, then the same conclusion as Theorem 4.5 holds, i.e., $v = w + C$ in $\mathbb{R}^N$ for some constant $C \in \mathbb{R}$. Moreover, since the strong maximum principle (see [BD]) still holds for viscosity subsolutions $u \in \mathrm{USC}(\mathbb{R}^N)$ of

$$-\Delta u + \alpha x \cdot Du - L_H|Du| \le 0$$

on any balls of $\mathbb{R}^N$, the above argument with minor modifications guarantees that if $v \in \mathcal{E}_\nu^+(\mathbb{R}^N) \cap \mathrm{USC}(\mathbb{R}^N)$ and $w \in \mathcal{E}_\nu^-(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ (or $v \in \mathcal{E}_\nu^+(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ and $w \in \mathcal{E}_\nu^-(\mathbb{R}^N) \cap \mathrm{LSC}(\mathbb{R}^N)$) are a viscosity (resp., classical) subsolution and a classical (resp., viscosity) supersolution of (4.1) with a common constant c, then $v - w = C$ for some constant $C \in \mathbb{R}$.

5 Long time behavior of solutions

In this section we study the long time behavior of solutions of (1.1). The main result in this section is stated as follows.

Theorem 5.1. *Let μ be a constant satisfying* (2.4). *Assume* (2.5), (2.8), *and that* (3.3) *and* (3.4) *hold for some constants $\gamma \in (0, 1]$, $C_f > 0$, and $C_0 > 0$. Let ν be a constant satisfying $\mu < \nu < \alpha/2$. Let $u \in \mathcal{E}_\mu(\overline{Q}) \cap C(\overline{Q}) \cap C^{2,1}(Q)$ be the unique solution of* (2.1) *and* (2.2) *and $(c, v) \in \mathbb{R} \times (C^2(\mathbb{R}^N) \cap \mathcal{E}_\nu(\mathbb{R}^N))$ a solution of* (4.1). *Then there is a constant $a \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \max_{B(0,R)} |u(x, t) - (ct + v(x) + a)| = 0 \quad \textit{for all } R > 0. \tag{5.1}$$

Remark 5.2. Of course, such solutions u and (c, v) as in the above theorem exist due to Theorems 2.2 and 4.4. Note that (2.9) follows from (3.3).

In this section we devote ourselves to proving Theorem 5.1, and we henceforth assume the hypotheses of the theorem.

As before, we write L_H for $\|DH\|_\infty$. Set

$$\begin{aligned} f_c(x) &= f(x) - c \quad \text{for } x \in \mathbb{R}^N, \\ u_c(x, t) &= u(x, t) - ct \quad \text{for } (x, t) \in \overline{Q}. \end{aligned}$$

Then u_c and (c, v) solve, respectively, the Cauchy problem (2.1) and (2.2) and the ergodic problem (4.1), with f_c in place of f. Assertion (5.1) now reads

$$\lim_{t\to\infty} \max_{B(0,R)} |u_c(x, t) - (v(x) + a)| = 0 \quad \text{for } R > 0.$$

Replacement of u and f by u_c and f_c, respectively, reduces the proof of Theorem 5.1 to the case of $c = 0$. Therefore we assume in the remainder of this section that $c = 0$.

In what follows we denote by η the function defined by

$$\eta(x,t) = u(x,t) - v(x) \qquad \text{on } \overline{Q}.$$

Since v satisfies

$$-\Delta v + \alpha x \cdot Dv + H(Dv) = f(x) \quad \text{in } \mathbb{R}^N, \tag{5.2}$$

we have

$$\eta_t - \Delta\eta + \alpha x \cdot D\varphi - L_H|D\eta| \leq 0 \quad \text{in } Q.$$

We write ϕ and ψ for ϕ_μ and ϕ_ν, respectively. Fix a constant $B > 1$ so that

$$-\Delta\psi(x) + \alpha x \cdot D\psi(x) - L_H|D\psi(x)| \geq \psi(x) - B \ \text{ for } x \in \mathbb{R}^N.$$

Let φ denote the function defined by $\varphi(x,t) = (\psi(x) - B)e^{-t}$ on $\overline{Q}$. It follows that

$$\varphi_t - \Delta\varphi + \alpha x \cdot D\varphi - L_H|D\varphi| \geq 0 \quad \text{in } Q.$$

We divide the proof of Theorem 5.1 into several lemmas.

Lemma 5.3. *There is a function $L : (0,1] \to (0,\infty)$ and for each $R > 0$ a modulus σ_R such that*

$$|u(x,t)| \leq \epsilon\phi_\nu(x) + L(\epsilon) \ \text{ for } (x,t) \in \overline{Q},\ \epsilon \in (0,1], \tag{5.3}$$

$$|u(x,t) - u(x,s)| \leq \sigma_R(|t-s|) \ \text{ for } x \in B(0,R),\ t,s \in [0,\infty),\ R > 0. \tag{5.4}$$

Proof. Once (5.3) has been shown, (5.4) is a consequence of Theorem 3.2. Thus we only need to prove (5.3).

Fix $\epsilon \in (0,1]$. Since $\eta \in \mathcal{E}_\nu(\overline{Q})$, we have

$$|\eta(x,0)| \leq \epsilon\psi(x) + M_\psi(\epsilon) \quad \text{for } x \in \mathbb{R}^N,$$
$$\lim_{|x|\to\infty} \sup_{0 \leq t \leq T} (\eta - \epsilon\varphi)(x,t) = -\infty \quad \text{for } T > 0$$

for some constant $M_\psi(\epsilon) > 0$. For each $T > 0$ we choose a constant $R > 0$, which depends on T and ϵ, so that

$$\eta \leq \epsilon\varphi \qquad \text{on } (\mathbb{R}^N \setminus B(0,R)) \times [0,T].$$

Note that

$$\epsilon\psi(x) + M_\psi(\epsilon) = \epsilon\varphi(x,0) + \epsilon B + M_\psi(\epsilon) \ \text{ for } x \in \mathbb{R}^N$$

and hence

$$\eta(x,t) \le \epsilon\varphi(x,t) + \epsilon B + M_\psi(\epsilon) \quad \text{for } (x,t) \in (\partial B(0,R) \times [0,T]) \cup (B(0,R) \times \{0\}).$$

Applying the maximum principle, we obtain

$$\eta(x,t) \le \epsilon\varphi(x,t) + \epsilon B + M_\psi(\epsilon) \quad \text{for } (x,t) \in B(0,R) \times [0,T].$$

Consequently we get

$$\eta(x,t) \le \epsilon\varphi(x,t) + \epsilon B + M_\psi(\epsilon) \quad \text{for } (x,t) \in \mathbb{R}^N \times [0,T],$$

and furthermore

$$\eta(x,t) \le \epsilon\varphi(x,t) + \epsilon B + M_\psi(\epsilon) \quad \text{for } (x,t) \in \overline{Q}.$$

Similarly, we get

$$\eta(x,t) \ge -\epsilon\varphi(x,t) - \epsilon B - M_\psi(\epsilon) \quad \text{for } (x,t) \in \overline{Q}.$$

Thus we have

$$|\eta(x,t)| \le \epsilon\psi(x) + \epsilon B + M_\psi(\epsilon) \quad \text{for } (x,t) \in \overline{Q},$$

which was to be shown. □

Lemma 5.4. *The sets* $\{u(\cdot,t) \mid t \ge 0\}$ *and* $\{u(\cdot,\cdot+t) \mid t \ge 0\}$ *are precompact in* $C(\mathbb{R}^N)$ *and* $C(\overline{Q})$*, respectively.*

Proof. By Theorem 3.1 and (5.3), the collection $\{u(\cdot,t) \mid t \ge 0\}$ is uniformly bounded and equi-continuous on bounded subsets of $\mathbb{R}^N$. Similarly, by Theorem 3.1, (5.3), and (5.4), the collection $\{u(\cdot,\cdot+t) \mid t \ge 0\}$ is uniformly bounded and equi-continuous on bounded subsets of $\overline{Q}$. Therefore, the Ascoli-Arzela theorem guarantees the needed precompactness. □

We define the functions $v^+, v^- : \mathbb{R}^N \to \mathbb{R}$ by

$$\begin{aligned} v^+(x) &= \limsup_{t\to\infty} u(x,t), \\ v^-(x) &= \liminf_{t\to\infty} u(x,t). \end{aligned}$$

Let $C_1 > 0$ be the constant from (3.5) and L a function from Lemma 5.3. It follows from (3.5) and (5.3) that for $g = v^+$ and v^-,

$$|g(x) - g(y)| \le C_1|x-y|^\gamma(\phi(x) + \phi(y)) \quad \text{for } x, y \in \mathbb{R}^N, \tag{5.5}$$

$$|g(x)| \le \epsilon\psi(x) + L(\epsilon) \quad \text{for } x \in \mathbb{R}^N \text{ and } \epsilon \in (0,1]. \tag{5.6}$$

In particular, we have $v^\pm \in C^\gamma(\mathbb{R}^N)$.

Lemma 5.5. *The functions v^+ and v^- are solutions in $C^2(\mathbb{R}^N)$ of* (5.2).

Proof. We only give the proof for v^+ since the proof for v^- is similar.

For $T > 0$ we define $u^T \in C(\overline{Q})$ by $u^T(x,t) = u(x, t+T)$. Note that the functions u^T are classical solutions of (2.1). Next, for $T > 0$ we define $v^T : \overline{Q} \to \mathbb{R}$ by

$$v^T(x,t) = \sup\{u^S(x,t) \mid S \geq T\}.$$

Then (5.3) and (5.4), with v^T in place of u, hold for all $T > 0$, which shows that $v^T \in C(\overline{Q})$ for all $T > 0$, and the functions v^T are viscosity subsolutions of (2.1). The convergence

$$v^T(x,t) \to v^+(x),$$

as $T \to \infty$, is monotone for all $(x,t) \in \overline{Q}$ and hence, by the Dini lemma, it is uniform on bounded subsets of $\overline{Q}$. Regarding v^+ as a function on $\overline{Q}$ which is independent of the t-variable, we see by the stability of viscosity subsolutions in the local uniform convergence that v^+ is a viscosity subsolution of (2.1). Since v^+ is independent of the t-variable, v^+ is a viscosity subsolution of (5.2).

Finally let $v \in C^2(\mathbb{R}^N) \cap \mathcal{E}_\nu(\mathbb{R}^N)$ be a solution of (5.2). As we have already remarked in Remark 4.7, $v^+(x) = v(x) + C$ for all $x \in \mathbb{R}^N$ and some constant $C \in \mathbb{R}$. Hence, v^+ is in $C^2(\mathbb{R}^N)$ and satisfies (5.2). □

We introduce the *ω-limit set* $\Omega(u)$ of u as the set of those functions $w \in C(\overline{Q})$ for which there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ such that, as $j \to \infty$, $t_j \to \infty$ and $u(x, t+t_j) \to w(x,t)$ uniformly on bounded subsets of $\overline{Q}$. By the stability of viscosity solutions of (2.1) in $C(Q)$, we see that any $w \in \Omega(u)$ is a viscosity solution of (2.1), which is a classical solution of (2.1) as well. By definition, it is obvious that

$$v^-(x) \leq w(x,t) \leq v^+(x) \quad \text{for } (x,t) \in \overline{Q} \text{ and } w \in \Omega(u).$$

Lemma 5.6. *There is a function $w^- \in \Omega(u)$ such that $w^-(0,1) = v^-(0)$.*

Proof. By the definition of $v^-(0)$, there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset (1,\infty)$ such that $u(0,t_j) \to v^-(0)$ as $j \to \infty$. In view of Lemma 5.4, there are a subsequence $\{s_j\}_{j\in\mathbb{N}} \subset \{t_j\}_{j\in\mathbb{N}}$ and a function $w^- \in C(\mathbb{R}^N)$ such that as $j \to \infty$,

$$u(\cdot, \cdot + s_j - 1) \to w^- \quad \text{in } C(\overline{Q}).$$

It is clear that $w^- \in \Omega(u)$ and $v^-(0) = w^-(0,1)$. □

Now, we are ready to prove Theorem 5.1

Proof of Theorem 5.1. 1. By Lemma 5.5, the functions v^+ and v^- are a viscosity subsolution and a viscosity supersolution of (5.2), respectively. Hence, Remark

4.7 ensures that there are constants $a, b \in \mathbb{R}$ such that $v^+(x) = v(x) + a$ and $v^-(x) = v(x) + b$ for all $x \in \mathbb{R}^N$. Since $v^+(x) \geq v^-(x)$ for all $x \in \mathbb{R}^N$, we have $a \geq b$. If $a = b$, then we find by the definition of $v^\pm$ and by the precompactness of $\{u(\cdot, t) \mid t \geq 0\}$ in $C(\mathbb{R}^N)$ (due to Lemma 5.4) that $u(\cdot, t) \to v + a$ in $C(\mathbb{R}^N)$ as $t \to \infty$.

2. Thus we only need to show that $a = b$.

Now, we prove that there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ such that

$$u(\cdot, t_j) \to v^- \quad \text{in } C(\mathbb{R}^N) \quad \text{as } j \to \infty. \tag{5.7}$$

By Lemma 5.6, there is a function $w^- \in \Omega(u)$ such that $w^-(0, 1) = v^-(0)$. Since $w^-(x, t) \geq v^-(x)$ for all $(x, t) \in \overline{Q}$, the function $\zeta \in C(\overline{Q}) \cap C^{2,1}(Q)$ defined by $\zeta(x, t) = v^-(x) - w^-(x, t)$ attains a maximum at the point $(0,\ 1) \in Q$. This function ζ satisfies

$$\zeta_t - \Delta\zeta + \alpha x \cdot D\zeta - L_H|D\zeta| \leq 0 \quad \text{in } Q.$$

By applying the strong maximum principle to ζ, we find that

$$\zeta(x, t) = \zeta(0, 1) = 0 \quad \text{for } (x, t) \in \mathbb{R}^N \times [0, 1).$$

That is, $w^-(x, t) = v^-(x)$ for all $(x, t) \in \mathbb{R}^N \times [0, 1)$. By the definition of $\Omega(u)$, there is a sequence $\{t_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ such that as $j \to \infty$, $t_j \to \infty$ and

$$u(\cdot, t_j) \to w^-(\cdot, 0) = v^- \quad \text{in } C(\mathbb{R}^N),$$

which shows (5.7).

3. For $T > 0$ we define the function $\eta^T \in C^{2,1}(\overline{Q})$ by $\eta^T(x, t) = u(x, t + T) - v(x) - b$. Note that for any $T > 0$ the function η^T satisfies

$$\eta^T_t - \Delta\eta^T + \alpha x \cdot D\eta^T - L_H|D\eta^T| \leq 0 \quad \text{in } Q.$$

Fix any $\epsilon > 0$ and note that

$$\epsilon\varphi(x, 0) + \epsilon B \geq 0 \ \text{ for } x \in \mathbb{R}^N.$$

Recalling (5.3), we may choose a constant $C \equiv C(\epsilon) > 0$ such that

$$|u(x, t)| \leq \frac{\epsilon}{2}\psi(x) + C \quad \text{for } (x, t) \in \overline{Q}.$$

We may choose a constant $R \equiv R(\epsilon) > 0$ such that

$$\frac{\epsilon}{2}\psi(x) + C - v(x) - b \leq \epsilon\varphi(x, 0) \quad \text{for } x \in \mathbb{R}^N \setminus B(0, R).$$

Then we get

$$\eta^T(x,0) \le \epsilon\varphi(x,0) \quad \text{for } x \in \mathbb{R}^N \setminus B(0,R),\ T > 0.$$

According to (5.7), we may fix a constant $T \equiv T(\epsilon) > 0$ so that

$$\eta^T(x,0) \le \epsilon \quad \text{for } x \in B(0,R).$$

We now have

$$\eta^T(x,0) \le \epsilon\varphi(x,0) + \epsilon B + \epsilon \quad \text{for } x \in \mathbb{R}^N.$$

By (5.3), we have

$$\lim_{|x|\to\infty} \sup_{0\le t\le S} (\eta^T(x,t) - \epsilon\varphi(x,t) - \epsilon(1+B)) = -\infty \quad \text{for } S > 0.$$

We apply the comparison principle to η^T and $\epsilon\varphi + \epsilon(1+B)$ as usual, to conclude that

$$\eta^T(x,t) \le \epsilon\varphi(x,t) + \epsilon(1+B) \quad \text{for } (x,t) \in \overline{Q}.$$

In particular, we have

$$u(0,t+T) \le v(0) + b + \epsilon\varphi(0,t) + \epsilon(1+B) \le v(0) + b + \epsilon(1+B) \quad \text{for } t \ge 0.$$

Here we used the fact that $\varphi(0,t) < 0$. Sending $t \to \infty$ along a sequence yields

$$v^+(0) \le v(0) + b + \epsilon(1+B),$$

which guarantees that $a \le b$ as $\epsilon > 0$ is arbitrary. This completes the proof. □

6 Locally Lipschitz Hamiltonian H

In this section, under the weaker assumption that H is locally Lipschitz continuous on $\mathbb{R}^N$ and the stronger assumption that f, $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$, we establish the comparison, existence, regularity results for the Cauchy problem (2.1) and (2.2) and ergodic problem (4.1), as well as and the long time behavior of solutions for the Cauchy problem. The proof of these results will be based on the results obtained in the previous sections.

We assume throughout this section that

(6.1) $$H \in C^{0+1}(\mathbb{R}^N),$$

(6.2) $$f \in \mathrm{Lip}(\mathbb{R}^N),$$

(6.3) $$u_0 \in \mathrm{Lip}(\mathbb{R}^N).$$

We set

$$M = \max\Big\{ \|Du_0\|_\infty,\ \frac{\|Df\|_\infty}{\alpha} \Big\}. \tag{6.4}$$

In this section we will frequently consider PDE

$$u_t - \Delta u + \alpha x \cdot Du + H_M(Du) = f(x) \quad \text{in } Q, \tag{6.5}$$

where H_M is a fixed function $H_M \in \mathrm{Lip}(\mathbb{R}^N)$ having the property:

$$H_M(p) = H(p) \quad \text{for } p \in B(0, M). \tag{6.6}$$

Theorem 6.1. *Problem* (2.1) and (2.2) *has a unique solution* $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ *such that*

$$\sup_{R_T} \frac{|u(x,t)|}{|x|+1} < \infty \quad \textit{for } T > 0. \tag{6.7}$$

Moreover,

$$\begin{aligned} |Du(x,t)| &\le M \quad \textit{for } (x,t) \in Q, && (6.8)\\ |u(x,t) - u(x,s)| &\le \omega_R(|t-s|) \quad \textit{for } x \in B(0,R),\ t,s \in [0,\infty), && (6.9) \end{aligned}$$

for all $R > 0$*, where* ω_R *is a modulus and* M *is the constant given by* (6.4)*.*

Proof. 1. We show first the existence of a solution $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ of (2.1) and (2.2) which satisfies (6.8). In view of Theorem 2.2, there is a solution $u \in C(\overline{Q}) \cap C^{2,1}(Q) \cap \mathcal{E}_\mu(\overline{Q})$ of (6.5) and (2.2). If u satisfies (6.8), then, because of (6.6), u satisfies (2.1) as well. Thus it is enough to show that u satisfies (6.8). For this we follow the proof of Theorem 3.1 with minor modifications.

2. Fix $0 < T < \infty$. We define $w \in C(S_T) \cap C^{2,1}(P_T)$, where $S_T = \mathbb{R}^{2N} \times [0,T)$ and $P_T = \mathbb{R}^{2N} \times (0,T)$, by $w(x,y,t) = u(x,t) - u(y,t)$, and observe that

$$\begin{aligned} &w_t - \Xi\, w + \alpha x \cdot D_x w + \alpha y \cdot D_y w - L|D_x w + D_y w| \\ &\qquad \le f(x) - f(y) \le \alpha M|x-y| \quad \text{in } P_T, \end{aligned} \tag{6.10}$$

where $L := \|DH_M\|_\infty$ and $\Xi : C^2(\mathbb{R}^N) \to C(\mathbb{R}^N)$ is the operator defined by

$$\Xi\, g(x,y) = \Delta_x g(x,y) + \Delta_y g(x,y) + 2\sum_{i=1}^N \frac{\partial^2 g}{\partial x_i \partial y_i}(x,y).$$

For $B > 1$ large enough, the function $\varphi(x,t) := (\phi_\mu(x) - B)e^{-t} + B$ on $\overline{Q}$ satisfies

$$\varphi_t - \Delta\varphi + \alpha x \cdot D\varphi - L|D\varphi| \ge 0 \quad \text{in } Q, \tag{6.11}$$

as we saw before, and $\varphi \geq 0$ on $\overline{Q}$. Fix any $\epsilon > 0$ and set

$$\eta(x, y, t) = M|x - y| + \epsilon\Big(\varphi(x, t) + \varphi(y, t) + \frac{1}{T - t}\Big) \quad \text{for } (x, y, t) \in S_T.$$

Using (6.11), we then calculate that for $(x, y, t) \in Q_T$, if $x \neq y$, then

$$\begin{aligned} \eta_t - \Xi\,\eta + \alpha x \cdot D_x\eta + \alpha y \cdot D_y\eta - L|D_x\eta + D_y\eta| \\ \geq \frac{\epsilon}{T^2} + \alpha M|x - y| > \alpha M|x - y|. \end{aligned} \tag{6.12}$$

By our choice of M, we have

$$w(x, y, 0) \leq M|x - y| \leq \eta(x, y, 0) \quad \text{for } (x, y) \in \mathbb{R}^{2N}.$$

Since $u \in \mathcal{E}_\mu(\overline{Q})$, we have

$$\lim_{R\to\infty} \sup\{w(x, y, t) - \eta(x, y, t) \mid (x, y, t) \in S_T,\ |x| + |y| \geq R\} = -\infty.$$

In view of (6.10) and (6.12), we may apply the maximum principle to $w - \eta$ on $B(0, R) \times B(0, R) \times [0, T)$, with $R > 0$ sufficiently large, to conclude that

$$w(x, y, t) \leq \eta(x, y, t) \quad \text{for } (x, y, t) \in S_T,$$

from which, since $\epsilon > 0$ and $T > 0$ are arbitrary, we obtain

$$u(x, t) - u(y, t) \leq M|x - y| \quad \text{for } (x, y, t) \in \mathbb{R}^{2N} \times [0, \infty).$$

This immediately yields (6.8).

3. Now we turn to the uniqueness assertion. Let $v \in C^{2,1}(Q) \cap C(\overline{Q})$ be another solution of (2.1) and (2.2) satisfying (6.7), and we show that $v = u$ on $\overline{Q}$. It is enough to show that $u = v$ on $\overline{Q}_T$ for all $0 < T < \infty$.

Fix any $0 < T < \infty$. We only show that $u \leq v$ on $\overline{Q}_T$ since the proof of the inequality $u \geq v$ on $\overline{Q}_T$ is similar. Let $A > 0$ be a constant to be fixed later. Fix any $0 < \epsilon < 1$ and set

$$w(x, t) = u(x, t) - v(x, t) - \epsilon(|x|^2 + At) \quad \text{for } (x, t) \in \overline{Q}_T.$$

In view of the growth condition (6.7) on u and v, we see that this function w attains a maximum over $\overline{Q}_T$. Fix a maximum point $(\bar{x}, \bar{t}) \in Q_T$ of w and observe that

$$0 = w(0, 0) \leq w(\bar{x}, \bar{t}) \leq C_1(|\bar{x}| + 1) - \epsilon|\bar{x}|^2 \leq -\frac{\epsilon}{2}|\bar{x}|^2 + \frac{C_1^2}{2\epsilon} + C_1,$$

where $C_1 > 0$ is a constant such that

$$|u(x,t)| + |v(x,t)| \le C_1(|x|+1) \quad \text{for } (x,t) \in \overline{Q}_T.$$

Consequently, we have

$$\epsilon^2|\bar{x}|^2 \le C_1(2\epsilon + C_1) \le 2C_1 + C_1^2. \tag{6.13}$$

We want to prove that

$$\max_{\overline{Q}_T} w = 0. \tag{6.14}$$

For this we need to show that $\bar{t} = 0$. Indeed, if $\bar{t} = 0$, then

$$0 \le w(0,0) \le \max_{\overline{Q}_T} w = w(\bar{x},0) \le u(\bar{x},0) - v(\bar{x},0) \le 0.$$

We argue by contradiction and thus suppose that $\bar{t} > 0$. By the maximum principle, we have

$$Dw(\bar{x},\bar{t}) = 0, \quad w_t(\bar{x},\bar{t}) \ge 0, \quad -\Delta w(\bar{x},\bar{t}) \ge 0. \tag{6.15}$$

In particular, we have

$$|Dv(\bar{x},\bar{t})| = |Du(\bar{x},\bar{t}) - 2\epsilon\bar{x}| \le M + R,$$

where $R := 2\sqrt{2C_1 + C_1^2}$. Here we have used that $|Du(x,t)| \le M$ for all $(x,t) \in Q$ by (6.8).

Now, setting $L_1 := \|DH\|_{L^\infty(B(0,M+R))}$, we compute that, at $(\bar{x},\bar{t})$,

$$\begin{aligned} 0 &= (u-v)_t - \Delta(u-v) + \alpha x \cdot D(u-v) + H(Du) - H(Dv) \\ &\ge w_t - \Delta w + \alpha x \cdot Dw - L_1|Du - Dv| + \epsilon(A - 2N + 2\alpha|x|^2) \\ &\ge w_t - \Delta w + \alpha x \cdot Dw - L_1|Dw| + \epsilon(A - 2N + 2\alpha|x|^2 - 2L_1|x|). \end{aligned}$$

We fix

$$A = 2\max_{r\in\mathbb{R}}(-\alpha r^2 + L_1 r + N) + 1 \equiv \frac{L_1^2}{2\alpha} + 2N + 1.$$

Then we have

$$w_t - \Delta w + \alpha x \cdot Dw - L_1|Dw| < 0 \quad \text{at } (\bar{x},\bar{t}).$$

On the other hand, by (6.15) we must have

$$w_t - \Delta w + \alpha x \cdot Dw - L_1|Dw| \ge 0 \quad \text{at } (\bar{x},\bar{t}).$$

These two inequalities are contradictory, which shows that $\bar{t} = 0$ and hence (6.14) is valid. Sending $\epsilon \to 0$ yields

$$u(x,t) \leq v(x,t) \quad \text{for } (x,t) \in \overline{Q}_T.$$

As noted before, we may conclude the desired uniqueness.

4. Since u satisfies (6.5), estimate (6.9) is a direct consequence of Theorem 3.2. □

Remark 6.2. In order to get (6.8), one may use the standard Bernstein method as well.

The uniqueness assertion in the above theorem can be extended to the following comparison theorem for viscosity sub- and supersolutions of (2.1).

Theorem 6.3. *Let $v \in \mathrm{USC}(Q)$ and $w \in \mathrm{LSC}(Q)$ be a viscosity subsolution and a viscosity supersolution of* (2.1)*, respectively. Assume that for each $T > 0$ there is a constant $C_T > 0$ such that*

$$v(x,t) \vee (-w(x,t)) \leq C_T(1 + |x|) \quad \textit{for } (x,t) \in Q_T, \tag{6.16}$$

and that $v(x,0) \leq u_0(x) \leq w(x,0)$ for all $x \in \mathbb{R}^N$. Then $u \leq v$ in $\overline{Q}$.

For the proof, we can easily adapt the argument for the uniqueness in Theorem 6.1 to show that $v \leq u \leq w$, where u is the unique solution of (2.1) and (2.2) satisfying (6.8). We omit here the details of the proof of Theorem 6.3.

Next, we discuss the ergodic problem (4.1).

Theorem 6.4. (a) *There exists a solution $(c, v) \in \mathbb{R} \times C^2(\mathbb{R}^N)$ of* (4.1) *such that*

$$\|Dv\|_\infty \leq \frac{\|f\|_\infty}{\alpha}. \tag{6.17}$$

(b) *Let $(c,v), (d,w) \in \mathbb{R} \times (C^2(\mathbb{R}^N) \cap \mathrm{Lip}(\mathbb{R}^N))$ be solutions of* (4.1)*. Then we have*

$$c = d \quad \textit{and} \quad v - w = C \quad \textit{on } \mathbb{R}^N$$

for some constant $C \in \mathbb{R}$.

Proof. 1. Let $R := \max\{\|Dv\|_\infty, \|Dw\|_\infty\}$. Assertion (b) is an immediate consequence of Theorem 4.5, since (c, v) and (d, w) are solutions of (4.1), with H replaced by a function H_R having the properties

$$H_R \in \mathrm{Lip}(\mathbb{R}^N) \quad \text{and} \quad H_R(p) = H(p) \ \text{ for } p \in B(0, R).$$

2. Due to Theorem 6.1, there is a (unique) solution $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ of (2.1) satisfying

$$\text{(6.18)} \quad u(x, 0) = 0 \ \text{ for } x \in \mathbb{R}^N \quad \text{and} \quad |Du(x, t)| \le \frac{\|Df\|_\infty}{\alpha} \ \text{ for } (x, t) \in Q.$$

Let $L = \frac{\|Df\|_\infty}{\alpha}$ and choose a function $H_L \in \mathrm{Lip}(\mathbb{R}^N)$ so that $H_L(p) = H(p)$ for all $p \in B(0, L)$. By Theorem 4.4, there is a solution $(c, v) \in \mathbb{R} \times C^2(\mathbb{R}^N)$ of (4.1), with H_L in place of H, such that $v \in \mathcal{E}_\nu(\mathbb{R}^N)$ for some $\nu \in (0, \alpha/2)$. By Theorem 5.1 there is a constant $a \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \max_{B(0,R)} |v(x) - (u(x, t) - ct - a)| = 0 \quad \text{for } R > 0.$$

This together with (6.18) yields

$$|v(x) - v(y)| = \lim_{t\to\infty} |(u(x, t) - ct - a) - (u(y, t) - ct - a)| \le L|x - y| \quad \text{for } x, y \in \mathbb{R}^N.$$

This shows that $\|Dv\|_\infty \le L$ and hence that (c, v) is a solution of (4.1). □

Finally we state the long time behavior of solutions of (2.1).

Theorem 6.5. *Let $u \in C(\overline{Q}) \cap C^{2,1}(\overline{Q})$ be the unique solution of* (2.1) *and* (2.2) *satisfying growth condition* (6.7)*. Let $(c, v) \in \mathbb{R} \times (C^2(\mathbb{R}^N) \cap \mathrm{Lip}(\mathbb{R}^N))$ be a solution of* (4.1)*. Then there is a constant $a \in \mathbb{R}$ such that*

$$\text{(6.19)} \qquad \lim_{t\to\infty} \max_{B(0,R)} |u(x, t) - (ct + v(x) + a)| = 0 \quad \textit{for all } R > 0.$$

Proof. Noting that u satisfies

$$|Du(x, t)| \le M \quad \text{for } (x, t) \in Q$$

by (6.8) and hence satisfies (6.5) and that (c, v) solves (4.1), with H_M in place of H, we conclude immediately from Theorem 5.1 that (6.19) is valid for some $a \in \mathbb{R}$. □

Appendix

Here we discuss briefly the solvability of the initial-boundary value problem

$$u_t - \Delta u + \alpha x \cdot Du + H(Du) = f(x,t) \quad \text{in } Q_T, \tag{A.1}$$

$$u = \psi \quad \text{on } \partial_p Q_T \tag{A.2}$$

in $C(\overline{Q}_T) \cap C^{2,1}(Q_T)$, where $Q_T := U \times (0,T)$, U is an open ball in $\mathbb{R}^N$, $0 < T < \infty$, $H \in \mathrm{Lip}(\mathbb{R}^N)$, $f \in C(\overline{Q}_T)$, and $\psi \in C(\overline{Q}_T)$ are given functions, and $\partial_p Q_T := (\partial U \times [0,T]) \cup (\overline{U} \times \{0\})$.

Our arguments will be based on Schauder theory for parabolic PDE presented in Chapter IV of [LSU].

Let $0 < \gamma < 1$. We first assume that

(A.3) $f \in C^{\gamma,\gamma/2}(\overline{Q}_T)$,

(A.4) $\psi \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$,

(A.5) $\psi_t(x,0) - \Delta\psi(x,0) + \alpha x \cdot D\psi(x,0) + H(D\psi(x,0)) = f(x,0) \quad \text{for } x \in \partial U.$

Theorem A.1. *Under the assumptions* (A.3)–(A.5) *there is a unique solution* $u \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$ *of* (A.1) *and* (A.2).

Outline of proof. We show that there is a constant $\tau > 0$, independent of T, f, and ψ, such that there exists a solution $u \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)$ of (A.1) and (A.2) in Q_S, with $S = \min\{\tau, T\}$, from which we conclude the proof by an induction argument.

Let $\tau \in (0, 1]$ be a constant to be fixed later on and set $S := \min\{\tau, T\}$.

Let X be the closed subset of $C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)$ consisting of all functions v satisfying $v = \psi$ on $\partial_p Q_S$. Fix any $v \in X$ and set $g(x,t) = f(x,t) - H(Dv(x,t))$ for $(x,t) \in \overline{Q}_S$. It is clear that $g \in C^{\gamma,\gamma/2}(\overline{Q}_S)$ and

$$\psi_t(x,0) - \Delta\psi(x,0) + \alpha x \cdot D\psi(x,0) = g(x,0) \quad \text{for } x \in \partial U.$$

This last equality is the first compatibility condition for the initial-boudary value problem

$$u_t - \Delta u + \alpha x \cdot Du = g(x,t) \quad \text{in } Q_S, \tag{A.6}$$

$$u = \psi \quad \text{on } \partial_p Q_S. \tag{A.7}$$

By virtue of Theorem 5.2 of Chapter IV in [LSU], there is a unique solution $u \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)$ of (A.6) and (A.7). Moreover there is a constant $C > 0$ independent of ψ and g such that

$$\|u\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} \le C(\|\psi\|_{C^{2+\gamma,1+\gamma/2}(\partial_p Q_S)} + \|g\|_{C^{\gamma,\gamma/2}(\overline{Q}_S)}). \tag{A.8}$$

Thus we may define the mapping $F : X \to X$ by setting $F(v) := u$.

Fix any $v_1, v_2 \in X$ and set $w := F(v_1) - F(v_2)$ and $g_i(x,t) := f(x,t) - H(Dv_i(x,t))$ for $(x,t) \in \overline{Q}_S$ and $i = 1, 2$. Noting that w satisfies (A.6) and (A.7), with $g = g_1 - g_2$ and $\psi = 0$, we get from (A.8)

$$\text{(A.9)} \qquad \|F(v_1) - F(v_2)\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} \leq C\|H(Dv_1) - H(Dv_2)\|_{C^{\gamma,\gamma/2}(\overline{Q}_S)}.$$

Since $v(x,t) := t\|g_1 - g_2\|_{L^\infty(Q_S)}$ and $-v$ are a supersolution and a subsolution of (A.6) and (A.7), with $g = g_1 - g_2$ and $\psi = 0$, we find by the maximum principle that

$$\text{(A.10)} \; \|F(v_1) - F(v_2)\|_{L^\infty(Q_S)} \leq S\|g_1 - g_2\|_{L^\infty(Q_S)} \leq SL\|D(v_1 - v_2)\|_{L^\infty(Q_S)},$$

where $L > 0$ is a positive constant such that $\|DH\|_\infty \leq L$. From (A.9), we get

$$\begin{aligned} \langle F(v_1) - F(v_2)\rangle_{Q_S}^{(2+\gamma)} &\leq C(\langle g_1 - g_2\rangle_{Q_S}^{(\gamma)} + \|g_1 - g_2\|_{L^\infty(Q_S)}) \\ &\leq CL(\langle D(v_1 - v_2)\rangle_{Q_S}^{(\gamma)} + \|D(v_1 - v_2)\|_{L^\infty(Q_S)}), \end{aligned}$$

where, for $\beta \in (0,1)$ and functions v on $\overline{Q}_S$,

$$\begin{aligned} \langle v\rangle_{Q_S}^{(2+\beta)} &= \langle D^2 v\rangle_{x,Q_S}^{(\beta)} + \langle Dv\rangle_{t,Q_S}^{(\frac{1+\beta}{2})} + \langle v_t\rangle_{Q_S}^{(\frac{\beta}{2})}, \\ \langle v\rangle_{Q_S}^{(\beta)} &= \langle v\rangle_{x,Q_S}^{(\gamma)} + \langle v\rangle_{t,Q_S}^{(\frac{\beta}{2})}, \\ \langle v\rangle_{x,Q_S}^{(\beta)} &= \sup\{\frac{|v(x,t) - v(y,t)|}{|x-y|^\beta} \mid (x,t), (y,t) \in \overline{Q}_S,\ x \neq y\}, \\ \langle v\rangle_{t,Q_S}^{(\beta)} &= \sup\{\frac{|v(x,t) - v(x,s)|}{|t-s|^\beta} \mid (x,t), (x,s) \in \overline{Q}_S,\ t \neq s\}. \end{aligned}$$

Hence, by using the interpolation lemma (see Lemma 3.2 of Chapter II in [LSU]), for any $w \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)$, we have

$$\begin{aligned} \langle Dw\rangle_{Q_S}^{(\gamma)} &\leq C_0(S^{\frac{1}{2}}\langle w\rangle_{Q_S}^{(2+\gamma)} + S^{-\frac{1+\gamma}{2}}\|w\|_{L^\infty(Q_S)}), \\ \|Dw\|_{L^\infty(Q_S)} &\leq C_0(S^{\frac{1+\gamma}{2}}\langle w\rangle_{Q_S}^{(2+\gamma)} + S^{-\frac{1}{2}}\|w\|_{L^\infty(Q_S)}). \end{aligned}$$

for some constant $C_0 > 0$ independent of S. These, (A.9), and (A.10) together yield

$$\text{(A.11)} \qquad \begin{aligned} \langle F(v_1) &- F(v_2)\rangle_{Q_S}^{(2+\gamma)} + S^{-1}\|F(v_1) - F(v_2)\|_{L^\infty(Q_S)} \\ &\leq LC_0\left(CS^{\frac{1}{2}} + (C+L)S^{\frac{1+\gamma}{2}}\right)\langle v_1 - v_2\rangle_{Q_S}^{(2+\gamma)} \\ &+ LC_0\left(CS^{\frac{1-\gamma}{2}} + (C+L)S^{\frac{1}{2}}\right)S^{-1}\|v_1 - v_2\|_{L^\infty(Q_S)}. \end{aligned}$$

Now, we fix $\tau \in (0, 1]$ so that

$$LC_0 \left(C\tau^{\frac{1}{2}} + (C+L)\tau^{\frac{1}{2}} \right) \leq \frac{1}{2}.$$

Since $S \leq \tau \leq 1$, from (A.11) we get

$$\text{(A.12)} \quad \begin{aligned} \langle F(v_1) - F(v_2) \rangle_{Q_S}^{(2+\gamma)} &+ S^{-1} \|F(v_1) - F(v_2)\|_{L^\infty(Q_S)} \\ &\leq \frac{1}{2} \left(\langle v_1 - v_2 \rangle_{Q_S}^{(2+\gamma)} + S^{-1} \|v_1 - v_2\|_{L^\infty(Q_S)} \right). \end{aligned}$$

This shows that F is a contraction mapping on X, equipped with the metric

$$\rho(v_1, v_2) = \langle v_1 - v_2 \rangle_{Q_S}^{(2+\gamma)} + S^{-1} \|v_1 - v_2\|_{L^\infty(Q_S)}.$$

By the contraction mapping theorem, there is a unique fixed point $u \in X$ of F. It is easily seen that u is a unique solution of (A.1) and (A.2) in Q_S. □

Remark A.2. We have the following estimate in the above theorem: if u is the solution of (A.1) and (A.2), then

$$\text{(A.13)} \qquad \|u\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)} \leq C(\|\psi\|_{C^{2+\gamma,1+\gamma/2}(\partial_p Q_T)} + \|f\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)})$$

for some constant $C > 0$, independent of ψ and f. We first note that for any $\varphi \in C^{2+\gamma,1+\gamma/2}(\partial_p Q_T)$ there is an extension of φ to $\overline{Q}_T$, denoted again by φ, which satisfies

$$\text{(A.14)} \qquad \|\varphi\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)} \leq C_0 \|\varphi\|_{C^{2;\gamma,1+\gamma/2}(\partial_p Q_T)},$$

where $C_0 > 0$ is a constant independent of φ. To prove (13), we may assume that (A.14) is satisfied with ψ in place of φ. Going back to the proof of Theorem A.1, the solution u of (A.1) and (A.2) in Q_S can be obtained as the limit of the sequence $\{u_n\}$ in $C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)$ defined by $u_1 = \psi$ and $u_{n+1} = F(u_n)$ for $n \in \mathbb{N}$. Then, using the interpolation inequalities, (A.8), and (A.12), we get

$$\begin{aligned} \|u_n\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} &\leq \sum_{k=1}^{n-1} \|u_{k+1} - u_k\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} + \|u_1\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} \\ &\leq C_1 \sum_{k=1}^{n-1} \rho(u_{k+1}, u_k) + \|u_1\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} \\ &\leq 2C_1 \rho(u_2, u_1) + \|u_1\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} \\ &\leq C_2 \|F(\psi) - \psi\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} + \|\psi\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} \\ &\leq C_2 \|F(\psi)\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} + (C_2 + 1)\|\psi\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} \\ &\leq CC_2(\|\psi\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} + \|f - H(D\psi)\|_{C^{\gamma,\gamma/2}(\overline{Q}_S)}) \\ &\quad +(C_2+1)\|\psi\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} \\ &\leq C_3(\|\psi\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_S)} + \|f\|_{C^{\gamma,\gamma/2}(\overline{Q}_S)}) \end{aligned}$$

for some positive constants C_1, C_2, and C_3. This guarantees (A.13).

Next, we just assume that

$$f \in C^{\gamma,\gamma/2}(\overline{Q}_T), \tag{A.15}$$

$$\psi \in C(\overline{Q}_T). \tag{A.16}$$

Theorem A.3. *Under the assumptions* (A.15) *and* (A.16) *there is a unique solution* $u \in C(\overline{Q}_T) \cap C^{2+\gamma,1+\gamma/2}(Q_T)$ *of* (A.1) *and* (A.2).

Outline of proof. We begin by noting that there is a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset C^\infty(\overline{Q}_T)$ such that

$$\psi_n \to \psi \quad \text{in } C(\overline{Q}_T) \quad \text{as } n \to \infty,$$

$$\psi_{n,t} - \Delta\psi_n + \alpha x \cdot D\psi_n + H(D\psi_n) = f \quad \text{on } \partial U \times \{0\} \text{ for } n \in \mathbb{N}.$$

Fix such a sequence $\{\psi_n\}$. By virtue of Theorem A.1, for each $n \in \mathbb{N}$ there is a unique solution $u_n \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$ of (A.1) and (A.2), with ψ_n in place of ψ.

Fix any $0 < T_0 < T$ and U_0 be an open ball such that $\overline{U}_0 \subset U$. Set $Q_0 = U_0 \times (T_0, T)$. We will show that

$$\|u_n\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_0)} \leq C_0 \quad \text{for } n \in \mathbb{N} \tag{A.17}$$

for some constant $C_0 \equiv C_0(U_0, T_0) > 0$, independent of n. By the comparison principle, we have

$$\|u_n - u_m\|_{L^\infty(Q_T)} \leq \|\psi_n - \psi_m\|_{C(\partial_p Q_T)} \quad \text{for } n, m \in \mathbb{N}. \tag{A.18}$$

Conceding for the moment that (A.17) is valid, we see by using (A.18) that for some function $u \in C(\overline{Q}_T) \cap C^{2+\gamma,1+\gamma/2}(\overline{Q}_0)$,

$$u_n \to u \quad \text{in } C(\overline{Q}_T) \cap C^{2,1}(\overline{Q}_0) \quad \text{as } n \to \infty.$$

In particular, u satisfies

$$\begin{cases} u_t - \Delta u + \alpha x \cdot Du + H(Du) = f & \text{in } Q_0, \\ (u - \psi)|_{\partial_p Q_T} = 0. \end{cases}$$

In view of the arbitrariness of U_0 and T_0, we conclude that u is in $C^{2+\gamma,1+\gamma/2}(Q_T)$ and is a solution of (A.1) and (A.2).

Thus we only need to show that (A.17) holds.

We follow the arguments at pp. 352–355 in [LSU]. To show (A.17), we fix $n \in \mathbb{N}$ and write u for u_n. Choose $R > 0$ and $x_0 \in \mathbb{R}^N$ so that $U = \{x \in \mathbb{R}^N \mid |x - x_0| < R\}$.

For $0 < \lambda \leq 1$ we set $Q^{(\lambda)} = \{x \in \mathbb{R}^N \mid |x - x_0| < R(1 - \frac{\lambda}{2})\} \times (\frac{\lambda^2 T}{2}, T)$. We may choose $\zeta^\lambda \in C^\infty(\overline{Q}_T)$ so that

$$\zeta^\lambda \geq 0, \quad \zeta^\lambda(x,t) = 1 \text{ for } (x,t) \in Q^{(\lambda)}, \quad \zeta^\lambda(x,t) = 0 \text{ for } (x,t) \in \overline{Q}_T \setminus Q^{(\frac{\lambda}{2})},$$

$$\|D^k \zeta^\lambda\|_{L^\infty(Q_T)} \leq C_1 \lambda^{-k} \text{ for } k = 0,1,2, \quad \|\zeta^\lambda_t\|_{L^\infty(Q_T)} \leq C_1 \lambda^{-2}$$

for some constant $C_1 > 0$ independent of λ.

Define $\mathcal{N} : C^{2,1}(Q_T) \to C(Q_T)$ by

$$\mathcal{N}(g)(x,t) = g_t(x,t) - \Delta g(x,t) + \alpha x \cdot Dg(x,t) + H(Dg(x,t)).$$

We may assume by replacing H and f by $H(p) - H(0)$ and $f(x) - H(0)$, respectively, that $H(0) = 0$. The function $v := \zeta^\lambda u$ satisfies

$$\begin{cases} v_t - \Delta v + \alpha x \cdot Dv + H(Dv) = \mathcal{N}(\zeta^\lambda u) - \zeta^\lambda(\mathcal{N}(u) - f(x)) & \text{in } Q_T, \\ v|_{\partial_p Q_T} = 0. \end{cases}$$

Setting $f_0 := \mathcal{N}(\zeta^\lambda u) - \zeta^\lambda \mathcal{N}(u) + \zeta^\lambda f$, in view of (A.13), we have

$$\|v\|_{C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)} \leq C\|f_0\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)}. \tag{A.19}$$

We have

$$\begin{aligned} \mathcal{N}(\zeta^\lambda u) - \zeta^\lambda \mathcal{N}(u) &= \zeta^\lambda_t u - (\Delta\zeta^\lambda)u - 2D\zeta^\lambda \cdot Du + \alpha(x \cdot D\zeta^\lambda)u \\ &\quad + H(uD\zeta^\lambda + \zeta^\lambda Du) - \zeta^\lambda H(Du). \end{aligned}$$

Following arguments in [LSU], we obtain

$$\begin{aligned} \|\zeta^\lambda_t u\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} &\leq C_2(\lambda^{-2-\gamma}\|u\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \lambda^{-2}\langle u\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}}), \\ \|(\Delta\zeta^\lambda)u\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} &\leq C_2(\lambda^{-2-\gamma}\|u\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \lambda^{-2}\langle u\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}}), \\ \|D\zeta^\lambda \cdot Du\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} &\leq C_2(\lambda^{-1-\gamma}\|Du\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \lambda^{-1}\langle Du\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}}), \\ \|\alpha x \cdot D\zeta^\lambda u\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} &\leq C_2(\lambda^{-1-\gamma}\|u\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \lambda^{-1}\langle u\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}}), \\ \|\zeta^\lambda f\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} &\leq C_2(\lambda^{-\gamma}\|f\|_{L^\infty(Q_T)} + \langle f\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}}) \end{aligned}$$

for some constant $C_2 > 0$, independent of λ. Also, noting that

$$\|H(uD\zeta^\lambda + \zeta^\lambda Du)\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} \leq \|DH\|_\infty \|uD\zeta^\lambda + \zeta^\lambda Du\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)}$$

and replacing C_2 by a larger constant if necessary, we get

$$\|H(uD\zeta^\lambda + \zeta^\lambda Du)\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)}$$
$$\le C_2(\lambda^{-1-\gamma}\|u\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \lambda^{-\gamma}\|Du\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \lambda^{-1}\langle u\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}} + \langle Du\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}}).$$

Similarly we have

$$\|\zeta^\lambda H(Du)\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} \le C_2(\lambda^{-\gamma}\|Du\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \langle Du\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}}).$$

Recalling that $0 < \lambda \le 1$ and summing up the above inequalities, we get

$$\text{(A.20)} \qquad \begin{aligned} \|f_0\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} \le\ & C_3(\lambda^{-2-\gamma}\|u\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \lambda^{-1-\gamma}\|Du\|_{L^\infty(Q^{(\frac{\lambda}{2})})} \\ & +\lambda^{-2}\langle u\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}} + \lambda^{-1}\langle Du\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}} \\ & +\lambda^{-\gamma}\|f\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \langle f\rangle^{(\gamma)}_{Q^{(\frac{\lambda}{2})}}) \end{aligned}$$

for some constant $C_3 > 0$ independent of λ. Hence, using (A.19) and interpolation inequalities (see Lemma 3.2 of Chapter II in [LSU]), we get

$$\langle v\rangle^{(2+\gamma)}_{Q_T} \le 2^{-3-\gamma}\langle u\rangle^{(2+\gamma)}_{Q^{(\frac{\lambda}{2})}} + C_4(\lambda^{-2-\gamma}\|u\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \lambda^{-\gamma}\|f\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)})$$

for some constant $C_4 > 0$, which yields

$$\lambda^{2+\gamma}\langle u\rangle^{(2+\gamma)}_{Q^{(\lambda)}} \le 2^{-3-\gamma}\lambda^{2+\gamma}\langle u\rangle^{(2+\gamma)}_{Q^{(\frac{\lambda}{2})}} + C_4(\|u\|_{L^\infty(Q^{(\frac{\lambda}{2})})} + \|f\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)}).$$

Recall that $u = u_n$ depends on $n \in \mathbb{N}$, and choose a constant $M > 0$ so that

$$\|u_n\|_{L^\infty(Q_T)} + \|f\|_{C^{\gamma,\gamma/2}(\overline{Q}_T)} \le M \quad \text{for } n \in \mathbb{N}.$$

Then

$$\lambda^{2+\gamma}\langle u_n\rangle^{(2+\gamma)}_{Q^{(\lambda)}} \le \frac{1}{2}\left(\frac{\lambda}{2}\right)^{2+\gamma}\langle u_n\rangle^{(2+\gamma)}_{Q^{(\frac{\lambda}{2})}} + C_4M \quad \text{for } n \in \mathbb{N}.$$

Since $u_n \in C^{2+\gamma,1+\gamma/2}(\overline{Q}_T)$, this yields

$$\lambda^{2+\gamma}\langle u_n\rangle^{(2+\gamma)}_{Q^{(\lambda)}} \le 2C_4M \quad \text{for } n \in \mathbb{N},\ \lambda \in (0,1],$$

which guarantees (A.17).

Uniqueness is a consequence of the standard maximum principle. □

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