Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space

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Abstract. We study the large time behavior of solutions of the Cauchy problem for the Hamilton-Jacobi equation $u_t + H(x, Du) = 0$ in $\mathbf{R}^n \times (0, \infty)$, where $H(x, p)$ is continuous on $\mathbf{R}^n \times \mathbf{R}^n$ and convex in p. We establish a general convergence result for viscosity solutions $u(x, t)$ of the Cauchy problem as $t \to \infty$.

1. Introduction and the main results

In recent years, there has been much interest on the asymptotic behavior of viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations or viscous Hamilton-Jacobi equations. Fathi [F2] was the first who established a fairly general convergence result for the Hamilton-Jacobi equation $u_t(x,t) + H(x, Du(x,t)) = 0$ on a compact manifold M with smooth strictly convex Hamiltonian H. His approach to this large time asymptotic problem is based on the weak KAM theory [F1, F3, FS1] which is concerned with the Hamilton-Jacobi equation as well as with the Lagrangian or Hamiltonian dynamical structures behind it. Barles and Souganidis [BS1, BS2] took another approach, based on PDE techniques, to the same asymptotic problem. The weak KAM approach due to Fathi to the asymptotic problem has been developed and further improved by Roquejoffre [R] and Davini-Siconolfi [DS]. Motivated by these developments the author jointly with Y. Fujita and P. Loreti (see [FIL1]) has recently investigated the asymptotic problem for viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator

$$u_t - \Delta u + \alpha x \cdot Du + H(Du) = f(x) \quad \text{in } \mathbf{R}^n \times (0, \infty),$$

and the corresponding Hamilton-Jacobi equations

$$u_t + \alpha x \cdot Du + H(Du) = f(x) \quad \text{in } \mathbf{R}^n \times (0, \infty),$$

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where H is a convex function on $\mathbf{R}^n$, Δ denotes the Laplace operator, and α is a positive constant, and has established a convergence result similar to those obtained by [BS, F2, R, DS].

In this paper we investigate the Cauchy problem

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty), \tag{1.1}$$

$$u(\cdot, 0) = u_0, \tag{1.2}$$

where H is a scalar function on $\mathbf{R}^n \times \mathbf{R}^n$, $u \equiv u(x,t)$ is the unknown scalar function on $\mathbf{R}^n \times [0, \infty)$, $u_t = \partial u/\partial t$, $Du = (\partial u/\partial x_1, ..., \partial u/\partial x_n)$, and u_0 is a given function on $\mathbf{R}^n$ describing the initial data. The function $H(x,p)$ is assumed here to be convex in p, and we call H the *Hamiltonian* and then the function L, defined by

$$L(x, \xi) = \sup_{p \in \mathbf{R}^n} (\xi \cdot p - H(x,p)),$$

the *Lagrangian*. We refer to [Rf] for general properties of convex functions.

We are also concerned with the *additive eigenvalue* problem:

$$H(x, Dv) = c \quad \text{in } \mathbf{R}^n, \tag{1.3}$$

where the unknown is a pair $(c, v) \in \mathbf{R} \times C(\mathbf{R}^n)$ for which v is a viscosity solution of (1.3). This problem is also called the *ergodic control* problem due to the fact that PDE (1.3) appears as the dynamic programming equation in ergodic control of deterministic optimal control theory. We remark that the additive eigenvalue problem (1.3) appears in the homogenization of Hamilton-Jacobi equations. See for this [LPV].

For notational simplicity, given $\phi \in C^1(\mathbf{R}^n)$, we will write $H[\phi](x)$ for $H(x, D\phi(x))$ or $H[\phi]$ for the function: $x \mapsto H(x, D\phi(x))$ on $\mathbf{R}^n$. For instance, (1.3) may be written as $H[v] = c$ in $\mathbf{R}^n$.

We make the following assumptions on the Hamiltonian H.

(A1) $H \in C(\mathbf{R}^n \times \mathbf{R}^n)$.

(A2) H is *coercive*, that is, for any $R > 0$,

$$\lim_{r \to \infty} \inf\{H(x,p) \mid x \in B(0,R),\ p \in \mathbf{R}^n \setminus B(0,r)\} = \infty.$$

(A3) For any $x \in \mathbf{R}^n$, the function: $p \mapsto H(x,p)$ is strictly convex in $\mathbf{R}^n$.

(A4) There are functions $\phi_i \in C^{0+1}(\mathbf{R}^n)$ and $\sigma_i \in C(\mathbf{R}^n)$, with $i = 0, 1$, such that for $i = 0, 1$,

$$H(x, D\phi_i(x)) \le -\sigma_i(x) \quad \text{almost every } x \in \mathbf{R}^n,$$

$$\lim_{|x| \to \infty} \sigma_i(x) = \infty, \qquad \lim_{|x| \to \infty} (\phi_0 - \phi_1)(x) = \infty.$$

By adding a constant to the function ϕ_0, we assume henceforth that

$$\phi_0(x) \ge \phi_1(x) \quad \text{for } x \in \mathbf{R}^n.$$

We introduce the class Φ_0 of functions by

$$\Phi_0 = \{u \in C(\mathbf{R}^n) \mid \inf_{\mathbf{R}^n}(u - \phi_0) > -\infty\}.$$

We call a *modulus* a function $m : [0, \infty) \to [0, \infty)$ if it is continuous and nondecreasing on $[0, \infty)$ and if $m(0) = 0$. The space of all absolutely continuous functions $\gamma : [S, T] \to \mathbf{R}^n$ will be denoted by $\mathrm{AC}([S, T], \mathbf{R}^n)$. For $x, y \in \mathbf{R}^n$ and $t > 0$, $\mathcal{C}(x, t)$ (resp., $\mathcal{C}(x, t; y, 0)$) will denote the spaces of all curves $\gamma \in \mathrm{AC}([0, t], \mathbf{R}^n)$ satisfying $\gamma(t) = x$ (resp., $\gamma(t) = x$ and $\gamma(0) = y$). For any interval $I \subset \mathbf{R}$ and $\gamma : I \to \mathbf{R}^n$, we call γ a curve if it is absolutely continuous on any compact subinterval of I.

We will establish the following theorems.

Theorem 1.1. *Let $u_0 \in \Phi_0$ and assume that* (A1)–(A4) *hold. Then there is a unique viscosity solution $u \in C(\mathbf{R}^n \times [0, \infty))$ of* (1.1) *and* (1.2) *satisfying*

$$\inf\{u(x, t) - \phi_0(x) \mid (x, t) \in \mathbf{R}^n \times [0, T]\} > -\infty \tag{1.4}$$

for any $T \in (0, \infty)$. Moreover the function u is represented as

$$u(x, t) = \inf\left\{\int_0^t L(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s + u_0(\gamma(0)) \mid \gamma \in \mathcal{C}(x, t)\right\} \quad \textit{for } (x, t) \in \mathbf{R}^n \times (0, \infty).$$

Theorem 1.2. *Let* (A1)–(A4) *hold. Then there is a solution $(c, v) \in \mathbf{R} \times \Phi_0$ of* (1.3)*. Moreover the constant c is unique in the sense that if $(d, w) \in \mathbf{R} \times \Phi_0$ is another solution of* (1.3)*, then $d = c$.*

The above theorem determines uniquely a constant c, which we will denote by c_H, for which (1.3) has a viscosity solution in the class Φ_0. The constant c_H is called the *critical value* or *additive eigenvalue* for the Hamiltonian H. This definition may suggest that c depends on the choice of (ϕ_0, ϕ_1). Actually, it depends only on H, but not on the choice of (ϕ_0, ϕ_1), as the characterization of c_H in Proposition 3.4 below shows. It is clear that if (c, v) is a solution of (1.3), then $(c, v + K)$ is a solution of (1.3) for any $K \in \mathbf{R}$. As is well-known, the structure of solutions of (1.3) is, in general, much more complicated than this one-dimensional structure.

Theorem 1.3. *Let* (A1)–(A4) *hold and $u_0 \in \Phi_0$. Let $u \in C(\mathbf{R}^n \times [0, \infty))$ be the viscosity solution of* (1.1) *and* (1.2) *satisfying* (1.4)*. Then there is a viscosity solution $v_0 \in \Phi_0$ of* (1.3)*, with $c = c_H$, such that as $t \to \infty$,*

$$u(x, t) + ct - v_0(x) \to 0 \quad \textit{uniformly on compact subsets of } \mathbf{R}^n.$$

We call the function $v_0(x) - ct$ obtained in the above theorem the *asymptotic solution* of (1.1) and (1.2). See Theorem 8.1 for a representation formula for the function v_0.

In order to prove the convergence result of Theorem 1.3, we follow the generalized dynamical approach introduced by Davini and Siconolfi [DS] in broad outline.

In the following we *always assume* that (A1)–(A4) hold.

The paper is organized as follows: in Section 2 we collect some basic observations needed in the following sections. Section 3 is devoted to the additive eigenvalue problem and to establishing Theorem 1.2. In Section 4 we establish a comparison theorem for (1.1) and (1.2), from which the uniqueness part of Theorem 1.1 follows. In Section 5 we study the Aubry set and critical curves for the Lagrangian L. Section 6 deals with the existence of a viscosity solution u of the Cauchy problem (1.1)–(1.2) together with some estimates on u. Section 7 combines the results in the preceding sections, to prove Theorem 1.3. In Section 8 we show a representation formula asymptotic solution for large time of (1.1) and (1.2). In Section 9 we give two sufficient conditions for H to satisfy (A4) and a two-dimensional example in which the Aubry set contains a non-empty disk, with positive radius, consisting of non-equilibrium points. In Appendix we show in a general setting that value functions associated with Hamiltonian H are viscosity solutions of the Hamilton-Jacobi equation $H = 0$.

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2. Preliminaries

In this section we collect some basic observations which will be needed in the following sections.

We will be concerned with functions f on $\mathbf{R}^n \times \mathbf{R}^n$. We write $D_1 f$ and $D_2 f$ for the gradients of f, respectively, in the first n variables and in the last n variables. Similarly, we use the symbols $D_1^{\pm} f$ and $D_2^{\pm} f$ to denote the sub- and superdifferentials of f in the first or last n variables.

We remark that, since $H(x, \cdot)$ is convex for any $x \in \mathbf{R}^n$, for any $u \in C^{0+1}(\Omega)$, where $\Omega \subset \mathbf{R}^n \times (0, \infty)$ is open, it is a viscosity subsolution of (1.1) in Ω if and only if it satisfies (1.1) almost everywhere (a.e. for short) in Ω. A similar remark holds true for the stationary problem (1.3).

Also, as is well-known, the coercivity assumption (A2) on H guarantees that if $v \in C(\Omega)$, where Ω is an open subset of $\mathbf{R}^n$, is a viscosity subsolution of (1.3) in Ω, then it is locally Lipschitz in Ω.

Another remark related to the convexity of H is that given nonempty, uniformly bounded, family $\mathcal{S}$ of subsolutions of (1.3) in Ω, where Ω is an open subset of $\mathbf{R}^n$, the pointwise infimum $u(x) := \inf\{v(x) \mid v \in \mathcal{S}\}$ gives a viscosity subsolution u of (1.3) in Ω. For instance, this can be checked by invoking the notion of semicontinuous viscosity solutions due to Barron-Jensen [BJ1, BJ2]. Indeed, due to this theory (see also [B, BC,I3]), $v \in C^{0+1}(\Omega)$ is a viscosity subsolution of (1.3) if and only if $H(x, p) \leq c$ for all $p \in D^- v(x)$ and all $x \in \Omega$. It is standard to see that if $p \in D^- u(x)$ for some $x \in \Omega$, then there are sequences $\{x_k\}_{k\in\mathbf{N}} \subset \Omega$, $\{v_k\}_{k\in\mathbf{N}} \subset \mathcal{S}$, and $\{p_k\}_{k\in\mathbf{N}} \subset \mathbf{R}^n$ such that $p_k \in D^- v_k(x_k)$ for all $k \in \mathbf{N}$ and $(x_k, p_k, v_k(x_k)) \to (x, p, u(x))$ as $k \to \infty$. Here, we have $H(x_k, p_k) \le c$ for all $k \in \mathbf{N}$ and conclude that $H(x,p) \le c$ for all $p \in D^- u(x)$ and all $x \in \Omega$. If, instead, $\mathcal{S}$ is a family of viscosity supersolutions of (1.3) in Ω, then a classical result in viscosity solutions theory assures that u, defined as the pointwise infimum of all functions $v \in \mathcal{S}$, is a viscosity supersolution of (1.3) in Ω. In particular, if $\mathcal{S}$ is a family of viscosity solutions of (1.3) in Ω, then the function u, defined as the pointwise infimum of $v \in \mathcal{S}$, is a viscosity solution of (1.3) in Ω.

Proposition 2.1. *For each $R > 0$ there exist constants $\delta_R > 0$ and $C_R > 0$ such that $L(x,\xi) \le C_R$ for all $(x,\xi) \in B(0,R) \times B(0,\delta_R)$.*

Proof. Fix any $R > 0$. By the continuity of H, there exists a constant $M_R > 0$ such that $H(x,0) \le M_R$ for all $x \in B(0,R)$. Also, by the coercivity of H, there exists a constant $\rho_R > 0$ such that $H(x,p) > M_R + 1$ for all $(x,p) \in B(0,R) \times \partial B(0,\rho_R)$. We set $\delta_R = \rho_R^{-1}$. Let $\xi \in B(0,\delta_R)$ and $x \in B(0,R)$. Let $q \in B(0,\rho_R)$ be the maximum point of the function: $f(p) := H(x,p) - \xi \cdot p$ on $B(0,\rho_R)$. Noting that $f(0) = H(x,0) \le M_R$ and $f(p) > M_R + 1 - \delta_R\rho_R = M_R$ for all $p \in \partial B(0,\rho_R)$, we see that $q \in \operatorname{int} B(0,\rho_R)$ and hence $\xi \in D_2^- H(x,q)$, which implies that $L(x,\xi) = \xi \cdot q - H(x,q)$. Consequently, we get

$$L(x,\xi) \le \delta_R\rho_R - \min_{p\in\mathbf{R}^n} H(x,p) = 1 - \min_{B(0,R)\times\mathbf{R}^n} H.$$

Now, choosing $C_R > 0$ so that $1 - \min_{B(0,R)\times\mathbf{R}^n} H \le C_R$, we obtain

$$L(x,\xi) \le C_R \quad \text{for all } (x,\xi) \in B(0,R) \times B(0,\delta_R). \qquad \square$$

Proposition 2.2. *Let $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$. Then $(x,\xi) \in \operatorname{int}\operatorname{dom} L$ if and only if $\xi \in D_2^- H(x,p)$ for some $p \in \mathbf{R}^n$.*

Proof. Fix $\hat{x}, \hat{\xi} \in \mathbf{R}^n$. Suppose first that $\hat{\xi} \in D_2^- H(\hat{x},\hat{p})$ for some $\hat{p} \in \mathbf{R}^n$. Define the function f on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$f(x,p) = H(x,p) - \hat{\xi} \cdot p + L(\hat{x},\hat{\xi}).$$

Note that the function $f(\hat{x},\cdot)$ attains the minimum value 0 at $\hat{p}$ and it is strictly convex on $\mathbf{R}^n$. Fix $r > 0$ and set

$$m = \min_{p\in\partial B(\hat{p},r)} f(\hat{x},p),$$

and note, because of the strict convexity of $f(\hat{x},\cdot)$, that $m > 0$. Note also that the function: $x \mapsto \min_{p\in\partial B(\hat{p},r)} f(x,p)$ is continuous on $\mathbf{R}^n$. Hence there is a constant $\delta > 0$ such that

$$\min\{f(x,p) \mid x \in B(\hat{x},\delta),\ p \in \partial B(\hat{p},r)\} > \frac{m}{2}, \tag{2.1}$$

$$\max\{f(x,\hat{p}) \mid x \in B(\hat{x},\delta)\} < \frac{m}{4}. \tag{2.2}$$

Fix any $(x,\xi) \in B(\hat{x},\delta) \times B(0,\frac{m}{4})$ and consider the affine function $g(p) := r^{-1}\xi \cdot (p-\hat{p}) + \frac{m}{4}$. We show that

$$f(x,p) > g(p) \quad \text{for all } p \in \mathbf{R}^n \setminus B(\hat{p},r). \tag{2.3}$$

To see this, we fix any $p \in \mathbf{R}^n \setminus B(\hat{p},r)$ and set $q = \hat{p} + r(p-\hat{p})/|p-\hat{p}| \in \partial B(\hat{p},r)$. Then, by (2.1), we have

$$f(x,q) > \frac{m}{2}.$$

Using the convexity of $f(x,\cdot)$ and noting that $q = (1-\frac{r}{|p-\hat{p}|})\hat{p} + \frac{r}{|p-\hat{p}|}p$, we get

$$f(x,q) \le \Big(1 - \frac{r}{|p-\hat{p}|}\Big) f(x,\hat{p}) + \frac{r}{|p-\hat{p}|} f(x,p)$$

and hence, by using (2.2), we get

$$\begin{aligned} f(x,p) \ge & \, r^{-1}|p-\hat{p}| f(x,q) + (1 - r^{-1}|p-\hat{p}|) f(x,\hat{p}) \\ > & \, r^{-1}|p-\hat{p}|\frac{m}{2} + (1 - r^{-1}|p-\hat{p}|)\frac{m}{4} = \frac{m}{4}(1 + r^{-1}|p-\hat{p}|). \end{aligned} \tag{2.4}$$

On the other hand, we have

$$g(p) \le \frac{m}{4}(r^{-1}|p-\hat{p}| + 1).$$

This combined with (2.4) shows that (2.3) is valid.

Next, observing that $f(x,\hat{p}) - g(\hat{p}) < \frac{m}{4} - g(\hat{p}) = 0$ by (2.2) and using (2.3), we see that the function: $p \mapsto f(x,p) - g(p)$ attains its global minimum at a point in $B(\hat{p},r)$. Fix such a minimum point $p_{x,\xi} \in B(\hat{p},r)$, which is indeed uniquely determined by the strict convexity of $f(x,\cdot)$. We have

$$0 \in D_2^- f(x,p_{x,\xi}) - Dg(p_{x,\xi}) = D_2^- H(x,p_{x,\xi}) - \hat{\xi} - r^{-1}\xi.$$

That is,

$$\hat{\xi} + r^{-1}\xi \in D_2^- H(x,p_{x,\xi}),$$

which is equivalent to saying that

$$p_{x,\xi} \in D_2^- L(x,\hat{\xi} + r^{-1}\xi).$$

In particular, we have $(x,\hat{\xi}+r^{-1}\xi) \in \operatorname{dom} L$ and $(\hat{x},\hat{\xi}) \in \operatorname{int}\operatorname{dom} L$.

Next, we suppose that $(\hat{x},\hat{\xi}) \in \operatorname{int}\operatorname{dom} L$. Then it is an easy consequence of the Hahn-Banach theorem that there is a $\hat{p} \in \mathbf{R}^n$ such that $\hat{\xi} \in D_2^- H(\hat{x},\hat{p})$. □

Remark. Let $(x,\xi) \in \operatorname{int}\operatorname{dom} L$. According to the above theorem (and its proof), there is a unique $p(x,\xi) \in D_2^- L(x,\xi)$. That is, on the set $\operatorname{int}\operatorname{dom} L$, the multi-valued map $D_2^- L$ can be identified with the single-valued function: $(x,\xi) \mapsto p(x,\xi)$. By the above proof, we see moreover that for each $r > 0$ there is a constant $\delta > 0$ such that

$p(y,\eta) \in B(p(x,\xi), r)$ for all $(y,\eta) \in B(x,\delta) \times B(\xi,\delta)$. From this observation, we easily see that the function: $(x,\xi) \mapsto p(x,\xi)$ is continuous on $\operatorname{int} \operatorname{dom} L$. Indeed, one can show that L is differentiable in the last n variables and $D_2 L$ is continuous on $\operatorname{int} \operatorname{dom} L$.

Proposition 2.3. *Let $K \subset \mathbf{R}^n \times \mathbf{R}^n$ be a compact set. Set*

$$S = \{(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n \mid \xi \in D_2^- H(x,p) \text{ for some } p \in \mathbf{R}^n \text{ such that } (x,p) \in K\}.$$

Then S is a compact subset of $\mathbf{R}^n \times \mathbf{R}^n$ and $S \subset \operatorname{int} \operatorname{dom} L$.

Proof. We choose a constant $R > 0$ so that $K = B(0,R) \times B(0,R)$.

To see that S is compact, we first check that $S \subset \mathbf{R}^{2n}$ is a closed set. Let $\{(x_k,\xi_k)\}_{k\in\mathbf{N}} \subset S$ be a sequence converging to $(x_0,\xi_0) \in \mathbf{R}^{2n}$. For each $k \in \mathbf{N}$ there corresponds a point $p_k \in B(0,R)$ such that

$$\xi_k \in D_2^- H(x_k, p_k).$$

This is equivalent to saying that

$$\xi_k \cdot p_k = L(x_k,\xi_k) + H(x_k,p_k). \tag{2.5}$$

We may assume by replacing the sequence $\{(x_k,\xi_k,p_k)\}$ by its subsequence if necessary that $\{p_k\}$ is convergent. Let $p_0 \in B(0,R)$ be the limit of the sequence $\{p_k\}$. Since L is lower semicontinuous, we get from (2.5) in the limit as $k \to \infty$,

$$\xi_0 \cdot p_0 \geq L(x_0,\xi_0) + H(x_0,p_0),$$

which implies that $\xi_0 \in D_2^- H(x_0,p_0)$. Hence, we have $(x_0,\xi_0) \in S$ and see that S is closed.

Next we show that S is bounded. Since $H \in C(\mathbf{R}^{2n})$ and the function: $p \mapsto H(x,p)$ is convex for any $x \in \mathbf{R}^n$, we see that there is a constant $M > 0$ such that the functions: $p \mapsto H(x,p)$, with $x \in B(0,R)$, is equi-Lipschitz continuous on $B(0,R)$ with a Lipschitz bound M. This implies that

$$|\xi| \leq M \quad \text{ for all } (x,\xi) \in S,$$

since if $(x,\xi) \in S$, then $\xi \in D_2^- H(x,p)$ for some $p \in B(0,R)$ and $|\xi| \leq M$. Thus we have seen that $S \subset B(0,R) \times B(0,M)$. The set S is bounded and closed in $\mathbf{R}^{2n}$ and therefore it is compact.

Finally, we apply Proposition 2.2 to $(x,\xi) \in S$, to see that $(x,\xi) \in \operatorname{int} \operatorname{dom} L$. □

Proposition 2.4. *Let $\phi \in C^{0+1}(\mathbf{R}^n)$ and $\gamma \in \mathrm{AC}([a,b],\mathbf{R}^n)$, where a, $b \in \mathbf{R}$ satisfy $a < b$. Then there is a function $q \in L^\infty(a,b,\mathbf{R}^n)$ such that*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi \circ \gamma(t) &= q(t) \cdot \dot\gamma(t) \quad && \text{a.e. } t \in (a,b), \\ q(t) &\in \partial_c \phi(\gamma(t)) \quad && \text{a.e. } t \in (a,b). \end{aligned}$$

Here $\partial_c\phi$ denotes the Clarke differential of ϕ (see [C]*), that is,*

$$\partial_c\phi(x) = \bigcap_{r>0} \overline{\mathrm{co}}\,\{D\phi(y) \mid y \in B(x,r),\ \phi \text{ is differentiable at } y\} \qquad \text{for } x \in \mathbf{R}^n.$$

Proof. Let $\rho \in C^\infty(\mathbf{R}^n)$ be a standard mollification kernel, i.e., $\rho \geq 0$, $\operatorname{spt}\rho \subset B(0,1)$, and $\int_{\mathbf{R}^n} \rho(x)\,\mathrm{d}x = 1$.

Set $\rho_k(x) := k^n\rho(kx)$ and $\phi_k(x) := \rho_k * \phi(x)$ for $x \in \mathbf{R}^n$ and $k \in \mathbf{N}$. Here the symbol "$*$" indicates the usual convolution of two functions. Set

$$\psi(t) = \phi\circ\gamma(t), \quad \psi_k(t) = \phi_k\circ\gamma(t), \quad \text{and} \quad q_k(t) = D\phi_k\circ\gamma(t) \text{ for } t \in [a,b],\ k \in \mathbf{N}.$$

We have $\dot\psi_k(t) = q_k(t)\cdot\dot\gamma(t)$ a.e. $t \in (a,b)$, and, by integration,

$$\psi_k(t) - \psi_k(a) = \int_a^t q_k(s)\cdot\dot\gamma(s)\,\mathrm{d}s \qquad \text{for all } t \in [a,b]. \tag{2.6}$$

Passing to a subsequence if necessary, we may assume that for some $q \in L^\infty(a,b,\mathbf{R}^n)$,

$$q_k \to q \qquad \text{weakly star in } L^\infty(a,b,\mathbf{R}^n) \text{ as } k \to \infty.$$

Therefore, from (2.6) we get in the limit as $k \to \infty$,

$$\psi(t) - \psi(a) = \int_a^t q(s)\cdot\dot\gamma(s)\,\mathrm{d}s \qquad \text{for all } t \in [a,b].$$

This shows that

$$\dot\psi(t) = q(t)\cdot\dot\gamma(t) \qquad \text{a.e. } t \in (a,b).$$

Noting that $\{q_k\}$ is weakly convergent to q in $L^2(a,b,\mathbf{R}^n)$, by Mazur's theorem, we may assume that there is a sequence $\{p_k\}$ such that

$$p_k \to q \qquad \text{strongly in } L^2(a,b,\mathbf{R}^n) \text{ as } k \to \infty,$$
$$p_k \in \mathrm{co}\,\{q_j \mid j \geq k\} \qquad \text{for all } k \in \mathbf{N}.$$

We may further assume that

$$p_k(t) \to q(t) \qquad \text{a.e. } t \in (a,b) \text{ as } k \to \infty.$$

We fix a set $I \subset (a,b)$ of full measure so that

$$p_k(t) \to q(t) \qquad \text{for all } t \in I \text{ as } k \to \infty. \tag{2.7}$$

Now, for any $x \in \mathbf{R}^n$ and any $k \in \mathbf{N}$, noting that

$$D\phi_k(x) = \int_{\mathbf{R}^n} \rho_k(x-y)D\phi(y)\,\mathrm{d}y,$$

we find that

$$D\phi_k(x) \in \overline{\mathrm{co}}\{D\phi(y) \mid y \in B(x, k^{-1}),\ \phi \text{ is differentiable at } y\}.$$

From this, we get

$$q_k(t) \in \overline{\mathrm{co}}\{D\phi(x) \mid x \in B(\gamma(t),\, k^{-1}),\ \phi \text{ is differentiable at } x\} \quad \text{for all } t \in [a, b],$$

and therefore

$$p_k(t) \in \overline{\mathrm{co}}\{D\phi(x) \mid x \in B(\gamma(t),\, k^{-1}),\ \phi \text{ is differentiable at } x\} \quad \text{for all } t \in [a, b]. \tag{2.8}$$

Combining (2.7) and (2.8), we get

$$q(t) \in \bigcap_{r>0} \overline{\mathrm{co}}\{D\phi(x) \mid x \in B(\gamma(t),\, r),\ \phi \text{ is differentiable at } x\} \quad \text{for all } t \in I.$$

That is, we have

$$q(t) \in \partial_c \phi(\gamma(t)) \quad \text{a.e. } t \in (a, b).$$

□

Proposition 2.5. *Let $w \in C^{0+1}(\mathbf{R}^n)$ be such that $H(x, Dw(x)) \le f(x)$ in $\mathbf{R}^n$ in the viscosity sense, where $f \in C(\mathbf{R}^n)$. Let $a, b \in \mathbf{R}$ be such that $a < b$ and let $\gamma \in$* AC$([a, b], \mathbf{R}^n)$. *Then*

$$\int_a^b L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s \ge w(\gamma(b)) - w(\gamma(a)) - \int_a^b f(\gamma(s))\,\mathrm{d}s.$$

Proof. By Proposition 2.4, there is a function $q \in L^\infty(a, b, \mathbf{R}^n)$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} w(\gamma(s)) &= q(s) \cdot \dot\gamma(s) \quad \text{a.e. } s \in (a, b), \\ q(s) &\in \partial_c w(\gamma(s)) \quad \text{a.e. } s \in (a, b). \end{aligned}$$

We calculate that

$$\begin{aligned} \int_a^b L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s &\ge \int_a^b [\dot\gamma(s) \cdot q(s) - H(\gamma(s), q(s))]\,\mathrm{d}s \\ &\ge \int_a^b \left[\frac{\mathrm{d}}{\mathrm{d}s} w(\gamma(s)) - f(\gamma(s))\right] \mathrm{d}s \\ &= w(\gamma(b)) - w(\gamma(a)) - \int_a^b f(\gamma(s))\,\mathrm{d}s. \end{aligned}$$

□

3. Additive eigenvalue problem

In this section we prove Theorem 1.2. Our proof below is parallel to that in [LPV].

Lemma 3.1. *There is a function $\psi_0 \in C^1(\mathbf{R}^n)$ such that*

$$H(x, D\psi_0(x)) \geq -C_0 \quad \textit{for all } x \in \mathbf{R}^n, \tag{3.1}$$
$$\psi_0(x) \geq \phi_0(x) \quad \textit{for all } x \in \mathbf{R}^n \tag{3.2}$$

for some constant $C_0 > 0$.

Proof. We choose a modulus ρ so that

$$H(x,p) \geq 0 \quad \text{for all } (x,p) \in B(0,r) \times [\mathbf{R}^n \setminus B(0,\rho(r))] \text{ and all } r \geq 1,$$
$$\|D\phi_0\|_{L^\infty(B(0,r))} \leq \rho(r) \quad \text{for all } r \geq 1.$$

Because of this choice, we have

$$\phi_0(x) - \phi_0(|x|^{-1}x) \leq \int_1^{|x|} \rho(r)\,\mathrm{d}r \quad \text{for all } x \in \mathbf{R}^n \setminus B(0,1).$$

We define the function $\psi_0 \in C^1(\mathbf{R}^n)$ by

$$\psi_0(x) = \max_{B(0,1)} \phi_0 + \int_0^{|x|} \rho(r)\,\mathrm{d}r.$$

Observe that

$$|D\psi_0(x)| = \rho(|x|) \quad \text{for all } x \in \mathbf{R}^n,$$
$$H(x, D\psi_0(x)) \geq 0 \quad \text{for all } x \in \mathbf{R}^n \setminus B(0,1),$$

and also that

$$\phi_0(x) \leq \phi_0(|x|^{-1}x) + \int_0^{|x|} \rho(r)\,\mathrm{d}r \leq \psi_0(x) \quad \text{for all } x \in \mathbf{R}^n \setminus B(0,1),$$

and therefore

$$\phi_0(x) \leq \psi_0(x) \quad \text{for all } x \in \mathbf{R}^n. \tag{3.3}$$

Choosing a constant $C_0 > 0$ so that

$$C_0 \geq \max_{x \in B(0,1)} |H(x, D\psi_0(x))|,$$

we have

$$H(x, D\psi_0(x)) \geq -C_0 \quad \text{for all } x \in \mathbf{R}^n.$$

This together with (3.3) completes the proof. □

We need the following comparison theorem, which generalizes comparison results in [A].

Theorem 3.2. *Let Ω be an open subset of $\mathbf{R}^n$. Let $f \in C(\overline{\Omega})$, $\lambda \geq 0$ and $\varepsilon \geq 0$. Assume that $\lambda + \varepsilon > 0$. Let u, $v : \overline{\Omega} \to \mathbf{R}$ be, respectively, an upper semicontinuous viscosity subsolution of*

$$\lambda u + H[u] \leq \lambda f - \varepsilon \quad \text{in } \Omega, \tag{3.4}$$

and a lower semicontinuous viscosity supersolution of

$$\lambda v + H[v] \geq \lambda f \quad \text{in } \Omega. \tag{3.5}$$

Assume that $f, v \in \Phi_0$ and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ on Ω.

The main idea in the following proof how to use the convexity property of H is similar to that in [I1].

Proof. It is enough to show that, for any $\mu > 0$, $u_\mu := u - \mu \leq v$ in $\overline{\Omega}$. Noting that u_μ is a viscosity subsolution of $\lambda u_\mu + H[u_\mu] \leq \lambda f - \varepsilon - \mu\lambda$ in Ω and that $\varepsilon + \mu\lambda > 0$, we may always assume by replacing u by u_μ if necessary that $\varepsilon > 0$.

Let $A > 0$ and define $u_A \in C(\overline{\Omega})$ by

$$u_A(x) = \min\{\phi_1(x) + A, u(x)\}.$$

Observe that there is a constant $R \equiv R(A) > 0$ such that

$$H(x, D\phi_1(x)) \leq -\varepsilon \quad \text{a.e. in } \Omega \setminus B(0,R),$$
$$f(x) \geq \phi_1(x) + A \quad \text{for all } x \in \Omega \setminus B(0,R).$$

Choose a constant $A_0 > 0$ so that

$$\phi_1(x) + A_0 > u(x) \quad \text{for all } x \in \overline{\Omega} \cap B(0,R),$$

and we assume henceforth that $A \geq A_0$.

For almost all $x \in \Omega$, we have

$$Du_A(x) = \begin{cases} Du(x) & \text{if } u(x) \leq \phi_1(x) + A, \\ D\phi_1(x) & \text{if } u(x) \geq \phi_1(x) + A. \end{cases}$$

Therefore, for almost all $x \in \Omega$, if $u(x) \leq \phi_1(x) + A$, then

$$\lambda u_A(x) + H(x, Du_A(x)) = \lambda u(x) + H(x, Du(x)) \leq \lambda f(x) - \varepsilon,$$

and if $u(x) \geq \phi_1(x) + A$, then $|x| > R$, $H(x, D\phi_1(x)) \leq -\varepsilon$, $\lambda u_A(x) \leq \lambda f(x)$, and therefore

$$\lambda u_A(x) + H(x, Du_A(x)) = \lambda f(x) + H(x, D\phi_1(x)) \leq \lambda f(x) - \varepsilon.$$

This observation assures that u_A is a viscosity subsolution of (3.4). The function u_A has the property that $u_A(x) = \phi_1(x) + A$ if $|x|$ is sufficiently large. Since

$$\lim_{|x|\to\infty} (v(x) - \phi_1(x)) = \infty,$$

we deduce that there is a constant $M > 0$ such that

$$u_A(x) \le v(x) \quad \text{ for all } x \in \overline{\Omega} \setminus B(0, M).$$

By a standard comparison theorem applied in $\overline{\Omega} \cap B(0, 2M)$, we obtain $u_A(x) \le v(x)$ for all $x \in \overline{\Omega} \cap B(0, 2M)$, from which it follows that $u_A(x) \le v(x)$ for all $x \in \overline{\Omega}$. For each $x \in \overline{\Omega}$, if we choose $A \ge A_0$ large enough, then $u_A(x) = u(x)$ and conclude that $u(x) \le v(x)$. □

Theorem 3.3. (1) *There is a solution* $(c, v) \in \mathbf{R} \times \Phi_0$ *of* (1.3). (2) *If* (c, v), $(d, w) \in \mathbf{R} \times \Phi_0$ *are solutions of* (1.3), *then* $c = d$.

Proof. We start by showing assertion (2). Let $(c, v), (d, w) \in \mathbf{R} \times \Phi_0$ be solutions of (1.3). Suppose that $c \ne d$. We may assume that $c < d$. Also, we may assume by adding a constant to v that $v(x_0) > w(x_0)$ at some point $x_0 \in \mathbf{R}^n$. On the other hand, by Theorem 3.2, we have $v \le w$ for all $x \in \mathbf{R}^n$, which is a contradiction. Thus we must have $c = d$.

In order to show existence of a solution of (1.3), we let $\lambda > 0$ and consider the problem

$$\lambda v_\lambda(x) + H(x, Dv_\lambda(x)) = \lambda \phi_0(x) \quad \text{ in } \mathbf{R}^n. \tag{3.6}$$

Let $\psi_0 \in C^1(\mathbf{R}^n)$ and $C_0 > 0$ be from Lemma 3.1. We may assume by replacing C_0 by a larger number if necessary that $\sigma_0(x) \ge -C_0$ for all $x \in \mathbf{R}^n$. Note that $H[\phi_0] \le C_0$ in $\mathbf{R}^n$ in the viscosity sense.

We define the functions $v_\lambda^\pm$ on $\mathbf{R}^n$ by

$$v_\lambda^+(x) = \psi_0(x) + \lambda^{-1} C_0 \quad \text{ and } \quad v_\lambda^-(x) = \phi_0(x) - \lambda^{-1} C_0.$$

It is easily seen that v_λ^+ and v_λ^- are viscosity supersolution and a viscosity subsolution of (3.6). In view of (3.2), we have $v_\lambda^-(x) < v_\lambda^+(x)$ for all $x \in \mathbf{R}^n$. By the Perron method in viscosity solutions theory, we find that the function v_λ on $\mathbf{R}^n$ given by

$$\begin{aligned} v_\lambda(x) = \sup\{w(x) \mid v_\lambda^- \le w \le v_\lambda^+ \ \text{ in } \ \mathbf{R}^n, \\ \lambda w + H[w] \le \lambda \phi_0 \ \text{ in } \ \mathbf{R}^n \ \text{ in the viscosity sense}\}. \end{aligned}$$

is a viscosity solution of (3.6). Because of the definition of v_λ, we have

$$\phi_0(x) - \lambda^{-1} C_0 \le v_\lambda(x) \le \psi_0(x) + \lambda^{-1} C_0 \quad \text{ for all } x \in \mathbf{R}^n. \tag{3.7}$$

Using the left hand side inequality of (3.7), we formally calculate that

$$\lambda \phi_0(x) = \lambda v_\lambda(x) + H(x, Dv_\lambda(x)) \ge \lambda \phi_0(x) - C_0 + H(x, Dv_\lambda(x)),$$

and therefore

$$H(x, Dv_\lambda(x)) \le C_0.$$

Indeed, this last inequality holds in the sense of viscosity solutions. This together with the coercivity of H yields the local equi-Lipschitz continuity of the family $\{v_\lambda\}_{\lambda>0}$. As a consequence, the family $\{v_\lambda - v_\lambda(0)\}_{\lambda>0} \subset C(\mathbf{R}^n)$ is locally uniformly bounded and locally equi-Lipschitz continuous on $\mathbf{R}^n$.

Going back to (3.7), we see that

$$\lambda\phi_0(x) - C_0 \leq \lambda v_\lambda(x) \leq \lambda\psi_0(x) + C_0 \quad \text{for all } x \in \mathbf{R}^n.$$

In particular, the set $\{\lambda v_\lambda(0)\}_{\lambda\in(0,1)} \subset \mathbf{R}$ is bounded. Thus we may choose a sequence $\{\lambda_j\}_{j\in\mathbf{N}}$ such that, as $j \to \infty$,

$$\lambda_j \to 0, \qquad -\lambda_j\psi_{\lambda_j}(0) \to c,$$
$$\psi_{\lambda_j}(x) - \psi_{\lambda_j}(0) \to v(x) \quad \text{on bounded sets } \subset \mathbf{R}^n$$

for some real number c and some function $v \in C^{0+1}(\mathbf{R}^n)$. Since

$$|\lambda(v_\lambda(x) - v_\lambda(0))| \leq \lambda L_R|x| \quad \text{for all } x \in B(0,R),$$

all $R > 0$, and some constants $L_R > 0$, we find that

$$-\lambda_j\psi_{\lambda_j}(x) \to c \quad \text{uniformly on bounded sets } \subset \mathbf{R}^n \text{ as } j \to \infty.$$

By a stability property of viscosity solutions, we deduce that v is a viscosity solution of (1.3) with c in hand.

Now, we show that $v \in \Phi_0$. Fix any $\lambda \in (0,1)$. As we have observed above, there is a constant $C_1 > 0$, independent of λ, such that $|\lambda v_\lambda(0)| \leq C_1$. Set $w_\lambda(x) = v_\lambda(x) - v_\lambda(0)$ for $x \in \mathbf{R}^n$. Note that w_λ is a viscosity solution of

$$\lambda w_\lambda + \lambda v_\lambda(0) + H(x, Dw_\lambda) = \lambda\phi_0 \quad \text{in } \mathbf{R}^n.$$

We may choose a constant $R > 0$ so that

$$H(x, D\phi_0(x)) \leq -C_1 \quad \text{a.e. } \mathbf{R}^n \setminus B(0,R),$$

and then a constant $C_2 \geq C_1$, independent of $\lambda \in (0,\,1)$, so that

$$\max\{|\phi_0(x)|, |w_\lambda(x)|\} \leq C_2 \quad \text{for all } x \in B(0,R).$$

Set $w = \phi_0 - 2C_2$. Obviously we have

$$w \leq w_\lambda \quad \text{in } B(0,R),$$

and

$$\lambda w + \lambda v_\lambda(0) + H(x, Dw(x)) \leq \lambda\phi_0 + C_1 - C_1 \leq \lambda\phi_0 \quad \text{a.e. } x \in \mathbf{R}^n \setminus B(0,R).$$

Noting that $w_\lambda \in \Phi_0$, we apply Theorem 3.2 to w and w_λ, to obtain

$$w \le w_\lambda \quad \text{ in } \mathbf{R}^n \setminus B(0,R).$$

Sending $\lambda \to 0$, we get

$$\phi_0 - 2C_2 \le v \quad \text{ in } \mathbf{R}^n \setminus B(0,R),$$

which shows that $v \in \Phi_0$, completing the proof. □

Proposition 3.4. *The critical value c_H is characterized as*

$$c_H = \inf\{a \in \mathbf{R} \mid \textit{there exists a viscosity solution } v \in C(\mathbf{R}^n) \textit{ of } H[v] \le a \textit{ in } \mathbf{R}^n\}.$$

Proof. We write d for the right hand side of the above formula. Let $\phi \in \Phi_0$ be a viscosity solution of $H[\phi] = c_H$ in $\mathbf{R}^n$. If $a \ge c_H$, then $H[\phi] \le a$ in $\mathbf{R}^n$ in the viscosity sense. Thus we have $d \le c_H$. Suppose that $d < c_H$. Then there is a constant $e \in (d,\, c_H)$ and a viscosity solution of $H[\psi] \le e$ in $\mathbf{R}^n$. By Theorem 3.2, we see that $\psi + C \le \phi$ in $\mathbf{R}^n$ for any $C \in \mathbf{R}$, which is clearly a contradiction. Thus we have $d = c_H$. □

4. A comparison theorem for the Cauchy problem

In this section we establish the following comparison theorem. Let $T \in (0, \infty)$.

Theorem 4.1. *Let Ω be an open subset of $\mathbf{R}^n$. Let $u,\, v : \overline{\Omega} \times [0,T) \to \mathbf{R}$. Assume that u, $-v$ are upper semicontinuous on $\overline{\Omega} \times [0,T)$ and that u and v are, respectively, a viscosity subsolution and a viscosity supersolution of*

$$u_t + H(x, Du) = 0 \quad \textit{in } \Omega \times (0,T). \tag{4.1}$$

Moreover, assume that

$$\lim_{r\to\infty} \inf\{v(x,t) - \phi_1(x) \mid (x,t) \in (\Omega \setminus B(0,r)) \times [0,T)\} = \infty, \tag{4.2}$$

and that $u \le v$ on $(\Omega \times \{0\}) \cup (\partial\Omega \times [0,T))$. Then $u \le v$ in $\overline{\Omega} \times [0,T)$.

Proof. We choose a constant $C > 0$ so that

$$H(x, D\phi_1(x)) \le C \quad \text{ a.e. } x \in \mathbf{R}^n,$$

and define the function $w \in C(\mathbf{R}^n \times \mathbf{R})$ by

$$w(x,t) := \phi_1(x) - Ct.$$

Observe that $w_t + H(x, Dw(x,t)) \le 0$ a.e. $(x,t) \in \mathbf{R}^{n+1}$.

We need only to show that for all $(x,t) \in \overline{\Omega}$ and all $A > 0$,

$$\min\{u(x,t),\, w(x,t) + A\} \le v(x,t). \tag{4.3}$$

Fix any $A > 0$. We set $w_A(x,t) = w(x,t) + A$ for $(x,t) \in \mathbf{R}^{n+1}$. The function w_A is a viscosity subsolution of (4.1). By the convexity of $H(x,p)$ in p, the function $\bar{u}$ defined by $\bar{u}(x,t) := \min\{u(x,t),\, w_A(x,t)\}$ is a viscosity subsolution of (4.1). Because of assumption (4.2), we see that there is a constant $R > 0$ such that $\bar{u}(x,t) \le v(x,t)$ for all $(x,t) \in (\overline{\Omega} \setminus B(0,R)) \times [0,T)$. We set $\Omega_R := \Omega \cap \operatorname{int} B(0,2R)$, so that $\bar{u}(x,t) \le v(x,t)$ for all $x \in \partial\Omega_R \times [0,T)$. Also, we have $\bar{u}(x,0) \le u(x,0) \le v(x,0)$ for all $x \in \Omega_R$.

Next we take the sup-convolution of $\bar{u}$ in the variable t. That is, for each $\varepsilon \in (0,1)$ we set

$$u^\varepsilon(x,t) := \sup_{s\in[0,T)} \left(\overline{u}(x,s) - \frac{(t-s)^2}{2\varepsilon} \right) \qquad \text{for all } (x,t) \in \overline{\Omega}_R \times \mathbf{R}.$$

For each $\delta > 0$, there is a $\gamma \in (0, \min\{\delta, T/2\})$ such that $\bar{u}(x,t) - \delta \le v(x,t)$ for all $(x,t) \in \overline{\Omega}_R \times [0,\gamma]$. As is well-known, there is an $\varepsilon \in (0,\delta)$ such that u^ε is a viscosity subsolution of (4.1) in $\Omega_R \times (\gamma,\, T-\gamma)$ and $u^\varepsilon(x,t) - 2\delta \le v(x,t)$ for all $(x,t) \in \big(\overline{\Omega}_R \times [0,\gamma]\big) \cup (\partial\Omega_R \times [\gamma,\, T-\gamma])$. Observe that the family of functions: $t \mapsto u^\varepsilon(x,t)$ on $[\gamma,\, T-\gamma]$, with $x \in \overline{\Omega}_R$, is equi-Lipschitz continuous, with a Lipschitz bound $C_\varepsilon > 0$, and therefore that for each $t \in [\gamma,\, T-\gamma]$, the function $z : x \mapsto u^\varepsilon(x,t)$ in Ω_R satisfies $H(x, Dz(x)) \le C_\varepsilon$ a.e., which implies that the family of functions: $x \mapsto u^\varepsilon(x,t)$, with $t \in [\gamma,\, T-\gamma]$, is equi-Lipschitz continuous in Ω_R.

Now, we may apply a standard comparison theorem, to get $u^\varepsilon(x,t) \le v(x,t)$ for all $(x,t) \in \Omega_R \times [\gamma,\, T-\gamma]$, from which we get $\bar{u}(x,t) \le v(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,T)$. This completes the proof. □

5. Aubry sets and critical curves

Let $c \equiv c_H$ be the critical value for H. In this and the following sections we assume without loss of generality that $c = 0$. Indeed, if we set $H_c(x,y) = H(x,y) - c$ and $L_c(x,y) = L(x,y) + c$ for $(x,y) \in \mathbf{R}^{2n}$, then the stationary Hamilton-Jacobi equation $H[v] = c$ for v is exactly $H_c[v] = 0$ for v and the evolution equation $u_t + H[u] = 0$ for u is the equation $w_t + H_c[w] = 0$ for $w(x,t) := u(x,t) + ct$. Note moreover that L_c is the Lagrangian of the Hamiltonian H_c, i.e., $L_c(x,\xi) = \sup\{\xi \cdot p - H_c(x,p) \mid p \in \mathbf{R}^n\}$ for all $x, \xi \in \mathbf{R}^n$. With these relations in mind, by replacing H and L by H_c and L_c, respectively, we may assume that $c = 0$.

We consider the Hamilton-Jacobi equation

$$H(x, Du(x)) = 0 \quad \text{in } \mathbf{R}^n \tag{5.1}$$

and study the (projected) Aubry set for the Lagrangian L (for the Hamiltonian H, or for (5.1)).

Henceforth $\mathcal{S}_H^-$, $\mathcal{S}_H^+$, and $\mathcal{S}_H$ denote the sets of continuous viscosity subsolutions, of continuous viscosity supersolutions, and of continuous viscosity solutions of (5.1), respectively.

Following the ideas in [FS2] with small variations in the presentation, we introduce the Aubry set $\mathcal{A}$ for (5.1) as follows. We define the function $d_H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$d_H(x, y) = \sup\{v(x) \mid v \in \mathcal{S}_H^-,\ v(y) = 0\}, \tag{5.2}$$

and then the *Aubry set* $\mathcal{A}$ for (5.1) by

$$\mathcal{A} = \{y \in \mathbf{R}^n \mid d_H(\cdot, y) \in \mathcal{S}_H\}. \tag{5.3}$$

Since the equation, $H[v] = 0$ in $\mathbf{R}^n$, has a viscosity solution in the class Φ_0 by Theorem 3.3 (or 1.2), the set

$$\{v \in \mathcal{S}_H^- \mid v(y) = 0\}$$

is nonempty and, because of the coercivity assumption on H, it is locally equi-Lipschitz continuous. Therefore, the function $d_H(\cdot, y)$ defined by (5.2) is locally Lipschitz continuous on $\mathbf{R}^n$ and vanishes at $x = y$ for any $y \in \mathbf{R}^n$. Since the pointwise supremum of a family of viscosity subsolutions of (5.1) defines a function which is a viscosity subsolution of (5.1), for any $y \in \mathbf{R}^n$, we have $d_H(\cdot, y) \in \mathcal{S}_H^-$. In view of the Perron method, we deduce that, for any $y \in \mathbf{R}^n$, the function $d_H(\cdot, y)$ is a viscosity solution of (5.1) in $\mathbf{R}^n \setminus \{y\}$. Thus we see that

$$y \in \mathbf{R}^n \setminus \mathcal{A} \quad \Longleftrightarrow \quad \exists\, p \in D_1^- d_H(y, y) \ \text{ such that } \ H(y, p) < 0. \tag{5.4}$$

For any $y, z \in \mathbf{R}^n$, the function $w(x) := d_H(x, y) - d_H(z, y)$ is a viscosity subsolution of (5.1) and satisfies $w(z) = 0$. Therefore we have $w(x) \le d_H(x, z)$. That is, we have the triangle inequality for d_H:

$$d_H(x, y) \le d_H(x, z) + d_H(z, y) \qquad \text{for all } x, y, z \in \mathbf{R}^n.$$

To continue, we make the following normalization. We fix a viscosity solution $\phi \in \Phi_0$ of $H[\phi] = 0$ in $\mathbf{R}^n$. We choose a constant $r > 0$ so that $\sigma_i(x) \ge 0$ for all $x \in \mathbf{R}^n \setminus B(0, r)$. There is a constant $M > 0$ such that $\phi(x) - M \le \phi_1(x)$ for all $x \in B(0, r)$. We set $\zeta_1(x) = \min\{\phi(x) - M, \phi_1(x)\}$ for $x \in \mathbf{R}^n$. Since $\lim_{|x|\to\infty}(\phi - \phi_1)(x) = \infty$, we have $\zeta_1(x) = \phi_1(x)$ for all $x \in \mathbf{R}^n \setminus B(0, R)$ and some $R > r$. Note that $H(x, D\zeta_1(x)) = H(x, D\phi(x)) = 0$ a.e. in $B(0, r)$, $H(x, D\zeta_1(x)) \le \max\{H(x, D\phi(x)), H(x, D\phi_1(x))\} \le 0$ a.e. in $B(0, R) \setminus B(0, r)$, and $H(x, D\zeta_1(x)) = H(x, D\phi_1(x)) = -\sigma_1(x)$ a.e. in $\mathbf{R}^n \setminus B(0, R)$. Therefore, by replacing ϕ_1 and σ_1 by ζ_1 and $\max\{\sigma_1, 0\}$, respectively, we may assume that $\sigma_1 \ge 0$ in $\mathbf{R}^n$. Similarly, we define the function $\zeta_0 \in C^{0+1}(\mathbf{R}^n)$ by setting $\zeta_0(x) = \min\{\phi(x) - M, \phi_0(x)\}$ and observe that $H[\zeta_0] \le 0$ in $\mathbf{R}^n$ in the viscosity sense and that $\sup_{\mathbf{R}^n} |\zeta_0 - \phi_0| < \infty$, which implies that $u \in \Phi_0$ if and only if $\inf_{\mathbf{R}^n}(u - \zeta_0) > -\infty$. Henceforth we write ϕ_0 for ζ_0. A warning is that the function $\sigma_0 = 0$ corresponds to the current ϕ_0 and does not have the property: $\lim_{|x|\to\infty} \sigma_0(x) = \infty$.

Proposition 5.1. *The following formula is valid for all $x,\ y \in \mathbf{R}^n$:*

$$d_H(x,y) = \inf\Big\{\int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \ \Big|\ t>0,\ \gamma\in\mathcal{C}(x,t;y,0)\Big\}. \tag{5.5}$$

Proof. We write $\rho(x,y)$ for the right hand side of (5.5) in this proof.

Let $x,y\in\mathbf{R}^n$, $t>0$, and $\gamma\in\mathcal{C}(x,t;y,0)$. Since $H[d_H(\cdot,y)]\le 0$ in $\mathbf{R}^n$ in the viscosity sense, by Proposition 2.5, we have

$$\int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \ge d_H(\gamma(t),y) - d_H(\gamma(0),y) = d_H(x,y).$$

From this we get

$$d_H(x,y)\le\rho(x,y) \quad \text{for all } x,y\in\mathbf{R}^n.$$

Next we show that for each $y\in\mathbf{R}^n$ the function $\rho(\cdot,y)$ is locally Lipschitz continuous on $\mathbf{R}^n$.

Fix any $R>0$. By Proposition 2.1, there are constants $\varepsilon_R>0$ and $C_R>0$ such that $L(x,\xi)\le C_R$ for all $(x,\xi)\in B(0,R)\times B(0,\varepsilon_R)$. Fix any $x,y\in B(0,R)$ and $\delta>0$, and set $T:=(\delta+|x-y|)/\varepsilon_R$ and $\xi=\varepsilon_R(x-y)/(\delta+|x-y|)$. Define the curve $\gamma\in\mathcal{C}(x,T;y,0)$ by $\gamma(s)=y+s\xi$. Noting that $\xi\in B(0,\varepsilon_R)$, we get

$$\rho(x,y)\le\int_0^T L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s=\int_0^T L(y+s\xi,\xi)\,\mathrm{d}s\le C_RT=\varepsilon_R^{-1}C_R(\delta+|x-y|).$$

Letting $\delta\to 0$ yields

$$\rho(x,y)\le\varepsilon_R^{-1}C_R|x-y|,$$

which, in particular, shows that $\rho(x,x)\le 0$. It is easy to see that for any $x,y,z\in\mathbf{R}^n$, $\rho(x,y)\le\rho(x,z)+\rho(z,y)$. Therefore, for any $x,y,z\in B(0,R)$, we have

$$|\rho(x,y)-\rho(z,y)|\le\varepsilon_R^{-1}C_R|x-z|.$$

In order to prove that $\rho(x,y)\le d_H(x,y)$ for all $x,y\in\mathbf{R}^n$, it is sufficient to show that for any $y\in\mathbf{R}^n$, the function $v:=\rho(\cdot,y)$ is a viscosity subsolution of $H[v]=0$ in $\mathbf{R}^n$. This is a consequence of a well-known observation on value functions like v. Indeed, Theorem A.1 in Appendix applied to the current v, with $S=\{y\}$ and $\Omega=\mathbf{R}^n$, assures that $v\in\mathcal{S}_H^-$. □

Proposition 5.2. *$\mathcal{A}$ is a closed subset of $\mathbf{R}^n$.*

Proof. Let $\{y_k\}\subset\mathcal{A}$ be a sequence converging to $y\in\mathbf{R}^n$. By (A2) the sequence $\{d_H(\cdot,y_k)\}$ is locally equi-Lipschitz on $\mathbf{R}^n$. In particular, there is a constant $C>0$ such that $\max\{d_H(y_k,y),d_H(y,y_k)\}\le C|y_k-y|$ for all $k\in\mathbf{N}$. By the triangle inequality for d_H, we have

$$|d_H(x,y)-d_H(x,y_k)|\le\max\{d_H(y_k,y),d_H(y,y_k)\}\le C|y_k-y| \quad \text{for all } x\in\mathbf{R}^n.$$

Consequently, as $k \to \infty$, we have $d_H(x, y_k) \to d_H(x, y)$ uniformly for $x \in \mathbf{R}^n$. By the stability of viscosity solutions under uniform convergence, we find that $d_H(\cdot, y) \in \mathcal{S}_H$, proving that $y \in \mathcal{A}$ and therefore that $\mathcal{A}$ is a closed set. □

Proposition 5.3. *For any compact $K \subset \mathbf{R}^n \setminus \mathcal{A}$ there are a function $\phi_K \in \Phi_0$ and a constant $\delta > 0$ such that, in the viscosity sense, $H[\phi_K] \le 0$ in $\mathbf{R}^n$ and $H[\phi_K] \le -\delta$ in a neighborhood of K.*

Proof. Let $y \in \mathbf{R}^n \setminus \mathcal{A}$. There is a function $\varphi \in C^1(\mathbf{R}^n)$ such that $\varphi(y) = 0$, $\varphi(x) < d_H(x, y)$ for all $x \in \mathbf{R}^n \setminus \{y\}$, and $H(y, D\varphi(y)) < 0$. With a sufficiently small constant $\delta > 0$, we set

$$\psi(x) = \max\{\varphi(x) + \delta, d_H(x, y)\} \quad \text{for all } x \in \mathbf{R}^n,$$

to get a function having the properties: (i) $H[\psi] \le 0$ in $\mathbf{R}^n$ in the viscosity sense, (ii) $H[\psi] \le -\varepsilon$ in $\operatorname{int} B(y, \varepsilon)$ in the viscosity sense, and (iii) $\psi \in \Phi_0$. Thus we see that for each $y \in \mathbf{R}^n \setminus \mathcal{A}$ there is a pair $(\psi_y, \varepsilon_y) \in \Phi_0 \times (0, \infty)$ such that $H[\psi_y] \le 0$ in $\mathbf{R}^n$ in the viscosity sense and $H[\psi_y] \le -\varepsilon_y$ in $\operatorname{int} B(y, \varepsilon_y)$ in the viscosity sense. By a compactness argument, we find a finite sequence $\{y_j\}_{j=1}^m$ such that $K \subset \bigcup_{j=1}^m \operatorname{int} B(y_j, \varepsilon_j)$, where $\varepsilon_j := \varepsilon_{y_j}$. We set $\varepsilon = \min\{\varepsilon_j \mid j = 1, 2, ..., m\}$ and

$$\phi_K(x) = \frac{1}{m} \sum_{j=1}^m \psi_j(x) \quad \text{for all } x \in \mathbf{R}^n, \text{ where } \psi_j := \psi_{y_j}.$$

It is easily seen that $H[\phi_K] \le 0$ in $\mathbf{R}^n$ in the viscosity sense, $H[\phi_K] \le -\varepsilon/m$ in a neighborhood of K in the viscosity sense, and $\phi_K \in \Phi_0$. □

Proposition 5.4. $\mathcal{A} \neq \emptyset$.

Proof. Suppose that $\mathcal{A} = \emptyset$. There is a constant $R > 0$ such that $H[\phi_1] \le -1$ in $\mathbf{R}^n \setminus B(0, R)$ in the viscosity sense. By Proposition 5.3, there are a function $\psi \in \Phi_0$ and a constant $\varepsilon \in (0, 1)$ such that $H[\psi] \le 0$ a.e. in $\mathbf{R}^n$ and $H[\psi] \le -\varepsilon$ a.e. in $B(0, R)$. By setting $v = \frac{1}{2}(\psi + \phi_1)$, we get a function $v \in C^{0+1}(\mathbf{R}^n)$ which satisfies $H[v] \le -\varepsilon/2$ a.e. in $\mathbf{R}^n$. Hence, by the definition of the additive eigenvalue c, we have $c \le -\varepsilon/2$. Since $c = 0$, we get a contradiction. □

Proposition 5.5. *Let $\phi \in C^{0+1}(\mathbf{R}^n)$ be a viscosity solution of $H[\phi] \le 0$ in $\mathbf{R}^n$, y a point in $\mathbf{R}^n$, and $\varepsilon > 0$ a constant. Assume that $H[\phi] \le -\varepsilon$ a.e. in $B(y, \varepsilon)$. Then $y \notin \mathcal{A}$.*

Proof. Let ϕ, y, and ε be as above. We argue by contradiction and suppose that $y \in \mathcal{A}$. Set $u = d_H(\cdot, y)$. By continuity, there is a constant $\delta > 0$ such that the function $v \in C^{0+1}(\mathbf{R}^n)$, defined by $v(x) = \phi(x) + \delta \min\{|x - y|, \varepsilon\}$, satisfies $H[v] \le 0$ a.e. in $\mathbf{R}^n$. By the definition of d_H, we have $u(x) \ge v(x) - v(y)$ for all $x \in \mathbf{R}^n$, which shows that

$u(x) > \phi(x) - \phi(y)$ for all $x \in \partial B(y, \varepsilon/2)$ and $u(y) = \phi(y) - \phi(y) = 0$. We approximate ϕ by a sequence of functions $\phi_k \in C^1(\mathbf{R}^n)$, with $k \in \mathbf{N}$, obtained by mollifying ϕ. Here, of course, the uniform convergence $\phi_k(x) \to \phi(x)$ is assumed on any compact subsets of $\mathbf{R}^n$ as $k \to \infty$. We may assume as well that $H[\phi_k] \le -\varepsilon/2$ on $B(y, \varepsilon/2)$. Noting that as $k \to \infty$,

$$\lim_{k\to\infty} \min_{x\in\partial B(y,\varepsilon/2)} (u(x) - \phi_k(x) - \phi_k(y)) \to \min_{x\in\partial B(y,\varepsilon/2)} (u(x) - \phi(x) - \phi(y)) > u(y) = 0,$$

we deduce that if k is sufficiently large, then $u - \phi_k$ attains a local minimum at a point $x_k \in B(y, \varepsilon/2)$. For such a k, since $H[u] \ge 0$ in $\mathbf{R}^n$ in the viscosity sense, we get

$$H(x_k, D\phi_k(x_k)) \ge 0.$$

On the other hand, by our choice of ϕ_k, we have

$$H(x, D\phi_k(x)) \le -\varepsilon/2 \quad \text{for all } x \in B(y, \varepsilon/2),$$

and, in particular, $H(x_k, D\phi_k(x_k)) \le -\varepsilon/2$. Thus we get a contradiction, which proves that $y \notin \mathcal{A}$. □

Proposition 5.6. *Let $y \in \mathbf{R}^n$. Then $y \notin \mathcal{A}$ if and only if there exist functions ϕ, $\sigma \in C(\mathbf{R}^n)$ such that $\sigma \ge 0$ in $\mathbf{R}^n$, $\sigma(y) > 0$, and ϕ is a viscosity subsolution of $H[\phi] \le -\sigma$ in $\mathbf{R}^n$.*

Proof. This is a consequence of Propositions 5.3 and 5.5. □

Proposition 5.7. *$\mathcal{A}$ is a compact subset of $\mathbf{R}^n$.*

Proof. Since $\lim_{|x|\to\infty} \sigma_1(x) = \infty$, ϕ_1 has the properties: $H[\phi_1] \le 0$ in $\mathbf{R}^n$ in the viscosity sense and $H[\phi_1] \le -1$ in $\mathbf{R}^n \setminus B(0, R)$ in the viscosity sense for some constant $R > 0$, which shows together with the previous proposition that $\mathcal{A} \subset B(0, R)$. Thus, in view of Proposition 5.2, we conclude that $\mathcal{A}$ is compact. □

Proposition 5.8. *Let $u \in \mathcal{S}_H$. Then*

$$u(x) = \inf\{u(y) + d_H(x, y) \mid y \in \mathcal{A}\} \quad \textit{for all } x \in \mathbf{R}^n. \tag{5.6}$$

Proof. We write $v(x)$ for the right hand side of (5.6). Since v is defined as the pointwise infimum of a family of viscosity solutions, the function v is a viscosity solution of $H[v] = 0$ in $\mathbf{R}^n$. Since $u(x) - u(y) \le d_H(x, y)$ for all $x, y \in \mathbf{R}^n$, we see that $u(x) \le v(x)$ for all $x \in \mathbf{R}^n$. On the other hand, for any $x \in \mathcal{A}$, we have $u(x) = u(x) + d_H(x, x) \ge v(x)$.

It remains to show that $u(x) \ge v(x)$ for all $x \in \mathbf{R}^n \setminus \mathcal{A}$. Fix any $\varepsilon > 0$. Choose a compact neighborhood V of $\mathcal{A}$ so that $v(x) \le u(x) + \varepsilon$ for all $x \in V$. Fix a constant $R > 0$ so that $H(x, D\phi_1(x)) \le -1$ a.e. $x \in \mathbf{R}^n \setminus B(0, R)$. By Proposition 5.3, there are a function $\psi \in C^{0+1}(\mathbf{R}^n)$ such that $H[\psi] \le 0$ a.e. in $\mathbf{R}^n$ and $H[\psi] \le -\delta$ a.e. in $B(0, R) \setminus V$

for some constant $\delta \in (0,1)$. We set $w(x) = \frac{1}{2}(\phi_1(x) + \psi(x))$ for all $x \in \mathbf{R}^n$ and observe that $H[w] \le -\frac{\delta}{2}$ a.e. in $\mathbf{R}^n \setminus V$. Let $\lambda \in (0,1)$ and set $v_\lambda(x) = (1-\lambda)v(x) + \lambda w(x) - 2\varepsilon$ for $x \in \mathbf{R}^n$. Observe that $H[v_\lambda] \le -\frac{\lambda\delta}{2}$ in $\mathbf{R}^n \setminus V$ and that for $\lambda \in (0,1)$ sufficiently small, $v_\lambda(x) \le u(x)$ for all $x \in V$. We apply Theorem 3.2, to get $v_\lambda(x) \le u(x)$ for all $x \in \mathbf{R}^n \setminus V$ and all λ sufficiently small. That is, if $\lambda \in (0,1)$ is sufficiently small, then we have $v_\lambda(x) \le u(x)$ for all $x \in \mathbf{R}^n$. From this, we find that $v(x) \le u(x)$ for all $x \in \mathbf{R}^n$. □

Proposition 5.9. *Let $S, T \in \mathbf{R}$ be such that $S < T$ and let $C > 0$ and $R > 0$. Let $\gamma \in \mathrm{AC}([S,T], \mathbf{R}^n)$ be such that*

$$\int_S^T L(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \le C, \qquad \gamma(S) \in B(0,R), \qquad \text{and} \qquad \gamma(T) \in B(0,R).$$

Then there is a constant $M_1 > 0$, depending only on C, R, $\min\{T-S, 1\}$, ϕ_0, ϕ_1, and σ_1, such that

$$|\gamma(t)| \le M_1 \quad \text{for all } t \in [S,T]. \tag{5.7}$$

Moreover, there is a constant $M_2 > 0$ and for each $\varepsilon > 0$ a constant $C_\varepsilon > 0$ such that for all $a, b \in [S,T]$ satisfying $a < b$,

$$\int_a^b |\dot\gamma(t)|\,\mathrm{d}t \le \varepsilon M_2 + C_\varepsilon(b-a).$$

Here the constants M_2 and C_ε depend only on M_1 and L.

Proof. We set $\tau = \frac{1}{2}\min\{T-S, 1\}$, so that $0 < 2\tau \le T - S$. We set

$$C_1 = C + 2\max_{B(0,R)} |\phi_1|,$$

and choose an $R_1 \ge R$ so that $\sigma_1(x) > \tau^{-1}C_1$ for all $x \in \mathbf{R}^n \setminus B(0,R_1)$. Next we set

$$C_2 = C + 3\max_{B(0,R_1)} |\phi_0| + 3\max_{B(0,R_1)} |\phi_1|,$$

and choose an $R_2 \ge R_1$ so that $(\phi_0 - \phi_1)(x) > C_2$ for all $x \in \mathbf{R}^n \setminus B(0,R_2)$.

Using Proposition 2.5, we get

$$\phi_1(\gamma(T)) - \phi_1(\gamma(S)) \le \int_S^T L(\gamma, \dot\gamma)\,\mathrm{d}t - \int_S^T \sigma_1(\gamma(t))\,\mathrm{d}t. \tag{5.8}$$

Hence we get

$$\int_S^T \sigma_1(\gamma)\,\mathrm{d}t \le C + 2\max_{B(0,R)} |\phi_1| = C_1.$$

Fix any $t \in [S+\tau, T]$. Noting that

$$\int_{t-\tau}^t \sigma_1(\gamma(s))\,\mathrm{d}s \le C_1,$$

we choose an $a \in [S,\, t]$ so that $\sigma_1(\gamma(a)) \le \tau^{-1}C_1$, which guarantees, by our choice of R_1, that $\gamma(a) \in B(0, R_1)$.

Using Proposition 2.5 again, we get

$$\begin{aligned} \phi_1(\gamma(T)) - \phi_1(\gamma(t)) &\le \int_t^T L(\gamma, \dot\gamma)\,\mathrm{d}s, \\ \phi_0(\gamma(t)) - \phi_0(\gamma(a)) &\le \int_a^t L(\gamma, \dot\gamma)\,\mathrm{d}s, \\ \phi_1(\gamma(a)) - \phi_1(\gamma(S)) &\le \int_S^a L(\gamma, \dot\gamma)\,\mathrm{d}s. \end{aligned}$$

Adding these, we get

$$\phi_0(\gamma(t)) - \phi_1(\gamma(t)) \le \int_S^T L(\gamma, \dot\gamma)\,\mathrm{d}t + 3 \max_{B(0,R_1)} |\phi_1| + \max_{B(0,R_1)} |\phi_0| \le C_2.$$

Therefore, by the choice of R_2, we obtain

$$\gamma(t) \in B(0, R_2).$$

Now let $t \in [S, S+\tau]$. Since

$$\int_t^{t+\tau} \sigma_1(\gamma(s))\,\mathrm{d}s \le C_1,$$

we may choose a $b \in [t,\, t+\tau]$ such that $\sigma_1(\gamma(b)) \le \tau^{-1}C_1$, which implies that $\gamma(b) \in B(0, R_1)$. As above, we get

$$\begin{aligned} \phi_0(\gamma(T)) - \phi_0(\gamma(b)) &\le \int_b^T L(\gamma, \dot\gamma)\,\mathrm{d}s, \\ \phi_1(\gamma(b)) - \phi_1(\gamma(t)) &\le \int_t^b L(\gamma, \dot\gamma)\,\mathrm{d}s, \\ \phi_0(\gamma(t)) - \phi_0(\gamma(S)) &\le \int_S^t L(\gamma, \dot\gamma)\,\mathrm{d}s, \end{aligned}$$

and moreover

$$\phi_0(\gamma(t)) - \phi_1(\gamma(t)) \le \int_S^T L(\gamma, \dot\gamma)\,\mathrm{d}t + 3 \max_{B(0,R_1)} |\phi_0| + \max_{B(0,R_1)} |\phi_1| \le C_2,$$

which guarantees that $\gamma(t) \in B(0, R_2)$. Thus, setting $M_1 = R_2$, we see that (5.7) holds.

Fix any $a, b \in [S,\, T]$ satisfying $a < b$ and any $\varepsilon > 0$. As before, we have

$$\phi_0(\gamma(T)) - \phi_0(\gamma(b)) + \int_a^b L(\gamma, \dot\gamma)\,\mathrm{d}t + \phi_0(\gamma(a)) - \phi_0(\gamma(S)) \le \int_S^T L(\gamma, \dot\gamma)\,\mathrm{d}t \le C,$$

from which we get

$$\int_a^b L(\gamma, \dot\gamma)\,\mathrm{d}t \le C + 4 \max_{B(0,M_1)} |\phi_0|. \tag{5.9}$$

Setting

$$C_3 := C + 4 \max_{B(0,M_1)} |\phi_0| \quad \text{and} \quad M_\varepsilon := \max_{(x,p)\in B(0,M_1)\times B(0,\varepsilon^{-1})} |H(x,p)|$$

and noting that for $(x,\xi) \in B(0,M_1) \times \mathbf{R}^n$,

$$L(x,\xi) \geq \max_{p\in B(0,\varepsilon^{-1})} [\xi \cdot p - H(x,p)] \geq \max_{p\in B(0,\varepsilon^{-1})} \xi \cdot p - M_\varepsilon = \varepsilon^{-1}|\xi| - M_\varepsilon,$$

we get from (5.9)

$$\varepsilon^{-1} \int_a^b |\dot\gamma(t)|\,\mathrm{d}t \leq C_3 + M_\varepsilon(b-a),$$

that is, we have

$$\int_a^b |\dot\gamma(t)|\,\mathrm{d}t \leq \varepsilon C_3 + \varepsilon M_\varepsilon(b-a).$$

This completes the proof. □

Proposition 5.10. *Let $y \in \mathbf{R}^n$. The following conditions are equivalent:*

(1) $y \in \mathcal{A}$.

(2) $\inf\Big\{\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \mid t \geq \delta,\ \gamma \in \mathcal{C}(y,t;y,0)\Big\} = 0$ *for some* $\delta > 0$.

(3) $\inf\Big\{\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \mid t \geq \delta,\ \gamma \in \mathcal{C}(y,t;y,0)\Big\} = 0$ *for any* $\delta > 0$.

We remark here in view of Proposition 5.1 that for any $y \in \mathbf{R}^n$,

$$0 = d_H(y,y) = \inf\Big\{\int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \ \Big|\ t > 0,\ \gamma \in \mathcal{C}(y,t;y,0)\Big\}.$$

In particular, we have $L(x,0) \geq 0$ for all $x \in \mathbf{R}^n$.

Proof. We start by observing that for any $y \in \mathbf{R}^n$, $t > 0$, and $\gamma \in \mathcal{C}(y,t;y,0)$,

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \geq \phi_1(\gamma(t)) - \phi_1(\gamma(0)) = 0.$$

It is easy to see that (2) and (3) are equivalent each other. Thus it is enough to prove that (1) implies (2) and that (3) implies (1).

We assume that $y \notin \mathcal{A}$, and will show that (3) does not hold. In view of Proposition 5.3, there is a function $\psi \in C^{0+1}(\mathbf{R}^n)$ and a constant $\delta > 0$ such that $H[\psi] \leq 0$ a.e. in $\mathbf{R}^n$ and $H[\psi] \leq -\delta$ a.e. in $B(y,2\delta)$. Let $t > 0$ and $\gamma \in \mathcal{C}(y,t;y,0)$ be such that

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s < 1.$$

We select a function $f \in C(\mathbf{R}^n)$ so that $0 \leq f \leq \delta$ in $\mathbf{R}^n$, $f(x) \geq \delta$ for all $x \in B(y,\delta)$, and $f(x) = 0$ for all $x \in \mathbf{R}^n \setminus B(y,2\delta)$. Then, noting that $H[\psi] \leq -f$ in $\mathbf{R}^n$ in the viscosity sense, by virtue of Proposition 2.5, we have

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \geq \psi(\gamma(t)) - \psi(\gamma(0)) + \int_0^t f(\gamma(s))\,\mathrm{d}s \geq \delta|I|,$$

where $I = \{s \in [0,t] \mid \gamma(s) \in B(y,\delta)\}$ and $|I|$ denotes the one-dimensional Lebesgue measure of I. By Proposition 5.9, there is a constant $C_\delta > 0$, depending only on δ, L, y, ϕ_0, and ϕ_1, such that

$$\int_0^t |\dot\gamma(s)|\,\mathrm{d}s \le \frac{\delta}{2} + C_\delta t.$$

Therefore, setting $\tau = \delta/(2C_\delta)$, we see that if $t \ge \tau$, then $\gamma(s) \in B(y,\delta)$ for all $s \in [0,\tau]$. Accordingly, if $t \ge \tau$, we have

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \ge \delta\tau.$$

This shows that (3) does not hold.

Next we suppose that (2) does not hold, and will show that $y \notin \mathcal{A}$. We see immediately from this assumption that $L(y,0) > 0$, which implies that $\min_{p\in\mathbf{R}^n} H(y,p) = H(y,q) < 0$ for some $q \in \mathbf{R}^n$. By Proposition 2.1, there are constants $\varepsilon > 0$ and $C > 0$ such that $L(x,p) \le C$ for all $(x,p) \in B(y,\varepsilon) \times B(0,\varepsilon)$. We may assume as well that

$$d_H(x,y) < 1 \quad \text{and} \quad H(x,q) \le 0 \qquad \text{for all } x \in B(y,\varepsilon).$$

Let $r \in (0,\varepsilon)$ be a constant to be fixed later on. Fix $x \in B(y,r) \setminus \{y\}$, $t > 0$, and $\gamma \in \mathcal{C}(x,t;y,0)$ so that

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s < 1.$$

According to Proposition 5.9, there is a constant $C_\varepsilon > 0$, independent of the choice of γ, such that

$$\int_0^t |\dot\gamma(s)|\,\mathrm{d}s < \frac{\varepsilon}{2} + C_\varepsilon t.$$

In particular, there is a constant $\tau > 0$ (for instance, we may choose $\tau = \varepsilon/(2C_\varepsilon)$) such that $\gamma(s) \in B(y,\varepsilon)$ for all $s \in [0, \min\{t,\tau\}]$.

Since (2) does not hold, we may choose a constant $a > 0$ so that

$$\inf\Big\{\int_0^T L(\eta,\dot\eta)\,\mathrm{d}s \mid T \ge \tau,\ \eta \in \mathcal{C}(y,t;y,0)\Big\} > a.$$

We divide our considerations into two cases. The first case is when $t \le \tau$. Then we have $\gamma(s) \in B(y,\varepsilon)$ for all $s \in [0,t]$ and hence

$$\begin{aligned} q\cdot(x-y) &= q\cdot(\gamma(t)-\gamma(0)) = \int_0^t q\cdot\dot\gamma(s)\,\mathrm{d}s \\ &\le \int_0^t [L(\gamma(s),\dot\gamma(s)) + H(\gamma(s),q)]\,\mathrm{d}s \le \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s. \end{aligned}$$

In the other case when $t > \tau$, we define $\eta \in \mathcal{C}(y, t+\varepsilon^{-1}|y-x|; y,0)$ by

$$\eta(s) = \begin{cases} \gamma(s) & \text{for } s \in [0,\, t], \\ x + (s-t)\varepsilon|y-x|^{-1}(y-x) & \text{for } s \in [t,\, t+\varepsilon^{-1}|x-y|]. \end{cases}$$

Noting that $(\eta(s), \dot\eta(s)) \in B(y,r) \times B(0,\varepsilon)$ for all $s \in (t,\, t+\varepsilon^{-1}|x-y|)$, we have

$$
\begin{aligned}
a &\le \int_0^{t+\varepsilon^{-1}|x-y|} L(\gamma,\dot\gamma)\,\mathrm{d}s = \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s + \int_t^{t+\varepsilon^{-1}|x-y|} L(\eta(s),\dot\eta(s))\,\mathrm{d}s \\
&\le \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s + C\varepsilon^{-1}|x-y| \le \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s + C\varepsilon^{-1}r.
\end{aligned}
$$

We fix $r \in (0,\varepsilon)$ so that $C\varepsilon^{-1}r \le \frac{a}{2}$. Consequently we get

$$
\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \ge \frac{a}{2}.
$$

Hence we have

$$
\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \ge \min\{p\cdot(x-y), a/2\},
$$

from which we get

$$
\min\{q\cdot(x-y), a/2\} \le d_H(x,y) \quad \text{ for all } x \in B(y,r).
$$

This shows that $q \in D_1^- d_H(y,y)$. Since $H(y,q) < 0$, we conclude that $y \notin \mathcal{A}$. □

We need the following theorem.

Theorem 5.11. *Let $y \in \mathcal{A}$. Then there is a curve $\gamma : \mathbf{R} \to \mathcal{A}$ such that $\gamma(0) = y$ and*

$$
d_H(\gamma(b),\gamma(a)) = \int_a^b L(\gamma(t),\dot\gamma(t))\,\mathrm{d}t = -d_H(\gamma(a),\gamma(b)). \tag{5.10}
$$

for any $a, b \in \mathbf{R}$, with $a < b$.

According to [DS], curves satisfying (5.10) are called *critical curves* for the Lagrangian L (or for the Hamiltonian H).

We need the following proposition to prove Theorem 5.11.

Proposition 5.12. *Let $S, T \in \mathbf{R}$ be such that $S < T$ and let $\{\gamma_k\}_{k\in\mathbf{N}} \subset \mathrm{AC}([S,T],\mathbf{R}^n)$ be a sequence converging to a function $\gamma \in C([S,T],\mathbf{R}^n)$ in the topology of uniform convergence. Assume that*

$$
\liminf_{k\to\infty} \int_S^T L(\gamma_k(t),\dot\gamma_k(t))\,\mathrm{d}t < \infty.
$$

Then $\gamma \in \mathrm{AC}([S,\, T],\mathbf{R}^n)$ and

$$
\int_S^T L(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \le \liminf_{k\to\infty} \int_S^T L(\gamma_k(t),\dot\gamma_k(t))\,\mathrm{d}t. \tag{5.11}
$$

Proof. We choose a constant $R > 0$ so that $|\gamma_k(t)| \le R$ for all $t \in [S,T]$ and all $k \in \mathbf{N}$. Passing to a subsequnece of $\{\gamma_k\}_{k\in\mathbf{N}}$ if necessary, we may assume that there is a constant $C > 0$ such that

$$
\int_S^T L(\gamma_k(t),\dot\gamma_k(t))\,\mathrm{d}t \le C \quad \text{ for all } k \in \mathbf{N}. \tag{5.12}
$$

We choose a constant $C_1 > 0$ so that $H(x, 0) \le C_1$ for all $x \in B(0, R)$, which guarantees that $L(x, \xi) \ge -C_1$ for all $(x, \xi) \in B(0, R) \times \mathbf{R}^n$. For each $\varepsilon > 0$ we set

$$M(\varepsilon) = \max\{|H(x,p)| \mid (x,p) \in B(0,R) \times B(0,\varepsilon^{-1})\},$$

so that for $(x, \xi) \in B(0, R) \times \mathbf{R}^n$,

$$L(x,\xi) \ge \max\{\xi \cdot p - H(x,p) \mid p \in B(0,\varepsilon^{-1})\} \ge \varepsilon^{-1}|\xi| - M(\varepsilon).$$

Now, we let $B \subset [S, T]$ be measurable and $k \in \mathbf{N}$, and observe by (5.12) that

$$\int_B (L(\gamma_k(t), \dot\gamma_k(t)) + C_1)\,\mathrm{d}t \le C + C_1(T - S),$$

and consequently

$$\int_B (\varepsilon^{-1}|\dot\gamma_k(t)| + C_1 - M(\varepsilon))\,\mathrm{d}t \le C + C_1(T - S).$$

Hence we have

$$\int_B |\dot\gamma_k(t)|\,\mathrm{d}t \le \varepsilon(C + C_1(T - S)) + \varepsilon M(\varepsilon)|B|.$$

Reselecting $M(\varepsilon) > 0$ if necessary, we may replace this estimate by

$$\int_B |\dot\gamma_k(t)|\,\mathrm{d}t \le \varepsilon + M(\varepsilon)|B|. \tag{5.13}$$

We deduce from (5.13) that for any $\varepsilon > 0$ and any mutually disjoint intervals $[a_i, b_i] \subset [S, T]$, with $i = 1, 2, ..., m$,

$$\sum_{i=1}^m |\gamma(b_i) - \gamma(a_i)| \le \varepsilon + M(\varepsilon)\sum_{i=1}^m (b_i - a_i),$$

which shows that $\gamma \in \mathrm{AC}([S, T], \mathbf{R}^n)$ and

$$\int_B |\dot\gamma(t)|\,\mathrm{d}t \le \varepsilon + M(\varepsilon)|B| \tag{5.14}$$

for any measurable subset B of $[S, T]$.

Next let $f \in \mathrm{AC}([S, T], \mathbf{R}^n)$ and observe by using integration by parts that as $k \to \infty$

$$\begin{aligned}\int_S^T f(t)\cdot\dot\gamma_k(t)\,\mathrm{d}t &= (f\cdot\gamma_k)(T) - (f\cdot\gamma_k)(S) - \int_S^T \dot f(t)\cdot\gamma_k(t)\,\mathrm{d}t\\ &\to (f\cdot\gamma)(T) - (f\cdot\gamma)(S) - \int_S^T \dot f(t)\cdot\gamma(t)\,\mathrm{d}t\\ &= \int_S^T f(t)\cdot\dot\gamma(t)\,\mathrm{d}t.\end{aligned}$$

Now, we introduce the Lagrangian L_α, with $\alpha > 0$, as follows. Fix $\alpha > 0$ and define the function $H_\alpha : \mathbf{R}^{2n} \to (0, \infty]$ by

$$H_\alpha(x,p) = H(x,p) + \delta_{B(0,\alpha)}(p),$$

where δ_C denotes the indicator function of $C \subset \mathbf{R}^n$ defined by $\delta_C(p) = 0$ if $p \in C$ and $= \infty$ otherwise, and the function $L_\alpha : \mathbf{R}^{2n} \to \mathbf{R}$ as the Lagrangian of H_α, that is, $L_\alpha(x,\xi) = \sup\{\xi \cdot p - H_\alpha(x,p) \mid p \in \mathbf{R}^n\}$ for $(x,\xi) \in \mathbf{R}^{2n}$. It is easy to see that, for all $(x,\xi) \in \mathbf{R}^{2n}$, $L_\alpha(x,\xi) \leq L_\beta(x,\xi) \leq L(x,\xi)$ if $\alpha < \beta$, that $\lim_{\alpha\to\infty} L_\alpha(x,\xi) = L(x,\xi)$ for all $(x,\xi) \in \mathbf{R}^{2n}$, and that for any $(x,\xi) \in \mathbf{R}^{2n}$, if $p \in D_2^- L_\alpha(x,\xi)$, then $|p| \leq \alpha$. Also, as is well-known, for any $\alpha > 0$, L_α is differentiable in the last n variables everywhere and L_α and $D_2 L_\alpha$ are continuous on $\mathbf{R}^{2n}$. In view of the monotone convergence theorem, in order to prove (5.11), we need only to show that for any $\alpha > 0$,

$$\int_S^T L_\alpha(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \leq \liminf_{k\to\infty} \int_S^T L(\gamma_k(t), \dot\gamma_k(t))\,\mathrm{d}t \tag{5.15}$$

To show (5.15), we fix $\alpha > 0$ and note by convexity that for a.e. $t \in (S,T)$ and any $k \in \mathbf{N}$,

$$L_\alpha(\gamma_k(t), \dot\gamma_k(t)) \geq L_\alpha(\gamma_k(t), \dot\gamma(t)) + D_2 L_\alpha(\gamma_k(t), \dot\gamma(t)) \cdot (\dot\gamma_k(t) - \dot\gamma(t)).$$

Since

$$|L_\alpha(\gamma_k(t), \dot\gamma(t))| \leq |L_\alpha(\gamma_k(t), 0)| + \alpha|\dot\gamma(t)| \leq \max_{x\in B(0,\alpha)} |L_\alpha(x,0)| + \alpha|\dot\gamma(t)| \in L^1(S,T),$$

by the Lebesgue dominated convergence theorem, we get

$$\lim_{k\to\infty} \int_S^T L_\alpha(\gamma_k(t), \dot\gamma(t))\,\mathrm{d}t = \int_S^T L_\alpha(\gamma(t), \dot\gamma(t))\,\mathrm{d}t.$$

Next, we set $f_k(t) = D_2 L_\alpha(\gamma_k(t), \dot\gamma(t))$ and $f(t) = D_2 L_\alpha(\gamma(t), \dot\gamma(t))$ for $t \in [S,T]$ and $k \in \mathbf{N}$. Then f_k, $f \in L^\infty(S,T,\mathbf{R}^n)$ for all $k \in \mathbf{N}$, and $|f_k(t)| \leq \alpha$ and $|f(t)| \leq \alpha$ a.e. $t \in (S,T)$ for all $k \in \mathbf{N}$. We may choose a sequence $\{g_j\}_{j\in\mathbf{N}} \subset \mathrm{AC}([S,T],\mathbf{R}^n)$ so that $g_j(t) \to f(t)$ a.e. $t \in (S,T)$ as $j \to \infty$ and $|g_j(t)| \leq \alpha$ for all $t \in [S,T]$, $j \in \mathbf{N}$. Note that $f_k(t) \to f(t)$ a.e. $t \in (S,T)$ as $k \to \infty$ and recall that the almost everwhere convergence implies the convergence in measure. For each $\varepsilon > 0$ we set

$$\mu(\varepsilon,k) = |\{t \in (S,T) \mid |(f_k - f)(t)| > \varepsilon\}| \quad \text{for } k \in \mathbf{N},$$
$$\nu(\varepsilon,j) = |\{t \in (S,T) \mid |(g_j - f)(t)| > \varepsilon\}| \quad \text{for } j \in \mathbf{N},$$

and observe that $\lim_{k\to\infty} \mu(\varepsilon,k) = \lim_{j\to\infty} \nu(\varepsilon,j) = 0$ for any $\varepsilon > 0$.

Fix any $\varepsilon > 0$, $\delta > 0$, and $k,j \in \mathbf{N}$. Observing that

$$|\{t \in (S,T) \mid |(f_k - g_j)(t)| > 2\varepsilon\}| \leq \mu(\varepsilon,k) + \nu(\varepsilon,j)$$

and using (5.14) with ε replaced by δ or 1, we get

$$\begin{aligned}\left|\int_S^T (f_k - g_j)(t) \cdot \dot\gamma_k(t)\,\mathrm{d}t\right| &\leq \int_{|f_k-g_j|>2\varepsilon} 2\alpha|\dot\gamma_k(t)|\,\mathrm{d}t + \int_{|f_k-g_j|\leq 2\varepsilon} 2\varepsilon|\dot\gamma_k(t)|\,\mathrm{d}t \\ &\leq 2\alpha[\delta + M(\delta)(\mu(\varepsilon,k) + \nu(\varepsilon,j))] + 2\varepsilon(1 + M(1)(T-S)).\end{aligned}$$

Similarly we get

$$\left|\int_S^T (g_j - f)(t)\cdot\dot\gamma(t)\,\mathrm{d}t\right| \le \int_{|g_j-f|>\varepsilon} 2\alpha|\dot\gamma(t)|\,\mathrm{d}t + \int_{|g_j-f|\le\varepsilon} \varepsilon|\dot\gamma(t)|\,\mathrm{d}t$$
$$\le 2\alpha(\delta + M(\delta)\nu(\varepsilon,j)) + \varepsilon(1+M(1)(T-S)).$$

Hence we have

$$\left|\int_S^T (f_k\cdot\dot\gamma_k - f\cdot\dot\gamma)\,\mathrm{d}t\right| \le 4\alpha(\delta + M(\delta)(\mu(\varepsilon,k)+\nu(\varepsilon,j))) + 3\varepsilon(1+M(1)(T-S))$$
$$+\left|\int_S^T g_j\cdot(\dot\gamma_k-\dot\gamma)\,\mathrm{d}t\right|.$$

Now, since $g_j \in \mathrm{AC}([S,T],\mathbf{R}^n)$, we have

$$\lim_{k\to\infty}\int_S^T g_j\cdot(\dot\gamma_k-\dot\gamma)\,\mathrm{d}t = 0,$$

and hence

$$\limsup_{k\to\infty}\left|\int_S^T (f_k\cdot\dot\gamma_k - f\cdot\dot\gamma)\,\mathrm{d}t\right| \le 4\alpha(\delta+M(\delta)\nu(\varepsilon,j)) + 3\varepsilon(1+M(1)(T-S))$$

for any $\varepsilon>0$, $\delta>0$, and $j\in\mathbf{N}$. Sending $j\to\infty$ and then ε, $\delta\to 0$, we see that

$$\lim_{k\to\infty}\int_S^T D_2L_\alpha(\gamma_k(t),\dot\gamma(t))\cdot\dot\gamma_k(t)\,\mathrm{d}t = \int_S^T D_2L_\alpha(\gamma(t),\dot\gamma(t))\cdot\dot\gamma(t)\,\mathrm{d}t.$$

Finally, noting by the Lebesgue dominated convergence theorem that

$$\lim_{k\to\infty}\int_S^T D_2L_\alpha(\gamma_k(t),\dot\gamma(t))\cdot\dot\gamma(t)\,\mathrm{d}t = \int_S^T D_2L_\alpha(\gamma(t),\dot\gamma(t))\cdot\dot\gamma(t)\,\mathrm{d}t,$$

we conclude that

$$\lim_{k\to\infty}\int_S^T \left(L_\alpha(\gamma_k(t),\dot\gamma(t)) + D_2L_\alpha(\gamma_k(t),\dot\gamma(t))\cdot(\dot\gamma_k(t)-\dot\gamma(t))\right)\,\mathrm{d}t = \int_S^T L(\gamma(t),\dot\gamma(t))\,\mathrm{d}t,$$

from which it follows that

$$\int_S^T L_\alpha(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \le \liminf_{k\to\infty}\int_S^T L_\alpha(\gamma_k(t),\dot\gamma_k(t))\,\mathrm{d}t \le \liminf_{k\to\infty}\int_S^T L(\gamma_k(t),\dot\gamma_k(t))\,\mathrm{d}t,$$

completing the proof. □

Proof of Theorem 5.11. Let $k\in\mathbf{N}$. By Proposition 5.10, we may find a curve $\gamma_k\in\mathrm{AC}([0,T_k],\mathbf{R}^n)$, where $T_k\ge k$, such that

$$\gamma_k(0)=\gamma_k(T_k)=y, \qquad \int_0^{T_k} L(\gamma_k(t),\dot\gamma_k(t))\,\mathrm{d}t < \frac{1}{k}.$$

Define the curve $\eta_k : [-T_k,\, T_k] \to \mathbf{R}^n$ by

$$\eta_k(t) = \begin{cases} \gamma_k(t) & \text{if } t \in [0, T_k], \\ \gamma_k(t + T_k) & \text{if } t \in [-T_k,\, 0], \end{cases}$$

and observe that

$$\eta_k(-T_k) = \eta_k(T_k) = \eta_k(0) = y,$$
$$\int_{-T_k}^{T_k} L(\eta_k(t), \dot\eta_k(t))\,\mathrm{d}t = 2\int_0^{T_k} L(\gamma_k(t), \dot\gamma_k(t))\,\mathrm{d}t < \frac{2}{k}. \tag{5.16}$$

From this, using Proposition 5.9, we see that the sequence $\{\eta_k\}_{k\geq j}$ is uniformly bounded and equi-continuous on $[-j, j]$ for any $j \in \mathbf{N}$. By the Ascoli-Arzela theorem, we may assume by passing to a subsequence if necessary that the sequence $\{\eta_k\}_{k\in\mathbf{N}}$ is convergent to a function $\eta \in C(\mathbf{R}, \mathbf{R}^n)$ in the topology of uniform convergence on bounded sets. Since $\eta_k(0) = y$ for all $k \in \mathbf{N}$, we have $\eta(0) = y$.

In view of Proposition 5.9, we may choose a constant $M > 0$ so that $\eta_k(t) \in B(0, M)$ for all $t \in [-T_k, T_k]$ and all $k \in \mathbf{N}$. Fix any $a, b \in \mathbf{R}$ such that $a < b$. Let $k \in \mathbf{N}$ be such a large number that $-k < a < b < k$. By Proposition 2.5, we have

$$d_H(\eta_k(T_k), \eta_k(b)) \leq \int_b^{T_k} L(\eta_k(t), \dot\eta_k(t))\,\mathrm{d}t,$$
$$d_H(\eta_k(a), \eta_k(-T_k)) \leq \int_{-T_k}^{a} L(\eta_k(t), \dot\eta_k(t))\,\mathrm{d}t.$$

Using these, we get

$$\int_{-T_k}^{T_k} L(\eta_k(t), \dot\eta_k(t))\,\mathrm{d}t \geq d_H(\eta_k(a), y) + \int_a^b L(\eta_k(t), \dot\eta_k(t))\,\mathrm{d}t + d_H(y, \eta_k(b)). \tag{5.17}$$

Due to the coercivity of H, the family $\{d_H(\cdot, y) \mid y \in B(0, M)\}$ is equi-Lipschitz continuous on $B(0, M)$. Thus, there is a constant $C > 0$ such that $|d_H(x, y)| \leq C$ for all $x, y \in B(0, M)$. We now get from (5.17)

$$\int_a^b L(\eta_k(t), \dot\eta_k(t))\,\mathrm{d}t \leq \frac{2}{k} + 2C.$$

Now, by Proposition 5.12, we see that $\eta \in \mathrm{AC}([-j, j], \mathbf{R}^n)$ for any $j \in \mathbf{N}$ and that for any $a, b \in \mathbf{R}$, with $a < b$,

$$\int_a^b L(\eta(t), \dot\eta(t))\,\mathrm{d}t \leq \liminf_{k\to\infty} \int_a^b L(\eta_k(t), \dot\eta_k(t))\,\mathrm{d}t. \tag{5.18}$$

Going back to (5.17) and using the inequality

$$d_H(\eta(b), \eta(a)) \leq \int_a^b L(\eta(t), \dot\eta(t))\,\mathrm{d}t,$$

we obtain

$$\begin{aligned}0 &\geq d_H(\eta(a), y) + \int_a^b L(\eta(t), \dot\eta(t))\,\mathrm{d}t + d_H(y, \eta(b)) \\ &\geq \int_a^b L(\eta(t), \dot\eta(t))\,\mathrm{d}t + d_H(\eta(a), \eta(b)) \\ &\geq d_H(\eta(b), \eta(a)) + d_H(\eta(a), \eta(b)) \geq d_H(\eta(a), \eta(a)) = 0.\end{aligned}$$

From this, we see that

$$\int_a^b L(\eta(t), \dot\eta(t))\,\mathrm{d}t = d_H(\eta(b), \eta(a)) = -d_H(\eta(a), \eta(b)).$$

To complete the proof, we show that $\eta(t) \in \mathcal{A}$ for all $t \in \mathbf{R}$. Fix any $z \in \mathbf{R}^n \setminus \mathcal{A}$ and, in view of Proposition 5.3, select a $\phi \in \mathcal{S}_H^-$ so that $H[\phi] \leq -\varepsilon$ in a neighborhood of z in the viscosity sense, where $\varepsilon > 0$ is a constant. We choose a function $\sigma \in C(\mathbf{R}^n)$, which satisfies $\sigma \geq 0$ in $\mathbf{R}^n$ and $\sigma(z) > 0$, so that $H[\phi] \leq -\sigma$ in $\mathbf{R}^n$ in the viscosity sense. By Proposition 2.5, we get

$$\begin{aligned}\int_{-T_k}^{T_k} \sigma(\eta_k(t))\,\mathrm{d}t &= \phi(\eta_k(T_k)) - \phi(\eta_k(-T_k)) + \int_{-T_k}^{T_k} \sigma(\eta_k(t))\,\mathrm{d}t \\ &\leq \int_{-T_k}^{T_k} L(\eta_k(t), \dot\eta_k(t))\,\mathrm{d}t < \frac{2}{k}.\end{aligned}$$

From this we deduce that

$$\int_{\mathbf{R}} \sigma(\eta(t))\,\mathrm{d}t = 0,$$

which guarantees that $\eta(t) \neq z$ for all $t \in \mathbf{R}$. This is enough to conclude that $\eta(t) \in \mathcal{A}$ for all $t \in \mathbf{R}$. □

Proposition 5.13. *Let $\gamma \in C(\mathbf{R}, \mathbf{R}^n)$ be a curve. The following three conditions are equivalent:*

(1) *γ is critical.*

(2) *For any $\phi \in \mathcal{S}^-$ and $a, b \in \mathbf{R}$ satisfying $a < b$,*

$$\phi(\gamma(b)) - \phi(\gamma(a)) = \int_a^b L(\gamma(t), \dot\gamma(t))\,\mathrm{d}t.$$

(3) *For any $a, b \in \mathbf{R}$ satisfying $a < b$,*

$$-d_H(\gamma(a), \gamma(b)) = \int_a^b L(\gamma(t), \dot\gamma(t))\,\mathrm{d}t.$$

Proof. We define the function $\rho_H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$\rho_H(x, y) = \inf\{v(x) \mid v \in \mathcal{S}_H^-,\ v(y) = 0\},$$

and observe that $\phi(x) - \phi(y) \geq \rho_H(x, y)$ for all $x, y \in \mathbf{R}^n$ and all $\phi \in \mathcal{S}_H^-$, that $\rho_H(\cdot, y) \in \mathcal{S}_H^-$ for all $y \in \mathbf{R}^n$, that for any $x, y \in \mathbf{R}^n$,

$$\rho_H(x, y) = \inf\{v(x) - v(y) \mid v \in \mathcal{S}_H^-\} = -\sup\{v(y) - v(x) \mid v \in \mathcal{S}_H^-\} = -d_H(y, x).$$

Now, we assume that (3) holds. Let $\phi \in \mathcal{S}_H^-$ and fix $a, b \in \mathbf{R}$ so that $a < b$. Using the above observations, we get

$$\int_a^b L(\gamma(t), \dot\gamma(t))\,\mathrm{d}t = -d_H(\gamma(a), \gamma(b)) = \rho_H(\gamma(b), \gamma(a)) \leq \phi(\gamma(b)) - \phi(\gamma(a)).$$

Combining this with Proposition 2.5, we see that (2) holds.

Next, it is clear that (2) implies (1) since $d_H(\cdot, \gamma(a)), -d_H(\gamma(a), \cdot) \in \mathcal{S}_H^-$. Also, it is clear by the definition of critical curves that (1) implies (3). □

Proposition 5.14. *Let γ be a critical curve. Then there is a function $q \in L^\infty(\mathbf{R}, \mathbf{R}^n)$ such that for any $\phi \in \mathcal{S}_H^-$,*

$$L(\gamma(t), \dot\gamma(t)) = q(t) \cdot \dot\gamma(t) \quad a.e.\ t \in \mathbf{R}, \tag{5.19}$$

$$H(\gamma(t), q(t)) = 0 \quad a.e.\ t \in \mathbf{R}, \tag{5.20}$$

$$q(t) \in \partial_c \phi(\gamma(t)) \quad a.e.\ t \in \mathbf{R}. \tag{5.21}$$

Proof. Fix any critical curve γ. It is enough to show that for each $k \in \mathbf{N}$ there is a function $q \in L^\infty(k, k+1, \mathbf{R})$ satisfying (5.19), (5.20), and (5.21), with $\mathbf{R}$ replaced by the interval $(k, k+1)$.

Fix any $k \in \mathbf{N}$ and any $\phi \in \mathcal{S}^-$. We write $a = k$ and $b = k+1$ for notational simplicity. By Proposition 2.4, there is a function $q \in L^\infty(a, b, \mathbf{R}^n)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t)) = q(t) \cdot \dot\gamma(t) \quad \text{a.e. } t \in (a, b), \tag{5.22}$$

$$q(t) \in \partial_c \phi(\gamma(t)) \quad \text{a.e. } t \in (a, b). \tag{5.23}$$

Since $\phi \in \mathcal{S}_H^-$, in view of (5.23) we get

$$H(\gamma(t), q(t)) \leq 0 \quad \text{a.e. } t \in (a, b). \tag{5.24}$$

Integrating (5.22) over (a, b) and using Proposition 5.13 yield

$$\begin{aligned}\phi(\gamma(b)) - \phi(\gamma(a)) &= \int_a^a q(t) \cdot \dot\gamma(t)\,\mathrm{d}t \leq \int_a^b [L(\gamma(t), \dot\gamma(t)) + H(\gamma(t), q(t))]\,\mathrm{d}t \\ &\leq \int_a^b L(\gamma(t), \dot\gamma(t))\,\mathrm{d}t = \phi(\gamma(b)) - \phi(\gamma(a)).\end{aligned}$$

This shows that

$$\int_a^b q(t) \cdot \dot\gamma(t)\,\mathrm{d}t = \int_a^b [L(\gamma(t), \dot\gamma(t)) + H(\gamma(t), q(t))]\,\mathrm{d}t = \int_a^b L(\gamma(t), \dot\gamma(t))\,\mathrm{d}t. \tag{5.25}$$

In particular, we get

$$\int_a^b H(\gamma(t), q(t))\,\mathrm{d}t = 0,$$

which together with (5.24) yields

$$H(\gamma(t), q(t)) = 0 \quad \text{ a.e. } t \in (a, b).$$

Similarly, since

$$q(t) \cdot \dot\gamma(t) \le L(\gamma(t), \dot\gamma(t)) + H(\gamma(t), q(t)) = L(\gamma(t), \dot\gamma(t)) \quad \text{ a.e. } t \in (a, b),$$

from (5.25) we see that

$$q(t) \cdot \dot\gamma(t) = L(\gamma(t), \dot\gamma(t)) \quad \text{ a.e. } t \in (a, b).$$

Thus the function q satisfies conditions (5.19), (5.20), and (5.21) in (a, b), with our current choice of ϕ. We need to show that for any $\psi \in \mathcal{S}_H^-$,

$$q(t) \in \partial_c \psi(\gamma(t)) \quad \text{ a.e. } t \in (a, b).$$

The argument above shows that there is a function $r \in L^\infty(a, b, \mathbf{R}^n)$ for which conditions (5.19), (5.20), and (5.21) in (a, b), with q and ϕ replaced by r and ψ, respectively. Then we have

$$r(t) \cdot \dot\gamma(t) = L(\gamma(t), \dot\gamma(t)) + H(\gamma(t), r(t)) \quad \text{ a.e. } t \in (a, b),$$

which implies that

$$r(t) = D_2 L(\gamma(t), \dot\gamma(t)) \quad \text{ a.e. } t \in (a, b).$$

By the same reasoning, we get

$$q(t) = D_2 L(\gamma(t), \dot\gamma(t)) \quad \text{ a.e. } t \in (a, b).$$

Therefore we have

$$q(t) = r(t) \in \partial_c \psi(\gamma(t)) \quad \text{ a.e. } t \in (a, b).$$

This completes the proof. □

6. Cauchy problem

In this section we prove Theorem 1.1 together with some estimates on the solution of (1.1) and (1.2) which satisfies (1.4).

Our strategy here for proving existence of a viscosity solution of (1.1) and (1.2) which satisfies (1.4) is to prove (i) the continuity of the function u on $\mathbf{R}^n \times [0, \infty)$ given by

$$u(x, t) = \inf\Big\{ \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)) \mid \gamma \in \mathcal{C}(x, t) \Big\} \tag{6.1}$$

and then (ii) to show that the function u is a viscosity solution of (1.1) and (1.2) by using the dynamic programming principle.

We assume henceforth by adding a constant to ϕ_0 and ϕ_1 if necessary that $u_0 \ge \phi_0$ in $\mathbf{R}^n$.

Lemma 6.1. *We have*

$$u(x,t) \geq \phi_0(x) \quad \textit{for all } (x,t) \in \mathbf{R}^n \times [0,\infty).$$

Proof. Fix any $(x,t) \in \mathbf{R}^n \times (0,\infty)$. For each $\varepsilon > 0$ there is a curve $\gamma \in \mathcal{C}(x,t)$ such that

$$u(x,t) + \varepsilon > \int_0^t L(\gamma(s),\dot{\gamma}(s))\,\mathrm{d}s + u_0(\gamma(0)).$$

By Proposition 2.5, since $H[\phi_0] \leq 0$ a.e., we have

$$u(x,t) + \varepsilon > \phi_0(\gamma(t)) - \phi_0(\gamma(0)) + u_0(\gamma(0)) \geq \phi_0(x),$$

which shows that $u(x,t) \geq \phi_0(x)$. □

Lemma 6.2. *We have*

$$u(x,t) \leq u_0(x) + L(x,0)t \quad \textit{for all } (x,t) \in \mathbf{R}^n \times (0,\infty).$$

Proof. Fix any $(x,t) \in \mathbf{R}^n \times (0,\infty)$. By choosing the curve $\gamma_x(t) \equiv x$ in formula (6.1), we find that

$$\begin{aligned} u(x,t) &\leq \int_0^t L(\gamma_x(s),\dot{\gamma}_x(s))\,\mathrm{d}s + u_0(\gamma_x(0)) \\ &= \int_0^t L(x,0)\,\mathrm{d}s + u_0(x) = u_0(x) + L(x,0)t. \end{aligned}$$ □

Proposition 6.3 (Dynamic Programming Principle). *For $t > 0$, $s > 0$, and $x \in \mathbf{R}^n$, we have*

$$u(x,s+t) = \inf\Big\{\int_0^t L(\gamma(r),\dot{\gamma}(r))\,\mathrm{d}r + u(\gamma(0),s) \;\Big|\; \gamma \in \mathcal{C}(x,t)\Big\}. \tag{6.2}$$

We omit giving the proof of this proposition and we refer to [L] for a proof in a standard case.

Lemma 6.4. *For any $R > 0$, there is a constant $C_R > 0$, depending only on R, u_0, and H, such that*

$$u(x,t) \leq C_R \quad \textit{for all } (x,t) \in B(0,R) \times (0,\infty).$$

Proof. Fix $R > 0$ so that $\mathcal{A} \subset B(0,R)$. According to Proposition 2.1, there are constants $\varepsilon_R > 0$ and $C_R > 0$ such that

$$L(x,\xi) \leq C_R \quad \text{for all } (x,\xi) \in B(0,R) \times B(0,\varepsilon_R).$$

It is clear that C_R and ε_R depends only on R and H. Set $T := 2R/\varepsilon_R$. Fix any $x_0 \in \mathcal{A}$ and any $(x,t) \in B(0,R) \times [T,\infty)$. Set $\tau := t - T$. Fix any critical curve γ_0 satisfying $\gamma_0(t-T) = x_0$. Define the curve $\gamma \in \mathcal{C}(x,t)$ by

$$\gamma(s) = \begin{cases} \dfrac{t-s}{T}x_0 + \left(1 - \dfrac{t-s}{T}\right)x & \text{for all } s \in [t-T,\, t], \\ \gamma_0(s) & \text{for all } s \in [0, t-T). \end{cases}$$

Observe that $\gamma(s) \in B(0,R)$ for $s \in [t-T,\, T]$, that $\dot\gamma(s) = T^{-1}(x - x_0)$ and hence $|\dot\gamma(s)| \le \varepsilon_R$ for $s \in (t-T,\, t)$, and therefore that $L(\gamma(s), \dot\gamma(s)) \le C_R$ for all $s \in (t-T,\, t)$. Thus, using Proposition 5.13, we get

$$\begin{aligned} u(x,t) &\le \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)) \\ &= \int_0^{t-T} L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + \int_{t-T}^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)) \\ &\le \int_0^{t-T} L(\gamma_0(s), \dot\gamma_0(s))\,\mathrm{d}s + \int_{t-T}^t C_R\,\mathrm{d}s + u_0(\gamma_0(0)) \\ &= \phi_0(\gamma_0(t-T)) + u_0(\gamma_0(0)) - \phi_0(\gamma_0(0)) + C_R T. \end{aligned}$$

Hence we get

$$u(x,t) \le \max_{B(0,R)} u_0 + 2 \max_{B(0,R)} |\phi_0| + C_R T \quad \text{for all } (x,t) \in B(0,R) \times [T, \infty).$$

On the other hand, by Lemma 6.2 we have

$$u(x,t) \le u_0(x) + L(x,0)T \quad \text{for all } (x,t) \in B(0,R) \times [0,T],$$

where we have used the fact that $L(x,0) \ge 0$ for all $x \in \mathbf{R}^n$. The proof is now complete. □

Lemma 6.5. *Let $R > 0$. Then there is a constant $C_R > 0$ having the following property: if $(x,t) \in B(0,R) \times (0,\infty)$ and $\gamma \in \mathcal{C}(x,t)$ satisfy*

$$u(x,t) + 1 > \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)),$$

then

$$|\gamma(s)| \le C_R \quad \text{for all } s \in (0,t].$$

Proof. Let $(x,t) \in B(0,R) \times (0,\infty)$ and $\gamma \in \mathcal{C}(x,t)$ be as in the above lemma. Due to Lemma 6.4, there is a constant $C_1 \equiv C_1(R) > 0$, independent of $t \in (0,\infty)$, such that

$$u(x,t) \le C_1 \quad \text{for all } x \in B(0,R).$$

Note that $C_1(R)$ may be chosen so as to depend on u_0 only through $\max_{B(0,R_1)} u_0$, where R_1 is a positive constant such that $R_1 \ge R$ and $\mathcal{A} \subset B(0,R_1)$.

Let $\tau \in (0,t]$. Using Proposition 2.5 and Lemma 6.1, we compute that

$$\begin{aligned} C_1(R) + 1 \ge u(x,t) + 1 &> \int_0^\tau L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s \\ &\quad + \int_\tau^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)) \\ &\ge \phi_1(\gamma(t)) - \phi_1(\gamma(\tau)) + u(\gamma(\tau), \tau) \\ &\ge \phi_1(x) - \phi_1(\gamma(\tau)) + \phi_0(\gamma(\tau)), \end{aligned}$$

which yields

$$\phi_0(\gamma(\tau)) - \phi_1(\gamma(\tau)) \le \phi_1(x) + C_1 + 1.$$

Since $\lim_{|x|\to\infty}(\phi_0 - \phi_1)(x) = \infty$, we find a constant $C_R > 0$ such that $\gamma(\tau) \in B(0, C_R)$ for all $\tau \in [0, t]$. □

Lemma 6.6. *For each $R > 0$ there is a modulus m_R such that for $(x, t) \in B(0, R) \times (0,\ 1]$ and $\gamma \in \mathcal{C}(x, t)$, if*

$$u(x, t) + 1 > \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)),$$

then

$$|\gamma(t - s) - x| \le m_R(s) \quad \textit{for all } s \in [0, t].$$

Proof. Let $R > 0$. Let $(x, t) \in B(0, R) \times (0,\ 1]$ and $\gamma \in \mathcal{C}(x, t)$ be as above. According to Lemmas 6.4 and 6.5, there is a constant $C \equiv C(R) \ge R$ such that

$$|\gamma(s)| \le C \quad \text{for all } s \in [0,\ t].$$

We choose a constant $C_1 \equiv C_1(R) > 0$ so that

$$\sup_{B(0,C)\times(0,\infty)} |u| \le C_1.$$

As we have already observed, for any $A > 0$ there is a constant $C_A \equiv C_A(R) > 0$ such that

$$L(x, \xi) \ge A|\xi| - C_A \quad \text{for all } (x, \xi) \in B(0, C) \times \mathbf{R}^n.$$

Fix $A > 0$, and we calculate that

$$\begin{aligned} C_1 + 1 &> \int_\tau^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u(\gamma(\tau), \tau) \\ &\ge \int_\tau^t [A|\dot\gamma(s)| - C_A]\,\mathrm{d}s - C_1. \end{aligned}$$

Hence we get

$$\int_\tau^t |\dot\gamma(s)|\,\mathrm{d}s \le A^{-1}(2C_1 + 1) + A^{-1}C_A(t - \tau).$$

There is a modulus m_R such that

$$\inf_{A>0} \left[A^{-1}(2C_1 + 1) + A^{-1}C_A r\right] \le m_R(r).$$

Fix such a modulus m_R, and we have

$$\int_\tau^t |\dot\gamma(s)|\,\mathrm{d}s \le m_R(t - \tau),$$

which implies that

$$|\gamma(t-s)-x| \le m_R(s) \quad \text{for all } s \in [0,t]. \qquad \square$$

Lemma 6.7. *For each $R > 0$ there is a modulus l_R such that for $(x,t) \in B(0,R)\times[0,1]$,*

$$|u(x,t)-u_0(x)| \le l_R(t). \tag{6.3}$$

Proof. Fix any $\varepsilon \in (0,1)$, $R > 0$, and $(x,t) \in B(0,R) \times (0,1]$. There is a curve $\gamma \in \mathcal{C}(x,t)$ such that

$$u(x,t)+\varepsilon > \int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)).$$

According to Lemma 6.6, there is a modulus m_R such that

$$|\gamma(t-s)-x| \le m_R(s) \quad \text{for all } s \in [0,t].$$

Recall that

$$\int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \ge \phi_0(\gamma(t)) - \phi_0(\gamma(0)).$$

Then we have

$$\begin{aligned} u(x,t)+\varepsilon &> u_0(x) + [\phi_0(x)-\phi_0(\gamma(0))] + [u_0(\gamma(0))-u_0(x)] \\ &\ge u_0(x) - \mu_{\phi_0,R}(m_R(t)) - \mu_{u_0,R}(m_R(t)), \end{aligned}$$

where $\mu_{\phi_0,R}$ and $\mu_{u_0,R}$ are the moduli of continuity of ϕ_0 and of u_0, respectively, on the set $B(0,R_1)$, with $R_1 = R + m_R(1)$. Since $\varepsilon \in (0,1)$ is arbitrary, the above inequality, together with Lemma 6.2, guarantees existence of a modulus l_R such that (6.3) holds. $\square$

Lemma 6.8. *For each $R > 0$ there is a modulus k_R such that for any $x, y \in B(0,R)$, $t > 0$,*

$$|u(x,t)-u(y,t)| \le k_R(|x-y|). \tag{6.4}$$

Proof. Let $R > 0$. Let $\varepsilon \equiv \varepsilon_R > 0$ and $C \equiv C_R > 0$ be constants such that $L(x,\xi) \le C$ for all $(x,\xi) \in B(0,R) \times B(0,\varepsilon)$. Let $x, y \in B(0,R)$ and $t > 0$. We may assume that $|x-y| \le \varepsilon \le 1$.

We first consider the case when $|x-y| \ge \varepsilon t$. By Lemma 6.7, there is a modulus l_R such that

$$|u(x,t)-u_0(x)| \le l_R(t), \qquad |u(y,t)-u_0(y)| \le l_R(t).$$

We may assume that

$$|u_0(x)-u_0(y)| \le l_R(|x-y|).$$

Consequently we have

$$\begin{aligned} |u(x,t)-u(y,t)| &\le |u_0(x)-u_0(y)| + |u(x,t)-u_0(x)| + |u(y,t)-u_0(y)| \\ &\le l_R(|x-y|) + 2l_R(t) \le 3l_R(\varepsilon^{-1}|x-y|). \end{aligned}$$

Next we consider the case when $|x-y| \le \varepsilon t$. Fix $\delta \in (0,1)$. We select $\gamma \in \mathcal{C}(x,t)$ so that

$$u(x,t)+\delta > \int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)).$$

We know that there is a constant $C_1 \equiv C_1(R) \ge R$ such that

$$|\gamma(s)| \le C_1 \quad \text{for all } s \in [0,t].$$

We may assume that $C \ge C_1$, so that $|\gamma(s)| \le C$ for all $s \in [0,t]$. Set $\tau := \varepsilon^{-1}|x-y|$ and note that $\tau \le \min\{1,\, t\}$. Define $\eta \in \mathcal{C}(y,t)$ by

$$\eta(s) = \begin{cases} \left(1 - \dfrac{t-s}{\tau}\right) y + \dfrac{t-s}{\tau} x & \text{for all } s \in [t-\tau,\, t], \\ \gamma(s+\tau) & \text{for all } s \in [0,\, t-\tau]. \end{cases}$$

Note that $|\dot\eta(s)| = \tau^{-1}|x-y| = \varepsilon$ for all $s \in (t-\tau,\, t)$. In view of Lemma 6.7, we may assume by replacing l_R by a larger modulus if necessary that

$$|u(\xi,\tau) - u_0(\xi)| \le l_R(\tau) \quad \text{for all } \xi \in B(0,C).$$

Then we have

$$\begin{aligned} u(y,t) \le& \int_{t-\tau}^t L(\eta(s),\dot\eta(s))\,\mathrm{d}s + \int_0^{t-\tau} L(\eta(s),\dot\eta(s))\,\mathrm{d}s + u_0(\eta(0)) \\ \le& C\tau + \int_\tau^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(\tau)), \\ <& C\tau + \delta + u(x,t) - \int_0^\tau L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s - u_0(\gamma(0)) + u_0(\gamma(\tau)) \\ \le& C\tau + \delta + u(x,t) - u(\gamma(\tau),\tau) + u_0(\gamma(\tau)) \le C\tau + \delta + u(x,t) + l_R(\tau). \end{aligned}$$

Therefore we get

$$u(y,t) - u(x,t) \le l_R(\tau) + C\tau = l_R(\varepsilon^{-1}|x-y|) + C\varepsilon^{-1}|x-y|.$$

By symmetry, we get

$$|u(y,t) - u(x,t)| \le l_R(\varepsilon^{-1}|x-y|) + C\varepsilon^{-1}|x-y|.$$

Thus we have in general

$$|u(y,t) - u(x,t)| \le 3l_R(\varepsilon^{-1}|x-y|) + C\varepsilon^{-1}|x-y|. \qquad \square$$

We extend the domain of definition of u to $\mathbf{R}^n \times [0,\infty)$ by setting

$$u(x,0) = u_0(x) \quad \text{for all } x \in \mathbf{R}^n. \tag{6.5}$$

Theorem 6.9. *$u \in C(\mathbf{R}^n \times [0,\infty))$ and moreover, for any $R > 0$, u is bounded and uniformly continuous on $B(0,R) \times [0,\infty)$.*

Proof. In view of Lemma 6.8, there is a pair of collections $\{C_R\}_{R>0}$ of positive constants and $\{k_R\}_{R>0}$ of moduli such that for any $R > 0$,

$$\|u\|_{L^\infty(B(0,R)\times(0,\infty))} \le C_R, \tag{6.6}$$

$$|u(x,s) - u(y,s)| \le k_R(|x-y|) \quad \text{for all } x, y \in B(0,R),\ s \in [0,\infty). \tag{6.7}$$

Fix any $s \ge 0$ and note by Proposition 6.3 that

$$u(x, s+t) = \inf\Big\{ \int_0^t L(\gamma(r), \dot\gamma(r))\,\mathrm{d}r + u(\gamma(0), s) \ \Big|\ \gamma \in \mathcal{C}(x,t) \Big\}.$$

Then we apply Lemma 6.7, with $u(\cdot, s)$ in place of u_0, to conclude that for each $R > 0$ there is a modulus l_R such that

$$|u(x, s+t) - u(x,s)| \le l_R(t) \quad \text{for all } (x,s) \in B(0,R) \times [0,\infty) \text{ and } t > 0. \tag{6.8}$$

That is, we have

$$|u(x,t) - u(x,s)| \le l_R(|t-s|) \quad \text{for all } x \in B(0,R) \text{ and } t, s \in [0,\infty).$$

This and Lemma 6.8 assure that u is uniformly continuous on $B(0,R) \times [0,\infty)$ for any $R > 0$. In particular, we see that $u \in C(\mathbf{R}^n \times [0,\infty))$. □

Theorem 6.10. *The function u is a viscosity solution of* (1.1) *and* (1.2).

Proof. Since $u \in C(\mathbf{R}^n \times [0,\infty))$ by Theorem 6.9, we only need to apply Theorems A.1 and A.2 together with Remark after these theorems to u, to conclude that u is a viscosity solution of (1.1). Moreover, it is obvious that u satisfies (1.2). □

7. Proof of Theorem 1.3

This section will be devoted to proving Theorem 1.3. As before, the critical value c_H is assume to be zero in this section.

Let $\{S_t\}_{t\ge0}$ be the semi-group of mappings on Φ_0 defined by $S_t u_0 = u(\cdot, t)$, where u is the unique viscosity solution of (1.1) and (1.2) satisfying (1.4).

In what follows we say that a sequence $\{f_j\}_{j\in\mathbf{N}} \subset C(\mathbf{R}^n)$ converges to f in $C(\mathbf{R}^n)$ if $f_j(x) \to f(x)$ on every compact subsets of $\mathbf{R}^n$ as $j \to \infty$.

For $u_0 \in \Phi_0$ we denote by $\omega(u_0)$ the ω-limit set in $C(\mathbf{R}^n)$ of the trajectory $\{S_t u_0\}_{t\ge0}$ issuing from u_0, which is defined as the set of those $w \in C(\mathbf{R}^n)$ such that

$$S_{t_j} u_0 \to w \quad \text{in } C(\mathbf{R}^n) \ \text{ as } \ j \to \infty$$

for some sequence $\{t_j\}_{j\in\mathbf{N}} \subset (0,\infty)$ diverging to infinity.

In view of Lemmas 6.1 and 6.4 and Theorem 6.9, the function $u(x,t) \equiv S_t u_0(x)$ is bounded and uniformly continuous on $B(0,R) \times [0,\infty)$ for any $R > 0$. By the Ascoli-Arzela theorem, for any sequence $\{t_j\}_{j\in\mathbf{N}} \subset (0,\infty)$ diverging to infinity, there is a subsequence $\{t_{j_k}\}_{k\in\mathbf{N}}$ such that as $k \to \infty$, $u(x, t_{j_k}) \to w(x)$ uniformly on bounded subsets of $\mathbf{R}^n$ for some function $w \in C(\mathbf{R}^n)$. By Lemma 6.1, we see that $w \in \Phi_0$. In particular, we have $\omega(u_0) \neq \emptyset$, $\omega(u_0) \subset \Phi_0$, and moreover, $\omega(w) \neq \emptyset$.

We denote the set of all critical curves by Γ. For any $\gamma \in \Gamma$, we write $\omega(\gamma)$ for the set of curves $\eta \in C(\mathbf{R}, \mathcal{A})$ for which there is a sequence $\{t_j\}_{j\in\mathbf{N}} \subset (0,\infty)$ such that as $j \to \infty$,

$$t_j \to \infty \quad \text{and} \quad \gamma(t+t_j) \to \eta(t) \quad \text{on bounded intervals } \subset \mathbf{R}.$$

Then we set $\omega(\Gamma) = \bigcup\{\omega(\gamma) \mid \gamma \in \Gamma\}$. Also, we set

$$\mathcal{M} = \{\eta(0) \mid \eta \in \omega(\Gamma)\}.$$

Here it is important to see as in the proof of Theorem 5.11 that $\omega(\Gamma) \subset \Gamma$.

Lemma 7.1. *Let $u, v \in \mathcal{S}_H^-$ and assume that $u \le v$ in $\mathcal{M}$. Then $u \le v$ in $\mathcal{A}$.*

Proof. Fix any $y \in \mathcal{A}$ and choose a curve $\gamma \in \Gamma$ so that $\gamma(0) = y$. By Proposition 5.13, for any $T > 0$ we get

$$u(\gamma(T)) - u(\gamma(0)) = d_H(\gamma(T), \gamma(0)) = v(\gamma(T)) - v(\gamma(0)). \tag{7.1}$$

Selecting $\{t_j\}_{j\in\mathbf{N}}$ diverging to infinity so that $\gamma(\cdot + t_j) \to \eta$ in $C(\mathbf{R}, \mathbf{R}^n)$ as $j \to \infty$ for some $\eta \in \omega(\gamma)$ and sending $j \to \infty$ in (7.1), with $T = t_j$, we get

$$u(y) - v(y) = u(\eta(0)) - v(\eta(0)) \le 0.$$

Thus we have $u(x) \le v(x)$ for all $x \in \mathcal{A}$. □

Lemma 7.2. *Let $u_0 \in \Phi_0$ and $\phi \in \mathcal{S}_H^-$. Then, for any critical curve γ, the function: $t \mapsto S_t u_0(\gamma(t)) - \phi(\gamma(t))$ is nonicreasing on $[0,\infty)$.*

Proof. Let $0 < a < b < \infty$ and set $u(x,t) = S_t u_0(x)$ for $(x,t) \in \mathbf{R}^n \times [0,\infty)$. By using the dynamic programming principle, we get

$$u(\gamma(b), b) \le \int_a^b L(\gamma(t), \dot\gamma(t))\, \mathrm{d}t + u(\gamma(a), a),$$

from which, since γ is a critical curve, we get

$$u(\gamma(b), b) \le \phi(\gamma(b)) - \phi(\gamma(a)) + u(\gamma(a), a),$$

completing the proof. □

Given functions $f, g : \mathbf{R}^n \to \mathbf{R}$, we write $[f,\, g]$ for the set of all functions $h : \mathbf{R}^n \to \mathbf{R}$ which satisfy $f \le h \le g$ in $\mathbf{R}^n$.

Lemma 7.3. *Let $f \in \mathcal{S}_H^- \cap \Phi_0$ and $g \in \mathcal{S}_H^+$, and assume that $f \le g$ in $\mathbf{R}^n$. Then for each $\varepsilon > 0$ there is a constant $\delta > 0$ such that for any $u_0, v_0 \in [f,\, g] \cap C(\mathbf{R}^n)$, if*

$$\max_{x \in B(0,\delta^{-1})} (u_0(x) - v_0(x)) \le \delta, \tag{7.2}$$

then

$$\sup_{(x,t) \in B(0,\varepsilon^{-1}) \times [0,\infty)} (S_t u_0(x) - S_t v_0(x)) \le \varepsilon.$$

Proof. Fix any $\varepsilon > 0$. We choose a constant $A_\varepsilon > 0$ so that

$$\phi_1(x) + A_\varepsilon \ge g(x) \quad \text{ for all } x \in B(0, \varepsilon^{-1}),$$

and then choose an $R_\varepsilon > 0$ so that

$$\phi_1(x) + A_\varepsilon \le f(x) \quad \text{ for all } x \in \mathbf{R}^n \setminus B(0, R_\varepsilon).$$

We set $\delta = \min\{\varepsilon, R_\varepsilon^{-1}\}$. We then have

$$\phi_1(x) + A_\varepsilon \le f(x) \quad \text{ for all } x \in \mathbf{R}^n \setminus B(0, \delta^{-1}). \tag{7.3}$$

Let $u_0, v_0 \in [f, g] \cap C(\mathbf{R}^n)$ satisfy (7.2). Set $u(x,t) = S_t u_0(x)$ and $v(x,t) = S_t v_0(x)$ for $(x,t) \in \mathbf{R}^n \times [0,\infty)$. By Theorem 4.1, we find that

$$f(x) \le u(x,t) \le g(x) \quad \text{and} \quad f(x) \le v(x,t) \le g(x) \text{ for } (x,t) \in \mathbf{R}^n \times [0,\infty). \tag{7.4}$$

Set $w(x,t) = \min\{u(x,t), \phi_1(x) + A_\varepsilon\}$ for $(x,t) \in \mathbf{R}^n \times [0,\infty)$. Observe that $w \in \mathcal{S}_H^-$ and by (7.3) and (7.4) that $w(x,t) = \phi_1(x) + A_\varepsilon \le f(x) \le v(x,t)$ for all $(x,t) \in (\mathbf{R}^n \setminus B(0,\delta^{-1})) \times [0,\infty)$. By Theorem 4.1, we get

$$w(x,t) \le v(x,t) + \varepsilon \quad \text{ for all } (x,t) \in B(0,\delta^{-1}) \times [0,\infty),$$

and conclude that $w \le v + \varepsilon$ in $\mathbf{R}^n \times [0,\infty)$.

Finally, we note that $u(x,t) \le g(x) \le \phi_1(x) + A_\varepsilon$ for $B(0,\varepsilon^{-1}) \times [0,\infty)$, to get

$$u(x,t) = w(x,t) \le v(x,t) + \varepsilon \quad \text{ for all } (x,t) \in B(0,\varepsilon^{-1}) \times [0,\infty). \qquad \square$$

Lemma 7.4. *For any function $f \in C(\mathbf{R}^n)$ there is a viscosity supersolution $g \in C^{0+1}(\mathbf{R}^n)$ of* (5.1) *which satisfies $g \ge f$ in $\mathbf{R}^n$.*

Proof. We may assume that $f \in C^1(\mathbf{R}^n)$ and $f \ge \phi_0$ in $\mathbf{R}^n$. We choose a constant $R > 0$ so that $\mathcal{A} \subset B(0,R)$ and choose a function $h \in C(\mathbf{R}^n)$ so that $h \ge 0$ in $\mathbf{R}^n$, $h(x) = 0$ for all $x \in B(0,R)$, and $h(x) \ge H(x, Df(x)) + 1$ for all $x \in \mathbf{R}^n \setminus B(0, R+1)$. We set $G(x,p) = H(x,p) - h(x)$ for $(x,p) \in \mathbf{R}^{2n}$ and consider the additive eigenvalue problem

$$G(x, Du) = a \quad \text{ in } \mathbf{R}^n. \tag{7.5}$$

Note that $G \le H$ in $\mathbf{R}^{2n}$ and that G satisfies (A1)–(A4), with the same choice of functions ϕ_0, ϕ_1. Theorem 3.3, applied to G in place of H, yields a solution $(a, \phi) \in \mathbf{R} \times (\Phi_0 \cap C^{0+1}(\mathbf{R}^n))$ of (7.5). Since $G \le H$ in $\mathbf{R}^{2n}$, we see that if $\psi \in \Phi_0$ is a viscosity subsolution of $H[\psi] = c_H$ in $\mathbf{R}^n$, then it is a viscosity subsolution of $G[\psi] \le c_H$ in $\mathbf{R}^n$ and therefore in view of Proposition 3.4 that $a \le c_H = 0$.

Fix a critical curve γ for H. Observe by Proposition 2.5 that for any $T > 0$,

$$\begin{aligned}\phi(\gamma(T)) - \phi(\gamma(0)) &\le \int_0^T (L(\gamma(t), \dot\gamma(t)) + a + h(\gamma(t)))\,\mathrm{d}t \\ &= \int_0^T L(\gamma, \dot\gamma)\,\mathrm{d}t + aT = d_H(\gamma(T), \gamma(0)) + aT.\end{aligned}$$

Sending $T \to \infty$ in the above yields $a \ge 0$, from which we see that $a = 0$. Thus, ϕ is a viscosity solution of $H[\phi] = h$ in $\mathbf{R}^n$.

By adding a constant to ϕ if necessary, we may assume that $\phi(x) \ge f(x)$ for all $x \in B(0, R+1)$. We apply Theorem 3.2, with $\Omega = \mathbf{R}^n \setminus B(0, R+1)$, $\lambda = 0$, $\varepsilon = 1$, and H replaced by G, to obtain $\phi(x) \ge f(x)$ for all $x \in \mathbf{R}^n \setminus B(0, R+1)$, which yields $\phi \ge f$ in $\mathbf{R}^n$. Since $h \ge 0$ in $\mathbf{R}^n$, we see that $\phi \in \mathcal{S}_H^+$. Thus, $g := \phi$ has the required properties. □

Lemma 7.5. *Let $v, w \in \omega(u_0)$, and let $\{t_j\}_{j\in\mathbf{N}}, \{r_j\}_{j\in\mathbf{N}} \subset (0, \infty)$ be sequences diverging to infinity such that $S_{t_j} u_0 \to v$ and $S_{t_j + r_j} u_0 \to w$ in $C(\mathbf{R}^n)$ as $j \to \infty$. Then $S_{r_j} v \to w$ in $C(\mathbf{R}^n)$ as $j \to \infty$.*

Proof. Fix any $\varepsilon > 0$. By Lemma 7.4, there is a function $g \in \mathcal{S}_H^+$ such that $g \ge u_0$ in $\mathbf{R}^n$. We may assume that $\phi_0 \le u_0$ in $\mathbf{R}^n$. By Theorem 4.1, we deduce that

$$S_t u_0, v \in [\phi_0, g],$$

and, in view of Lemma 7.3, we find a constant $\delta > 0$ such that if

$$\max_{B(0,\delta^{-1})} |S_{t_j} u_0 - v| \le \delta, \tag{7.6}$$

then

$$\sup_{(x,t)\in B(0,\varepsilon^{-1})\times[0,\infty)} |S_{t_j+t} u_0(x) - S_t v(x)| \le \varepsilon. \tag{7.7}$$

We may choose a $J \in \mathbf{N}$ such that (7.6) holds for all $j \ge J$. Thus we see that (7.7) holds for all $j \ge J$ and conclude that as $j \to \infty$,

$$S_{r_j} v = S_{t_j + r_j} u_0 + (S_{t_j} v - S_{t_j + r_j} u_0) \to w \quad \text{in } C(\mathbf{R}^n). \qquad \square$$

Proposition 7.6. *There exist a constant $\delta \in (0, 1)$ and a modulus m such that for any $u_0 \in \Phi_0$, $\gamma \in \Gamma$, $\varepsilon \in (-\delta, \delta)$, and $T > 0$,*

$$S_T u_0(\gamma(T)) \le u_0(\gamma(\varepsilon T)) + \int_{\varepsilon T}^T L(\gamma(t), \dot\gamma(t))\,\mathrm{d}t + |\varepsilon| T m(|\varepsilon|).$$

For the proof of the above proposition we need the following lemma.

Lemma 7.7. *There exist a constant $\delta \in (0,1)$ and a modulus m such that for any $\gamma \in \Gamma$ and any $\lambda \in [1-\delta, 1+\delta]$,*

$$\lambda^{-1} L(\gamma(t), \lambda\dot\gamma(t)) \le L(\gamma(t), \dot\gamma(t)) + |\lambda - 1| m(|\lambda - 1|) \quad \textit{a.e. } t \in \mathbf{R}.$$

Proof. Choose a constant $R_1 > 0$ so that $\mathcal{A} \subset B(0, R_1)$ and choose a constant $R_2 > 0$ so that for any $x \in B(0, R_1)$ and any $p \in \mathbf{R}^n$, if $H(x,p) \le 0$, then $p \in B(0, R_2)$. Define the set $S \subset \mathbf{R}^n \times \mathbf{R}^n$ by

$$S := \{(x, \xi) \in B(0, R_1) \times \mathbf{R}^n \mid \xi \in D_2^- H(x,p) \text{ for some } p \in B(0, R_2)\}.$$

By Proposition 2.3, the set S is a compact subset of $\operatorname{int}\operatorname{dom} L$. Thus we may choose a constant $\varepsilon > 0$ so that

$$S_\varepsilon := \{(x,\xi) \in \mathbf{R}^{2n} \mid \operatorname{dist}((x,\xi), S) \le \varepsilon\} \subset \operatorname{int}\operatorname{dom} L.$$

We fix an $R_3 > 0$ so that $S \subset B(0, R_3)$ (the ball on the right hand side is a ball in $\mathbf{R}^{2n}$) and set $\delta = \min\{1/2,\, \varepsilon/R_3\}$, so that for any $(x, \xi) \in S$ and any $\lambda \in (1-\delta, 1+\delta)$, $(x, \lambda\xi) \in S_\varepsilon$. Let m_0 be a modulus of continuity of the uniformly continuous function $D_2 L$ on S_ε.

Fix any critical curve γ. Since $\gamma(t) \in \mathcal{A}$ for all $t \in \mathbf{R}$, we have $\gamma(t) \in B(0, R_1)$ for all $t \in \mathbf{R}$. According to Proposition 5.14, there is a function $q \in L^\infty(\mathbf{R}, \mathbf{R}^n)$ such that

$$H(\gamma(t), q(t)) = 0 \quad \text{and} \quad \dot\gamma(t) \in D_2^- H(\gamma(t), q(t)) \quad \text{a.e. } t \in \mathbf{R}. \tag{7.8}$$

Therefore we have $(\gamma(t), \dot\gamma(t)) \in S$ a.e. $t \in \mathbf{R}$. Hence, for $\lambda \in (1-\delta,\, 1+\delta)$, we have

$$(\gamma(t), \lambda\dot\gamma(t)) \in S_\varepsilon \quad \text{a.e. } t \in \mathbf{R}.$$

Consequently, for $\mu \in (1-\delta,\, 1+\delta)$, we have

$$|D_2 L(\gamma(t), \dot\gamma(t)) - D_2 L(\gamma(t), \mu\dot\gamma(t))| \le m_0(|1-\mu||\dot\gamma|) \quad \text{a.e. } t \in \mathbf{R}.$$

In view of (7.8), we have

$$D_2 L(\gamma(t), \dot\gamma(t)) \cdot \dot\gamma(t) = L(\gamma(t), \dot\gamma(t)) \quad \text{a.e. } t \in \mathbf{R}.$$

Now we compute that for any $\lambda \in (1-\delta,\, 1+\delta)$ and a.e. $t \in \mathbf{R}$,

$$L(\gamma(t), \lambda\dot\gamma(t)) = L(\gamma(t), \dot\gamma(t)) + (\lambda - 1) D_2 L(\gamma(t), (1 + \theta_t(\lambda - 1))\,\dot\gamma(t)) \cdot \dot\gamma(t)$$

(for some $\theta_t \in (0,1)$, and furthermore)

$$\begin{aligned} &\le L(\gamma(t), \dot\gamma(t)) + (\lambda - 1) D_2 L(\gamma(t), \dot\gamma(t)) \cdot \dot\gamma(t) \\ &\quad + |\lambda - 1||\dot\gamma(t)| m_0(|\lambda - 1||\dot\gamma(t)|) \\ &= \lambda L(\gamma(t), \dot\gamma(t)) + |\lambda - 1||\dot\gamma(t)| m_0(|\lambda - 1||\dot\gamma(t)|). \end{aligned}$$

Setting $m(r) = 2R_3 m_0(R_3 r)$, for all $\lambda \in (1-\delta,\, 1+\delta)$ and a.e. $t \in \mathbf{R}$, we have

$$\lambda^{-1} L(\gamma(t), \lambda\dot{\gamma}(t)) \le L(\gamma(t), \dot{\gamma}(t)) + |\lambda - 1| m(|\lambda - 1|). \qquad \square$$

Proof of Proposition 7.6. Let $\delta \in (0,\, 1)$ and m be those from Lemma 7.7. Fix any $u_0 \in \Phi_0$, $\gamma \in \Gamma$, and $T > 0$. Set $u(x,t) = S_t u_0(x)$ for $(x,t) \in \mathbf{R}^n \times [0,\infty)$. Since

$$u(\gamma(T), T) = \inf\Big\{ \int_0^T L(\eta(t), \dot{\eta}(t))\, \mathrm{d}t + u(\eta(0), 0) \ \Big|\ \eta \in \mathcal{C}(\gamma(T), T) \Big\},$$

choosing $\eta(t) := \gamma(\lambda t + (1-\lambda)T)$ in the above formula, we get

$$u(\gamma(T), T) \le \int_0^T L(\gamma(\lambda t + (1-\lambda)T), \lambda\dot{\gamma}(\lambda t + (1-\lambda)T))\, \mathrm{d}t + u(\gamma((1-\lambda)T), 0).$$

By making the change of variables $s = \lambda t + (1-\lambda)T$ in the above inequality, we get

$$u(\gamma(T), T) \le \int_{(1-\lambda)T}^T \lambda^{-1} L(\gamma(s), \lambda\dot{\gamma}(s))\, \mathrm{d}t + u(\gamma((1-\lambda)T), 0).$$

Using Lemma 7.7, we see immediately that

$$u(\gamma(T), T) \le \int_{(1-\lambda)T}^T L(\gamma(s), \dot{\gamma}(s))\, \mathrm{d}t + u(\gamma((1-\lambda)T), 0) + |1-\lambda| m(|1-\lambda|) T.$$

Setting $\varepsilon = 1 - \lambda$ yields the desired inequality. $\square$

Henceforth we fix any $u_0 \in \Phi_0$ and define functions $v^\pm : \mathbf{R}^n \to \mathbf{R}$ by

$$v^+(x) = \limsup_{t\to\infty} S_t u_0(x), \qquad v^-(x) = \liminf_{t\to\infty} S_t u_0(x).$$

Since the function $u(x,t) := S_t u_0(x)$ is bounded and uniformly continuous on $B(0,R) \times [0,\infty)$ for any $R > 0$, we see that $v^\pm \in C(\mathbf{R}^n)$ and that $v^+(x) = \limsup^*_{t\to\infty} u(x,t)$ and $v^-(x) = \liminf{}_{*\,t\to\infty} u(x,t)$ for all $x \in \mathbf{R}^n$. As is standard in viscosity solutions theory, we have $v^+ \in \mathcal{S}_H^-$ and $v^- \in \mathcal{S}_H^+$. Moreover, by the convexity of $H(x,\cdot)$, we have $v^- \in \mathcal{S}_H^-$. Also, from Lemma 6.1 we see that $v^\pm \in \Phi_0$.

Lemma 7.8. *Let $w \in \omega(u_0)$ and $\gamma \in \Gamma$. Then*

$$\liminf_{t\to\infty} \big[w(\gamma(t)) - v^-(\gamma(t)) \big] = 0.$$

Proof. From the definition of v^-, we deduce that $v^- \le w$ in $\mathbf{R}^n$. We choose sequences $\{t_j\}_{j\in\mathbf{N}}, \{\tau_j\}_{j\in\mathbf{N}} \subset (0,\infty)$ diverging to infinity so that as $j \to \infty$,

$$S_{t_j} u_0(\gamma(0)) \to v^-(\gamma(0)) \quad \text{and} \quad S_{t_j+\tau_j} u_0 \to w \quad \text{in } C(\mathbf{R}^n).$$

We then observe that for $j \in \mathbf{N}$,

$$\begin{aligned} w(\gamma(\tau_j)) - v^-(\gamma(\tau_j)) &= [w(\gamma(\tau_j)) - S_{t_j+\tau_j} u_0(\gamma(\tau_j))] + [S_{t_j+\tau_j} u_0(\gamma(\tau_j)) - v^-(\gamma(\tau_j))] \\ &\le [w(\gamma(\tau_j)) - S_{t_j+\tau_j} u_0(\gamma(\tau_j))] + [S_{t_j} u_0(\gamma(0)) - v^-(\gamma(0))], \end{aligned}$$

which yields

$$\liminf_{t\to\infty}[w(\gamma(t)) - v^-(\gamma(t))] \le 0.$$

Since $w - v^- \ge 0$ in $\mathbf{R}^n$, we thus conclude that

$$\liminf_{t\to\infty}[w(\gamma(t)) - v^-(\gamma(t))] = 0. \qquad \square$$

Proof of Theorem 1.3. We show that $w \le v^-$ on $\mathcal{M}$ for any $w \in \omega(u_0)$. Once this is done, we have $v^+ \le v^-$ on $\mathcal{M}$ and therefore $v^+ = v^-$ on $\mathcal{M}$. Furthermore, by Lemma 7.1, we get $v^+ = v^-$ on $\mathcal{A}$, and by Proposition 5.9, we see that $v^+ \le v^-$ in $\mathbf{R}^n$. Since $v^+ \ge v^-$ by the definition of $v^\pm$, we conclude that $v^- = v^+$ in $\mathbf{R}^n$ and that $u(\cdot, t) \to v^-$ in $C(\mathbf{R}^n)$ as $t \to \infty$.

It remains to show that $w \le v^-$ on $\mathcal{M}$ for all $w \in \omega(u_0)$. Let $w \in \omega(u_0)$ and $y \in \mathcal{M}$. We choose a curve $\gamma \in \Gamma$ so that there is a sequence $\{a_j\}_{j\in\mathbf{N}} \subset (0,\infty)$ diverging to infinity such that $\lim_{j\to\infty} \gamma(a_j) = y$. Let $\delta \in (0,1)$ and m be those from Proposition 7.6, so that for any $T > 0$, $s \in \mathbf{R}$, and $\varepsilon \in (-\delta, \delta)$,

$$\begin{aligned} S_T w(\gamma(T+s)) - w(\gamma(\varepsilon T + s)) &\le \int_{\varepsilon T}^{T} L(\gamma(t+s), \dot\gamma(t+s))\, \mathrm{d}t + \varepsilon T m(|\varepsilon|) \\ &= v^-(\gamma(T+s)) - v^-(\gamma(\varepsilon T + s)) + \varepsilon T m(|\varepsilon|). \end{aligned} \tag{7.9}$$

In view of Lemma 7.5, there is a sequence $\{b_j\}_{j\in\mathbf{N}}$ diverging to infinity such that $S_{b_j} w \to w$ in $C(\mathbf{R}^n)$ as $j \to \infty$. We may assume by reselecting $\{a_j\}_{j\in\mathbf{N}}$ if necessary that $a_j > b_j$ for all $j \in \mathbf{N}$ and $c_j := a_j - b_j \to \infty$ as $j \to \infty$. Also, we may assume that $\gamma(\cdot + c_j) \to \eta$ in $C(\mathbf{R})$ as $j \to \infty$. Fix any $t \ge 0$. Setting $T = b_j$, $s = c_j$, and $\varepsilon = t/b_j$ in (7.9) and sending $j \to \infty$, we get

$$w(y) - v^-(y) \le w(\eta(t)) - v^-(\eta(t)).$$

From this, using Lemma 7.8, we find that $w(y) - v^-(y) \le 0$, which guarantees that $w(y) = v^-(y)$. $\square$

8. A formula for asymptotic solutions

In the previous section we show that the viscosity solution u of (1.1) and (1.2) satisfying (1.4) approaches to $v^-(x) - ct$ in $C(\mathbf{R}^n)$ as $t \to \infty$, where (c, v^-) is a solution of (1.3).

In this section we give a formula for the function v^-. Let $c = c_H$. We define the function $d_{H,c} \in C(\mathbf{R}^n \times \mathbf{R}^n)$ by

$$d_{H,c}(x,y) = \sup\{v(x) \mid v \in C(\mathbf{R}^n),\ H[v] \le c \text{ in the viscosity sense in } \mathbf{R}^n,\ v(y) = 0\}$$

and $\mathcal{A}_c$ as the set of those $y \in \mathbf{R}^n$ for which the function $d_{H,c}(\cdot, y)$ is a viscosity solution of $H[u] = c$ in $\mathbf{R}^n$. We set $H_c = H - c$ and $L_c = L + c$ as before. Note that d_H and $\mathcal{A}$ defined for $H = H_c$ in Section 5 are the same as $d_{H,c}$ and $\mathcal{A}_c$, respectively. Recall that

$$v^-(x) = \liminf_{t\to\infty}(u(x,t) + ct) \quad \text{for all } x \in \mathbf{R}^n.$$

Theorem 8.1. *We have*

$$v^-(x) = \inf\{d_{H,c}(x,y) + d_{H,c}(y,z) + u_0(z) \mid y \in \mathcal{A}_c,\ z \in \mathbf{R}^n\} \quad \textit{for any } x \in \mathbf{R}^n.$$

Proof. We may assume that $c = 0$, so that $\mathcal{A}_c = \mathcal{A}$ and $d_{H,c} = d_H$. We write $V(x)$ for the right hand side of the above equality. We fix any $x \in \mathbf{R}^n$.

Fix any $\varepsilon > 0$ and choose points $y, z \in \mathbf{R}^n$ so that

$$V(x) + \varepsilon > d_H(x,y) + d_H(y,z) + u_0(z). \tag{8.1}$$

By the definition of d_H, there are $T > 0$, $S > 0$, $\xi \in \mathcal{C}(x,T;y,0)$, and $\eta \in \mathcal{C}(y,S;z,0)$ such that

$$d_H(x,y) + \varepsilon > \int_0^T L(\xi,\dot{\xi})\,\mathrm{d}t, \tag{8.2}$$

$$d_H(y,z) + \varepsilon > \int_0^S L(\eta,\dot{\eta})\,\mathrm{d}t. \tag{8.3}$$

By Proposition 5.10, for each $k \in \mathbf{N}$, there are $\tau_k > 0$ and $\zeta_k \in \mathcal{C}(y,\tau_k;y,0)$ such that

$$\varepsilon > \int_0^{\tau_k} L(\zeta_k,\dot{\zeta}_k)\,\mathrm{d}t. \tag{8.4}$$

We concatenate these three different kind of curves by setting

$$\gamma_k(t) = \begin{cases} \eta(t) & \text{for } t \in [0,\ S], \\ \zeta_k(t-S) & \text{for } t \in [S,\ S+\tau_k], \\ \xi(t-S-\tau_k) & \text{for } t \in [S+\tau_k,\ S+\tau_k+T]. \end{cases}$$

Note that $\gamma_k \in \mathcal{C}(x, S+\tau_k+T; z, 0)$ for all $k \in \mathbf{N}$. Adding (8.1)–(8.4), for any $k \in \mathbf{N}$, we get

$$V(x) + 4\varepsilon > \int_0^{T+\tau_k+S} L(\gamma_k,\dot{\gamma}_k)\,\mathrm{d}t + u_0(\gamma_k(0)) \ge u(x, T+\tau_k+S).$$

Therefore we obtain

$$V(x) + 4\varepsilon \ge \liminf_{t\to\infty} u(x,t) = v^-(x),$$

and conclude that $V(x) \ge v^-(x)$.

Again let $\varepsilon > 0$, and we choose a neighborhood U of $\mathcal{A}$ so that for any $x \in U$ there exists a point $y \in \mathcal{A}$ for which $\max\{d_H(x,y), d_H(y,x)\} \le \varepsilon$. Choose an $R > 0$ so that $\sigma_1(x) \ge 4$ for all $x \in \mathbf{R}^n \setminus B(0,R)$ and in view of Proposition 5.3, a function $\psi \in C^{0+1}(\mathbf{R}^n)$ so that $H[\psi] \le 0$ a.e in $\mathbf{R}^n$ and $H[\psi] \le -4\delta$ a.e. in $B(0,R) \setminus U$ for some $\delta \in (0,1)$. Setting $\phi_\varepsilon = \frac{1}{2}(\phi_1 + \psi)$, we find functions ϕ_ε and $\sigma \in C(\mathbf{R}^n)$ such that $H[\phi_\varepsilon] \le -\sigma$ a.e. in $\mathbf{R}^n$, $\sigma \ge 0$ in $\mathbf{R}^n$, and $\sigma(x) \ge \delta$ for all $x \in \mathbf{R}^n \setminus U$. We may assume that $\phi_\varepsilon \ge u_0$ in $\mathbf{R}^n$. By the definition of $v^-(x)$, there are $T > 0$ and $\gamma \in \mathcal{C}(x,T)$ so that

$$T > \delta^{-1}(v^-(x) + \varepsilon - \phi_\varepsilon(x)) \quad \text{and} \quad v^-(x) + \varepsilon > \int_0^T L(\gamma,\dot{\gamma})\,\mathrm{d}t + u_0(\gamma(0)). \tag{8.5}$$

Using Proposition 2.5, we get

$$\int_0^T L(\gamma,\dot\gamma)\,\mathrm{d}t \ge \phi_\varepsilon(x) - \phi_\varepsilon(\gamma(0)) + \int_0^T \sigma(\gamma(t))\,\mathrm{d}t,$$

and hence

$$v^-(x) + \varepsilon - \phi_\varepsilon(x) > \int_0^T \sigma(\gamma(t))\,\mathrm{d}t.$$

This together with our choice of T guarantees that there is a $\tau \in (0,T)$ such that $\gamma(\tau) \in U$. Then, by (8.5), we have

$$\begin{aligned} v^-(x) + \varepsilon &> \int_0^\tau L(\gamma,\dot\gamma)\,\mathrm{d}t + \int_\tau^T L(\gamma,\dot\gamma)\,\mathrm{d}t + u_0(\gamma(0)) \\ &\ge d_H(\gamma(\tau),\gamma(0)) + d_H(x,\gamma(\tau)) + u_0(\gamma(0)) \end{aligned}$$

By our choice of U, there is a $y \in \mathcal{A}$ such that

$$d_H(\gamma(\tau),y) \le \varepsilon \quad \text{and} \quad d_H(y,\gamma(\tau)) \le \varepsilon.$$

Thus we have

$$\begin{aligned} v^-(x) + 3\varepsilon &> d_H(\gamma(\tau),\gamma(0)) + d_H(x,\gamma(\tau)) + u_0(\gamma(0)) + d_H(\gamma(\tau),y) + d_H(y,\gamma(\tau)) \\ &\ge d_H(x,y) + d_H(y,\gamma(0)) + u_0(\gamma(0)) \ge V(x), \end{aligned}$$

from which we obtain $v^-(x) \ge V(x)$. □

9. Examples

We give two sufficient conditions for H to satisfy (A4).

Let $H_0 \in C(\mathbf{R}^n \times \mathbf{R}^n)$ and $f \in C(\mathbf{R}^n)$. Set $H(x,p) = H_0(x,p) - f(x)$ for $(x,p) \in \mathbf{R}^n \times \mathbf{R}^n$. We assume that

$$\lim_{|x|\to\infty} f(x) = \infty, \tag{9.1}$$

and that there exists a $\delta > 0$ such that

$$\sup_{\mathbf{R}^n \times B(0,\delta)} |H_0| < \infty. \tag{9.2}$$

Fix such a $\delta > 0$ and set

$$C_\delta = \sup_{\mathbf{R}^n \times B(0,\delta)} |H_0|.$$

Then we define $\phi_i \in C^{0+1}(\mathbf{R}^n)$, with $i = 0, 1$, by setting

$$\phi_0(x) = -\frac{\delta}{2}|x| \quad \text{and} \quad \phi_1(x) = -\delta|x|,$$

and observe that for $i = 0, 1$,

$$H_0(x, D\phi_i(x)) \le C_\delta \qquad \text{for all } x \in \mathbf{R}^n \setminus \{0\}.$$

Hence, for $i = 0, 1$, we have

$$H_0(x, D\phi_i(x)) \le \frac{1}{2} f(x) + C_\delta - \frac{1}{2} \min_{\mathbf{R}^n} f \qquad \text{for all } x \in \mathbf{R}^n \setminus \{0\}.$$

If we set

$$\sigma_i(x) = \frac{1}{2} f(x) - C_\delta + \frac{1}{2} \min_{\mathbf{R}^n} f \quad \text{for } x \in \mathbf{R}^n \text{ and } i = 0, 1,$$

then H satisfies (A4) with these ϕ_i and σ_i, $i = 0, 1$. It is clear that if H_0 satisfies (A1)–(A3), then so does H.

A smaller ϕ_0 yields a larger space Φ_0, and in applications of Theorems 1.1–1.3, it is important to have a larger Φ_0. We are thus interested in finding a smaller ϕ_0. A method better than the above in this respect is as follows. We assume that (9.1), (9.2), and (A2) with H_0 in place of H hold and that for each $x \in \mathbf{R}^n$ the function: $p \mapsto H_0(x, p)$ is convex in $\mathbf{R}^n$. We fix a function $\theta \in C^1(\mathbf{R}^n)$ so that

$$\lim_{|x| \to \infty} \theta(x) = \infty \quad \text{and} \quad \lim_{|x| \to \infty} |D\theta(x)| = 0.$$

For instance, the function $\theta(x) = \log(|x|^2 + 1)$ has these properties. Fix an $\varepsilon > 0$ so that $\varepsilon |D\theta(x)| \le \delta/2$ for all $x \in \mathbf{R}^n$. Fix any $\lambda \in (0, 1)$. Define the function $G \in C(\mathbf{R}^n \times \mathbf{R}^n)$ by

$$G(x, p) = \max\{H_0(x, p), H_0(x, p - \varepsilon D\theta(x))\} - (1 - \lambda) f(x) - C_\delta + (1 - \lambda) \min_{\mathbf{R}^n} f.$$

We note that for each $x \in \mathbf{R}^n$ the function: $p \mapsto G(x, p)$ is convex in $\mathbf{R}^n$. Define the function $\psi \in C^{0+1}(\mathbf{R}^n)$ by

$$\psi(x) = \inf\{v(x) \mid v \in C^{0+1}(\mathbf{R}^n),\ G[Dv] \le 0 \ \text{ a.e. in } \mathbf{R}^n,\ v(0) = 0\}.$$

Note that $v(x) := -\frac{\delta}{2}|x|$ has the properties: $G(x, Dv(x)) \le 0$ a.e. $x \in \mathbf{R}^n$ and $v(0) = 0$. Hence we have $\psi(x) \le -\frac{\delta}{2}|x|$ for all $x \in \mathbf{R}^n$. Because of the convexity of $G(x, p)$ in p, we see that ψ is a viscosity solution of $G[\psi] \le 0$ in $\mathbf{R}^n$. This implies that ψ and $\psi - \varepsilon\theta$ are both viscosity solutions of

$$H(x, Dv) \le -\lambda f(x) + C_\delta - (1 - \lambda) \min_{\mathbf{R}^n} f \qquad \text{in } \mathbf{R}^n.$$

With functions $\phi_0 := \psi$, $\phi_1 := \psi - \varepsilon\theta$, and $\sigma_0 = \sigma_1 := \lambda f - C_\delta + (1 - \lambda) \min_{\mathbf{R}^n} f$, the function H satisfies all the conditions of (A4). As is already noted, the function ψ satisfies the inequality $\psi(x) \le -\frac{\delta}{2}|x|$ for all $x \in \mathbf{R}^n$. Moreover, for any $\gamma \in (1/2, 1)$, the function $v(x) := -\gamma\delta|x|$ satisfies

$$G(x, Dv(x)) \le 0 \qquad \text{a.e. } x \in \mathbf{R}^n \setminus B(0, R)$$

for some constant $R \equiv R(\gamma) > 0$. It is now easy to see that if $A > 0$ is large enough, then

$$\psi(x) \le \min\{-\frac{\delta}{2}|x|, -\gamma\delta|x| + A\} \qquad \text{for all } x \in \mathbf{R}^n.$$

Now we examine another class of Hamiltonians H. Let $\alpha > 0$ and let $H_0 \in C(\mathbf{R}^n)$ be a strictly convex function satisfying the coercivity condition

$$\lim_{|p|\to\infty} H_0(p) = \infty.$$

Let $f \in C(\mathbf{R}^n)$. We set

$$H(x,p) = \alpha x \cdot p + H_0(p) - f(x) \quad \text{ for } (x,p) \in \mathbf{R}^n \times \mathbf{R}^n.$$

This class of Hamiltonians H is very close to that treated in [FIL2].

Clearly, this function H satisfies (A1), (A2), and (A3). Let L_0 denote the convex conjugate H_0^* of H_0. By the strict convexity of H_0, we see that $L_0 \in C^1(\mathbf{R}^n)$. Define the function $\psi \in C^1(\mathbf{R}^n)$ by

$$\psi(x) = -\frac{1}{\alpha} L_0(-\alpha x).$$

Then we have $D\psi(x) = DL_0(-\alpha x)$ and therefore, by the convex duality, $H_0(D\psi(x)) = D\psi(x) \cdot (-\alpha x) - L(-\alpha x)$ for all $x \in \mathbf{R}^n$. Consequently, for all $x \in \mathbf{R}^n$, we have

$$H(x, D\psi(x)) = \alpha x \cdot D\psi(x) + H_0(D\psi(x)) - f(x) = -L_0(-\alpha x) - f(x).$$

Now we assume that there is a convex function $l \in C(\mathbf{R}^n)$ such that

$$\lim_{|x|\to\infty} (l(-\alpha x) + f(x)) = \infty, \tag{9.3}$$

$$\lim_{|\xi|\to\infty} (L_0 - l)(\xi) = \infty. \tag{9.4}$$

Let h denote the convex conjugate of l. We define $\phi \in C^{0+1}(\mathbf{R}^n)$ by $\phi(x) = -\frac{1}{\alpha} l(-\alpha x)$ for $x \in \mathbf{R}^n$. This function ψ is almost everywhere differentiable. Let $x \in \mathbf{R}^n$ be any point where ϕ is differentiable. By a computation similar to the above for ψ, we get

$$\alpha x \cdot D\phi(x) + h(D\phi(x)) - f(x) \le -l(-\alpha x) - f(x). \tag{9.5}$$

By assumption (9.4), there is a constant $C > 0$ such that $L_0(\xi) \ge l(\xi) - C$ for all $\xi \in \mathbf{R}^n$. This inequality implies that $H_0 \le h + C$ in $\mathbf{R}^n$. Hence, from (9.5), we get

$$H(x, D\phi(x)) \le -l(-\alpha x) - f(x) + C.$$

We now conclude that the function H satisfies (A4), with the functions $\phi_0 = \phi$, $\phi_1 = \psi$, $\sigma_0(x) = l(-\alpha x) + f(x) - C$, and $\sigma_1(x) = L(-\alpha x) + f(x)$.

In [FIL2], it is assumed that H_0 has a superlinear growth at infinity, while here it is only assumed that H_0 is coercive. It is assumed here that H_0 is strictly convex in $\mathbf{R}^n$, while it is only assumed in [FIL2] that H_0 is just convex in $\mathbf{R}^n$, so that L_0 may not be a C^1 function. The reason why the strict convexity of H_0 is not needed in [FIL2] is in the fact that Hamiltonians H in this class have a simple structure of the Aubry sets. Indeed, if c is the additive eigenvalue of H, then $\min_{p\in\mathbf{R}^n} H(x,p) = c$ for all $x \in \mathcal{A}_c$. Given such a simple property of the Aubry set, the proof of Theorem 1.3 can be simplified greatly and does not require the C^1 regularity of L_0, while such a regularity is needed in the proof of Lemma 7.8 in the general case. Any $x \in \mathcal{A}_c$ is called an *equilibrium point* if $\min_{p\in\mathbf{R}^n} H(x,p) = c$. A characterization of an equilibrium point $x \in \mathcal{A}_c$ is given by the condition that $L(x,0) = -c$. The property of Aubry sets $\mathcal{A}$ mentioned above can be stated that the set $\mathcal{A}$ comprises only of equilibrium points.

The following example tells us that such a nice property of Aubry sets is not always the case. Let $n = 2$ and here we write (x,y) for a generic point in $\mathbf{R}^2$. We choose a function $g \in C(\mathbf{R}^2)$ so that $g \ge 0$ in $\mathbf{R}^2$, $g(x,y) = 0$ for all $(x,y) \in \mathbf{R}^2 \setminus B((0,0),1)$, and $g(x,y) > 0$ for all $(x,y) \in B((0,0),1)$. Also, we choose a function $h \in C(\mathbf{R}^2)$ so that $h(x,y) \ge 0$ for all $(x,y) \in \mathbf{R}^2$, $h(x,y) = 0$ for all $(x,y) \in B((0,0),2)$, and $h(x,y) \ge x^2 + y^2 - 4$ for all $(x,y) \in \mathbf{R}^2$. We define the Hamiltonian $H \in C(\mathbf{R}^4)$ by

$$H(x,y,p,q) = (p - g(x,y))^2 + q^2 - g(x,y)^2 - h(x,y).$$

It is clear that this Hamiltonian H satisfies (A1)–(A3). Note that (9.1) and (9.2) are satisfied with $H_0(x,y,p,q) = (p-g(x,y))^2+q^2-g(x,y)^2$ and $f = h$. Thus we see that H satisfies (A4) as well. Note moreover that we may take the function: $(x,y) \mapsto \delta|(x,y)|$, with any $\delta > 0$, as ϕ_0 in (A4).

Note that the zero function $z = 0$ is a viscosity solution of $H[z] \le 0$ in $\mathbf{R}^2$ and that $\min_{(p,q)\in\mathbf{R}^2} H(x,y,p,q) = 0$ for all $(x,y) \in B((0,0),2)$. Therefore, in view of Proposition 3.4, we deduce that the additive eigenvalue c for H is zero.

Now we claim that $\mathcal{A} = B((0,0),2)$. Since the zero function $z = 0$ satisfies $H[z] = -h(x,y) < 0$ in $\mathbf{R}^2 \setminus B((0,0),2)$, we see by Proposition 5.5 that $\mathcal{A} \subset B((0,0),2)$. Let $\phi \in C^{0+1}(\mathbf{R}^2)$ be any viscosity subsolution of $H[\phi] = 0$ in $\mathbf{R}^2$. Then, since $H(x,y,p,q) = (p-g(x,y))^2+q^2-g(x,y)^2$ for any $(x,y,p,q) \in \mathbf{R}^2 \times B((0,0),2)$, for almost all $(x,y) \in B((0,0),2)$ we have

$$0 \le \frac{\partial \phi}{\partial x}(x,y) \le 2g(x,y). \tag{9.6}$$

Since $g(x,y) = 0$ for all $(x,y) \in B((0,0),2) \setminus B((0,0),1)$, we find that $D\phi = 0$ a.e. in $B((0,0),2) \setminus B((0,0),1)$ and therefore that $\phi(x,y) = a$ for all $(x,y) \in B((0,0),2) \setminus B((0,0),1)$ and some constant $a \in \mathbf{R}$. The first inequality in (9.6) guarantees that for each $y \in (-1,1)$ the function: $x \mapsto \phi(x,y)$ is nondecreasing in $(-1,1)$. These observations obviously implies that $\phi(x,y) = a$ for all $(x,y) \in B((0,0),2)$. This shows that for any $(x_0,y_0) \in \operatorname{int} B((0,0),2)$, the function $d_H(x,y) \equiv 0$ in a neighborhood of (x_0,y_0) and hence it is a viscosity solution of $H[u] = 0$ in $\mathbf{R}^2$. Thus we see that $\operatorname{int} B((0,0),2) \subset \mathcal{A}$. By the fact that $\mathcal{A}$ is a closed set, we conclude that $\mathcal{A} = B((0,0),2)$.

Finally we remark that $H(x,y,g(x,y),0) = -g(x,y)^2 < 0$ for all $(x,y) \in \operatorname{int} B((0,0),1)$, which shows that any $(x,y) \in \operatorname{int} B((0,0),1)$ is an element of $\mathcal{A}$, but not an equilibrium point.

Next we examine another example whose Aubry set does not contain any equilibrium points. As before we consider the two-dimensional case. We fix $\alpha, \beta \in \mathbf{R}$ so that $0 < \alpha < \beta$ and choose a function $g \in C([0,\infty))$ so that $g(r) = 0$ for all $r \in [\alpha,\beta]$, $g(r) > 0$ for all $r \in [0,\alpha) \cup (\beta,\infty)$, and $\lim_{r\to\infty} g(r)/r^2 = \infty$. We define the functions $H_0, H \in C(\mathbf{R}^4)$ by

$$\begin{aligned} H_0(x,y,p,q) &= (p-y)^2 - y^2 + (q+x)^2 - x^2, \\ H(x,y,p,q) &= H_0(x,y,p,q) - g(\sqrt{x^2+y^2}). \end{aligned}$$

It is easily seen that this function H satisfies (A1)–(A3). Let $\delta > 0$ and set $\psi(x,y) = \delta(x^2+y^2)$ for $(x,y) \in \mathbf{R}^2$. Writing $\psi_x = \partial\psi/\partial x$ and $\psi_y = \partial\psi/\partial x$, we observe that $\psi_x(x,y) = 2\delta x$, $\psi_y(x,y) = 2\delta y$, and $H_0(x,y,\psi_x,\psi_y) = 4\delta^2(x^2+y^2)$ for all $(x,y) \in \mathbf{R}^2$. Therefore, for any $\delta > 0$, if we set $\phi_0(x,y) = -\delta(x^2+y^2)$ and $\phi_1(x,y) = -2\delta(x^2+y^2)$ for $(x,y) \in \mathbf{R}^2$, then (A4) holds with these ϕ_0 and ϕ_1.

Noting that the zero function $z = 0$ is a viscosity subsolution of $H[z] = 0$ in $\mathbf{R}^2$, we find that the additive eigenvalue c for H is nonpositive. We fix any $r \in [\alpha,\beta]$ and consider the curve $\gamma \in \mathrm{AC}([0,2\pi])$ given by $\gamma(t) \equiv (x(t),y(t)) := r(\cos t, \sin t)$. We denote by U the open annulus $\operatorname{int} B((0,0),\beta) \setminus B((0,0),\alpha)$ for notational simplicity. Let $\phi \in C^{0+1}(\mathbf{R}^2)$ be a viscosity solution of $H[\phi] = c$ in $\mathbf{R}^n$. Such a viscosity solution indeed exists according to Theorem 3.3. Due to Proposition 2.4, there are functions $p, q \in L^\infty(0,2\pi,\mathbf{R}^2)$ such that for almost all $t \in (0,2\pi)$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t)) = r(-p(t)\sin t + q(t)\cos t), \\ &(p(t),q(t)) \in \partial_c\phi(\gamma(t)). \end{aligned}$$

The last inclusion guarantees that $H(x(t),y(t),p(t),q(t)) \le c$ a.e. $t \in (0,2\pi)$. Hence, recalling that $\alpha \le r \le \beta$, we get

$$c \ge H_0(x(t),y(t),p(t),q(t)) = p(t)^2 - 2y(t)p(t) + q(t)^2 + 2x(t)q(t) \quad \text{a.e. } t \in (0,2\pi).$$

We calculate that

$$\begin{aligned} \phi(\gamma(T)) - \phi(\gamma(0)) &= r\int_0^T (-p(t)\sin t + q(t)\cos t)\ \mathrm{d}t \\ &\le \frac{1}{2}\int_0^T \left(c - p(t)^2 - q(t)^2\right)\,\mathrm{d}t \le cT \quad \text{for all } T \in [0,2\pi]. \end{aligned}$$

This clearly implies that $c = 0$ and also that the function: $t \mapsto \phi(\gamma(t))$ is a constant. Thus we find that $\phi(x,y) = h(x^2+y^2)$ for some function $h \in C^{0+1}([\alpha,\beta])$.

Next, we show that ϕ is a constant function in U. At any $r \in (\alpha,\beta)$ and any $(x,y) \in \partial B((0,0),r)$, we have

$$\phi_x(x,y) = 2xh'(x^2+y^2) \quad \text{and} \quad \phi_y(x,y) = 2yh'(x^2+y^2),$$

and, in particular, $y\phi_x(x,y) - x\phi_y(x,y) = 0$. Therefore, for almost all $(x,y) \in U$, we have

$$0 \geq H_0(x,y,\phi_x,\phi_y) = (\phi_x - y)^2 - y^2 + (\phi_y + x)^2 - x^2 = \phi_x^2 + \phi_y^2.$$

That is, we have

$$\phi_x(x,y) = \phi_y(x,y) = 0 \quad \text{a.e. } (x,y) \in U,$$

which assures that ϕ is a constant in U.

Now we know that for any $y \in U$, the function: $x \mapsto d_H(x,y)$ is a constant in a neighborhood of y, which guarantees that $U \subset \mathcal{A}$ and moreover that $\mathcal{A} = \overline{U}$.

Finally, we note that $H(x,y,y,-x) = H_0(x,y,y,-x) = -x^2 - y^2 < 0$ for all $(x,y) \in \overline{U}$, and conclude that any $(x,y) \in \mathcal{A} = \overline{U}$ is not an equilibrium points.

The following two propositions give sufficient conditions for points of the Aubry set $\mathcal{A}$ to be equilibrium points. Here, of course, we assume that $c_H = 0$.

Proposition 9.1. *If y is an isolated point of $\mathcal{A}$, then it is an equilibrium point.*

Proof. Let $\gamma \in \Gamma$ be such that $\gamma(0) = y$. Since y is an isolated point of $\mathcal{A}$ and $\gamma(t) \in \mathcal{A}$ for all $t \in \mathbf{R}$, we see that $\gamma(t) = y$ for all $t \in \mathbf{R}$. Hence we have

$$0 = d_H(y,y) = \int_0^1 L(\gamma(t),\dot\gamma(t))\,\mathrm{d}t = L(y,0),$$

which shows that y is an equilibrium point. □

Proposition 9.2. *Assume that there exists a viscosity solution $w \in C(\mathbf{R}^n)$ of $H(x,Dw) = \min_{p\in\mathbf{R}^n} H(x,p)$ in $\mathbf{R}^n$. Then $\mathcal{A}$ consists only of equilibrium points.*

For instance, if $H(x,0) \leq H(x,p)$ for all $(x,p) \in \mathbf{R}^{2n}$, then $w = 0$ satisfies $H(x,Dw(x)) = \min_{p\in\mathbf{R}^n} H(x,p)$ for all $x \in \mathbf{R}^n$ in the viscosity sense. If H has the form $H(x,p) = \alpha x\cdot p + H_0(p) - f(x)$ as before, then H attains a minimum as a function of p at any point p satisfying $\alpha x + D^- H_0(p) \ni 0$ and therefore

$$\min_{p\in\mathbf{R}^n} H(x,p) = \alpha x\cdot q + H_0(q) - f(x),$$

where $q \in D^- L_0(-\alpha x)$ and L_0 denotes the convex conjugate H_0^* of H_0. Therefore, in this case, the function $w(x) := -(1/\alpha)L_0(-\alpha x)$ is a viscosity solution of $H[w] = \min_{p\in\mathbf{R}^n} H(x,p)$ in $\mathbf{R}^n$. In these two cases, the Aubry sets consist only of equilibrium points.

Proof. Since $c_H = 0$, we have $\min_{p\in\mathbf{R}^n} H(x,p) \leq 0$ for all $x \in \mathbf{R}^n$. Note that the function $\sigma(x) := -\min_{p\in\mathbf{R}^n} H(x,p)$ is continuous on $\mathbf{R}^n$ and that w is a viscosity solution of $H[w] = -\sigma$ in $\mathbf{R}^n$. Applying Proposition 5.6 (or 5.5), we see that if $y \in \mathbf{R}^n$ and $\min_{p\in\mathbf{R}^n} H(y,p) < 0$, then $y \notin \mathcal{A}$. That is, if $y \in \mathcal{A}$, then $\min_{p\in\mathbf{R}^n} H(y,p) = 0$, which is equivalent to say that y is an equilibrium point. □

Appendix

We show here that value functions, associated with given Hamiltonian H or its Lagrangian L, are viscosity solutions of $H = 0$.

Let $H \in C(\mathbf{R}^n \times \mathbf{R}^n)$ be a function such that for each $x \in \mathbf{R}^n$ the function: $p \mapsto H(x,p)$ is convex in $\mathbf{R}^n$, and let L be its Lagrangian. Let S be a nonempty subset of $\mathbf{R}^n$ and v_0 a real-valued function on S. We define the function $v : \mathbf{R}^n \to [-\infty, \infty]$ by

$$v(x) = \inf\Big\{\int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + v_0(\gamma(0)) \;\Big|\; t > 0,\ \gamma \in \mathcal{C}(x,t),\ \gamma(0) \in S\Big\}.$$

Theorem A.1. *Let Ω be an open subset of $\mathbf{R}^n$, and assume that $v \in C(\Omega)$. Then v is a viscosity subsolution of $H[v] = 0$ in Ω.*

Proof. Let $(\varphi, x) \in C^1(\Omega) \times \Omega$ and assume that $v - \varphi$ attains a maximum at x. We may assume without loss of generality that $v(x) = \varphi(x)$. Define the multi-function $F : \Omega \to 2^{\mathbf{R}^n}$ by

$$F(x) = \{\xi \in \mathbf{R}^n \mid D\varphi(x)\cdot\xi \geq L(x,\xi) + H(x, D\varphi(x))\}.$$

Since, for any $x \in \mathbf{R}^n$, the function: $p \mapsto H(x,p)$ is a real-valued convex function in $\mathbf{R}^n$, it is subdifferentiable everywhere, which shows that $F(x) \neq \emptyset$ for all $x \in \Omega$. Also, it is easily seen that $F(x)$ is a closed convex set for any $x \in \Omega$ and that the multi-function F is upper semicontinuous in Ω. By the standard existence result for the differential inclusion (see, e.g., [AC, Theorem 2.1.3]), we see that there are a constant $T > 0$ and a function $\eta \in \mathrm{Lip}([0,T], \mathbf{R}^n)$ such that $\dot\eta(s) \in -F(\eta(s))$ a.e. $s \in (0,T)$ and $\eta(0) = x$.

Fix any $\varepsilon \in (0,T)$, $t > 0$, and $\gamma \in \mathcal{C}(\eta(\varepsilon), t)$ such that $\gamma(0) \in S$. We define the curve $\zeta \in \mathcal{C}(x, t+\varepsilon)$ by

$$\zeta(s) = \begin{cases} \gamma(s) & \text{for } s \in [0,t] \\ \eta(\varepsilon + t - s) & \text{for } s \in (t, t+\varepsilon]. \end{cases}$$

It is obvious that $\zeta(0) \in S$. Noting that

$$\dot\zeta(s) = -\dot\eta(\varepsilon + t - s) \in F(\eta(\varepsilon + t - s)) = F(\zeta(s)) \quad \text{a.e. } s \in (t, t+\varepsilon),$$

we have

$$D\varphi(\zeta(s))\cdot\dot\zeta(s) = L(\zeta(s), \dot\zeta(s)) + H(\zeta(s), D\varphi(\zeta(s))) \quad \text{a.e. } s \in (t, t+\varepsilon).$$

Hence we get

$$\begin{aligned} v(x) &= \varphi(x) = \varphi(\zeta(t+\varepsilon)) = \varphi(\zeta(t)) + \int_t^{t+\varepsilon} D\varphi(\zeta(s))\cdot\dot\zeta(s)\,\mathrm{d}s \\ &\geq v(\gamma(t)) + \int_t^{t+\varepsilon} [L(\zeta(s), \dot\zeta(s)) + H(\zeta(s), D\varphi(\zeta(s)))]\,\mathrm{d}s \\ &\geq v_0(\zeta(0)) + \int_0^t L(\zeta(s), \dot\zeta(s))\,\mathrm{d}s + \int_t^{t+\varepsilon} H(\zeta(s), D\varphi(\zeta(s)))\,\mathrm{d}s \\ &\geq v(x) + \int_t^{t+\varepsilon} H(\zeta(s), D\varphi(\zeta(s)))\,\mathrm{d}s. \end{aligned}$$

That is, we have

$$\int_t^{t+\varepsilon} H(\zeta(s), D\varphi(\zeta(s)))\,\mathrm{d}s \le 0.$$

Dividing this by ε and sending $\varepsilon \to 0$, we obtain $H(x, D\varphi(x)) \le 0$. □

Theorem A.2. *Let Ω be an open subset of $\mathbf{R}^n$ such that $S \cap \Omega = \emptyset$, and assume that $v \in C(\Omega)$. Then v is a viscosity supersolution of $H[v] = 0$ in Ω.*

Proof. Let $(\varphi, y) \in C^1(\Omega) \times \Omega$ be such that $v - \varphi$ has a strict minimum at y. We will show that $H(y, D\varphi(y)) \ge 0$. To do this, we argue by contradicition and thus suppose that $H(y, D\varphi(y)) < 0$. We may assume as usual that $v(y) = \varphi(y)$. We choose a constant $r > 0$ so that $B(y, r) \subset \Omega$ and $H(x, D\varphi(x)) \le 0$ for all $x \in B(y, r)$. We set $m = \min_{\partial B(y,r)}(v - \varphi)$. Note that $m > 0$ and $v(x) \ge \varphi(x) + m$ for all $x \in \partial B(y, r)$.

Pick any $t > 0$ and $\gamma \in \mathcal{C}(y, t)$ such that $\gamma(0) \in S$. Since $\gamma(0) \notin \Omega$, there is a constant $\tau \in (0, t)$ such that $\gamma(\tau) \in \partial B(y, r)$ and $\gamma(s) \in B(y, r)$ for all $s \in [\tau, t]$. We now compute that

$$\begin{aligned} v(x) =& \varphi(\gamma(t)) = \varphi(\gamma(\tau)) + \int_\tau^t D\varphi(\gamma(s)) \cdot \dot\gamma(s)\,\mathrm{d}s \\ &\le v(\gamma(\tau)) - m + \int_\tau^t [L(\gamma(s), \dot\gamma(s)) + H(\gamma(s), D\varphi(\gamma(s)))]\,\mathrm{d}s \\ &\le v_0(\gamma(0)) + \int_0^\tau L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + \int_\tau^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s - m \\ &\le v_0(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s - m. \end{aligned}$$

Taking the infimum over $\gamma \in \mathcal{C}(x, t)$, with $\gamma(0) \in S$, and $t > 0$ in the above inequality, we get $v(x) \le v(x) - m$, which is a contradiction. This proves that $H(y, D\varphi(y)) \ge 0$. □

Remark. We may apply above theorems to (1.1) as follows. We introduce the Hamiltonian $\widetilde{H} \in C(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ defined by $\widetilde{H}(x, t, p, q) = q + H(x, p)$. The corresponding Lagrangian $\widetilde{L}$ is given by $\widetilde{L}(x, t, \xi, \eta) = L(x, \xi) + \delta_{\{1\}}(\eta)$, where L is the Lagrangian of H and $\delta_{\{1\}}$ denotes the indicator function of the set $\{1\} \subset \mathbf{R}$. We set $S = \mathbf{R}^n \times \{0\}$ and $\Omega = \mathbf{R}^n \times (0, \infty)$. Also, for given $u_0 \in C(\mathbf{R}^n)$, we define the function $v_0 \in C(S)$ by $v_0(x, 0) = u_0(x)$. We then observe that

$$\begin{aligned} &\inf\Big\{ \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s + u_0(\gamma(0)) \;\Big|\; \gamma \in \mathcal{C}(x, t) \Big\} \\ &= \inf\Big\{ \int_0^T \widetilde{L}(\zeta(s), \dot\zeta(s))\,\mathrm{d}s + v_0(\zeta(0)) \;\Big|\; T > 0,\ \zeta \in \mathcal{C}((x, t), T),\ \zeta(0) \in S \Big\}. \end{aligned}$$

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