The classical inequality holds at the boundary points for subsolutions of Hamilton-Jacobi equations with coercive Hamiltonians

We consider the BVP

$$\begin{cases} u_t + H(x,t,u,Du) = 0 & \text{in } Q := \Omega \times (a,b), \\ u = g(x,t) & \text{on } \partial_l Q := \partial\Omega \times (a,b). \end{cases} \tag{BVP}$$

Here $\Omega \subset \mathbf{R}^n$, and H and g are assumed to be continuous.

CLAIM. *Let $u \in \mathrm{USC}(\overline{\Omega}\times(a,b))$ be a subsolution of* (BVP). *Assume that H is coercive. Then $u(x,t) \le g(x,t)$ for all $(x,t) \in \partial_l Q$.*

Proof. Fix any $(\hat{x},\hat{t}) \in \partial_l Q$. Choose a sequence $\{x_k\} \subset \overline{\Omega}^c := \mathbf{R}^n \setminus \overline{\Omega}$ so that $x_k \to \hat{x}$ as $k \to \infty$. Let $r > 0$ be such that $[\hat{t} - r,\, \hat{t} + r] \subset (a,\, b)$. We consider the function

$$\phi(x,t) := u(x,t) - \alpha|x - x_k| - \beta|t - \hat{t}|^2$$

on $K := B((\hat{x},\hat{t}),\, r) \cap (\overline{\Omega} \times (a,\, b))$. Since $\phi \in \mathrm{USC}(K)$, ϕ attains a maximum at a point (ξ,τ), which depends on (α,β,k). Since $\phi(\xi,\tau) \ge \phi(\hat{x},\hat{t})$, we have

$$\alpha|\xi - \hat{x}| + \beta|\tau - \hat{t}|^2 \le u(\xi,\tau) - u(\hat{x},\hat{t}) + \alpha|\hat{x} - x_k|. \tag{2}$$

If we choose a sequence of (α_k,β_k) so that $(\alpha_k,\beta_k) \to (\infty,\infty)$ and $\alpha_k|\hat{x} - x_k| \to 0$ as $k \to \infty$, then we see from (2) that $(\xi,\tau) \to (\hat{x},\hat{t})$ as $k \to \infty$ and

$$u(\hat{x},\hat{t}) \le \liminf u(\xi,\tau) \le \limsup u(\xi,\tau) \ (\le u(\hat{x},\hat{t}) \text{ by USC of } u).$$

This sequence of inequalities shows that $\lim u(\xi,\tau) = u(\hat{x},\hat{t})$.

We may assume that $|\hat{x} - x_k| \le 1/k^2$ for all k. We set $\alpha_k = k$, so that $\alpha_k|\hat{x} - x_k| \le 1/k \to 0$ as $k \to \infty$. Set $p_k := \alpha_k(\xi - x_k)/|\xi - x_k|$ and $q_k = 2\beta_k(\tau - \hat{t})$, where β_k is to be fixed. Note that $\beta_k|\tau - \hat{t}|^2 \le u(\xi,\tau) - u(\hat{x},\hat{t}) + 1/k \le C^2$ for some constant C independent of k. Hence, we have $2\beta_k|\tau - \hat{t}| \le C\sqrt{2\beta_k}$. We select β_k so that

$$-C\sqrt{2\beta_k} + \min_{K\times I\times\partial B(0,\alpha_k)} H > 0,$$

where $I := [u(\hat{x}, \hat{t}) - 1,\, u(\hat{x}, \hat{t}) + C^2]$. Note that $u(\xi, \tau) \in I$.

Assume that $k \gg 1$, so that $(\xi, \tau) \in B((\hat{x}, \hat{t}), r/2)$. We have either

$$q_k + H(\xi, \tau, u(\xi, \tau), p_k) \leq 0 \tag{3}$$

or $u(\xi, \tau) \leq g(\xi, \tau)$. Since $q_k \geq -C\sqrt{2\beta_k}$ and $|p_k| = \alpha_k$, we cannot have (3) and must have $u(\xi, \tau) \leq g(\xi, \tau)$. Sending $k \to \infty$, we find that $u(\hat{x}, \hat{t}) \leq g(\hat{x}, \hat{t})$ $\square$