Motion of a graph by R-curvature

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October 24, 2001

Abstract

We show the existence of weak solutions to the PDE which describes the motion by R-curvature in $\mathbf{R}^d$, by the continuum limit of a class of infinite particle systems. We also show that weak solutions of the PDE are viscosity solutions and give the uniqueness result on both weak and viscosity solutions.

1 Introduction

In [8] Firey proposed a mathematical model of the wearing process of a convex stone rolling on a beach. In his model a stone evolves according to the Gauss curvature flow. (see, e. g., [1, 4, 5, 17] for the mathematical developments regarding the Gauss curvature flow).

The crystalline approximation of a simple closed convex curve which evolves according to the curvature flow was considered by Girão and Kohn and is useful in the numerical analysis (see [10, 11]). We refer to [9] and the references therein for the recent development of this topics.

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In [14] we proposed and studied a two-dimensional random version of [10]. We also generalized Firey's argument to the case when the stone does not necessarily have a convex shape and when the boundary of the stone is given by the graph of an evolving function (see [15]).

In this paper we propose and study the stochastic approximations of evolving functions which are generalizations of Gauss curvature flow considered in [15].

For $u \in C(\mathbf{R}^d : \mathbf{R})$ and $x \in \mathbf{R}^d$, the following set is called the subdifferential of u at x:

$$\partial u(x) \equiv \{z \in \mathbf{R}^d : u(y) - u(x) \geq < z, y - x > \text{ for all } y \in \mathbf{R}^d\}, \tag{1.1}$$

where $< \cdot, \cdot >$ denotes the inner product in $\mathbf{R}^d$. For a set $A \subset \mathbf{R}^d$ and a function $v : A \mapsto \mathbf{R}$, let $\mathrm{epi}(v)$ and $\hat{v}$ denote, respectively, the epi-graph of v, i.e., the set $\{(x, y) : x \in A,\ y \geq v(x)\}$, and the convex envelope of v, i.e., the function whose epi-graph is the convex hull of $\mathrm{epi}(v)$, provided that it exists.

R-curvature which can be defined as follows plays a crucial role in this paper.

Definition 1 (R-curvature) *Let $R \in L^1(\mathbf{R}^d : [0, \infty), dx)$. For $u \in C(\mathbf{R}^d : \mathbf{R})$, we define the R-curvature of u as the finite Borel measure $w(R, u, dx)$ on $\mathbf{R}^d$ given by (see e.g. [2, section 9.6]):*

$$w(R, u, A) \equiv \int_{\cup_{x \in A} \partial u(x)} R(y) dy \quad \textit{for all Borel } A \subset \mathbf{R}^d. \tag{1.2}$$

Remark 1 *(i) w in (1.2) has another expression:*

$$w(R, u, A) = \int_{\cup_{x \in A} \partial \hat{u}(x)} R(y) dy \quad \textit{for all Borel } A \subset \mathbf{R}^d \tag{1.3}$$

since the Lebesgue measure of the set $\cup_{x \in \mathbf{R}^d} \{p \in \partial \hat{u}(x) : p \text{ is singular}\}$ is zero ($p \in \partial \hat{u}(x)$ is called singular if $\{(y, \hat{u}(x) + < p, y - x >) : y \in \mathbf{R}^d\} \cap epi(\hat{u})$ contains at least two different points.) (see [2, section 9.4]). (ii) $w(1, u, dx)$ is also called the Monge-Ampère measure associated with u and is useful in the study of the Monge-Ampère equation (see [2, 12]).

By the continuum limit of a class of infinite particle systems, we first show the existence of the solution to the following equation (see Theorem 1 in Sect. 2).

Definition 2 (Motion by R-curvature) *The graph of $u \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$ is called the motion by R-curvature if the following holds: for any $\varphi \in C_o(\mathbf{R}^d : \mathbf{R})$ and any $t \geq 0$,*

$$\begin{aligned} & \int_{\mathbf{R}^d} \varphi(x)u(t,x)dx - \int_{\mathbf{R}^d} \varphi(x)u(0,x)dx \\ = & \int_0^t ds \int_{\mathbf{R}^d} \varphi(x)w(R, u(s,\cdot), dx). \end{aligned} \tag{1.4}$$

Roughly speaking, our infinite particle systems $\{(Z_n(t,z))_{z\in\mathbf{Z}^d/n}\}_{0\leq t}$ satisfy that for any $t \geq 0$ and any $z \in \mathbf{Z}^d/n$,

$$P(Z_n(t+\Delta t, z) - Z_n(t,z) > 0) \propto E[w(R, \hat{Z}_n(t,\cdot), \{z\})]\Delta t + o(\Delta t)$$

as $\Delta t \to 0$ $(n \geq 1)$, where $\hat{Z}_n(t,\cdot)$ denotes a convex envelope of the function $z \mapsto Z_n(t,z)$ (see Sect. 2).

We also show the uniqueness result on and elementary properties of the solutions to (1.4) (see Theorems 1 and 2 in Sect. 2).

Theorem 3 in Sect. 2 shows that a continuous solution to (1.4) is a viscosity solution of the following PDE:

$$\partial u(t,x)/\partial t = \chi(u, Du(t,x), t, x)\mathrm{Det}_+(D^2u(t,x))R(Du(t,x)), \tag{1.5}$$

where $Du(t,x) \equiv (\partial u(t,x)/\partial x_i)_{i=1}^d$, $D^2u(t,x) \equiv (\partial^2 u(t,x)/\partial x_i \partial x_j)_{i,j=1}^d$,

$$\chi(u,p,t,x) \equiv \begin{cases} 1 & \text{if } p \in \partial u(t,x), \\ 0 & \text{otherwise,} \end{cases}$$

$\partial u(t,x)$ denotes the subdifferential of the function $x \mapsto u(t,x)$, and for a real $d \times d$-symmetric matrix X,

$$\mathrm{Det}_+ X \equiv \begin{cases} \mathrm{Det}X & \text{if } X \text{ is nonnegative definite,} \\ 0 & \text{otherwise.} \end{cases}$$

We give the definition of the viscosity solution to (1.5) for the reader's convenience.

Definition 3 (Viscosity solution) *(see [15] and also [6]).*
(i). We say that $u \in C((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *is a viscosity subsolution of (1.5) if the following holds: whenever* $\varphi \in C^2((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *and* $u-\varphi$ *attains its maximum at* $(t_o, x_o) \in (0,\infty)\times \mathbf{R}^d$,

$$\partial\varphi(t_o,x_o)/\partial t \le \chi(u, D\varphi(t_o,x_o), t_o, x_o) Det_+(D^2\varphi(t_o,x_o))R(D\varphi(t_o,x_o)).$$

(ii) We say that $u \in C((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *is a viscosity supersolution of (1.5) if the following holds: whenever* $\varphi \in C^2((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *and* $u-\varphi$ *attains its minimum at* $(t_o, x_o) \in (0,\infty)\times \mathbf{R}^d$,

$$\partial\varphi(t_o,x_o)/\partial t \ge \chi^-(u, D\varphi(t_o,x_o), t_o, x_o) Det_+(D^2\varphi(t_o,x_o))R(D\varphi(t_o,x_o)).$$

Here $\chi^-(v,p,t,x) = 1$ *if* $p \in \partial v(t,x)$ *and is not singular and if there exists* $\varepsilon > 0$ *such that for all* $(s,y) \in (0,\infty)\times \mathbf{R}^d$ *satisfying* $|y| > \varepsilon^{-1}$ *and* $|s-t| < \varepsilon$,

$$v(s,y) >< p, y > +\varepsilon|y|,$$

and $\chi^-(v,p,t,x) = 0$, *otherwise.*
(iii) We say that a function $u \in C((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *is a viscosity solution of (1.5) if it is both a viscosity subsolution and supersolution of (1.5).*

Theorem 4 in Sect. 2 shows that a continuous viscosity solution to (1.5) in the space of continuous functions $v : [0,\infty)\times \mathbf{R}^d \mapsto \mathbf{R}$ for which

$$\sup\{|v(t,x) - v(0,x)| : (t,x) \in [0,T]\times \mathbf{R}^d\} < \infty \text{ for all } T > 0 \qquad (1.6)$$

is also a solution to (1.4). Theorem 5 in Sect. 2 shows that a continuous viscosity solution to (1.5) is also a solution to (1.4) under the stronger assumption than that in Theorem 4.

In Sect. 2 we give our main result which will be proved in Sect. 4. In Sect. 3 we state and prove technical lemmas. Sect. 5 is the appendix.

We give the following for the reader's convenience. For any metric space A and B, the topology of $C(A : B)$ is induced by the uniform convergence on every compact subsets of A, and for f and $g \in C(\mathbf{R}^d : \mathbf{R})$, we put

$$d_{C(\mathbf{R}^d:\mathbf{R})}(f,g) \equiv \sum_{m\ge 1} 2^{-m}\min(\sup_{|x|\le m}|f(x)-g(x)|, 1).$$

2 Main Result

We fix a sequence $\{\varepsilon_n\}_{n\geq 1}$ of positive real numbers which converge to zero as $n \to \infty$, and introduce assumptions.
(A.1). $R \in L^1(\mathbf{R}^d : [0,\infty), dx)$, $||R||_{L^1} \equiv \int_{\mathbf{R}^d} R(y)dy > 0$ and $h \in C(\mathbf{R}^d : \mathbf{R})$.
(A.2). The set $\partial h(\mathbf{R}^d) \equiv \cup_{x\in\mathbf{R}^d}\partial h(x)$ has a positive Lebesgue measure, i.e., $\partial h(\mathbf{R}^d)$ has a non-empty interior.

Under (A.1)-(A.2), for any $n \geq 1$ and $v : \mathbf{Z}^d/n \mapsto \mathbf{R}$, put

$$\begin{aligned}\mu_{n,v}(d\beta) \;\equiv\; & n^d\varepsilon_n^{-1}\{\sum_{z\in\mathbf{Z}^d/n} w(R,\hat{v},\{z\})\delta_{v_{n,z}}(d\beta) \\ & +(||R||_{L^1} - w(R,\hat{v},\mathbf{R}^d))\delta_v(d\beta)\},\end{aligned} \tag{2.1}$$

where

$$v_{n,z}(x) \equiv \begin{cases} v(x) + \varepsilon_n & \text{if } x = z, \\ v(x) & \text{if } x \in (\mathbf{Z}^d/n)\backslash\{z\}\end{cases}$$

(Notice that $w(R,\hat{v},\mathbf{R}^d\backslash(\mathbf{Z}^d/n)) = 0$ (see Remark 1, (i)).).

Put also

$$\begin{aligned}S_n \;\equiv\; & \{v : \mathbf{Z}^d/n \mapsto \mathbf{R} | \sum_{z\in\mathbf{Z}^d/n} (v(z) - h(z)) < \infty, \\ & (v(z) - h(z))/\varepsilon_n \in \mathbf{N}\cup\{0\} \text{ for all } z \in \mathbf{Z}^d/n\}.\end{aligned} \tag{2.2}$$

S_n is countable and complete by the metric $d_{S_n}(u,v) \equiv \sum_{z\in\mathbf{Z}^d/n} |u(z) - v(z)|$ $(u,\, v \in S_n)$.

For a bounded function $f : S_n \mapsto \mathbf{R}$ and $v \in S_n$, put

$$A_n f(v) \equiv \int_{S_n} (f(\beta) - f(v))\mu_{n,v}(d\beta). \tag{2.3}$$

Then A_n generates jump-type Markov processes on S_n under (A.1)-(A.2) (see e.g. [7, p. 162]).

Let $\{Z_n(t,\cdot)\}_{0\leq t}$ denote a Markov process on S_n, with a generator A_n and with an initial condition $Z_n(0,z) = h(z)$ $(z \in \mathbf{Z}^d/n)$. For $t \geq 0$ and $x \in \mathbf{R}^d$, put

$$X_n(t,x) \equiv \max(\hat{Z}_n(t,x), h(x)). \tag{2.4}$$

We introduce additional assumptions before we state our first result.
(A.3). The closure of the set $\{x \in \mathbf{R}^d : \hat{h}(x) < h(x)\}$ does not contain any line which is unbounded in two different directions.
(A.4). For any $p \notin \partial h(\mathbf{R}^d)$ and $C \in \mathbf{R}$,

$$\int_{\mathbf{R}^d} \max(< p, x > +C - h(x), 0) dx = \infty. \tag{2.5}$$

Then the following holds.

Theorem 1 *Suppose that (A.1) and (A.3)-(A.4) hold. Then there exists a unique continuous solution u to (1.4) with $u(0,\cdot) = h$. Suppose in addition that (A.2) holds. Then the following holds: for any $\gamma > 0$ and $T > 0$,*

$$\lim_{n\to\infty} P(\sup_{0\le t\le T} d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(t,\cdot), u(t,\cdot)) \ge \gamma) = 0. \tag{2.6}$$

Remark 2 *(i) (A.3) holds when $d = 1$. (ii) For any $C \in \mathbf{R}$, if $p \notin \partial h(\mathbf{R}^d)$, then the set $\{x \in \mathbf{R}^d :< p, x > +C > h(x)\}$ is unbounded. (iii) If h is convex, then (A.4) holds by (i). Indeed, the set $\{x \in \mathbf{R}^d :< p, x > +C > h(x)\}$ is convex for all $(p, C) \in \mathbf{R}^d \times \mathbf{R}$ and is nondecreasing in C for any $p \in \mathbf{R}^d$.*

The following theorem collects some of elementary properties of solutions to (1.4) and hence those of the motion by R-curvature.

Theorem 2 *Suppose that (A.1) holds. Let $u \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$ be a solution of (1.4) with $u(0,\cdot) = h$. Then:*
(a) For each $x \in \mathbf{R}^d$, the function $t \mapsto u(t,x)$ is nondecreasing in $[0,\infty)$.
(b) If $\hat{u}(t,x) < u(t,x)$ for some $(t,x) \in (0,\infty) \times \mathbf{R}^d$, then $u(s,x) = h(x)$ for all $s \in [0,t]$. In particular, $u = \max(\hat{u}, h)$ on $[0,\infty) \times \mathbf{R}^d$.
(c) For any $(t,x) \in [0,\infty) \times \mathbf{R}^d$,

$$u(t,x) - \hat{u}(t,x) \le h(x) - \hat{h}(x). \tag{2.7}$$

In particular, if $h(x) = \hat{h}(x)$ for some $x \in \mathbf{R}^d$, then $u(t,x) = \hat{u}(t,x)$. Or equivalently, if $\partial h(x) \neq \emptyset$ for some $x \in \mathbf{R}^d$, then $\partial u(t,x) \neq \emptyset$.

Suppose in addition that (A.4) holds. Then:

(d) For any $t>0$, $\partial u(t,\mathbf{R}^d)=\partial h(\mathbf{R}^d)$. In particular,

$$\int_{\mathbf{R}^d}(u(t,x)-h(x))dx=t\cdot w(R,h,\mathbf{R}^d). \tag{2.8}$$

(e) Let $\overline{u}\in C([0,\infty)\times\mathbf{R}^d:\mathbf{R})$ be the solution of (1.4) with $\overline{u}(0,\cdot)=\hat{h}$. If $u(s,\cdot)-\hat{u}(s,\cdot)\neq h-\hat{h}$ for some $s\in(0,\infty)$, then $\overline{u}(t,\cdot)-\hat{u}(t,\cdot)\neq 0$ for all $t\geq s$.

According to the above theorem, (a) any graph moves upward by R-curvature, (b) those points on any graph moving by R-curvature do not move as far as they stay in its cavities, (c) the height between any graph moving by R-curvature and its convex envelope is nonincreasing as it evolves, (d) any graph moving by R-curvature sweeps in time $t>0$ a region with volume given by $t\cdot w(R,h,\mathbf{R}^d)$, and (e) for the motion of a graph by R-curvature, taking its convex envelope at time $t>0$ and evolving up to time t starting with the convex envelope of the initial graph give different graphs in general, if the initial graph is not convex.

We introduce
(A.5). $R\in C(\mathbf{R}^d:[0,\infty))$,
and give the relation between the motion by R-curvature and the viscosity solution of (1.5).

Theorem 3 *Suppose that (A.1) and (A.5) hold. Then a continuous solution u to (1.4) with $u(0,\cdot)=h$ is a viscosity solution to (1.5).*

We introduce more assumptions to show that the viscosity solution to (1.5) in the flamework of [15] is the motion by R-curvature.
(A.6). $R(x)\geq R(rx)$ for any $r\geq 1$ and $x\in\mathbf{R}^d$.
(A.7). $\inf_{x\neq o}h(x)/|x|>0$.
(A.8). There exists a constant $C>0$ such that $h(x+y)+h(x-y)-2h(x)\leq C$ for all $(x,y)\in\mathbf{R}^d\times U_1(o)$, where $U_1(o)\equiv\{y\in\mathbf{R}^d:|y|<1\}$.

Theorem 4 *Suppose that (A.1) and (A.3)-(A.8) hold. Then there exists a unique continuous viscosity solution u to (1.5) with $u(0,\cdot)=h$ in the space of continuous functions v for which (1.6) holds. u is also a unique continuous solution to (1.4) with $u(0,\cdot)=h$.*

We give the uniqueness result for the viscosity solution to (1.5) in a different flamework from that of [15], when the solution is a Gauss curvature flow. As a corollary, we show that a continuous viscosity solution to (1.5) is the motion by R-curvature. Put
(A.1)'. $R(y) = (1+|y|^2)^{-(d+1)/2}$ and $h \in C(\mathbf{R}^d : \mathbf{R})$.
For $r > 0$, define $h^r : \mathbf{R}^d \mapsto \mathbf{R}$ by

$$h^r(x) = \inf\{y \in \mathbf{R} \mid U_r((x,y)) \subset \mathrm{epi}(h)\} \quad (x \in \mathbf{R}^d), \tag{2.9}$$

and we introduce
(A.2)'.

$$\liminf_{\theta\downarrow 1}\{\liminf_{r\to\infty}[\liminf_{|x|\to\infty}(h(\theta x) - h^r(x))]\} > 0,$$

$$\lim_{\theta\downarrow 1}\{\sup_{x\in\mathbf{R}^d}(h(x) - h(\theta x))\} = 0.$$

Then we have

Theorem 5 *Suppose that (A.1)'-(A.2)' hold. Then for any viscosity subsolution u and supersolution v, of (1.5) in the space $C([0,\infty)\times\mathbf{R}^d : \mathbf{R})$, such that $u(0,\cdot) \le h \le v(0,\cdot)$, $u \le v$.*

Remark 3 *(A.2)' holds if there exists a convex function $h_0 : \mathbf{R}^d \mapsto \mathbf{R}$ such that $h_0(x) \to \infty$ as $|x| \to \infty$ and that*

$$\lim_{|x|\to\infty}[h(x) - h_0(x)] = 0. \tag{2.10}$$

(see Sect. 5 for the proof).

We easily obtain the following, from Theorems 1, 3 and 5, which should be compared with Theorem 4.

Corollary 1 *Suppose that (A.1)'-(A.2)' and (A.3)-(A.4) hold. Then there exists a unique continuous viscosity solution u to (1.5) with $u(0,\cdot) = h$. u is also a unique continuous solution to (1.4) with $u(0,\cdot) = h$.*

3 Lemmas

In this section we state and prove technical lemmas.

Lemma 1 *Suppose that $f : U_{2m}(o) \mapsto [0,\infty)$ is convex for some $m \geq 1$. Then*

$$\sup_{|x|<m} f(x) \leq |U_m(o)|^{-1} \int_{|x|<2m} f(x)dx. \tag{3.1}$$

(Proof). For x and $y \in U_m(o)$, by the convexity of f,

$$2f(x) \leq f(x+y) + f(x-y).$$

Integrating the both sides on the set $U_m(o)$ with respect to dy, the proof is over since f is nonnegative.

Q. E. D.

Lemma 2 *Suppose that (A.1)-(A.2) hold. Then for any $r \in (0,1/2)$, $n \geq d^{1/2}/r$, $m \geq 1$ and s and t for which $0 \leq s \leq t$, the following holds almost surely:*

$$\begin{aligned} &\sup_{|x|\leq m} (X_n(t,x) - X_n(s,x)) \qquad (3.2)\\ \leq\ & 2\{4r \sup_{|x|\leq m+2, u=s,t} |\hat{Z}_n(u,x)| + \sup_{|x|,|y|\leq m+1, |x-y|<2r} |h(x)-h(y)|\} \\ & + \sum_{z\in \mathbf{Z}^d/n, |z|\leq m+1} (Z_n(t,z) - Z_n(s,z))n^{-d}/|U_r(o)| + \varepsilon_n. \end{aligned}$$

(Proof). For $x = (x_i)_{i=1}^d \in \mathbf{R}^d$, put $[x] \equiv ([x_i])_{i=1}^d$, where $[x_i]$ denotes an integer part of x_i. Then for any $t \geq 0$ and $x \in \mathbf{R}^d$,

$$0 \leq Z_n(t,[nx]/n) - X_n(t,[nx]/n) < \varepsilon_n. \tag{3.3}$$

This is true, since

$$Z_n(t,[nx]/n) \begin{cases} \in [\hat{Z}_n(t,[nx]/n), \hat{Z}_n(t,[nx]/n) + \varepsilon_n) \\ \qquad \text{if } \hat{Z}_n(t,[nx]/n) \geq h([nx]/n), \\ = h([nx]/n) \quad \text{if } \hat{Z}_n(t,[nx]/n) < h([nx]/n). \end{cases}$$

For $x \in U_m(o)$,

$$\begin{aligned} 0 &\leq X_n(t,x) - X_n(s,x) \qquad (3.4)\\ &\leq \sup_{|x-y|\leq r} \{|X_n(t,x) - X_n(t,[ny]/n)| + |X_n(s,[ny]/n) - X_n(s,x)|\} \\ &\qquad + \int_{U_r(x)} (X_n(t,[ny]/n) - X_n(s,[ny]/n))dy/|U_r(o)|. \end{aligned}$$

Hence (3.2) holds by (3.3), the intermediate value theorem, and by the following: for a convex function $f : \mathbf{R}^d \mapsto \mathbf{R}$ and $r > 0$,

$$\sup\{|f(x) - f(y)|/|x-y| : x \neq y, x, y \in U_r(o)\} \leq 2 \sup_{|z|\leq r+1} |f(z)| \qquad (3.5)$$

(see e.g. [2, p. 20, Lemma 3.1]).

Q. E. D.

Lemma 3 *Suppose that (A.1)-(A.2) hold. Then for any* $T > 0$*,* $n \geq 2d^{1/2}$ *and* $m \geq 1$*,*

$$\begin{aligned} &P(\sup_{0\leq t\leq T} \int_{|x|\leq m} \{\hat{Z}_n(t,x) + 2 \sup_{|y|\leq m+2} |\hat{h}(y)|\}dx \qquad (3.6)\\ &\qquad > 2^{d+1}m(\sup_{|x|\leq m+2} \{2|\hat{h}(x)| + |h(x)|\}|U_{m+2}(o)| + T||R||_{L^1})) \\ &\leq \varepsilon_n n^{-d} m^{-2} (\sup_{|x|\leq m+2} \{2|\hat{h}(x)| + |h(x)|\}|U_{m+2}(o)| + T||R||_{L^1})^{-1}. \end{aligned}$$

(Proof). For $t \in [0,T]$,

$$\begin{aligned} &\int_{|x|\leq m} \{\hat{Z}_n(t,x) + 2 \sup_{|y|\leq m+2} |\hat{h}(y)|\}dx \qquad (3.7)\\ &\leq 2^d \sum_{z\in \mathbf{Z}^d/n, |z|\leq m+1} \{Z_n(t,z) + 2 \sup_{|y|\leq m+2} |\hat{h}(y)|\}n^{-d} \\ &\leq 2^d\{ \sum_{z\in \mathbf{Z}^d/n, |z|\leq m+1} (Z_n(t,z) - Z_n(0,z))n^{-d} \end{aligned}$$

$$
\begin{aligned}
&-\int_0^t ds \sum_{z\in \mathbf{Z}^d/n, |z|\le m+1} w(R,\hat{Z}_n(s,\cdot),\{z\})\} \\
&+2^d(\sup_{|x|\le m+2}\{2|\hat{h}(x)|+|h(x)|\}|U_{m+2}(o)|+T||R||_{L^1}),
\end{aligned}
$$

since $w(R,\hat{Z}_n(s,\cdot),U_{m+2}(o))\le ||R||_{L^1}$. Here we used the fact that a convex function takes its maximum on the boundary of the set where it is defined and that $Z_n(t,z)\ge -|\hat{h}(z)|$.

Hence by Doob-Kolmogorov's inequality (see [13, p. 34]), the following completes the proof: by Itô's formula (see [7, p. 162] or [13, p. 66]), for any s and t for which $0\le s\le t$ and any $\varphi\in C_b(\mathbf{R}^d:\mathbf{R})$,

$$
\begin{aligned}
&E[|\sum_{z\in\mathbf{Z}^d/n}\varphi(z)(Z_n(t,z)-Z_n(s,z))n^{-d} \\
&\quad -\int_s^t\sum_{z\in\mathbf{Z}^d/n}\varphi(z)w(R,\hat{Z}_n(u,\cdot),\{z\})du|^2] \\
=\ & E[\int_s^t\sum_{z\in\mathbf{Z}^d/n}\varphi(z)^2 w(R,\hat{Z}_n(u,\cdot),\{z\})du]\varepsilon_n n^{-d}.
\end{aligned}
\tag{3.8}
$$

Q. E. D.

Lemma 4 *Suppose that (A.1)-(A.2) hold. Then for any η and $T>0$, there exists $\delta>0$ such that*

$$
\lim_{n\to\infty}P(\max_{1\le i\le[T/\delta]+1}d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(i\delta,\cdot),X_n((i-1)\delta,\cdot))\ge\eta)=0. \tag{3.9}
$$

(Proof). Take m for which $2^{-(m+1)}<\eta/2$, and take also $r\in(0,1/2)$ such that

$$
\begin{aligned}
r\ \le\ & (\eta/64)|U_{m+2}(o)|\{2^{d+2}(m+2)(\sup_{|x|\le 2(m+2)+2}\{2|\hat{h}(x)| \\
& +|h(x)|\}|U_{2(m+2)+2}(o)|+(T+1)||R||_{L^1})\}^{-1},
\end{aligned}
$$

$$
2\sup_{|x|,|y|\le m+1,|x-y|<2r}|h(x)-h(y)|<\eta/8.
$$

Then for any $\delta \in (0, \min(1, \eta|U_r(o)|/(8||R||_{L^1})))$ and any $n \geq d^{1/2}/r$, by Lemmas 1-2,

$$\begin{aligned} & P(\max_{1\leq i\leq [T/\delta]+1} d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(i\delta,\cdot), X_n((i-1)\delta,\cdot)) \geq \eta) \quad (3.10) \\ \leq\ & P(\sup_{0\leq t\leq T+1}\int_{|x|\leq 2(m+2)}\{\hat{Z}_n(t,x) + 2\sup_{|y|\leq 2(m+3)}|\hat{h}(y)|\}dx > 2^{d+2}(m+2) \\ & \quad \times(\sup_{|x|\leq 2(m+3)}\{2|\hat{h}(x)| + |h(x)|\}|U_{2(m+3)}(o)| + (T+1)||R||_{L^1})) \\ & + \sum_{1\leq i\leq [T/\delta]+1} P(|\sum_{z\in\mathbf{Z}^d/n, |z|\leq m+1}(Z_n(i\delta,z) - Z_n((i-1)\delta,z))n^{-d} \\ & \quad - \int_{(i-1)\delta}^{i\delta}\sum_{z\in\mathbf{Z}^d/n,|z|\leq m+1} w(R,\hat{Z}_n(s,\cdot),\{z\})ds| \geq |U_r(o)|(\eta/8 - \varepsilon_n)), \end{aligned}$$

since for $t \geq 0$,

$$\sup_{|x|\leq m+2}|\hat{Z}_n(t,x)| \leq \sup_{|x|\leq m+2}\{\hat{Z}_n(t,x) + 2\sup_{|y|\leq 2(m+3)}|\hat{h}(y)|\}, \quad (3.11)$$

and since

$$\inf_{|x|\leq 2(m+2)}\{\hat{Z}_n(t,x) + 2\sup_{|y|\leq 2(m+3)}|\hat{h}(y)|\} \geq 0. \quad (3.12)$$

(3.10) together with Lemma 3 and (3.8) completes the proof.

Q. E. D

Lemma 5 *Suppose that (A.1)-(A.2) hold. Then for any $n \geq 1$, the following holds almost surely: for any $t \geq 0$ and $x \in \mathbf{R}^d$,*

$$X_n(t,x) = \max(\hat{X}_n(t,x), h(x)). \quad (3.13)$$

(Proof). By (2.4),

$$\begin{aligned} & \hat{Z}_n(t,x) \leq \hat{X}_n(t,x) \quad (3.14) \\ =\ & \inf\{\sum_{i=0}^{j}\lambda_i X_n(t,x_i) : j \geq 0, \lambda_i \geq 0, x_i \in \mathbf{R}^d (i = 0,\cdots,j) \end{aligned}$$

$$
\begin{aligned}
&, \sum_{i=0}^{j} \lambda_i = 1, \sum_{i=0}^{j} \lambda_i x_i = x\} \\
\leq & \inf\{\sum_{i=0}^{j} \lambda_i Z_n(t, x_i) : j \geq 0, \lambda_i \geq 0, x_i \in \mathbf{Z}^d/n (i = 0, \cdots, j) \\
&, \sum_{i=0}^{j} \lambda_i = 1, \sum_{i=0}^{j} \lambda_i x_i = x\} = \hat{Z}_n(t, x)
\end{aligned}
$$

(see [16, p. 53, Th. 2. 27]).

Q. E. D.

Lemma 6 *Suppose that (A.1) and (A.3)-(A.4) hold. Then for any functions f and $f_n \in C(\mathbf{R}^d : \mathbf{R})$ $(n \geq 1)$ such that $f_n = \max(\hat{f}_n, h)$ $(n \geq 1)$, and that $f_n \to f$ in $C(\mathbf{R}^d : \mathbf{R})$ as $n \to \infty$, and that $\int_{\mathbf{R}^d}(f(x) - h(x))dx$ is finite, $\partial f(\mathbf{R}^d) = \partial h(\mathbf{R}^d)$ and $\hat{f}_n \to \hat{f}$ in $C(\mathbf{R}^d : \mathbf{R})$ as $n \to \infty$. In particular, $f = \max(\hat{f}, h)$ and the following holds: for any $\varphi \in C_o(\mathbf{R}^d : \mathbf{R})$,*

$$
\lim_{n\to\infty} \int_{\mathbf{R}^d} \varphi(x) w(R, f_n, dx) = \int_{\mathbf{R}^d} \varphi(x) w(R, f, dx). \tag{3.15}
$$

(Proof). We first show that $\partial f(\mathbf{R}^d) = \partial h(\mathbf{R}^d)$. Indeed, $\partial h(\mathbf{R}^d) \subset \partial f(\mathbf{R}^d)$ since $f \geq h$, and $\partial f(\mathbf{R}^d) \subset \partial h(\mathbf{R}^d)$ by (A.4) since for any $x_0 \in \mathbf{R}^d$ for which $\partial f(x_0) \neq \emptyset$ and any $p \in \partial f(x_0)$,

$$
\int_{\mathbf{R}^d} \max(< p, x - x_0 > + f(x_0) - h(x), 0) dx \leq \int_{\mathbf{R}^d} (f(x) - h(x)) dx < \infty.
$$

Since $f_n \geq \hat{f}_n \geq \hat{h}$ and $\{f_n\}_{n\geq 1}$ is uniformly bounded on every compact subset of $\mathbf{R}^d$, there exists a convergent subsequence $\{\hat{f}_{n_k}\}_{k\geq 1}$ in $C(\mathbf{R}^d : \mathbf{R})$ (see [2, p. 21, Theorem 3.2]). We denote by g the limit of $\hat{f}_{n_k}$ as $k \to \infty$. It is easy to see that $f = \max(g, h)$ and that $f \geq \hat{f} \geq g \geq \hat{h}$.

Suppose that $\hat{f} \neq g$. Then there exist x_1 and $x_2 \in \mathbf{R}^d$ and $p \in \partial \hat{f}(x_1) \cap \partial g(x_2)$ such that

$$
< p, x - x_1 > + \hat{f}(x_1) >< p, x - x_2 > + g(x_2) \quad \text{for all } x \in \mathbf{R}^d \tag{3.16}
$$

since $\partial \hat{f}(\mathbf{R}^d) = \partial g(\mathbf{R}^d)$ from the above argument.

For sufficiently large $k \geq 1$, one can take $(\lambda_{k,i}, x_{k,i})_{i=0}^{d} \in ([0,1] \times \mathbf{R}^d)^{d+1}$ such that $\sum_{i=0}^{d} \lambda_{k,i} = 1$, $\sum_{i=0}^{d} \lambda_{k,i} x_{k,i} = x_2$, $h(\sum_{i=0}^{d} t_i x_{k,i}) > \hat{f}_{n_k}(\sum_{i=0}^{d} t_i x_{k,i})$ for all $t_i \geq 0$ $(i = 0, \cdots, d)$ for which $\sum_{i=0}^{d} t_i = 1$ and for which $t_i = 0$ if $\lambda_{k,i} = 0$, and that

$$0 \leq \sum_{i=0}^{d} \lambda_{k,i} h(x_{k,i}) - \hat{f}_{n_k}(x_2) < 1/k.$$

Indeed, since $f(x_2) = h(x_2) \geq \hat{f}(x_2) > g(x_2)$ by (3.16), $f_{n_k}(x_2) = h(x_2) > \hat{f}_{n_k}(x_2)$ for sufficiently large $k \geq 1$, and henceforth

$$\begin{aligned} \hat{f}_{n_k}(x_2) &= \inf\{\sum_{i=0}^{d} \lambda_i f_{n_k}(y_i) : \lambda_i \geq 0, y_i \in \mathbf{R}^d (i = 0, \cdots, d) \\ &\quad , \sum_{i=0}^{d} \lambda_i = 1, \sum_{i=0}^{d} \lambda_i y_i = x_2\} \quad \text{(see [16, p. 56, Prop. 2.31])} \\ &= \inf\{\sum_{i=0}^{d} \lambda_i h(y_i) : \lambda_i \geq 0, y_i \in \mathbf{R}^d (i = 0, \cdots, d) \\ &\quad , \sum_{i=0}^{d} \lambda_i = 1, \sum_{i=0}^{d} \lambda_i y_i = x_2, h(\sum_{i=0}^{d} t_i y_i) > \hat{f}_{n_k}(\sum_{i=0}^{d} t_i y_i) \\ &\quad (t_i \geq 0,\ t_i = 0 \text{ if } \lambda_i = 0 (i = 0, \cdots, d), \sum_{i=0}^{d} t_i = 1)\}. \end{aligned} \tag{3.17}$$

Here we used the following argument. It is easy to see that, in the first infimum of (3.17), one can assume that $f_{n_k}(y_i) = h(y_i) > \hat{f}_{n_k}(y_i)$ if $\lambda_i > 0$. Suppose that there exist $j \in \{1, \cdots, d\}$ and $\lambda_i > 0$, $y_i \in \mathbf{R}^d (i = 0, \cdots, j)$ for which $h(y_i) > \hat{f}_{n_k}(y_i)(i = 0, \cdots, j)$, $\sum_{i=0}^{j} \lambda_i = 1$ and $\sum_{i=0}^{j} \lambda_i y_i = x_2$. Suppose also that there exists $t_i \geq 0 (i = 0, \cdots, j)$ for which $\sum_{i=0}^{j} t_i = 1$, $\tilde{y} \equiv \sum_{i=0}^{j} t_i y_i (\neq x_2)$ and $\hat{f}_{n_k}(\tilde{y}) = f_{n_k}(\tilde{y})$. Put $\lambda \equiv \min_{i=0,\cdots,j}(\lambda_i/t_i)$ $(\in (0,1))$. Put also $\overline{\lambda}_i \equiv (\lambda_i - \lambda t_i)/(1 - \lambda)$. Then the following holds:

$$\lambda + (1 - \lambda) \sum_{i=0,\cdots,j,\lambda_i/t_i > \lambda} \overline{\lambda}_i = 1,$$

$$\lambda \tilde{y} + (1 - \lambda) \sum_{i=0,\cdots,j,\lambda_i/t_i > \lambda} \overline{\lambda}_i y_i = x_2,$$

$$\lambda f_{n_k}(\tilde{y}) + (1-\lambda) \sum_{i=0,\cdots,j,\lambda_i/t_i>\lambda} \overline{\lambda}_i h(y_i) < \sum_{i=0}^{j} \lambda_i h(y_i),$$

since $\hat{f}_{n_k}(\tilde{y}) = f_{n_k}(\tilde{y})$ and $h(y_i) > \hat{f}_{n_k}(y_i)(i = 0, \cdots, j)$. Hence one can take $\{\lambda, (1-\lambda)\overline{\lambda}_i, y, y_i : \lambda_i/t_i > \lambda, i = 0, \cdots, j\}$ as a better set of points in the infimum of (3.17) than $\{\lambda_i, y_i : i = 0, \cdots, j\}$, which implies the second equality in (3.17).

Since $\hat{f}(x_2) > g(x_2)$, $\{x_{k,i} : \lambda_{k,i} > 0, i = 0, \cdots, d\}_{k\geq 1}$ is not bounded. Therefore, by (A.3), there exists a sequence $\{m(k)\}_{k\geq 1} \subset \mathbf{N}$ such that $\{x_{m(k),i}, \lambda_{m(k),i}^{-1}\}_{k\geq 1}$ is bounded and that $|\sum_{j\neq i, j=0,\cdots,d,} \lambda_{m(k),j} x_{m(k),j}| \to \infty$ as $k \to \infty$, for some $i \in \{0, \cdots, d\}$. Hence epi(g) contains a line which is infinite in one direction, which intersects with epi$(\hat{f})$ in another one, and which contains $(x_2, g(x_2))$. This contradicts (3.16).

(3.15) can be proved by [2, p. 119, Th. 9.1].

Q. E. D.

Lemma 7 *Suppose that* $R \in L^1(\mathbf{R}^d : [0,\infty), dx)$. *Then for any continuous solutions* u_1 *and* u_2, *to (1.4), for which* $u_1(0,\cdot) \leq u_2(0,\cdot)$ *and for which* $\partial u_1(t, \mathbf{R}^d) \subset \partial u_2(t, \mathbf{R}^d)$ *for all* $t \geq 0$, $u_1 \leq u_2$.

(Proof). Suppose that $u_1 \leq u_2$ is not true. For $\varepsilon > 0$, put

$$V_\varepsilon = \{(t,x) \in (0, \varepsilon^{-1}) \times \mathbf{R}^d; u_2(t,x) + \varepsilon t < u_1(t,x)\}.$$

Then for sufficiently small $\varepsilon > 0$, the set V_ε is open and $|V_\varepsilon| > 0$.

If $(\tau, \xi) \in V_\varepsilon$, and if $p \in \partial u_1(\tau, \xi)$, then $p \in \partial u_2(\tau, \eta)$ for some η for which $(\tau, \eta) \in V_\varepsilon$. Indeed, $u_2(\tau, x) - l(x)$ attains the minimum over $\mathbf{R}^d$ at a point η for which $p \in \partial u_2(\tau, \eta)$, where

$$l(x) := u_1(\tau, \xi) + < p, x - \xi > \qquad \text{for all } x \in \mathbf{R}^d.$$

We also have $(\tau, \eta) \in V_\varepsilon$ since

$$u_2(\tau, x) \geq u_1(\tau, x) - \varepsilon\tau \geq l(x) - \varepsilon\tau \tag{3.18}$$

if $(\tau, x) \notin V_\varepsilon$, and since

$$u_2(\tau, \xi) < u_1(\tau, \xi) - \varepsilon\tau = l(\xi) - \varepsilon\tau. \tag{3.19}$$

An immediate consequence is that

$$\int_{V_\varepsilon} w(R, u_1(t,\cdot), dx)dt \le \int_{V_\varepsilon} w(R, u_2(t,\cdot), dx)dt. \tag{3.20}$$

Take a nondecreasing sequence $\{\eta_n\}_{n\ge 1}$ of nondecreasing C^1-functions such that

$$\eta_n(r) = 0 \quad \text{for all } r \le 0, \quad \eta_n(r) = 1 \quad \text{for all } r \ge \frac{1}{n}, \tag{3.21}$$

and for $r \in \mathbf{R}$, put

$$\zeta_n(r) = \int_0^r \eta_n(s)ds. \tag{3.22}$$

Then for any $x \in \mathbf{R}^d$ and $t \in (0, \varepsilon^{-1})$,

$$\begin{aligned} 0 &\le \zeta_n(u_1(t,x) - u_2(t,x) - \varepsilon t) \qquad (3.23) \\ &= \int_0^t \eta_n(u_1(s,x) - u_2(s,x) - \varepsilon s)(u_1(ds,x) - u_2(ds,x) - \varepsilon ds). \end{aligned}$$

(Notice that the function $t \mapsto u_i(t,x)$ is nondecreasing for $i = 1,\ 2$ and $x \in \mathbf{R}^d$.) Hence by (1.4), for any $r > 0$

$$\begin{aligned} 0 &\le \int_{|x|\le r}\int_0^{\varepsilon^{-1}} \eta_n(u_1(s,x) - u_2(s,x) - \varepsilon s) \qquad (3.24) \\ &\quad \times (u_1(ds,x) - u_2(ds,x) - \varepsilon ds)dx \\ &= \int_0^{\varepsilon^{-1}}\int_{|x|\le r} \eta_n(u_1(s,x) - u_2(s,x) - \varepsilon s) \\ &\qquad \times (w(R, u_1(s,\cdot), dx) - w(R, u_2(s,\cdot), dx) - \varepsilon dx)ds \\ &\to \int_{V_\varepsilon} w(R, u_1(t,\cdot), dx)dt - \int_{V_\varepsilon} w(R, u_2(t,\cdot), dx)dt - \varepsilon|V_\varepsilon| \end{aligned}$$

as $r \to \infty$ and then $n \to \infty$. This together with (3.20) implies that $|V_\varepsilon| = 0$, which is a contradiction.

Q. E. D.

Lemma 8 *Suppose that* $u \in C((0,\infty)\times\mathbf{R}^d : \mathbf{R})$ *and* $\psi \in C^2((0,\infty)\times\mathbf{R}^d : \mathbf{R})$ *satisfy the following: for some* $(s,y) \in (0,\infty)\times\mathbf{R}^d$,

$$\chi^-(u, D\psi(s,y), s, y) = 1, \quad u(s,y) = \psi(s,y), \quad D^2\psi(s,y) > 0,$$

and there exists $A > 0$ such that for any $\varepsilon > 0$,

$$U_\varepsilon^- \equiv \{(t,x) \in (0,\infty) \times \mathbf{R}^d | \psi(t,x) + \varepsilon > u(t,x)\} \subset U_{(\varepsilon/A)^{1/2}}((s,y)). \quad (3.25)$$

Then for sufficiently small $\varepsilon > 0$ and any $(\tau, \xi) \in U_\varepsilon^-$, $D\psi(\tau,\xi) \in \partial u(\tau, z)$ for some z for which $(\tau, z) \in U_\varepsilon^-$.

(Proof). Take $r \in (0, s)$ such that

$$D^2\psi(t,x) > 0 \quad \text{for all } (t,x) \in [s-r, s+r] \times U_r(y). \quad (3.26)$$

Since $\chi^-(u, D\psi(s,y), s, y) = 1$, by the continuity of u, there exists a constant $\delta \in (0, r]$ such that for all $p \in U_\delta(D\psi(s,y))$, and $(t,x) \in [s-\delta, s+\delta] \times (\mathbf{R}^d \setminus U_r(y))$,

$$u(t,x) \geq \psi(s,y) + \delta + < p, x - y > \quad (3.27)$$

(see Definition 3). Take a constant $\gamma \in (0, \delta]$ so that

$$\begin{aligned} \psi(s,y) + \delta &\geq \psi(t,y) + \gamma \quad \text{for all } t \in [s-\gamma, s+\gamma], & (3.28) \\ D\psi(t,x) &\in U_\delta(D\psi(s,y)) \quad \text{for all } (t,x) \in [s-\gamma, s+\gamma] \times U_\gamma(y). & (3.29) \end{aligned}$$

Take $\varepsilon \in (0, \gamma]$ sufficiently small so that $U_\varepsilon^- \subset [s-\gamma, s+\gamma] \times U_\gamma(y)$. Then for $(\tau, \xi) \in U_\varepsilon^-$, from (3.26), we see that

$$\psi(\tau, x) \geq \tilde{l}(x) := \psi(\tau,\xi) + < D\psi(\tau,\xi), x - \xi > \quad \text{for all } x \in U_r(y). \quad (3.30)$$

In particular, we have from (3.28),

$$\psi(s,y) + \delta \geq \psi(\tau, y) + \varepsilon \geq \tilde{l}(y) + \varepsilon. \quad (3.31)$$

Hence for all $x \in \mathbf{R}^d \setminus U_r(y)$, by (3.27) and (3.29),

$$\begin{aligned}\tilde{l}(x)+\varepsilon &= \tilde{l}(y)+\varepsilon+<D\psi(\tau,\xi),x-y> \\ &\leq \psi(s,y)+\delta+<D\psi(\tau,\xi),x-y>\leq u(\tau,x).\end{aligned} \tag{3.32}$$

We also have, by (3.30), for all $x \in U_r(y)$ for which $(\tau, x) \notin U_\varepsilon^-$,

$$\tilde{l}(x)+\varepsilon \leq \psi(\tau,x)+\varepsilon \leq u(\tau,x). \tag{3.33}$$

Since

$$u(\tau,\xi)-\tilde{l}(\xi)<\varepsilon,$$

the function $x \mapsto u(\tau,x)-\tilde{l}(x)$ attains a minimum at z for which $(\tau, z) \in U_\varepsilon^-$, which means that $D\psi(\tau,\xi) \in \partial u(\tau, z)$.

Q. E. D.

The following two lemmas can be shown by the arguments in the proof of [15, Theorem 1], and we omit the proof.

Lemma 9 *Suppose that (A.5)-(A.7) hold, and that u_1 and u_2 are continuous viscosity solutions, to (1.5) with $u_1(0,\cdot)=u_2(0,\cdot)$, for which*

$$\sup\{|u_i(t,x)-u_i(0,x)| : i=1,2,(t,x)\in[0,T]\times\mathbf{R}^d\}<\infty \quad \textit{for all } T>0. \tag{3.34}$$

Then $u_1 = u_2$.

Lemma 10 *Suppose that v and u are viscosity supersolution and continuous viscosity subsolution of (1.5), respectively, and that $u \leq v$ on the set $(\{0\} \times U_r(o)) \cup ((0,T) \times (\mathbf{R}^d \setminus U_r(o)))$ for some $r > 0$ and $T > 0$. Then $u \leq v$ in $(0,T) \times \mathbf{R}^d$.*

4 Proof of Main Result

In this section we prove theorems given in Sect. 2.
(Proof of Theorem 1). First of all, we point out that one can show, from (A.4), that $\partial v(t,\mathbf{R}^d) = \partial h(\mathbf{R}^d)$ for all $t \geq 0$, for any continuous solution v to (1.4) with $v(0,\cdot) = h(\cdot)$, by the argument of the first part of the proof of

Lemma 6. In particular, a continuous solution v to (1.4) with $v(0,\cdot) = h(\cdot)$ is unique by Lemma 7.

Suppose first that (A.2) does not hold. Then $u(t,\cdot) \equiv h(\cdot)$ for all $t \geq 0$ is a unique solution to (1.4) with $u(0,\cdot) = h(\cdot)$.

Suppose next that (A.2) holds. Take $m_0 > 0$ such that

$$\sup_{|x|\leq 2m_0+2} \{2|\hat{h}(x)| + |h(x)|\} > 0.$$

For $t \geq 0$, put

$$\begin{aligned} \Gamma_t \quad \equiv \quad & \{\max(f,h)(x) : f \text{ is convex from } \mathbf{R}^d \text{ to } \mathbf{R}, \text{ and} \\ & \sup_{|x|\leq m} |f(x)| \leq 2^{d+2} m |U_m(o)|^{-1} (\sup_{|x|\leq 2m+2} \{2|\hat{h}(x)| \\ & +|h(x)|\} |U_{2m+2}(o)| + t||R||_{L^1}) \text{ for all } m \geq m_0\}. \end{aligned} \tag{4.1}$$

Then Γ_t is compact in $C(\mathbf{R}^d : \mathbf{R})$ (see [2, section 3.3]).

By Lemmas 1, 3 and (3.11)-(3.12), the following holds: for any $t \geq 0$,

$$\lim_{n\to\infty} P(X_n(t,\cdot) \in \Gamma_t) = 1. \tag{4.2}$$

This together with Lemma 4 implies the tightness of $\{X_n(t,\cdot)\}_{0\leq t, n\geq 1}$ in $D([0,\infty) : C(\mathbf{R}^d : \mathbf{R}))$, since for any $\delta > 0$ and any t and s for which $[t/\delta] = [s/\delta]$,

$$d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(t,\cdot), X_n(s,\cdot)) \leq d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(([t/\delta]+1)\delta,\cdot), X_n([t/\delta]\delta,\cdot))$$

(see [7, p. 129, Corollary 7.4]).

One can also show, by Lemma 4, that any weak limit point of $\{X_n(t,\cdot)\}_{0\leq t}$, as $n \to \infty$, belongs to the set $C([0,\infty) : C(\mathbf{R}^d : \mathbf{R}))$, since for any $t > 0$ and $\delta > 0$,

$$d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(t,\cdot), X_n(t-,\cdot)) \leq \max_{1\leq i\leq [t/\delta]+1} d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(i\delta,\cdot), X_n((i-1)\delta,\cdot))$$

(see [7, p. 148, Theorem 10.2, (a)]).

Let $\{X_{n_k}\}_{k\geq 1}$ be a weakly convergent subsequence of $\{X_n\}_{n\geq 1}$, and X be the weak limit of $\{X_{n_k}\}_{k\geq 1}$. Then by Skorohod's theorem (see [7, p. 102, Theorem 1.8]), taking a new probability space, one can assume that for any $T \geq 0$, $X_{n_k}(t, \cdot)$ converges, as $k \to \infty$, to $X(t, \cdot)$ in $C(\mathbf{R}^d : \mathbf{R})$ uniformly in $t \in [0, T]$ a.s.. By (3.8), we have

$$E[\int_{\mathbf{R}^d} (X(t,x) - h(x))dx] \leq t||R||_{L^1}, \tag{4.3}$$

since

$$X_n(0, [nx]/n) = h([nx]/n) \to X(0, x) = h(x) \quad (\text{as } n \to \infty) \tag{4.4}$$

for $x \in \mathbf{R}^d$, and since for $\varphi \in C_o(\mathbf{R}^d : \mathbf{R})$

$$\begin{aligned} & | \sum_{z \in \mathbf{Z}^d/n_k} \varphi(z)(Z_{n_k}(t,z) - Z_{n_k}(0,z))n_k^{-d} \qquad (4.5) \\ & \qquad - \int_{\mathbf{R}^d} \varphi([n_k x]/n_k)(X_{n_k}(t, [n_k x]/n_k) - X_{n_k}(0, [n_k x]/n_k))dx| \\ \leq \;& 2\varepsilon_{n_k} \int_{\mathbf{R}^d} |\varphi([n_k x]/n_k)| dx \to 0 \quad (\text{as } k \to \infty) \text{ a.s.} \end{aligned}$$

by (3.3). Hence by (1.3), (4.3), Lemmas 5 and 6, for any $\varphi \in C_o(\mathbf{R}^d : \mathbf{R})$ and any $t \geq 0$,

$$\begin{aligned} & \sum_{z \in \mathbf{Z}^d/n_k} \varphi(z) w(R, \hat{Z}_{n_k}(t, \cdot), \{z\}) = \int_{\mathbf{R}^d} \varphi(x) w(R, X_{n_k}(t, \cdot), dx) \qquad (4.6) \\ \to \;& \int_{\mathbf{R}^d} \varphi(x) w(R, X(t, \cdot), dx) \end{aligned}$$

as $k \to \infty$ a.s.. (3.8) and (4.5)-(4.6) imply that $X(t,x)$ is a unique continuous solution to (1.4) with $X(0,x) = h(x)$. In particular, X is nonrandom and henceforth (2.6) holds.

Q. E. D.

(Proof of Theorem 2). The proof of (a) is standard and is omitted.

(b) Suppose that $\hat{u}(t,x) < u(t,x)$. Then there exists a constant $r > 0$ such that

$$\hat{u}(s,y) < u(s,y) \qquad \text{for all } (s,y) \in [t-r, t] \times U_r(x), \tag{4.7}$$

since the function $s \mapsto \hat{u}(s,y)$ is non-decreasing in $[0,\infty)$ for each $y \in \mathbf{R}^d$, and $\hat{u}(s,\cdot) \in C(\mathbf{R}^d : \mathbf{R})$ for each $s \in [0,\infty)$, and $u \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$. Therefore, we have

$$\int_{[t-r,t]\times U_r(x)} dsw(R, u(s,\cdot), dy) = 0. \tag{4.8}$$

This together with (1.4) implies the following:

$$u(s,y) = u(t,y) \qquad \text{for all } (s,y) \in [t-r,t] \times U_r(x). \tag{4.9}$$

Since the function $s \mapsto u(s,y)$ is nondecreasing in $[0,\infty)$ for each $y \in \mathbf{R}^d$, this implies that $u(t,\cdot) = u(0,\cdot)$ in the set $U_r(x)$.

(c) If $u(t,x) = \hat{u}(t,x)$, then $u(t,x) - \hat{u}(t,x) \le h(x) - \hat{h}(x)$ since $h(x) \ge \hat{h}(x)$. Suppose that $\hat{u}(t,x) < u(t,x)$. Then $u(t,x) = h(x)$ by (b), from which we conclude that $u(t,x) - \hat{u}(t,x) \le h(x) - \hat{h}(x)$ since $\hat{u}(t,x) \ge \hat{h}(x)$.

(d) $\partial u(t,\mathbf{R}^d) = \partial h(\mathbf{R}^d)$ by the first argument of the proof of Theorem 1. Plugging functions $\varphi_n \in C_o(\mathbf{R}^d : \mathbf{R})$ $(n \ge 1)$ into (1.4), where the sequence $\{\varphi_n\}_{n\ge 1}$ is nondecreasing and approximates the constant function $\varphi(x) \equiv 1$, and sending $n \to \infty$, we get

$$\int_{\mathbf{R}^d} (u(t,x) - h(x))dx = \int_0^t ds \int_{\mathbf{R}^d} w(R, u(s,\cdot), dx) = t \cdot w(R, h, \mathbf{R}^d). \tag{4.10}$$

(e) We argue by contradiction. Suppose that $\hat{u}(t,\cdot) = \overline{u}(t,\cdot)$ for some $t \ge s$. Then we have

$$\int_{\mathbf{R}^d} (u(t,x) - h(x))dx < \int_{\mathbf{R}^d} (\hat{u}(t,x) - \hat{h}(x))dx \tag{4.11}$$

since

$$u(t,\cdot) - \hat{u}(t,\cdot) \le u(s,\cdot) - \hat{u}(s,\cdot) \ne h - \hat{h},$$

in view of (c). Hence

$$\int_{\mathbf{R}^d} (u(t,x) - h(x))dx < \int_{\mathbf{R}^d} (\overline{u}(t,x) - \hat{h}(x))dx, \tag{4.12}$$

which is a contradiction in view of (d), since (A.4) with h replaced by $\hat{h}$ holds (see Remark 2, (iii)).

Q. E. D.

(Proof of Theorem 3). By (a) in Theorem 2, (1.4) is equivalent to

$$\int_{[0,\infty)\times\mathbf{R}^d} \varphi(t,x)[u(dt,x)dx - dtw(R,u(t,\cdot),dx)] = 0 \tag{4.13}$$

for all $\varphi \in C_o([0,\infty)\times\mathbf{R}^d : \mathbf{R})$.
(Step I). We first show that u is a viscosity subsolution of (1.5).
Let $\psi \in C^2((0,\infty)\times\mathbf{R}^d : \mathbf{R})$ and assume that $u-\psi$ attains a maximum at $(s,y) \in (0,\infty)\times\mathbf{R}^d$. We may assume that $u(s,y) = \psi(s,y)$, so that $u(t,x) < \psi(t,x)$ for all $(t,x) \in (0,\infty)\times\mathbf{R}^d \setminus \{(s,y)\}$ (see [6]).
(i). Consider first the case when $\hat{u}(s,y) = u(s,y)$.
By adding to ψ the function $(t,x) \mapsto A\{|t-s|^2 + |x-y|^2\}$, with a suitable $A > 0$, if necessary, we may assume that $D^2\psi(s,y) > 0$ and that the following set

$$U_\varepsilon^+ \equiv \{(t,x) \in (0,\infty)\times\mathbf{R}^d | \psi(t,x) - \varepsilon < u(t,x)\} \quad (\varepsilon > 0) \tag{4.14}$$

is contained in the set $U_{(\varepsilon/A)^{1/2}}((s,y))$.

In the same way as in (3.18)-(3.20), considering $(u,\psi,-\varepsilon)$ instead of $(u_1,u_2,\varepsilon t)$, by the compactness of the closure of the set U_ε^+, one can show that if $(\tau,\xi) \in U_\varepsilon^+$ and $p \in \partial u(\tau,\xi)$, then $p \in \partial\psi(\tau,z) = \{D\psi(\tau,z)\}$ for some z for which $(\tau,z) \in U_\varepsilon^+$ and that

$$\int_{U_\varepsilon^+} w(R,u(t,\cdot),dx)dt \le \int_{U_\varepsilon^+} w(R,\psi(t,\cdot),dx)dt. \tag{4.15}$$

We argue by contradiction. Assume that the following holds:

$$\partial\psi(s,y)/\partial t > R(D\psi(s,y)) \det{}_+ D^2\psi(s,y), \tag{4.16}$$

since by the definition of χ, we have

$$\chi(u, D\psi(s,y), s, y) = 1.$$

By reselecting $\varepsilon > 0$ if necessary, we may assume that

$$\partial\psi(t,x)/\partial t > \varepsilon + R(D\psi(t,x)) \det{}_+ D^2\psi(t,x), \quad \det{}_+ D^2\psi(t,x) > 0 \tag{4.17}$$

for all $(t,x) \in U_{\varepsilon/A}((s,y))$. Hence in the same way as in (3.21)-(3.24), considering $u - \psi + \varepsilon$ instead of $u_1 - u_2 - \varepsilon t$, we have

$$\varepsilon |U_\varepsilon^+| \le \int_{U_\varepsilon^+} w(R, u(t,\cdot), dx)dt - \int_{U_\varepsilon^+} R(D\psi(t,x)) \det{}_+ D^2\psi(t,x) dx dt \le 0, \tag{4.18}$$

by (4.15), which is a contradiction.
(ii). Consider next the case when $\hat{u}(s,y) < u(s,y)$.
We have

$$\chi(u, D\psi(s,y), s, y) = 0,$$

from which we only have to show that

$$\partial\psi(s,y)/\partial t \le 0. \tag{4.19}$$

(4.19) is true, since from (b) in Theorem 2, we have

$$u(t,y) = u(s,y) \quad \text{for all } t \in (0,s),$$

from which we have

$$\psi(s,y) < \psi(t,y) \quad \text{for all } t \in (0,s).$$

(Step II). Next, we show that u is a viscosity supersolution of (1.5).
Let $\psi \in C^2((0,\infty) \times \mathbf{R}^d : \mathbf{R}^d)$ and assume that $u - \psi$ attains a minimum at $(s,y) \in (0,\infty) \times \mathbf{R}^d$. We may assume as well that $u(s,y) = \psi(s,y)$, so that $u(t,x) > \psi(t,x)$ for all $(t,x) \in (0,\infty) \times \mathbf{R}^d \setminus \{(s,y)\}$ (see [6]).

By (a) in Theorem 2, we see that

$$\partial\psi(s,y)/\partial t \ge 0.$$

Hence we only have to consider the case when the following holds:

$$\chi^-(u, D\psi(s,y), s, y) = 1, \quad \det{}_+ D^2\psi(s,y) > 0.$$

By subtracting from ψ the function $(t,x) \mapsto A\{|t-s|^2 + |x-y|^2\}$, with a sufficiently small $A > 0$, if necessary, we may assume that $D^2\psi(s,y)$ is positive definite and that (3.25) holds. By Lemma 8, if $\varepsilon > 0$ is sufficiently small, then for any $(t,x) \in U_\varepsilon^-$, $D\psi(t,x) \in \partial u(t,z)$ for some z for which $(t,z) \in U_\varepsilon^-$.

As in (Step I), we argue by contradiction. Suppose that

$$\partial\psi(s,y)/\partial t < R(D\psi(s,y))\,\mathrm{det}_+ D^2\psi(s,y). \tag{4.20}$$

Reselecting $\varepsilon > 0$ sufficiently small, we may assume that

$$\partial\psi(t,x)/\partial t + \varepsilon < R(D\psi(t,x))\,\mathrm{det}_+ D^2\psi(t,x), \quad \mathrm{det}_+ D^2\psi(t,x) > 0 \tag{4.21}$$

for all $(t,x) \in U_{(\varepsilon/A)^{1/2}}((s,y))$. Then in the same way as in (Step I), we get

$$\varepsilon|U_\varepsilon^-| \le \int_{U_\varepsilon^-} [R(D\psi(t,x))\det D^2\psi(t,x)dxdt - w(R,u(t,\cdot),dx)dt] \le 0, \tag{4.22}$$

which is a contradiction.

Q. E. D.

(Proof of Theorem 4). By Lemma 9 and Theorems 1 and 3, we only have to show the following: for a solution u to (1.4) with $u(0,\cdot) = h(\cdot)$ and any $T > 0$,

$$\sup\{|u(t,x) - h(x)| : (t,x) \in [0,T] \times \mathbf{R}^d\} < \infty. \tag{4.23}$$

This is true, since

$$\begin{aligned} & u(t,x) - h(x) & (4.24)\\ \le\ & |U_1(o)|^{-1}\int_{|y|\le 1} (u(t,x+y) - h(x+y) + u(t,x-y) - h(x-y))dy/2 \\ & +|U_1(o)|^{-1}\int_{|y|\le 1} (h(x+y) + h(x-y) - 2h(x))dy/2 \\ \le\ & t|U_1(o)|^{-1}\int_{\mathbf{R}^d} R(y)dy + C/2 < \infty, \end{aligned}$$

by (A.1) and (A.8). Here we used the following. If $u(t,x) - h(x) > 0$, then by (c) in Theorem 2, for any $y \in \mathbf{R}^d$,

$$u(t,x) = \hat{u}(t,x) \le (\hat{u}(t,x+y) + \hat{u}(t,x-y))/2 \le (u(t,x+y) + u(t,x-y))/2.$$

Q. E. D.

(Proof of Theorem 5). In $\mathbf{R}^{d+1}$, the moving ball with a fixed center and with radius given by

$$r(t) = (r_0 - (d+1)t)^{1/(d+1)} \qquad t \in [0, r_0/(d+1)), \tag{4.25}$$

with $r_0 > 0$, is a Gauss curvature flow. In particular, for fixed $a \in \mathbf{R}^d$ and $b \in \mathbf{R}$, put

$$w(t,x;a,b,r_0) := \begin{cases} b - ((r_0 - (d+1)t)^{2/(d+1)} - |x-a|^2)^{1/2} \\ \qquad\qquad \text{if } |x-a| \le (r_0 - (d+1)t)^{1/(d+1)}, \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{4.26}$$

Then $w(\cdot;a,b,r_0)$ is a viscosity supersolution of (1.5) with $R(p) = (1 + |p|^2)^{-(d+1)/2}$. Applying Lemma 10 to u and $w(\cdot;a,b,r)$, where $r > 0$ and $a \in \mathbf{R}^d$ are chosen arbitrarily, and $b = h^r(a)$, we find that

$$u(t,x) \le w(t,x;a,b,r) \quad \text{for all } (t,x) \in [0, r/(d+1)) \times \mathbf{R}^d. \tag{4.27}$$

In particular,

$$u(t,a) \le w(t,a;a,b,r) < h^r(a) \quad \text{for all } (t,a) \in [0, r/(d+1)) \times \mathbf{R}^d. \tag{4.28}$$

Fix any $\theta > 1$ and put

$$\varepsilon := \sup_{x \in \mathbf{R}^d} (h(x) - h(\theta x))(\ge h(o) - h(\theta o) = 0), \tag{4.29}$$

and define $z : [0,\infty) \times \mathbf{R}^d \mapsto \mathbf{R}$ by

$$z(t,x) = \varepsilon + v(\theta^{2d} t, \theta x). \tag{4.30}$$

Then z is a viscosity supersolution of (1.5) with $R(p) = (1 + |p|^2)^{-(d+1)/2}$ and satisfies

$$z(0,x) \ge h(x) \qquad \text{for all } x \in \mathbf{R}^d. \tag{4.31}$$

By (A.2)' and (4.29), for any θ for which $\theta - 1(> 0)$ is sufficiently small, and for $r > 0$ which is sufficiently large, depending on θ, there exists a constant $L > 0$ such that

$$z(0,x) \geq h(\theta x) > h^r(x) \qquad \text{if } |x| \geq L. \tag{4.32}$$

Hence by (4.28) and (4.32), if $0 \leq t < r/(d+1)$ and $|x| \geq L$, then

$$u(t,x) < h^r(x) < z(t,x), \tag{4.33}$$

since for each $x \in \mathbf{R}^d$ the function $t \mapsto v(t,x)$ is non-decreasing in $[0,\infty)$.

Again by Lemma 10, we have from (4.31) and (4.33),

$$u(t,x) \leq z(t,x) \equiv \varepsilon + v(\theta^{2d}t, \theta x) \qquad \text{in } (0, r/(d+1)) \times \mathbf{R}^d. \tag{4.34}$$

Let $r \to \infty$ and $\theta \downarrow 1$. Then we have, by (A.2)',

$$u(t,x) \leq v(t,x) \qquad \text{in } (0,\infty) \times \mathbf{R}^d. \tag{4.35}$$

Q. E. D.

5 Appendix

In this section we prove Remark 3.

Proposition 1 *Let $h \in C(\mathbf{R}^d : \mathbf{R})$. Suppose that there exists a convex function $h_0 : \mathbf{R}^d \mapsto \mathbf{R}$ such that $h_0(x) \to \infty$ as $|x| \to \infty$ and that (2.10) holds. Then*

$$\lim_{|x|\to\infty} [h(\theta x) - h^r(x)] = \infty \quad \textit{for all } \theta > 1, r > 0, \tag{5.1}$$

$$\lim_{\theta \downarrow 1} \{ \sup_{x \in \mathbf{R}^d} [h(x) - h(\theta x)] \} = 0. \tag{5.2}$$

(Proof). Without loss of generality, we may assume that $h_0 \geq 0$.

For any $\theta > 1$ and $r > 0$, we have

$$h_0(\theta x) - \max_{|y| \leq r} h_0(x+y) \to \infty \qquad \text{as } |x| \to \infty, \tag{5.3}$$

since by the convexity of h_0, for any x and $y \in \mathbf{R}^d$, with $|y| \leq r$,

$$h_0(x+y) = h_0(\theta^{-1}\theta x + (1-\theta^{-1})\theta(\theta-1)^{-1}y) \tag{5.4}$$
$$\leq \theta^{-1}h_0(\theta x) + (1-\theta^{-1})h_0(\theta(\theta-1)^{-1}y) \leq \theta^{-1}h_0(\theta x) + C(r,\theta),$$

where $C(r,\theta)$ is a constant. We also have

$$h^r(x) \leq \sup_{|z|\leq r} h(x+z) + r \qquad \text{for all } x \in \mathbf{R}^d \tag{5.5}$$

since $U_r((x,y)) \subset \mathrm{epi}(h)$ for all $y \geq \sup_{|z|\leq r} h(x+z) + r$. (2.10), (5.3) and (5.5) implies (5.1).

We also have, by (5.4) with $y = o$, for any $\theta > 1$,

$$h_0(x) - h_0(\theta x) \leq \theta h_0(x) - h_0(\theta x) \leq (\theta - 1)h_0(0). \tag{5.6}$$

This together with (2.10) and (4.29) implies (5.2).

Q. E. D.

Acknowledgement: We would like to thank Prof. K. Ishii for informing us that Theorem 5 is similar to [3, Theorem 4.1] where they considered the mean curvature flow with a convex coercive initial function by a different approach from ours.

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