ASYMPTOTIC ANALYSIS FOR A CLASS OF INFINITE SYSTEMS OF FIRST-ORDER PDE : nonlinear parabolic PDE in the singular limit

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ABSTRACT

We study the asymptotic behavior of solutions of the Cauchy problem for a functional partial differential equation with a small parameter as the parameter tends to zero. We establish a convergence theorem in which the limit problem is identified with the Cauchy problem for a nonlinear parabolic partial differential equation. We also present comparison and existence results for the Cauchy problem for the functional partial differential equation and the limit problem.

1. INTRODUCTION

In this paper we study the asymptotic behavior of solutions of the Cauchy problem for the functional partial differential equation

$$\begin{cases} (\mathrm{E})_\varepsilon \quad u_t^\varepsilon(x,t,\xi) = \dfrac{1}{\varepsilon} H(Du^\varepsilon(x,t,\xi),\xi) \\ \qquad\qquad + \dfrac{1}{\varepsilon^2}\displaystyle\int_I k(\xi,\eta)\,[u^\varepsilon(x,t,\eta) - u^\varepsilon(x,t,\xi)]\,d\eta \\ \qquad\qquad\qquad \text{for } (x,t,\xi) \in \mathbf{R}^n \times (0,\infty) \times I, \\ u^\varepsilon(x,0,\xi) = g(x,\xi) \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I, \end{cases} \tag{CP$_\varepsilon$}$$

where ε is a positive parameter, I is a given finite interval of the real line, H is a Borel function on $\mathbf{R}^n \times I$ such that for each $\xi \in I$ the function $H(\cdot,\xi)$ is continuous on $\mathbf{R}^n$, and k is a bounded, positive, Borel measurable function on $I \times I$.

The functional partial differential equation $(E)_\varepsilon$ may be regarded as an infinite system of first order partial differential equations. Indeed, one of our motivations to study $(CP)_\varepsilon$ is to extend an asymptotic result obtained in Evans [3] for a finite system of partial differential equations to that for $(CP)_\varepsilon$. Prior to [3] there are many contributions to the asymptotic behavior of solutions of systems of differential equations related to the problems treated in [3] and we refer for these to [6, 7, 3] and the references therein.

The functional partial differential equation $(E)_\varepsilon$ arises as a fundamental equation for the optimal control of the system whose states are described by ordinary differential equations, subject to random changes of states in I and to control which induce the integral term in $(E)_\varepsilon$ and the nonlinearity of H, respectively.

Other than the extension to infinite systems, new features in this paper beyond [3] are: (i) the treatment of the initial layer, i.e., the case when the initial data $g(x, \xi)$ depends on ξ and (ii) the nonlinearity of the term H.

In our asymptotic analysis of $(CP)_\varepsilon$, we use the perturbed test function method developed in [3], which is based on the notion of viscosity solution and the stability properties of viscosity solutions. The extension from finite systems to infinite systems was not trivial and, as we will see in section 5, we need to take into account of terms up to order ε^2 when we build the perturbed test function.

The problem of the initial layer in our analysis is resolved by constructing appropriate barrier functions, a result of which is stated in Lemma 5.1 below. On the other hand, the extension to the nonlinear term H is rather straightforward.

In view of viscosity solutions theory, our treatment of functional partial differential equation is new in that the function $H(p, \xi)$ is not assumed to be continuous and it is assumed to be continuous in p and Borel measurable in ξ and in that the solution $u^\varepsilon(x, t, \xi)$ is not assumed to be continuous and is assumed to be continuous in (x, t) and Borel measurable in ξ. We modify the standard definition of viscosity solution to that situation and then the problem is how to prove the existence of viscosity solution of $(CP)_\varepsilon$. This is done by employing an argument based on monotone classes of functions.

The paper is organized as follows: In section 2 we prepare our notation and then state our main results. In section 3 we prove a comparison theorem for subsolutions and supersolutions of

$$\begin{cases} \text{(E)} \quad u_t(x,t,\xi) = H(Du(x,t,\xi),\xi) + \displaystyle\int_I k(\xi,\eta)\,[u(x,t,\eta) - u(x,t,\xi)]\,d\eta \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (x,t,\xi) \in \mathbf{R}^n \times (0,\infty) \times I, \\ \qquad u(x,0,\xi) = g(x,\xi) \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I, \end{cases} \tag{CP}$$

which readily yields a comparison assertion for $(CP)_\varepsilon$. In section 4, we establish existence and stability theorems for (CP). In section 5, we prove our main convergence result, which roughly says that, as $\varepsilon \to 0$, the solutions $u^\varepsilon(x,t,\xi)$ converge to the unique viscosity solution u of

$$\begin{cases} (\mathrm{E})_0 \quad u_t(x,t) = \operatorname{tr}[\bar{A}(Du(x,t))D^2u(x,t)] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (x,t) \in \mathbf{R}^n \times (0,\infty), \\ \qquad\quad u(x,0) = g(x) \quad \text{for } x \in \mathbf{R}^n, \end{cases} \tag{CP$_0$}$$

where $g = \bar{g} \in \mathrm{BUC}(\mathbf{R}^n)$ and $\bar{A} \in C(\mathbf{R}^n, \mathcal{S}^n)$, and $\bar{g}$ and $\bar{A}$ will be specified later in section 2 (see (2.15) and (2.16)). Here and henceforth, $\mathcal{S}^n$ denotes the space of real symmetric matrices of order n and for any subset Ω of $\mathbf{R}^m$, $\mathrm{BUC}(\Omega)$ denotes the set of all bounded, uniformly continuous functions on Ω. Section 6 is devoted to the proof of two basic lemmas which are used in the previous sections.

2. PRELIMINARIES AND MAIN RESULTS

We use the following notation: $Q_T = \mathbf{R}^n \times (0,T)$, $R_T = \mathbf{R}^n \times [0,T)$ for $0 < T \le \infty$, and for function $f : S \to \mathbf{R}^m$ we write $\|f\|_\infty = \sup_S |f|$. I denotes a fixed finite interval, with length $|I| > 0$, and also the identity operator on a given space.

For any $k \in \mathbf{Z}_+ := \mathbf{N} \cup \{0\}$ and $\Omega \subset \mathbf{R}^m$, $C^k(\Omega) \otimes \mathcal{B}(I)$ denotes the set of functions f on $\Omega \times I$ such that for each $x \in \Omega$ the function $f(x,\cdot)$ is Borel measurable in I and for each $\xi \in I$ the function $f(\cdot,\xi)$ is k times continuously differentiable on Ω. We write also $C(\Omega) \otimes \mathcal{B}(I)$ for $C^0(\Omega) \otimes \mathcal{B}(I)$. For any Borel subset $\Omega \subset \mathbf{R}^m$, $\mathcal{B}(\Omega)$ denotes the space of all Borel functions on Ω, and $\mathcal{B}^\infty(\Omega)$ denotes the Banach space of bounded Borel functions f on Ω with norm $\|f\|_\infty$.

Throughout this paper we fix positive numbers κ_0, κ_1, with $\kappa_0 < \kappa_1$, and consider the class $\mathcal{D}_0$ of Borel functions k on $I \times I$ such that $\kappa_0 \le k(\xi,\eta) \le \kappa_1$ for all $\xi, \eta \in I$.

We call a continuous function $\omega : [0,\infty) \to [0,\infty)$ a modulus if ω is non-decreasing in $[0,\infty)$ and $\omega(0) = 0$.

Let G_1 and G_2 denote the sets, respectively, of all pairs (ω, L) of a modulus ω and a positive constant L and of all pairs of a collection $\{\omega_R\}_{R>0}$ of moduli and a collection $\{L_R\}_{R>0}$ of positive constants. We write $G = G_1 \times G_2$.

For $\gamma_1 \equiv (\omega, L) \in G_1$ let $\mathcal{D}_1(\gamma_1)$ denote the set of all functions $g \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ such that

$$|g(x,\xi) - g(y,\xi)| \le \omega(|x-y|), \quad |g(x,\xi)| \le L \quad \text{for all } x, y \in \mathbf{R}^n,\ \xi \in I. \tag{D1}$$

For $\gamma_2 \equiv (\{\omega_R\}_{R>0}, \{L_R\}_{R>0}) \in G_2$ let $\mathcal{D}_2(\gamma_2)$ denote the set of all functions $H \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ such that

$$|H(p,\xi) - H(q,\xi)| \le \omega_R(|p-q|), \quad |H(p,\xi)| \le L_R \\ \text{for all } p, q \in B(0,R),\ \xi \in I,\ R > 0, \tag{D2}$$

where $B(0, R)$ denotes the closed ball with radius R centered at the origin. For $\gamma \equiv (\gamma_1, \gamma_2) \in G$ we write

$$\mathcal{D}(\gamma) = \mathcal{D}_0 \times \mathcal{D}_1(\gamma_1) \times \mathcal{D}_2(\gamma_2),$$

and set

$$\mathcal{D}_i = \bigcup \{\mathcal{D}_i(\gamma) \mid \gamma \in G_i\} \quad \text{for} \quad i = 1, 2 \qquad \text{and} \qquad \mathcal{D} = \bigcup \{\mathcal{D}(\gamma) \mid \gamma \in G\}.$$

We often consider the subclass of functions $k \in \mathcal{D}_0$ for which

$$\int_I k(\xi, \eta) d\eta = 1 \qquad \text{for all } \xi \in I. \tag{K1}$$

For such a function k, we define the continuous linear operator $K : \mathcal{B}^\infty(I) \to \mathcal{B}^\infty(I)$ by

$$Kf(\xi) = \int_I k(\xi, \eta) f(\eta) d\eta \qquad \text{for } \xi \in I. \tag{2.1}$$

Note that this formula extends the domain of definition of K to the space of (Lebesgue) measurable functions $f : I \to \mathbf{R}$ which are integrable. Associated with this operator, we define the compact linear operator $\bar{K} : L^2(I) \to L^2(I)$ by

$$\bar{K} f(\xi) = \int_I k(\xi, \eta) f(\eta) d\eta \qquad \text{for } f \in L^2(I). \tag{2.2}$$

As usual and in the above formula, we often identify elements of $L^2(I)$ with measurable functions on I, the square of which are integrable. The precise meaning of (2.2) is the following: for function $f : I \to \mathbf{R}$ which is measurable and such that $|f|^2$ is integrable, let

$$[f] := \{g : I \to \mathbf{R} \mid g \text{ measurable}, \ g(\xi) = f(\xi) \text{ a.e. } \xi \in I\}.$$

With this notation, $\bar{K}$ is defined by

$$\bar{K}[f] = [Kf].$$

By hypothesis (K1), the operator $\bar{K}$ has unity as its eigenvalue and the function $\mathbf{1} \in L^2(I)$ defined by $\mathbf{1}(\xi) \equiv 1$ as a corresponding eigenfunction. By the Perron-Frobenius theory, we see that the kernel $\operatorname{Ker}(I - \bar{K})$ is one-dimensional, i.e.,

$$\operatorname{Ker}(I - \bar{K}) = \operatorname{span}\{\mathbf{1}\}.$$

(See the proof of Lemma 5.2 in section 6.)

By the Fredholm-Riesz-Schauder theory (see, e.g., [8]), the kernel $\mathrm{Ker}\,(I-\bar{K}^*)$, where $\bar{K}^*$ denotes the adjoint operator of $\bar{K}$, is a one–dimensional subspace of $L^2(I)$. Hence, there exists a unique vector $r \in L^2(I)$ such that

$$\int_I r(\xi)k(\xi,\eta)d\xi = r(\eta) \quad \text{a.e. } \eta \in I, \tag{2.3}$$

$$\int_I r(\xi)d\xi = 1. \tag{2.4}$$

When we regard the vector r as a function, we may assume by replacing r by the function defined by the left hand side of (2.3) if necessary that $r \in \mathcal{B}^\infty(I)$ and that

$$\int_I r(\xi)k(\xi,\eta)d\xi = r(\eta) \quad \text{for all } \eta \in I. \tag{2.5}$$

Moreover, by the Perron-Frobenius theory, we see that $r(\xi) > 0$ for all $\xi \in I$. Then form (2.5) we get

$$\kappa_0|I| \le r(\xi) \le \kappa_1|I| \quad \text{for } \xi \in I. \tag{2.6}$$

By the Fredholm-Riesz-Schauder theory, there is a bounded linear operator $\bar{S} : \{r\}^\perp \to \{\mathbf{1}\}^\perp$, where $B^\perp$ denotes the orthogonal complement of B in $L^2(I)$, such that

$$\bar{S}f - \bar{K}\bar{S}f = f \quad \text{for } f \in \{r\}^\perp. \tag{2.7}$$

For any integrable function $h : I \to \mathbf{R}$, we define

$$\{h\}^{\perp,\infty} = \{f \in \mathcal{B}^\infty(I) \mid \int_I h(\xi)f(\xi)d\xi = 0\}.$$

Associated with $\bar{S}$, we define a continuous linear operator $S : \{r\}^{\perp,\infty} \to \{\mathbf{1}\}^{\perp,\infty}$ by

$$Sf = f + Kg, \quad \text{with } g \in \bar{S}[f].$$

Here note that Kg does not depend on the choice of $g \in \bar{S}[f]$ and that for $f \in \{r\}^{\perp,\infty}$ and $g \in \bar{S}[f]$,

$$\begin{aligned}|Kg(\xi)| &\le \int_I k(\xi,\eta)|g(\eta)|d\eta \le \left(\int_I |k(\xi,\eta)|^2 d\eta\right)^{1/2} \|\bar{S}[f]\|_2 \\ &\le \kappa_1|I|^{1/2}\|\bar{S}\|\|f\|_2 \le \kappa_1|I|\|\bar{S}\|\|f\|_\infty,\end{aligned}$$

where $\|f\|_2 = \left(\int_I |f(\xi)|^2 d\xi\right)^{1/2}$.

Now, (2.7) reads

$$(I-K)Sf = f \quad \text{for } f \in \{r\}^{\perp,\infty}. \tag{2.8}$$

Let $H \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ satisfy (D2) for some $(\{\omega_R\}, \{L_R\}) \in G_2$ and

$$\int_I H(p,\xi) r(\xi) d\xi = 0 \qquad \text{for } p \in \mathbf{R}^n. \tag{H1}$$

We define $a \in C(\mathbf{R}^n) \otimes \mathcal{B}(I)$ by

$$a(p,\cdot) = SH(p,\cdot). \tag{2.9}$$

Observe that if, in addition, we assume that $H \in C^m(\mathbf{R}^n) \otimes \mathcal{B}(I)$ for some $m \in \mathbf{N}$ and that for each $R > 0$ there are a constant $C_R > 0$ and a modulus ω_R such that for any multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{Z}_+^n$ with $\alpha_1 + \cdots \alpha_n \leq m$,

$$|D_p^\alpha H(p,\xi)| \leq C_R \qquad \text{for } (p,\xi) \in B(0,R) \times I,\ \xi \in I, \tag{2.10}$$
$$|D_p^\alpha H(p,\xi) - D_p^\alpha H(q,\xi)| \leq \omega_R(|p-q|) \qquad \text{for } p,q \in B(0,R),\ \xi \in I,\ R > 0, \tag{2.11}$$

and if we set $f(p,\xi) = SH(p,\cdot)(\xi)$ for $(p,\xi) \in \mathbf{R}^n \times I$, then $f \in C^m(\mathbf{R}^n) \otimes \mathcal{B}(I)$ and furthermore for each $R > 0$ there exist a constant $M_R > 0$ and a modulus μ_R such that for any multi-index $\alpha \in \mathbf{Z}_+^n$, with $\alpha_1 + \cdots + \alpha_n \leq m$,

$$|D_p^\alpha f(p,\xi)| \leq M_R \qquad \text{for } (p,\xi) \in B(0,R) \times I,\ R > 0, \tag{2.12}$$
$$|D_p^\alpha f(p,\xi) - D_p^\alpha f(q,\xi)| \leq \mu_R(|p-q|) \qquad \text{for } p,q \in B(0,R),\ \xi \in I,\ R > 0. \tag{2.13}$$

In addition to (D2) and (H1), we assume that $H \in C^1(\mathbf{R}^n) \otimes \mathcal{B}(I)$ and that H satisfies (2.10) and (2.11) with $m = 1$. We define $A : \mathbf{R}^n \times I \to \mathcal{S}^n$ and $\bar{A} : \mathbf{R}^n \to \mathcal{S}^n$ by

$$A(p,\xi) = \frac{1}{2}\left(D_p H(p,\xi) \otimes D_p a(p,\xi) + D_p a(p,\xi) \otimes D_p H(p,\xi)\right), \tag{2.14}$$
$$\bar{A}(p) = \int_I r(\xi) A(p,\xi) d\xi. \tag{2.15}$$

The components of the matrix-valued function A belong to $C(\mathbf{R}^n) \otimes \mathcal{B}(I)$. Also, in view of (2.13), we see that $\bar{A}$ is continuous on $\mathbf{R}^n$. We claim that $\bar{A}(p)$ is a non-negative definite matrix for any $p \in \mathbf{R}^n$. To see this, we first observe that

$$D_p H(p,\xi) = D_p a(p,\xi) - \int_I k(\xi,\eta) D_p a(p,\eta) d\eta \qquad \text{for all } (p,\xi) \in \mathbf{R}^n \times I.$$

Let $y \in \mathbf{R}^n$ and compute that for $p \in \mathbf{R}^n$,

$$
\begin{aligned}
&\langle \bar{A}(p)y, y\rangle \\
&= \int_I r(\xi)\langle D_pH(p,\xi), y\rangle\langle D_pa(p,\xi), y\rangle d\xi \\
&= \int_I r(\xi)\langle D_pa(p,\xi), y\rangle^2 d\xi - \iint_{I\times I} r(\xi)k(\xi,\eta)\langle D_pa(p,\eta), y\rangle\langle D_pa(p,\xi), y\rangle d\xi d\eta \\
&\geq \int_I r(\xi)\langle D_pa(p,\xi), y\rangle^2 d\xi \\
&- \Big(\iint_{I\times I} r(\xi)k(\xi,\eta)\langle D_pa(p,\eta), y\rangle^2 d\xi d\eta\Big)^{1/2}\Big(\iint_{I\times I} r(\xi)k(\xi,\eta)\langle D_pa(p,\xi), y\rangle^2 d\xi d\eta\Big)^{1/2} \\
&= 0,
\end{aligned}
$$

which was to be proven. Here and henceforth we write $\langle p, q\rangle$ for the Euclidean inner product of $p, q \in \mathbf{R}^n$.

Let $\Omega \subset R_\infty$ and $(\nu, M) \in G_1$. We denote by $\mathcal{U}(\nu, M) \equiv \mathcal{U}(\Omega \times I; \nu, M)$ the set of functions $u \in C(\Omega) \otimes \mathcal{B}(I)$ such that

$$|u(x,t,\xi) - u(y,s,\xi)| \leq \nu(|x-y| + |t-s|) \tag{U1}$$

$$|u(x,0,\xi)| \leq M \tag{U2}$$

for all $(x,t) \in \Omega$ and $\xi \in I$. We denote

$$\mathcal{U} \equiv \mathcal{U}(\Omega \times I) = \bigcup\{\mathcal{U}(\lambda) \mid \lambda \in G_1\}.$$

We write

$$\mathcal{U}_c(\Omega \times I; \lambda) = \mathcal{U}(\Omega \times I; \lambda) \cap C(\Omega \times I), \quad \mathcal{U}_c(\Omega \times I) = \mathcal{U}(\Omega \times I) \cap C(\Omega \times I).$$

We denote by $\mathcal{U}^+(\Omega \times I)$ the set of those functions u on $\Omega \times I$ such that for each $(x,t) \in \Omega$ the function $u(x,t,\cdot)$ is Borel measurable and integrable in I and for each $\xi \in I$ the function $u(\cdot,\xi)$ is upper semicontinuous in Ω. We set $\mathcal{U}^-(\Omega\times I) = -\mathcal{U}^+(\Omega\times I)$.

Next, we give the definition of viscosity solutions of (E).

Definition 2.1. *Let $\Omega \subset Q_\infty$ be an open subset and $(k, H) \in \mathcal{D}_0 \times \mathcal{D}_2$.* (i) *We call $u \in \mathcal{U}^+(\Omega \times I)$ a viscosity subsolution of* (E) *in $\Omega \times I$ if whenever $\varphi \in C^1(\Omega)$, $\xi \in I$, and $u(\cdot,\xi) - \varphi$ attains its local maximum at $(\hat{x}, \hat{t})$, then*

$$\varphi_t(\hat{x},\hat{t}) \leq H(D\varphi(\hat{x},\hat{t}), \xi) + \int_I k(\xi,\eta)[u(\hat{x},\hat{t},\eta) - u(\hat{x},\hat{t},\xi)]d\eta.$$

(ii) *Similarly we call* $u \in \mathcal{U}^-(\Omega \times I)$ *a viscosity supersolution of* (E) *in* $\Omega \times I$ *if whenever* $\varphi \in C^1(\Omega)$, $\xi \in I$, *and* $u(\cdot, \xi) - \varphi$ *attains its local minimum at* $(\hat{x}, \hat{t})$, *then*

$$\varphi_t(\hat{x}, \hat{t}) \geq H(D\varphi(\hat{x}, \hat{t}), \xi) + \int_I k(\xi, \eta)[u(\hat{x}, \hat{t}, \eta) - u(\hat{x}, \hat{t}, \xi)]d\eta.$$

(iii) *Finally, we call* $u \in C(\Omega) \otimes \mathcal{B}(I)$ *a viscosity solution of* (E) *in* $\Omega \times I$ *if it is both a viscosity sub- and supersolution of* (E) *in* $\Omega \times I$.

For the definition of viscosity solutions of $(\mathrm{E})_0$, we use the standard definition, for which we refer to [1].

Now, we state our main results.

Theorem 2.2. *Let* $(k, g, H) \in \mathcal{D}$. *Then there is a unique viscosity solution* $u \in \mathcal{U}(R_\infty \times I)$ *of* (CP).

Of course, $u \in \mathcal{U}(R_\infty \times I)$ is defined to be a viscosity solution of (CP) if it is a viscosity solution of (E) in $Q_\infty \times I$ and it satisfies the initial condition: $u(x, 0, \xi) = g(x, \xi)$ for all $(x, \xi) \in \mathbf{R}^n \times I$.

Theorem 2.3. *Let* $k \in \mathcal{D}_0$, $g \in \mathrm{BUC}(\mathbf{R}^n)$, *and* $H \in C^1(\mathbf{R}^n) \otimes \mathcal{B}(I)$. *Assume that* (K1) *and* (H1) *hold and that* (2.10) *and* (2.11), *with* $m = 1$, *hold for some* $(\{\omega_R\}, \{C_R\}) \in G_2$. *Then there is a unique viscosity solution* $u \in \mathrm{BUC}(R_\infty)$ *of* $(\mathrm{CP})_0$.

The assumptions on k and H in the above theorem are made just to make sure that the function $\bar{A}$ is continuous on $\mathbf{R}^n$.

Theorem 2.4. *Let* $(k, g, H) \in \mathcal{D}$. *Assume that* (K1) *and* (H1) *hold and that* H *satisfies* (2.10) *and* (2.11), *with* $m = 1$, *for some* $(\{\omega_R\}, \{C_R\}) \in G_2$. *Set*

$$\bar{g}(x) = \int_I r(\xi) g(x, \xi) d\xi \quad \text{for } x \in \mathbf{R}^n. \tag{2.16}$$

Let $u^\varepsilon \in \mathcal{U}(R_\infty \times I)$ *be the viscosity solution of* $(\mathrm{CP})_\varepsilon$. *Let* $u \in \mathrm{BUC}(R_\infty)$ *be the viscosity solution of* $(\mathrm{CP})_0$ *with* $\bar{g}$ *in place of* g. *Then, for each* $\delta \in (0, 1)$,

$$\lim_{\varepsilon \searrow 0} \sup\{|u^\varepsilon(x, t, \xi) - u(x, t)| \mid (x, t, \xi) \in \mathbf{R}^n \times [\delta, \delta^{-1}] \times I\} = 0.$$

In addition, if $g(x, \xi)$ *is independent of* ξ, *then for each* $T > 0$

$$\lim_{\varepsilon \searrow 0} \sup\{|u^\varepsilon(x, t, \xi) - u(x, t)| \mid (x, t, \xi) \in \mathbf{R}^n \times [0, T] \times I\} = 0.$$

We remark here that in case when $|I| = 1$ and $k(\xi, \eta) \equiv 1$, we have

$$r(\xi) \equiv 1, \qquad S = I : \{\mathbf{1}\}^{\perp,\infty} \to \{\mathbf{1}\}^{\perp,\infty},$$

and hence

$$\bar{A}(p) = \int_I D_p H(p, \xi) \otimes D_p H(p, \xi) d\xi.$$

In general, it is rather difficult to find an explicit formula for $\bar{A}$.

Another remark we make here concerns the finite system

$$u^i_t(x,t) = \frac{1}{\varepsilon} H_i(Du^i(x,t)) + \frac{1}{\varepsilon^2} \sum_{j=1}^m c_{ij} \left(u^j(x,t) - u^i(x,t) \right)$$
$$\text{for } (x,t) \in Q_T,\ i = 1, ..., m. \tag{2.17}$$

Here $m \in \mathbf{N}$, $H_i \in C(\mathbf{R}^n)$, with $i \in \{1, ..., m\}$, and c_{ij}, with $i, j \in \{1, ..., m\}$, are positive constants. This system can be regarded as a special case of $(\mathrm{E})_\varepsilon$. To see this, we set

$$I = [0, 1),$$

and

$$k(\xi, \eta) = c_{ij} \quad \text{for } (\xi, \eta) \in [(i-1)/m, i/m) \times [(j-1)/m, j/m),\ i, j \in \{1, ..., m\}.$$

For given initial data $g^i \in \mathrm{BUC}(\mathbf{R}^n)$, with $i \in \{1, ..., m\}$, defining the function g on $\mathbf{R}^n \times I$ by

$$g(x, \xi) = g^i(x) \quad \text{for } x \in \mathbf{R}^n,\ \xi \in [(i-1)/m, i/m), \text{ and } i \in \{1, ..., m\}$$

and solving $(\mathrm{E})_\varepsilon$ (see Theorem 2.2), because of the uniqueness of viscosity solutions of $(\mathrm{CP})_\varepsilon$ (thanks to Theorem 2.2), we observe that the viscosity solution $u^\varepsilon(x, t, \xi)$ of $(\mathrm{CP})_\varepsilon$ is piecewise constant as a function of ξ and indeed it is independent of ξ for $\xi \in [(i-1)/m, im)$ and for all $i \in \{1, ..., m\}$. Furthermore, defining functions $u^1, ..., u^m$ by

$$u^i(x, t) = u^\varepsilon(x, t, (i-1)/m) \quad \text{for } (x, t) \in R_T,\ i \in \{1, ..., m\},$$

we observe that $\{u^i\}$ is a viscosity solution of (2.17). Accordingly, Theorem 2.4 above recovers the asymptotic result [3, Theorem 3.2] for systems of first-order PDE although the generality here is slightly different from that in [3].

3. COMPARISON THEOREM

In this section we establish the following theorem.

Theorem 3.1. *Let $T \in (0, \infty)$ and $(k, H) \in \mathcal{D}_0 \times \mathcal{D}_2$. Let $u \in \mathcal{U}^+(Q_T \times I)$ and $v \in \mathcal{U}^-(Q_T \times I)$ be, respectively, a viscosity subsolution and a viscosity supersolution of* (E) *in $Q_T \times I$. Assume that u and $-v$ are bounded above on $Q_T \times I$ and that*

$$\lim_{r \searrow 0} \sup\{u(x, t, \xi) - v(y, s, \xi) \mid (x, t, y, s) \in Q_T^2,\ |x - y| < r,$$
$$t, s \in (0, r),\ \xi \in I\} \le 0. \tag{3.1}$$

Then $u \le v$ *in* $Q_T \times I$.

Proof. By adding v and the function : $(x,t,\xi) \mapsto \varepsilon/(T-t)$, with $\varepsilon > 0$, we may assume that

$$\lim_{t \nearrow T} \sup\{v(x,t,\xi) \mid (x,\xi) \in \mathbf{R}^n \times I\} = \infty \tag{3.2}$$

and that v is a viscosity supersolution of

$$v_t(x,t,\xi) - \mu = H(Dv(x,t,\xi),\xi) + \int_I k(\xi,\eta)[v(x,t,\eta) - v(x,t,\xi)]d\eta \quad \text{for } (x,t,\xi) \in Q_T \times I \tag{3.3}$$

for some constant $\mu > 0$.

We suppose that $\theta_0 := \sup\{u(x,t,\xi) - v(x,t,\xi) \mid (x,t,\xi) \in Q_T \times I\} > 0$ and will show a contradiction.

For $\alpha > 0$ and $\delta > 0$ we consider the function

$$\Phi(x,t,y,s,\xi;\alpha,\delta) = u(x,t,\xi) - v(y,s,\xi) - \alpha|x-y|^2 - \alpha(t-s)^2 - \delta|x|^2$$

on $Q_T^2 \times I$.

For $\alpha > 0$, $\delta > 0$ set

$$\theta(\alpha,\delta) = \sup\{\Phi(x,t,y,s,\xi;\alpha,\delta) \mid (x,t,y,s) \in R_T^2,\ \xi \in I\}.$$

It is easy to see that the limit

$$\theta := \lim_{\alpha \to \infty} \lim_{\delta \searrow 0} \theta(\alpha,\delta)$$

exists and $\theta \ge \theta_0 > 0$.

We note that for $(x,t,y,s,\xi) \in Q_T^2 \times I$, if $\Phi(x,t,y,s,\xi;\alpha,\delta) \ge 0$, then

$$\alpha|x-y|^2 + \alpha(t-s)^2 + \delta|x|^2 \le \sup_{Q_T \times I} u + \sup_{Q_T \times I} (-v), \tag{3.4}$$

Fix any $\beta \in (0, \theta/2)$. By the definition of θ, there is a constant $\alpha_0 > 0$ and a function $\delta_0 : [\alpha_0, \infty) \to (0,1)$ such that for any $\alpha \ge \alpha_0$ and $\delta \in (0, \delta_0(\alpha))$,

$$\theta(\alpha,\delta) > \theta - 2\beta.$$

Fix such an $\alpha \ge \alpha_0$ and a $\delta \in (0, \delta_0(\alpha))$. Then fix a $\xi \in I$ so that

$$\sup\{\Phi(x,t,y,s,\xi;\alpha,\delta) \mid (x,t,y,s) \in Q_T^2\} > \theta(\alpha,\delta) - \beta. \tag{3.5}$$

Noting that $\theta(\alpha, \delta) - \beta > \theta - 2\beta > 0$, using (3.2) and assumption (3.1), and replacing α_0 by a larger number if necessary, we may assume that the function $\Phi(x, t, y, s, \xi; \alpha, \delta)$ of $(x, t, y, s) \in Q_T^2$ attains a maximum at some point $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$.

Observe from (3.4) that $\delta|\hat{x}|^2$ is bounded as $\delta \searrow 0$. Observe as well from (3.5) that

$$u(\hat{x}, \hat{t}, \eta) - v(\hat{y}, \hat{s}, \eta) - \beta \le u(\hat{x}, \hat{t}, \xi) - v(\hat{y}, \hat{s}, \xi) \quad \text{for all } \eta \in I. \tag{3.6}$$

Now, since u is a viscosity subsolution of (E) in $Q_T \times I$ and v is a viscosity supersolution of (3.3), we obtain

$$2\alpha(\hat{t} - \hat{s}) \le H(2\alpha(\hat{x} - \hat{y}) + 2\delta\hat{x}, \xi) + \int_I k(\xi, \eta)(u(\hat{x}, \hat{t}, \eta) - u(\hat{x}, \hat{t}, \xi))d\eta,$$

$$2\alpha(\hat{t} - \hat{s}) - \mu \ge H(2\alpha(\hat{x} - \hat{y}), \xi) + \int_I k(\xi, \eta)(v(\hat{y}, \hat{s}, \eta) - v(\hat{y}, \hat{s}, \xi))d\eta.$$

From the former of these together with (3.6), we get

$$2\alpha(\hat{t} - \hat{s}) \le H(2\alpha(\hat{x} - \hat{y}) + 2\delta\hat{x}, \xi) + \int_I k(\xi, \eta)(v(\hat{y}, \hat{s}, \eta) - v(\hat{y}, \hat{s}, \xi) + \beta)d\eta.$$

Hence we get

$$\mu \le H(2\alpha(\hat{x} - \hat{y}) + 2\delta\hat{x}, \xi) - H(2\alpha(\hat{x} - \hat{y}), \xi) + \beta \int_I k(\xi, \eta)d\eta.$$

In view of (3.4), reselecting δ if necessary, we may assume that

$$H(2\alpha(\hat{x} - \hat{y}) + 2\delta\hat{x}, \xi) - H(2\alpha(\hat{x} - \hat{y}), \xi) \le \beta.$$

Then we obtain

$$\mu \le (1 + \int_I k(\xi, \eta)d\eta)\beta.$$

Since $\beta \in (0, \theta/2)$ is arbitrary, this yields a contradiction, which completes the proof. □

4. EXISTENCE AND STABILITY THEOREMS

In the following discussions it is convenient to introduce the Cauchy problem for the PDE

$$\widehat{\text{(CP)}} \quad \begin{cases} (\widehat{\text{E}}) \quad u_t(x, t, \xi) = H(Du(x, t, \xi), \xi) + \int_I k_1(\xi, \eta)\,(u(x, t, \eta) - u(x, t, \xi))_+\,d\eta \\ \qquad + \int_I k_2(\xi, \eta)\,(u(x, t, \eta) - u(x, t, \xi))_-\,d\eta \quad \text{for } (x, t, \xi) \in Q_\infty \times I, \\ u(x, 0, \xi) = g(x, \xi) \quad \text{for } (x, \xi) \in \mathbf{R}^n \times I, \end{cases}$$

where r_+ and r_- denote the positive and negative parts of $r \in \mathbf{R}$, respectively, $k_1 \in \mathcal{D}_0$, and $k_2 \in -\mathcal{D}_0 \equiv \{-k \mid k \in \mathcal{D}_0\}$. We remark here that PDE (E) is the special case of ($\widehat{\mathrm{E}}$) having $k_1 = k$ and $k_2 = -k$.

Definition 2.1 does not cover equation ($\widehat{\mathrm{E}}$), but viscosity sub-, super-, and solutions of ($\widehat{\mathrm{E}}$) are defined in the same spirit as those of (E).

We use the notation:

$$\widehat{\mathcal{D}}_0 = \mathcal{D}_0 \times (-\mathcal{D}_0), \quad \widehat{\mathcal{D}} = \widehat{\mathcal{D}}_0 \times \mathcal{D}_1 \times \mathcal{D}_2, \quad \widehat{\mathcal{D}}(\gamma) = \widehat{\mathcal{D}}_0 \times \mathcal{D}_1(\gamma_1) \times \mathcal{D}_2(\gamma_2),$$
$$\widehat{\mathcal{D}}_c(\gamma) = \widehat{\mathcal{D}}(\gamma) \cap \left(C(I \times I)^2 \times C(\mathbf{R}^n \times I)^2\right), \quad \widehat{\mathcal{D}}_c = \widehat{\mathcal{D}} \cap \left(C(I \times I)^2 \times C(\mathbf{R}^n \times I)^2\right),$$

where $\gamma \equiv (\gamma_1, \gamma_2) \in G$.

Before going into the discussions about stability of solutions of ($\widehat{\mathrm{E}}$), we state the following theorem. We leave it to the reader to check that a straightforward adaptation of the proof of Theorem 3.1 works in the present case.

Theorem 4.1. *Let $T \in (0, \infty)$ and $(k_1, k_2, H) \in \widehat{\mathcal{D}}_0 \times \mathcal{D}_2$. Let $u \in \mathcal{U}^+(Q_T \times I)$ and $v \in \mathcal{U}^-(Q_T \times I)$ be, respectively, a viscosity subsolution and a viscosity supersolution of* ($\widehat{\mathrm{E}}$) *in $Q_T \times I$. Assume that u and $-v$ are bounded above on $Q_T \times I$ and that*

$$\lim_{r \searrow 0} \sup\{u(x,t,\xi) - v(y,s,\xi) \mid (x,t,y,s) \in Q_T^2,\ |x-y| < r,\ t,s \in (0,r),\ \xi \in I\} \leq 0.$$

Then $u \leq v$ in $Q_T \times I$.

Now, we are concerned with stability assertions for solutions of ($\widehat{\mathrm{E}}$). We remark that if u is a viscosity subsolution of ($\widehat{E}$), then $v := -u$ is a viscosity supersolution of ($\widehat{\mathrm{E}}$), with $k_1(\xi)$, $k_2(\xi)$, and $H(p,\xi)$ replaced by $-k_2(\xi)$, $-k_1(\xi)$, and $-H(-p,\xi)$, respectively. Any assertions for viscosity subsolutions can be rephrased as assertions for viscosity supersolutions. In this view point, we state assertions only for subsolutions.

Theorem 4.2. *Let $(k_1, k_2, H) \in \widehat{\mathcal{D}}_0 \times \mathcal{D}_2$ and let Ω be an open subset of Q_∞. Let $\mathcal{S}$ be a collection of viscosity subsolutions of* ($\widehat{\mathrm{E}}$) *in $\Omega \times I$. Set*

$$u(x,t,\xi) = \sup\{v(x,t,\xi) \mid v \in \mathcal{S}\} \quad \text{for } (x,t,\xi) \in \Omega \times I.$$

Assume that for each compact subset V of Ω the function u is bounded above on $V \times I$ and that the upper semicontinuous envelope $u^\natural$ with respect to (x,t), i.e.,

$$u^\natural(x,t,\xi) = \lim_{r \searrow 0} \sup\{u(y,s,\xi) \mid (y,s) \in \Omega,\ |y-x| + |s-t| < r\} \quad \text{for } (x,t,\xi) \in \Omega \times I,$$

belongs to $\mathcal{U}^+(\Omega \times I)$. Then $u^\natural$ is a viscosity subsolution of (E) *in $\Omega \times I$.*

Proof. Let $(x,t,\xi) \in \Omega \times I$ and $\varphi \in C^1(\Omega)$. Assume that $u^\natural(\cdot,\xi) - \varphi$ attains a strict maximum at (x,t).

As in the proof of standard stability properties for viscosity solutions, we find sequences $\{v_j\}_{j\in\mathbf{N}} \subset \mathcal{S}$ and $\{(x_j,t_j)\}_{j\in\mathbf{N}} \subset \Omega$ such that

$$(x_j,t_j) \to (x,t) \quad \text{and} \quad v_j(x_j,t_j,\xi) \to u^\natural(x,t,\xi) \quad \text{as } j \to \infty,$$

and for each $j \in \mathbf{N}$, $v_j(\cdot,\xi) - \varphi$ attains a local maximum at (x_j,t_j).

Then we have

$$\begin{aligned}
\varphi_t(x_j,t_j) \le & H(D\varphi(x_j,t_j),\xi) + \int_I k_1(\xi,\eta)\,(v_j(x_j,t_j,\eta) - v_j(x_j,t_j,\xi))_+\,d\eta \\
& + \int_I k_2(\xi,\eta)\,(v_j(x_j,t_j,\eta) - v_j(x_j,t_j,\xi))_-\,d\eta \\
\le & H(D\varphi(x_j,t_j),\xi) + \int_I k_1(\xi,\eta)\,\big(u^\natural(x_j,t_j,\eta) - v_j(x_j,t_j,\xi)\big)_+\,d\eta \\
& + \int_I k_2(\xi,\eta)\,\big(u^\natural(x_j,t_j,\eta) - v_j(x_j,t_j,\xi)\big)_-\,d\eta
\end{aligned}$$

for all $j \in \mathbf{N}$. Sending $j \to \infty$ and using Fatou's lemma, we get

$$\begin{aligned}
\varphi_t(x,t) \le & H(D\varphi(x,t),\xi) + \int_I k_1(\xi,\eta)\,\big(u^\natural(x,t,\eta) - u^\natural(x,t,\xi)\big)_+\,d\eta \\
& + \int_I k_2(\xi,\eta)\,\big(u^\natural(x,t,\eta) - u^\natural(x,t,\xi)\big)_-\,d\eta,
\end{aligned}$$

completing the proof. □

The following theorem is a stability assertion similar to the above theorem formulated in a flavor of standard viscosity solutions theory, which can be proved in a way parallel to the above theorem with minor modifications.

Theorem 4.3. *Let $(k_1,k_2,H) \in \widehat{\mathcal{D}}_0 \times \mathcal{D}_2$ and let Ω be an open subset of Q_∞. Let $\mathcal{S}$ be a collection of viscosity subsolutions of* $(\widehat{\mathrm{E}})$ *in $\Omega \times I$. Assume that $k_1, k_2 \in C(I)$ and $H \in C(\mathbf{R}^n \times I)$ and that $\mathcal{S} \subset \mathrm{USC}(\mathbf{R}^n \times I)$, i.e., every $v \in \mathcal{S}$ is upper semicontinuous on $\mathbf{R}^n \times I$. Set*

$$u(x,t,\xi) = \sup\{v(x,t,\xi) \mid v \in \mathcal{S}\} \quad \text{for } (x,t,\xi) \in \Omega \times I.$$

Assume that for each compact subset V of Ω the function u is bounded above on $V \times I$, and let u^* be the upper semicontinuous envelope of u with respect to (x,t,ξ), i.e., for $(x,t,\xi) \in \Omega \times I$,

$$u^*(x,t,\xi) = \lim_{r \searrow 0} \sup\{u(y,s,\eta) \mid (y,s,\eta) \in \Omega \times I,\ |y-x| + |s-t| + |\eta - \xi| < r\}.$$

Then u^* is a viscosity subsolution of (E) *in* $\Omega \times I$.

Theorem 4.4. *Let $(k_1, k_2, H) \in \widehat{\mathcal{D}}_0 \times \mathcal{D}_2$ and let Ω be an open subset of Q_∞. Let $\{u_j\}_{j\in\mathbf{N}}$ be a sequence of viscosity subsolutions of* $(\widehat{\mathrm{E}})$ *in $\Omega \times I$. Assume that the sequence $\{u_j\}$ is non-increasing and convergent pointwise, i.e,*

$$u_j(x,t,\xi) \geq u_{j+1}(x,t,\xi) \quad \textit{for all } (x,t,\xi) \in \Omega \times I,\ j \in \mathbf{N},$$

and

$$u(x,t,\xi) = \lim_{j\to\infty} u_j(x,t,\xi) \quad \textit{for } (x,t,\xi) \in \Omega \times I$$

for some function $u : \Omega \to \mathbf{R}$. Assume that for each compact subset V of Ω the function u_1 is bounded above on $V \times I$ and that for each $(x,t) \in \Omega$ the function $u(x,t,\cdot)$ is integrable in I. Then u is a viscosity subsolution of $(\widehat{\mathrm{E}})$ *in $\Omega \times I$.*

Proof. First of all, we remark that $u \in \mathcal{U}^+(\Omega \times I)$.

Let $(x,t,\xi) \in \Omega \times I$ and $\varphi \in C^1(\Omega)$. Assume that $u(\cdot,\xi) - \varphi$ attains a strict maximum at (x,t).

Fix a compact neighborhood $V \subset \Omega$ of (x,t), and for each $j \in \mathbf{N}$ let $(x_j,t_j) \in V$ be a maximum point of $u_j(\cdot,\xi) - \varphi$ over V. By taking a subsequence if necessary, we may assume that $(x_j,t_j) \to (y,s)$ for some $(y,s) \in V$ as $j \to \infty$.

Since

$$\max_V (u_j(\cdot,\xi) - \varphi) \geq \max_V (u(\cdot,\xi) - \varphi) \quad \text{for all } j \in \mathbf{N},$$

we have

$$\max_V (u(\cdot,\xi) - \varphi) \leq u_j(x_j,t_j,\xi) - \varphi(x_j,t_j) \leq u_m(x_j,t_j,\xi) - \varphi(x_j,t_j) \quad \text{if } m \leq j. \tag{4.1}$$

Sending $j \to \infty$, thanks to the upper semicontinuity of $u_m(\cdot,\xi)$, we get

$$\max_V (u(\cdot,\xi) - \varphi) \leq u_m(y,s,\xi) - \varphi(y,s) \quad \text{for all } m \in \mathbf{N},$$

and furthermore,

$$\max_V (u(\cdot,\xi) - \varphi) \leq u(y,s,\xi) - \varphi(y,s).$$

From this we see that (y,s) is a maximum point of $u(\cdot,\xi) - \varphi$ and hence that $(y,s) = (x,t)$.

The monotonicity of $\{u_j\}$ and the semicontinuity of $u_m(\cdot,\xi)$ ensure that

$$\begin{aligned}\limsup_{j\to\infty}(u_j(x_j,t_j,\xi) - \varphi(x_j,t_j)) &\leq \limsup_{j\to\infty}(u_m(x_j,t_j,\xi) - \varphi(x_j,t_j)) \\ &\leq u_m(x,t,\xi) - \varphi(x,t) \quad \text{for } m \in \mathbf{N}.\end{aligned}$$

This yields immediately that

$$\limsup_{j\to\infty}(u_j(x_j,t_j,\xi)-\varphi(x_j,t_j))\le u(x,t,\xi)-\varphi(x,t),$$

and therefore, we get

$$\limsup_{j\to\infty}(u_j(x_j,t_j,\xi)-\varphi(x_j,t_j))\le \max_V(u(\cdot,\xi)-\varphi)\le \liminf_{j\to\infty}(u_j(x_j,t_j,\xi)-\varphi(x_j,t_j)).$$

The latter inequality above is a direct consequence of (4.1). Thus we see that

$$\lim_{j\to\infty} u_j(x_j,t_j,\xi)=u(x,t,\xi).$$

We may assume that every (x_j,t_j) are in the interior of V. Since u_j is a viscosity subsolution of $(\widehat{\mathrm{E}})$, we have

$$\begin{aligned}\varphi_t(x_j,t_j)\le H(D\varphi(x_j,t_j),\xi)&+\int_I k_1(\xi,\eta)\,(u_j(x_j,t_j,\eta)-u_j(x_j,t_j,\xi))_+\,d\eta\\ &+\int_I k_2(\xi,\eta)\,(u_j(x_j,t_j,\eta)-u_j(x_j,t_j,\xi))_-\,d\eta.\end{aligned}$$

Hence, for $j\ge m$, we have

$$\begin{aligned}\varphi_t(x_j,t_j)\le H(D\varphi(x_j,t_j),\xi)&+\int_I k_1(\xi,\eta)\,(u_m(x_j,t_j,\eta)-u_j(x_j,t_j,\xi))_+\,d\eta\\ &+\int_I k_2(\xi,\eta)\,(u_m(x_j,t_j,\eta)-u_j(x_j,t_j,\xi))_-\,d\eta.\end{aligned}$$

Sending $j\to\infty$, in view of Fatou's lemma, we get

$$\begin{aligned}\varphi_t(x,t)\le H(D\varphi(x,t),\xi)&+\int_I k_1(\xi,\eta)\,(u_m(x,t,\eta)-u(x,t,\xi))_+\,d\eta\\ &+\int_I k_2(\xi,\eta)\,(u_m(x,t,\eta)-u(x,t,\xi))_-\,d\eta.\end{aligned}$$

Since $u(x,t,\cdot)$ is integrable in I, by the monotone convergence theorem, we obtain

$$\begin{aligned}\varphi_t(x,t)\le H(D\varphi(x,t),\xi)&+\int_I k_1(\xi,\eta)\,(u(x,t,\eta)-u(x,t,\xi))_+\,d\eta\\ &+\int_I k_2(\xi,\eta)\,(u(x,t,\eta)-u(x,t,\xi))_-\,d\eta.\end{aligned}$$

This completes the proof. □

Theorem 4.5. *Let $\gamma \in G$ and $(k_1, k_2, g, H) \in \widehat{\mathcal{D}}_c(\gamma)$. Then there is a unique viscosity solution $u \in \mathcal{U}_c(R_\infty)$ of $(\widehat{\mathrm{CP}})$ satisfying the initial condition, $u(x,0,\xi) = g(x,\xi)$ for $(x,\xi) \in \mathbf{R}^n \times I$. Moreover there is a $\lambda \in G_1$ depending only on γ for which $u \in \mathcal{U}_c(\lambda)$.*

The uniqueness and existence assertion above seems to be somehow a standard observation. See, for instance, [5, 1, 2] for related topics.

Proof. The uniqueness assertion follows from Theorem 4.1.

We may utilize the standard Perron method in viscosity solutions theory, in order to show the existence of a viscosity solution of (CP).

To do this, first of all fix $\gamma \equiv (\omega, L, \{\omega_R\}, \{L_R\}) \in G$ so that $(g, H) \in \mathcal{D}(\gamma)$.

Fix $\varepsilon \in (0,1)$, and choose a function $\psi_\varepsilon \in C^1([0,\infty))$ so that

$$\omega(r) \wedge (2L) \le \psi_\varepsilon(r) \le 2L \quad \text{for } r \ge 0, \quad 0 \le \psi_\varepsilon'(r) \le B_\varepsilon \quad \text{for } r \ge 0,$$
$$\psi_\varepsilon'(0) = 0, \quad \text{and} \quad \psi_\varepsilon(0) < \varepsilon,$$

where B_ε is a constant depending on ε. Then we have

$$g(x,\xi) \le g(y,\xi) + \psi_\varepsilon(|x-y|) \quad \text{for } x, y \in \mathbf{R}^n,\ \xi \in I.$$

Let $y \in \mathbf{R}^n$ and set

$$h_\varepsilon^+(x;y) = \psi_\varepsilon(|x-y|) \quad \text{for } x \in \mathbf{R}^n,$$

and note that

$$Dh_\varepsilon^+(x;y) \in B(0, B_\varepsilon) \quad \text{for all } x \in \mathbf{R}^n.$$

We set

$$f^+(x,t,\xi;\varepsilon,y) = g(y,\xi) + h_\varepsilon^+(x;y) + M_\varepsilon t \quad \text{for } (x,t,\xi) \in R_\infty \times I,$$

where $M_\varepsilon = L_{B_\varepsilon} + 2L\kappa_1|I|$, and observe that $f^+(\cdot\,;\varepsilon,y)$ is a viscosity supersolution of $(\widehat{\mathrm{E}})$ and that $g(x,\xi) \le f^+(x,0,\xi;\varepsilon,y)$ and $f^+(x,0,\xi;\varepsilon,x) \le g(x,\xi) + \varepsilon$ for all $(x,\xi) \in \mathbf{R}^n \times I$.

Define $w^+ : R_\infty \times I \to \mathbf{R}$ by

$$w^+(x,t,\xi) = \inf\{f^+(x,t,\xi;\varepsilon,y) \mid \varepsilon \in (0,1),\ y \in \mathbf{R}^n\}.$$

It is easy to see that w^+ is upper semicontinuous in $R_\infty \times I$, that $w^+(x,0,\xi) = g(x,\xi) \le w^+(x,t,\xi)$ for $(x,t,\xi) \in R_\infty \times I$, and, by Theorem 4.3, that the lower semicontinuous envelope $(w^+)_* := -(-w^+)^*$ is a viscosity supersolution of $(\widehat{\mathrm{E}})$.

Similarly, we define $w^- : R_\infty \times I \to \mathbf{R}$ by

$$w^-(x,t,\xi) = \sup\{f^-(x,t,\xi;\varepsilon,y) \mid \varepsilon \in (0,1),\ y \in \mathbf{R}^n\},$$

where

$$f^-(x,t,\xi;\varepsilon,y) = g(y,\xi) - h_\varepsilon^+(x;y) - M_\varepsilon t,$$

and observe that w^- is lower semicontinuous in $R_\infty \times I$, that $w^-(x,0,\xi) = g(x,\xi) \geq w^-(x,t,\xi)$ for all $(x,t,\xi) \in R_\infty \times I$, and that $(w^-)^*$ is a viscosity subsolution of $(\widehat{\mathrm{E}})$.

Now, set

$$u(x,t,\xi) = \sup\{v(x,t,\xi) \mid v \text{ a viscosity subsolution of } (\widehat{\mathrm{E}}), \\ (w^-)^* \leq v \leq (w^+)_* \text{ in } R_\infty \times I\}$$

for $(x,t,\xi) \in R_\infty \times I$.

Note that the map $F : \mathcal{B}^\infty(I) \to \mathcal{B}^\infty(I)$ defined by

$$F(\psi)(\xi) = \int_I k_1(\xi,\eta)(\psi(\eta) - \psi(\xi))_+ d\eta + \int_I k_2(\xi,\eta)(\psi(\eta) - \psi(\xi))_- d\eta$$

is quasi-monotone in the sense that for any $\psi_1, \psi_2 \in \mathcal{B}^\infty(I)$ and $\xi \in I$, if $\psi_1 \leq \psi_2$ in I and $\psi_1(\xi) = \psi_2(\xi)$, then $F(\psi_1)(\xi) \leq F(\psi_2)(\xi)$. Using this property, as in the standard proof of the Perron method (see [1, 5]), we infer that u_* is a viscosity supersolution of $(\widehat{\mathrm{E}})$. We see as well by Theorem 4.3 that u^* is a viscosity subsolution of $(\widehat{\mathrm{E}})$. By Theorem 4.1, we see that $u^* \leq u_*$ in $R_\infty \times I$, which shows that $u \in C(R_\infty \times I)$ and it is a viscosity solution of $(\widehat{\mathrm{E}})$.

Next, we want to show that $u \in \mathcal{U}(\nu, M)$ for some modulus ν and a constant $M > 0$. It is clear that

$$|u(x,0,\xi)| = |g(x,\xi)| \leq L \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I.$$

Noting that for each $y \in \mathbf{R}^n$, the function $w(x,t,\xi) = u(x+y,t,\xi) + \omega(|y|)$ of (x,t,ξ) is a viscosity solution of $(\widehat{\mathrm{E}})$ and that

$$u(x,0,\xi) = g(x,\xi) \leq g(x+y,\xi) + \omega(|y|) \leq w(x,0,\xi) \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I,$$

we apply Theorem 4.1, to conclude that

$$u(x,t,\xi) - u(x+y,t,\xi) \leq \omega(|y|) \quad \text{for } (x,t,\xi) \in R_\infty \times I.$$

That is, we have

$$|u(x,t,\xi) - u(y,t,\xi)| \leq \omega(|x-y|) \quad \text{for } x,y \in \mathbf{R}^n,\ (t,\xi) \in [0,\infty) \times I.$$

In view of this uniform continuity of u, by using the same construction as w^+ and w^-, we deduce that for each $\varepsilon \in (0,1)$,

$$|u(x,t+s,\xi) - u(x,t,\xi)| \leq \varepsilon + M_\varepsilon s \quad \text{for } (x,t,\xi) \in R_\infty \times I,\ s \geq 0,$$

where the choice of M_ε is same as before.

Setting

$$\mu(r) = \inf\{\varepsilon + M_\varepsilon r \mid \varepsilon \in (0,1)\} \quad \text{for } r \geq 0,$$

we see that μ is a modulus and

$$|u(x,t,\xi) - u(y,s,\xi)| \leq \omega(|x-y|) + \mu(|t-s|) \quad \text{for } (x,y) \in \mathbf{R}^n,\ t,s \in [0,\infty),\ \xi \in I.$$

Thus we conclude that $u \in \mathcal{U}(\omega + \mu, L)$. □

The arguments in the last half of the above proof applied to viscosity solutions of ($\widehat{\mathrm{E}}$), yield the following theorem. We leave the details of its proof to the reader.

Theorem 4.6. *For each $\gamma \in G$ there is a $\lambda \in G_1$ such that if $(k_1, k_2, g, H) \in \widehat{\mathcal{D}}(\gamma)$ and $u \in \mathcal{U}$ is a viscosity solution of ($\widehat{\mathrm{E}}$) which is bounded on $R_T \times I$ for each $T > 0$, then $u \in \mathcal{U}(\lambda)$.*

In order to prove Theorem 2.2, we utilize an argument concerning monotone classes of functions. Let Ω be a set and $\mathcal{F}$ be a collection of functions $f : \Omega \to \mathbf{R}^q$. We call $\mathcal{F}$ a monotone class of functions if whenever $\{f_j\}_{j\in\mathbf{N}} \subset \mathcal{F}$ satisfying either $f_j(x) \leq f_{j+1}(x)$ for all $(x,j) \in \Omega \times \mathbf{N}$ or $f_j(x) \geq f_{j+1}(x)$ for all $(x,j) \in \Omega \times \mathbf{N}$, and

$$\lim f_j(x) = f(x) \quad \text{for all } x \in \Omega$$

for some $f : \Omega \to \mathbf{R}^q$, then $f \in \mathcal{F}$. Here the inequalities $f_j(x) \leq f_{j+1}(x)$ and $f_j(x) \geq f_{j+1}(x)$ should be understood in the component-wise sense. It is clear that if Λ is a nonempty set and for each $\lambda \in \Lambda$, $\mathcal{F}_\lambda$ is a monotone class of functions on Ω with values in $\mathbf{R}^q$, then so is $\cap_{\lambda\in\Lambda}\mathcal{F}_\lambda$. This observation allows us to define, for any nonempty collection $\mathcal{G}$ of functions on Ω with values in $\mathbf{R}^q$, the smallest monotone class of functions containing $\mathcal{G}$. We denote by $m(\mathcal{G})$ the smallest monotone class of functions containing $\mathcal{G}$.

Lemma 4.7. *For any $\gamma \in G$, $m(\widehat{\mathcal{D}}_c(\gamma)) = \widehat{\mathcal{D}}(\gamma)$.*

Assuming the validity of this lemma, whose proof will be given in section 6, we now proceed to state and prove the following theorem, from which Theorem 2.2 follows immediately.

Theorem 4.8. *Let $(k_1, k_2, g, H) \in \widehat{\mathcal{D}}$. Then there is a unique viscosity solution $u \in \mathcal{U}(R_\infty \times I)$ of ($\widehat{\mathrm{CP}}$).*

Proof. Fix $\gamma \equiv (\omega, L, \{\omega_R\}, \{L_R\}) \in G$. Define $\mathcal{M}$ as the subset of $\widehat{\mathcal{D}}(\gamma)$ consisting of those (k, g, H), where $k = (k_1, k_2)$, for which ($\widehat{\mathrm{CP}}$) has a viscosity solution $u \in \mathcal{U}$.

By Theorem 4.5, we see that

$$\widehat{\mathcal{D}}_c(\gamma) \subset \mathcal{M}. \tag{4.2}$$

We intend to show that $\mathcal{M}$ is a monotone class of functions. To this end, let $\{(k^j, g^j, H^j)\}_{j\in\mathbf{N}} \subset \mathcal{M}$ satisfy either

$$k^j(\xi,\eta) \le k^{j+1}(\xi,\eta), \quad g^j(x,\xi) \le g^{j+1}(x,\xi) \quad \text{and} \quad H^j(p,\xi) \le H^{j+1}(p,\xi) \tag{4.3}$$

for all $x, p \in \mathbf{R}^n$, $\xi, \eta \in I$, and $j \in \mathbf{N}$, or

$$k^j(\xi,\eta) \le k^{j+1}(\xi,\eta), \quad g^j(x,\xi) \ge g^{j+1}(x,\xi) \quad \text{and} \quad H^j(p,\xi) \ge H^{j+1}(p,\xi) \tag{4.4}$$

for all $x, p \in \mathbf{R}^n$, $\xi, \eta \in I$, and $j \in \mathbf{N}$, and

$$k(\xi,\eta) = \lim_{j\to\infty} k^j(\xi,\eta), \quad g(x,\xi) = \lim_{j\to\infty} g^j(x,\xi), \quad \text{and} \quad H(p,\xi) = \lim_{j\to\infty} H^j(p,\xi) \tag{4.5}$$

for all $x, p \in \mathbf{R}^n$ and $\xi, \eta \in I$ and for some functions $k : I^2 \to \mathbf{R}^2$ and $g,\ H : \mathbf{R}^n \times I \to \mathbf{R}$.

Noting that the pointwise limit of a sequence of Borel functions is also a Borel function, we see easily that $\widehat{\mathcal{D}}(\gamma)$ is a monotone class of functions. In particular, the triple (k, g, H), where k, g, and H are defined by (4.5), belongs to $\widehat{\mathcal{D}}(\gamma)$.

Observe as well that for any $\lambda \in G_1$, the set $\mathcal{U}(\lambda)$ is a monotone class of functions.

To show that $\mathcal{M}$ is a monotone class of functions, we fix a sequence $\{(k^j, g^j, H^j)\}_{j\in\mathbf{N}} \subset \mathcal{M}$ satisfying either (4.3) or (4.4), and for each $j \in \mathbf{N}$ let $u^j \in \mathcal{U}$ be the unique viscosity solution of $(\widehat{\mathrm{CP}})$, with k^j, g^j, and H^j in place of (k_1, k_2), g, and H, respectively.

We first consider the case when (4.3) is satisfied. Since u^j is a viscosity subsolution of $(\widehat{\mathrm{E}})$ with (k_1, k_2) and H replaced by k^{j+1} and H^{j+1}, respectively, by Theorem 4.1 we have

$$u^j(x,t,\xi) \le u^{j+1}(x,t,\xi) \quad \text{for all } (x,t,\xi) \in R_\infty \times I,\ j \in \mathbf{N}.$$

Theorem 4.6 guarantees that there is a $\lambda \in G_1$ such that $u^j \in \mathcal{U}(\lambda)$ for all $j \in \mathbf{N}$. Since $\mathcal{U}(\lambda)$ is a monotone class of functions, we see that the pointwise limit

$$u(x,t,\xi) := \lim_{j\to\infty} u^j(x,t,\xi) \quad \text{for } (x,t,\xi) \in R_\infty \times I$$

exists and this function u belongs to $\mathcal{U}(\lambda)$.

By the stability results, Theorems 4.2 and 4.4, of viscosity solutions of $(\widehat{\mathrm{E}})$, we see that $u \in \mathcal{U}$ is the unique viscosity solution of $(\widehat{\mathrm{CP}})$, with $(k_1, k_2) = k$, and conclude that $(k, g, H) \in \mathcal{M}$. (We remark here that for each $j \in \mathbf{N}$ u^j is a viscosity subsolution of $(\widehat{\mathrm{E}})$, with k^j and H^j in place of (k_1, k_2) and H, respectively, and that for each $i, j \in \mathbf{N}$, if $i \le j$, then u^j is a viscosity supersolution of $(\widehat{\mathrm{E}})$ with (k_1, k_2, H) replaced by (k^i, H^i).)

Arguments parallel to the above guarantee that $(k, g, H) \in \mathcal{M}$, where k, g, and H are defined by (4.5), also in case when $\{(k^j, g^j, H^j)\}_{j\in\mathbf{N}}$ satisfies (4.4). Thus we see that $\mathcal{M}$ is a monotone class of functions.

By Lemma 4.7, we know that $m(\widehat{\mathcal{D}}_c(\gamma)) = \widehat{\mathcal{D}}(\gamma)$. Furthermore, since $\mathcal{M}$ is a monotone class of functions, in view of (4.2), we see that $m(\widehat{\mathcal{D}}_c(\gamma)) \subset \mathcal{M}$. Hence, we have

$$\widehat{\mathcal{D}}(\gamma) \subset \mathcal{M} \subset \widehat{\mathcal{D}}(\gamma).$$

The last inclusion follows from the definition of $\mathcal{M}$. Thus we conclude that $\mathcal{M} = \widehat{\mathcal{D}}(\gamma)$ and by the definition of $\mathcal{M}$ that for any $(k, g, H) \in \widehat{\mathcal{D}}(\gamma)$ there is a viscosity solution $u \in \mathcal{U}$ of $(\widehat{\mathrm{CP}})$, with $(k_1, k_2) = k$, which was to be proven. □

5. PROOF OF THEOREMS 2.3 AND 2.4

Outline of proof of Theorem 2.3. Since the function $\bar{A}$ is continuous and for each $p \in \mathbf{R}^n$ $\bar{A}(p)$ is non-negative definite, using the standard techniques to deal with second order parabolic equations (see, for instance, [1]) and the arguments for proving Theorem 3.1, we see that there is at most one viscosity solution of $(\mathrm{CP})_0$. Indeed, we get a comparison theorem similar to Theorem 3.1. By applying the standard Perron procedure, we see that there exists a viscosity solution $u \in C(R_\infty)$ of $(\mathrm{CP})_\infty$ which is bounded on R_∞. The comparison principle for $(\mathrm{CP})_0$ together with translation invariance in the space variable x yields the uniform continuity of $u(x,t)$ in x. More precisely, we find a modulus ω_1 such that

$$|u(x,t) - u(y,t)| \le \omega_1(|x-y|) \quad \text{for all } x, y \in \mathbf{R}^n,\ t \ge 0.$$

Then, by constructing barrier functions and using the comparison principle, we see that for some modulus ω_2,

$$|u(x,t) - u(x,s)| \le \omega_2(|t-s|) \quad \text{for all } x \in \mathbf{R}^n,\ t, s \in [0, \infty).$$

Thus we see that $u \in \mathrm{BUC}(R_\infty)$, completing the proof. □

Now we intend to prove Theorem 2.4.

Let $(k, g, H) \in \mathcal{D}$, $\bar{g}$, $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$, and u be as in Theorem 2.4.

Note that, by (K1), $(\mathrm{E})_\varepsilon$ reads

$$u_t^\varepsilon(x,t,\xi) = \frac{1}{\varepsilon} H(Du^\varepsilon(x,t,\xi), \xi) + \frac{1}{\varepsilon^2}\Big(\int_I k(\xi,\eta) u^\varepsilon(x,t,\eta)d\eta - u^\varepsilon(x,t,\xi)\Big)$$
$$\text{for } (x,t,\xi) \in \mathbf{R}^n \times (0,\infty) \times I.$$

We set $h(x,\xi) = g(x,\xi) - \bar{g}(x)$ for $(x,\xi) \in \mathbf{R}^n \times I$, and note that

$$\int_I r(\xi) h(x,\xi) d\xi = 0 \quad \text{for all } x \in \mathbf{R}^n.$$

To prove Theorem 2.4, we use the so-called relaxed limits. We define

$$u^+(x,t) = \lim_{r\searrow 0} \sup\{u^\varepsilon(y,s,\eta) \mid (y,s,\eta) \in R_\infty \times I, |y-x| + |s-t| < r\}$$
$$u^-(x,t) = \lim_{r\searrow 0} \inf\{u^\varepsilon(y,s,\eta) \mid (y,s,\eta) \in R_\infty \times I, |y-x| + |s-t| < r\}$$

for $(x,t) \in R_\infty \times I$.

Lemma 5.1. *There is a modulus* μ *such that*

$$\bar g(x) - \mu(t) \le u^-(x,t) \le u^+(x,t) \le \bar g(x) + \mu(t) \quad \text{for } (x,t) \in Q_\infty.$$

In addition, if $h = 0$*, then the above inequalities hold for all* $(x,t) \in R_\infty$*.*

Lemma 5.2. *There are constants* $\delta > 0$ *and* $C_0 > 0$ *such that for any* $h \in \{r\}^{\perp,\infty}$*,*

$$\|e^{t(K-I)}h\|_\infty \le C_0 e^{-\delta t}\|h\|_\infty \quad \text{for all } t \ge 0.$$

We shall give a proof of this lemma in the next section and, assuming the validity of Lemma 5.2 in this section, we continue the proof of Theorem 2.4.

Proof of Lemma 5.1. Using the standard mollification, for each $\gamma \in (0,1)$ we may choose functions $\bar g_\gamma \in C^2(\mathbf{R}^n)$ and $h_\gamma, H_\gamma \in C^1(\mathbf{R}^n) \otimes \mathcal{B}(I)$ such that

$$|\bar g_\gamma(x)| \vee |h_\gamma(x,\xi)| \le C, \qquad |D\bar g_\gamma(x)| \vee \|D^2\bar g_\gamma(x)\| \vee |Dh_\gamma(x,\xi)| \le C_\gamma,$$
$$|H_\gamma(p,\xi)| \vee |DH_\gamma(p,\xi)| \le L_R,$$

for all $(x,p,\xi) \in \mathbf{R}^n \times B(0,R) \times I$ and $R > 0$ and for some constants $C > 0$, $C_\gamma > 0$,and $L_R > 0$. Here C does not depend on either γ or R, C_γ does not depend on R, but may depend on γ, etc. We may assume further that

$$\int_I r(\xi)h_\gamma(x,\xi)d\xi = 0 \quad \text{for } x \in \mathbf{R}^n, \tag{5.1}$$
$$\int_I r(\xi)H_\gamma(p,\xi)d\xi = 0 \quad \text{for } p \in \mathbf{R}^n, \tag{5.2}$$

and

$$g(x,\xi) \le \bar g_\gamma(x) + h_\gamma(x,\xi) \quad \text{and} \quad \bar g_\gamma(x) \le \bar g(x) + \sigma(\gamma) \quad \text{for all } (x,\xi) \in \mathbf{R}^n \times I,$$

where $\sigma(\gamma) \to 0$ as $\gamma \searrow 0$,

Fix $\gamma \in (0,1)$. In what follows we write $\bar g$ and h for $\bar g_\gamma$ and h_γ, respectively. This abuse of notation hopefully does not cause any confusion.

Fix $\varepsilon \in (0,1)$, and we define $f_\varepsilon \in C^1(\mathbf{R}^{n+1}) \otimes \mathcal{B}(I)$ by

$$f_\varepsilon(x,t,\cdot) = e^{\frac{t}{\varepsilon^2}(K-I)} h(x,\cdot) \quad \text{for } (x,t) \in \mathbf{R}^n \times \mathbf{R}.$$

Of course, we have

$$\begin{cases} \dfrac{\partial}{\partial t} f_\varepsilon(x,t,\xi) = \dfrac{1}{\varepsilon^2}(K-I) f_\varepsilon(x,t,\cdot)(\xi) \\ f_\varepsilon(x,0,\xi) = h(x,\xi) \end{cases}$$

for all $(x,t,\xi) \in \mathbf{R}^n \times \mathbf{R} \times I$. By Lemma 5.2, since (5.1) holds, we have

$$\|f_\varepsilon(x,t,\cdot)\|_\infty \le C_0 e^{-\frac{\delta t}{\varepsilon^2}} \|h(x,\cdot)\|_\infty \le C C_0 e^{-\frac{\delta t}{\varepsilon^2}},$$
$$\|D f_\varepsilon(x,t,\cdot)\|_\infty \le \sqrt{n} C_\gamma C_0 e^{-\frac{\delta t}{\varepsilon^2}}$$

for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$, where δ and C_0 are positive constants from Lemma 5.2.

We set

$$\varphi_\varepsilon(x,\cdot) = S H_\varepsilon(D\bar{g}(x),\cdot)$$

in view of (5.2), and

$$w(x,t,\xi) = \bar{g}(x) + f_\varepsilon(x,t,\xi) + B_1 t + \varepsilon(\varphi_\varepsilon(x,\xi) + B_2) + \varepsilon B_3 (1 - e^{-\frac{\delta t}{\varepsilon^2}}) \tag{5.3}$$

for $(x,t,\xi) \in R_\infty \times I$, where B_1, B_2, and B_3 are positive constants to be fixed later. Recall that

$$(I-K)\varphi_\varepsilon(x,\cdot) = H_\varepsilon(D\bar{g}(x),\cdot) \quad \text{for } x \in \mathbf{R}^n,$$

and

$$D\varphi_\varepsilon(x,\cdot) = S\left(D^2\bar{g}(x) D_p H_\varepsilon(D\bar{g}(x),\cdot)\right) \quad \text{for } x \in \mathbf{R}^n.$$

The last identity guarantees that

$$|D\varphi_\varepsilon(x,\xi)| \le C_1 \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I$$

for some constant $C_1 > 0$ independent of ε. We may assume as well that

$$|\varphi_\varepsilon(x,\xi)| \le C_1 \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I.$$

We calculate that

$$\begin{aligned} J :=& w_t(x,t,\xi) - \frac{1}{\varepsilon} H(Dw(x,t,\xi),\xi) - \frac{\delta}{\varepsilon^2}\Big(\int_I k(\xi,\eta) w(x,t,\eta)\,d\eta - w(x,t,\xi)\Big) \\ =& \frac{1}{\varepsilon^2}(K-I) f_\varepsilon(x,t,\cdot)(\xi) + B_1 + \frac{\delta}{\varepsilon} B_3 e^{-\frac{\delta t}{\varepsilon^2}} \\ &- \frac{1}{\varepsilon} H(D\bar{g}(x) + Df_\varepsilon(x,t,\xi) + \varepsilon D\varphi_\varepsilon(x,\xi),\xi) \\ &- \frac{1}{\varepsilon^2}(K-I)\left(f_\varepsilon(x,t,\cdot) + \varepsilon\varphi_\varepsilon(x,\cdot)\right)(\xi) \\ =& B_1 + \frac{\delta}{\varepsilon} B_3 e^{-\frac{\delta t}{\varepsilon^2}} - \frac{1}{\varepsilon} H(D\bar{g}(x) + Df_\varepsilon(x,t,\xi) + \varepsilon D\varphi_\varepsilon(x,\xi),\xi) \\ &+ \frac{1}{\varepsilon} H(D\bar{g}(x),\xi). \end{aligned}$$

Noting that as $\varepsilon \to 0$,

$$H(D\bar{g}(x) + Df_\varepsilon(x,t,\xi) + \varepsilon D\varphi_\varepsilon(x,\xi),\xi) = H(D\bar{g}(x),\xi) + O(\varepsilon + e^{-\frac{\delta t}{\varepsilon^2}}),$$

we see that

$$J \geq B_1 + \frac{\delta B_3}{\varepsilon} e^{-\frac{\delta t}{\varepsilon^2}} - M\Big(1 + \frac{\delta}{\varepsilon} e^{-\frac{\delta t}{\varepsilon^2}}\Big)$$

for some constant $M > 0$ which does not depend on ε.

We fix $B_1 = B_3 = M$, so that w is a viscosity supersolution of $(\mathrm{E})_\varepsilon$. Moreover, we fix $B_2 = C_1$ so that

$$u^\varepsilon(x,0,\xi) \leq w(x,0,\xi) \quad \text{for } (x,\xi) \in \mathbf{R}^n \times I.$$

It is obvious that $w \in \mathcal{U}$. Thus, by Theorem 3.1, we see that

$$u^\varepsilon(x,t,\xi) \leq w(x,t,\xi) \quad \text{for } (x,t,\xi) \in R_\infty \times I.$$

Sending $\varepsilon \searrow 0$, we see that

$$u^+(x,t) \leq \bar{g}_\gamma(x) + Mt \quad \text{for } (x,t) \in Q_\infty.$$

Writing $M(\gamma)$ for M in view of its dependence on γ and setting

$$\mu(t) = \inf\{\sigma(\gamma) + M(\gamma)t \mid \gamma \in (0,1)\} \quad \text{for } t \geq 0,$$

we get a modulus μ such that

$$u^+(x,t) \leq \bar{g}(x) + \mu(t) \quad \text{for } (x,t) \in Q_\infty.$$

Similar arguments ensure that for some modulus μ,

$$u^-(x,t) \geq \bar{g}(x) - \mu(t) \quad \text{for } (x,t) \in Q_\infty.$$

In case when $h = 0$, we use the same function w defined by (5.3) with $f_\varepsilon = 0$ and $B_2 = 0$ and argue in the same way as above, to conclude that

$$\bar{g}(x) - \mu(t) \leq u^-(x,t) \leq u^+(x,t) \leq \bar{g}(x) + \mu(t) \quad \text{for } (x,t) \in R_\infty$$

for some modulus μ. This completes the proof. □

Lemma 5.3. *The functions u^+ and u^- are a viscosity subsolution and a viscosity supersolution of* $(\mathrm{E})_0$ *in Q_∞, respectively.*

We need the following proposition in the proof of Lemma 5.3.

Lemma 5.4. *There are a collection* $\{H_\varepsilon\}_{\varepsilon\in(0,1)} \subset C^2(\mathbf{R}^n)\otimes\mathcal{B}(I)$ *and a* $(\{\omega_R\}_{R>0},\{C_R\}_{R>0}) \in G_2$ *such that for each* $\varepsilon\in(0,1)$, H_ε *satisfies* (H1) *and such that for all* $(x,\xi)\in B(0,R)\times I$, $\varepsilon\in(0,1)$, *and* $R>0$,

$$|H_\varepsilon(p,\xi)-H(p,\xi)|\le\omega_R(\varepsilon)\varepsilon,\quad |D_pH_\varepsilon(p,\xi)-D_pH_\varepsilon(p,\xi)|\le\omega_R(\varepsilon),$$
$$|H_\varepsilon(p,\xi)|\vee|D_pH_\varepsilon(p,\xi)|\le C_R,\quad \|D_p^2H_\varepsilon(p,\xi)\|\le\frac{\omega_R(\varepsilon)}{\varepsilon}.$$

Proof. By the standard mollification techniques, for each $\varepsilon>0$ we find a function $H_\varepsilon\in\mathcal{D}_2\cap C^2(\mathbf{R}^n)\otimes\mathcal{B}(I)$ such that for all $(x,\xi)\in B(0,R)\times I$, $\varepsilon\in(0,1)$, and $R>0$,

$$|H_\varepsilon(p,\xi)-H(p,\xi)|\le C_R\varepsilon,\quad |D_pH_\varepsilon(p,\xi)-D_pH(p,\xi)|\le\omega_R(\varepsilon),$$
$$|H_\varepsilon(p,\xi)|\vee|D_pH_\varepsilon(p,\xi)|\le C_R,\quad \|D_p^2H_\varepsilon(p,\xi)\|\le\frac{\omega_R(\varepsilon)}{\varepsilon},$$

where ω_R is a modulus and $C_R>0$ is a constant, which can be chosen independently of ε.

Fix $R>0$ and fix such ω_R and C_R. Set

$$\sigma_R(r)=\inf\{(C_Rs)\vee\omega_R(sr)\vee\frac{\omega_R(sr)}{s}\mid 0<s<1\}\quad\text{ for } r\ge 0.$$

Then it is clear that σ_R is a non-decreasing, upper semicontinuous, real-valued function on $[0,\infty)$ and that $\sigma_R(0)=0$.

By definition, for each $\varepsilon>0$ there is an $s\equiv s(\varepsilon)\in(0,1)$ such that

$$\sigma_R(\varepsilon)+\varepsilon>(C_Rs)\vee\omega_R(s\varepsilon)\vee\frac{\omega_R(s\varepsilon)}{s}.$$

Then the function $\widetilde{H}_\varepsilon(p,\xi):=H_{s\varepsilon}(p,\xi)$ and $\tilde\sigma_R(r)=\sigma_R(r)+r$ satisfy

$$|\widetilde{H}_\varepsilon(p,\xi)-H(p,\xi)|\le C_Rs\varepsilon\le\varepsilon\tilde\sigma_R(\varepsilon),\quad |D_p\widetilde{H}_\varepsilon(p,\xi)-D_pH(p,\xi)|\le\omega_R(s\varepsilon)\le\tilde\sigma_R(\varepsilon),$$
$$|\widetilde{H}_\varepsilon(p,\xi)|\vee|D_p\widetilde{H}_\varepsilon(p,\xi)|\le C_R,\quad \|D_p^2\widetilde{H}_\varepsilon(p,\xi)\|\le\frac{\omega_R(s\varepsilon)}{s\varepsilon}\le\frac{\tilde\sigma_R(\varepsilon)}{\varepsilon}$$

for all $(x,\xi)\in B(0,R)\times I$ and $\varepsilon>0$. In the above inequalities one may replace $\tilde\sigma_R$ by a modulus. Thus the collection $\{\widetilde{H}_\varepsilon\}_{\varepsilon\in(0,1)}$ together with appropriate choice of collections of moduli and of positive constants has the required properties. □

Proof of Lemma 5.3. We begin by showing that u^+ is a viscosity subsolution of $(\mathrm{E})_0$. Let $\varphi\in C^3(Q_\infty)$, and assume that $u^+-\varphi$ attains a strict maximum at some point $(\hat x,\hat t)\in Q_\infty$.

Let $\{H_\varepsilon\}_{\varepsilon\in(0,1)}$ be a collection of functions from Lemma 5.4. For $\varepsilon\in(0,1)$, we define the function $\Phi(\cdot,\varepsilon)$ on $Q_\infty\times I$ by

$$\Phi(x,t,\xi,\varepsilon)=u^\varepsilon(x,t,\xi)-\varphi(x,t)-\varepsilon\varphi_1^\varepsilon(x,t,\xi)-\varepsilon^2\varphi_2^\varepsilon(x,t,\xi),$$

where

$$\begin{aligned}
\varphi_1^\varepsilon(x,t,\cdot) &= SH_\varepsilon(x,t,\cdot), \\
b^\varepsilon(x,t,\cdot) &= \langle D_pH_\varepsilon(D\varphi(x,t),\cdot), D\varphi_1^\varepsilon(x,t,\cdot)\rangle, \\
\bar{b}^\varepsilon(x,t) &= \int_I r(\xi) b^\varepsilon(x,t,\xi)d\xi, \\
\varphi_2^\varepsilon(x,t,\cdot) &= S\left(b^\varepsilon(x,t,\cdot) - \bar{b}^\varepsilon(x,t)\right)
\end{aligned}$$

for $(x,t) \in Q_\infty$.

Note that

$$\varphi_1^\varepsilon,\ b^\varepsilon,\ \varphi_2^\varepsilon \in C^1(Q_\infty) \otimes \mathcal{B}(I),$$

and

$$\begin{aligned}
D\varphi_1^\varepsilon(x,t,\cdot) &= S\left[D^2\varphi(x,t) D_pH_\varepsilon(D\varphi(x,t),\cdot)\right], \\
\frac{\partial}{\partial t}\varphi_1^\varepsilon(x,t,\cdot) &= S\left[\langle D\varphi_t(x,t), D_pH_\varepsilon(D\varphi(x,t),\cdot)\rangle\right]
\end{aligned}$$

for $(x,t) \in Q_\infty$.

Fix a compact neighborhood $V \subset Q_\infty$ of $(\hat{x},\hat{t})$. Using Lemma 5.4, we deduce that

$$\sup_{0<\varepsilon<1} \sup_{V\times I} \left(|\varphi_1^\varepsilon| + |D\varphi_1^\varepsilon| + \left|\frac{\partial \varphi_1^\varepsilon}{\partial t}\right| + |b^\varepsilon| + |\varphi_2^\varepsilon|\right) < \infty,$$

$$\sup_{V\times I}\left(|D\varphi_2| + \left|\frac{\partial \varphi_2^\varepsilon}{\partial t}\right|\right) \le \frac{\omega_V(\varepsilon)}{\varepsilon},$$

where ω_V is a modulus.

By the definition of u^+, there is a sequence $\varepsilon_j \searrow 0$ such that

$$\theta_j := \sup\{\Phi(x,t,\xi,\varepsilon_j) \mid (x,t) \in V,\ \xi \in I\} \to (u^+ - \varphi)(\hat{x},\hat{t}) \quad \text{as } j \to \infty.$$

Then we choose a sequence of points $(x_j,t_j,\xi_j) \in V \times I$ such that for each $j \in \mathbf{N}$, the function $\Phi(x,t,\xi_j,\varepsilon_j)$ attains a maximum over V at $(x_j,t_j) \in V$ and

$$\Phi(x_j,t_j,\xi_j,\varepsilon_j) \ge \theta_j - \varepsilon_j^3. \tag{5.4}$$

It is easily seen that

$$(x_j,t_j) \to (\hat{x},\hat{t}) \quad \text{as } j \to \infty.$$

Since u^ε is a viscosity subsolution of $(\mathrm{E})_\varepsilon$ in $Q_\infty \times I$, we have

$$\begin{aligned}
\varphi_t(x_j,t_j) \le{} & \frac{1}{\varepsilon_j} H(D\varphi(x_j,t_j) + \varepsilon_j D\varphi_1^{\varepsilon_j}(x_j,t_j,\xi_j) + \varepsilon_j^2 D\varphi_2^{\varepsilon_j}(x_j,t_j,\xi_j),\xi_j) \\
& + \frac{1}{\varepsilon_j^2}\left(\int_I k(\xi_j,\eta) u^{\varepsilon_j}(x_j,t_j,\eta)d\eta - u^{\varepsilon_j}(x_j,t_j,\xi_j)\right) \\
& + O(\varepsilon_j) \quad \text{as } j \to \infty.
\end{aligned} \tag{5.5}$$

Note that as $j \to \infty$,

$$
\begin{aligned}
&H(D\varphi(x_j,t_j) + \varepsilon_j D\varphi_1^{\varepsilon_j}(x_j,t_j,\xi_j) + \varepsilon_j^2 D\varphi_2^{\varepsilon_j}(x_j,t_j,\xi_j),\xi_j) \\
&= H_{\varepsilon_j}(D\varphi(x_j,t_j),\xi_j) + \varepsilon_j \langle D_p H_{\varepsilon_j}(D\varphi(x_j,t_j),\xi_j), D\varphi_1^{\varepsilon_j}(x_j,t_j,\xi_j)\rangle + o(\varepsilon_j) \\
&= H_{\varepsilon_j}(D\varphi(x_j,t_j),\xi_j) + \varepsilon_j b^{\varepsilon_j}(x_j,t_j,\xi_j) + o(\varepsilon_j).
\end{aligned}
\tag{5.6}
$$

From (5.4), we have

$$
\begin{aligned}
&u^{\varepsilon_j}(x_j,t_j,\xi) - u^{\varepsilon_j}(x_j,t_j,\xi_j) \\
&\le \varepsilon_j^3 + \varepsilon_j\left[\varphi_1^{\varepsilon_j}(x_j,t_j,\xi) - \varphi_1^{\varepsilon_j}(x_j,t_j,\xi_j)\right] + \varepsilon_j^2\left[\varphi_2^{\varepsilon_j}(x_j,t_j,\xi) - \varphi_2^{\varepsilon_j}(x_j,t_j,\xi_j)\right]
\end{aligned}
$$

for all $\xi \in I,\ j \in \mathbf{N}$. Hence, in view of the definition of the operator S, we have

$$
\begin{aligned}
&\int_I k(\xi_j,\eta) u^{\varepsilon_j}(x_j,t_j,\eta)d\eta - u^{\varepsilon_j}(x_j,t_j,\xi_j) \\
&= \varepsilon_j^3 + \varepsilon_j (K-I)\varphi_1^{\varepsilon_j}(x_j,t_j,\cdot)(\xi_j) + \varepsilon_j^2 (K-I)\varphi_2^{\varepsilon_j}(x_j,t_j,\cdot)(\xi_j) \\
&= \varepsilon_j^3 - \varepsilon_j H_{\varepsilon_j}(D\varphi(x_j,t_j),\xi_j) - \varepsilon_j^2\left[b^{\varepsilon_j}(x_j,t_j,\xi_j) - \bar{b}^{\varepsilon_j}(x_j,t_j)\right]
\end{aligned}
$$

Combining this with (5.5) and (5.6), we get

$$
\varphi_t(x_j,t_j) \le \bar{b}^{\varepsilon_j}(x_j,t_j) + o(1) \quad \text{as } j \to \infty. \tag{5.7}
$$

Since

$$
\begin{aligned}
\bar{b}^{\varepsilon_j}(x_j,t_j) &= \int_I r(\xi)\langle D_p H(D\varphi(x_j,t_j),\xi), D^2\varphi(x_j,t_j) D_p a(D\varphi(x_j,t_j),\xi)\rangle d\xi + o(1) \\
&= \operatorname{tr}\left[\bar{A}(D\varphi(x_j,t_j)) D^2\varphi(x_j,t_j)\right] + o(1)
\end{aligned}
$$

as $j \to \infty$, we conclude from (5.7) that

$$
\varphi_t(\hat{x},\hat{t}) \le \operatorname{tr}[\bar{A}(D\varphi(\hat{x},\hat{t})) D^2\varphi(\hat{x},\hat{t})],
$$

which shows that u^+ is a viscosity subsolution of $(\mathrm{E})_0$.

Arguments similar to the above prove that u^- is a viscosity supersolution of $(\mathrm{E})_0$. □

Proof of Theorem 2.4. In view of Lemma 5.1, we see that

$$
\lim_{r \searrow 0} \sup\{u^+(x,t) - u^-(y,s) \mid (x,t),(y,s) \in Q_r,\ |x-y| + |t-s| < r\} = 0.
$$

By Lemma 5.3, we know that u^+ and u^- are a viscosity subsolution and a viscosity supersolution of $(\mathrm{E})_0$. Thus, by using Theorem 3.1, we see that $u^+ \le u \le u^-$ in Q_∞, from which we deduce easily that as $\varepsilon \searrow 0$,

$$
u^\varepsilon(x,t,\xi) \to u(x,t) \quad \text{locally uniformly in } Q_\infty \times I.
$$

Since $(\mathrm{E})_\varepsilon$ and $(\mathrm{E})_0$ are translation invariant in x, we conclude from the above that for any collection $\{y_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathbf{R}^n$, as $\varepsilon \searrow 0$,

$$u^\varepsilon(x+y_\varepsilon,t,\xi) - u(x+y_\varepsilon,t) \to 0 \quad \text{locally uniformly in } Q_\infty \times I.$$

Now a simple argument by contradiction shows that, for any $\delta \in (0,1)$, as $\varepsilon \searrow 0$,

$$u^\varepsilon(x,t,\xi) \to u(x,t) \quad \text{uniformly in } \mathbf{R}^n \times [\delta,\, \delta^{-1}] \times I. \tag{5.8}$$

Finally, if $g(x,\xi)$ is independent of ξ, then (5.8) and the last assertion of Lemma 5.1 yield the uniform convergence of $u^\varepsilon(x,t,\xi)$ to $u(x,t)$ in $R_T \times I$ for any $T \in (0,\infty)$ as $\varepsilon \searrow 0$. $\square$

6. PROOF OF LEMMAS 4.7 AND 5.2

This section is devoted to the proof of Lemmas 4.7 and 5.2.

Proof of Lemma 4.7. Fix $\gamma \in G$. Since $\widehat{\mathcal{D}}(\gamma)$ is a monotone class of functions and $\widehat{\mathcal{D}}_c(\gamma) \subset \widehat{\mathcal{D}}(\gamma)$, we see that

$$m(\widehat{\mathcal{D}}_c(\gamma)) \subset \widehat{\mathcal{D}}(\gamma).$$

Fix $\hat{u} \in \widehat{\mathcal{D}}(\gamma)$ and will show that $\hat{u} \in m(\widehat{\mathcal{D}}_c(\gamma))$.

We write $\mathcal{M} = m(\widehat{\mathcal{D}}_c(\gamma))$. Given $u \in \widehat{\mathcal{D}}(\gamma)$, since

$$\widehat{\mathcal{D}}(\gamma) \subset \mathcal{B}(I)^2 \times \mathcal{B}(\mathbf{R}^n \times I) \times \mathcal{B}(\mathbf{R}^n \times I),$$

we may regard u as a function : $\mathbf{R}^n \times \mathbf{R}^n \times I \to \mathbf{R}^4$ and write

$$u(x,p,\xi) = (u_1(\xi), u_2(\xi), u_3(x,\xi), u_4(p,\xi)) \quad \text{for } x,p \in \mathbf{R}^n,\ \xi \in I.$$

We first show that if $\{u^k\}_{k\in\mathbf{N}} \subset \mathcal{M}$ and for each $z \in \Omega := \mathbf{R}^{2n} \times I$, $u^k(z) \to u(z)$ as $k \to \infty$, where $u : \Omega \to \mathbf{R}^4$, then $u \in \mathcal{M}$, i.e., $\mathcal{M}$ is closed under the pointwise convergence.

To see this, fix $u \in \widehat{\mathcal{D}}_c(\gamma)$ and define

$$\mathcal{M}_1 = \{v \in \mathcal{M} \mid u \vee v \in \mathcal{M}\}.$$

Here $u \vee v$ is defined by $u \vee v = (u_1 \vee v_1, ..., u_4 \vee v_4)$, where $u = (u_1, ..., u_4)$ and $v = (v_1, ..., v_4)$. It is easily seen that $\mathcal{M}_1$ is a monotone class of functions and that $\widehat{\mathcal{D}}_c(\gamma) \subset \mathcal{M}_1$. Hence we see that $\mathcal{M} \subset \mathcal{M}_1$ and conclude that if $u \in \widehat{\mathcal{D}}_c(\gamma)$ and $v \in \mathcal{M}$, then $u \vee v \in \mathcal{M}$.

We now fix $v \in \mathcal{M}$ and define

$$\mathcal{M}_2 = \{u \in \mathcal{M} \mid u \vee v \in \mathcal{M}\}.$$

Again, it is easy to see that $\mathcal{M}_2$ is a monotone class of functions. Since $\widehat{\mathcal{D}}_c(\gamma) \subset \mathcal{M}_2$, we have $\mathcal{M} \subset \mathcal{M}_2$, and conclude that if $u, v \in \mathcal{M}$, then $u \vee v \in \mathcal{M}$.

Similarly we see that if $u, v \in \mathcal{M}$, then $u \wedge v \in \mathcal{M}$, where $u \wedge v$ is defined by $u \wedge v = (u_1 \wedge v_1, ..., u_4 \wedge v_4)$.

Let $\{u^k\}_{k\in\mathbf{N}} \subset \mathcal{M}$ and $u : \Omega \to \mathbf{R}^4$. Assume that for each $z \in \Omega$,

$$u(z) = \lim_{k\to\infty} u^k(z).$$

For each $k, l \in \mathbf{N}$, we have

$$u^{k,l} := u^k \vee u^{k+1} \vee \cdots \vee u^{k+l} \in \mathcal{M}.$$

For each $z \in \Omega$ the sequence $\{u^{k,l}(z)\}_{l\in\mathbf{N}}$ is bounded and non-decreasing. Therefore, the limit

$$v^k(z) := \lim_{l\to\infty} u^{k,l}(z)$$

exists for any $z \in \Omega$ and $k \in \mathbf{N}$. Since $\mathcal{M}$ is a monotone class of functions, we have $v^k \in \mathcal{M}$ for all $k \in \mathbf{N}$. Noting that for each $z \in \Omega$ the sequence $\{v^k(z)\}$ is non-increasing and converges to $u(z)$, we see that $u \in \mathcal{M}$. Thus we conclude that $\mathcal{M}$ is closed under the pointwise convergence.

It is enough to show that $\hat{u}$ can be approximated, in the pointwise sense, by a sequence of functions in $\mathcal{M}$.

Set $X = \mathbf{R}^2 \times C(\mathbf{R}^n) \times C(\mathbf{R}^n)$. Define the distance d on the space X by

$$d(f, g) = |f_1 - g_1| \vee |f_2 - g_2| \vee \sup_{i\in\mathbf{N}} \Big(\max_{|x|\le i} |f_3(x) - g_3(x)| \vee \max_{|p|\le i} |f_4(p) - g_4(p)| \Big) \wedge i^{-1},$$

where $f = (f_1, ..., f_4)$ and $g = (g_1, ..., g_4)$. Let $\gamma = (\omega, L, \{\omega_R\}, \{L_R\})$ and define D as the set of those $f \equiv (f_1, ..., f_4) \in X$ which satisfy

$$\begin{gathered} f_1 \in [\kappa_0, \kappa_1], \quad f_2 \in [-\kappa_1, -\kappa_0], \\ |f_3(x)| \le L, \quad |f_3(x) - f_3(y)| \le \omega(|x - y|), \\ |f_4(p)| \le L_R, \quad |f_4(p) - f_4(q)| \le \omega_R(|p - q|) \end{gathered}$$

for all $x, y \in \mathbf{R}^n$, $p, q \in B(0, R)$, and $R > 0$.

Note that for $u \in \mathcal{B}(\mathbf{R}^{2n} \times I, \mathbf{R}^4)$, we have $u \in \mathcal{D}(\gamma)$ if and only if $u(\cdot, \xi) \in D$ for all $\xi \in I$.

It is standard to see that D is a compact subset of the metric space (X, d).

Fix $\varepsilon > 0$. We choose a finite sequence $\{f^k\}_{k=1}^N \subset D$ so that

$$D \subset \bigcup_{k=1}^N B_k,$$

where

$$B_k = \{f \in X \mid d(f, f^k) \le \varepsilon\}.$$

We define $\varphi : I \to \{1, ..., N\}$ by

$$\varphi(\xi) = j \quad \text{if and only if} \quad \hat{u}(\cdot, \xi) \in B_j \;\text{ and }\; \hat{u}(\cdot, \xi) \notin B_i \;\text{ for all } i < j,$$

and claim that $\varphi \in \mathcal{B}(I)$.

To see this, we need to show that the sets $C_k := \{\xi \in I \mid \hat{u}(\cdot, \xi) \in B_k\}$ are all Borel subsets of I. Fix any $k \in \{1, ..., N\}$ and a dense subset $\{y_j\}_{j\in\mathbf{N}}$ of $\mathbf{R}^n$. Observe that for $\xi \in I$, $\hat{u}(\cdot, \xi) \in B_k$ if and only if

$$\begin{gathered} |\hat{u}_1(\xi) - f_1^k| \le \varepsilon, \quad |\hat{u}_2(\xi) - f_2^k| \le \varepsilon, \\ |\hat{u}_3(y_j, \xi) - f_3^k(y_j)| \le \varepsilon, \\ |\hat{u}_4(y_j, \xi) - f_4^k(y_j)| \le \varepsilon \end{gathered}$$

for all $i, j \in \mathbf{N}$ satisfying $i\varepsilon < 1$ and $|y_j| \le i$. Since for each $j \in \mathbf{N}$ the functions $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3(y_j, \cdot)$, and $\hat{u}_4(y_j, \cdot)$ are Borel measurable, we see that the sets of $\xi \in I$ defined by each of inequalities above are Borel subsets of I and hence that C_k is a Borel subset of I.

Now, we define the function $F \in C(\mathbf{R}^{2n} \times \mathbf{R}, \mathbf{R}^4)$ by

$$F(z, s) = \begin{cases} (k + 1 - s)f^k(z) + (s - k)f^{k+1}(z) & \text{if } s \in [k, k + 1] \text{ and } 1 \le k < N, \\ f^1(z) & \text{if } s \le 1, \\ f^N(z) & \text{if } s \ge N. \end{cases}$$

Noting that $F(\cdot, \varphi(\xi)) = F(\cdot, k) = f^k$ for all $\xi \in C_k \setminus \cup_{j<k} C_j$ and $k \in \{1, ..., N\}$, we deduce that

$$d(F(\cdot, \varphi(\xi)), \hat{u}(\cdot, \xi)) \le \varepsilon \quad \text{for all } \xi \in I.$$

Since $F(\cdot, s) \in D$ for all $s \in \mathbf{R}$ by the definition, we infer that the function $F \circ \varphi$ defined by $F \circ \varphi(z, \xi) = F(z, \varphi(\xi))$ is an element of $\mathcal{M}$. Indeed, if we set

$$\mathcal{B} = \{h : I \to \mathbf{R} \mid F \circ h \in \mathcal{M}\},$$

then $C(I) \subset \mathcal{B}$ and $\mathcal{B}$ is a monotone class of functions. Therefore, any Borel measurable function on I is an element of $\mathcal{B}$ and, consequently, $F \circ \varphi \in \mathcal{M}$. Here the monotonicity of $\mathcal{B}$ is a consequence of the fact that $\mathcal{M}$ is closed under the pointwise convergence.

Thus, we see that there is a sequence $\{u^k\}_{k\in\mathbf{N}} \subset \mathcal{M}$ which converges to $\hat{u}$ in the metric d, i.e., in the topology of locally uniform convergence on $\mathbf{R}^{2n} \times I$. Since $\mathcal{M}$ is closed under the pointwise convergence, we conclude that $\hat{u} \in \mathcal{M}$, which completes the proof. □

Proof of Lemma 5.2. In this proof we regard $L^2(I)$, $\mathcal{B}^\infty(I)$, etc. as the vector spaces with complex scalar field.

We first prove that

$$\text{if } \mu \in \mathbf{C} \text{ is an eigenvalue of } \bar{K} \text{ and } |\mu| \geq 1, \text{ then } \mu = 1. \tag{6.1}$$

To show this, we fix $\mu \in \mathbf{C}$ and $\phi \in L^2(I)$ so that $|\mu| \geq 1$, $\phi \neq 0$, and $\bar{K}\phi = \mu\phi$. Identifying ϕ with the function h defined by

$$h(\xi) = \mu^{-1} \int_I k(\xi, \eta) g(\eta) d\eta,$$

where g is a function in the equivalence class ϕ, we may regard ϕ as a function in $\mathcal{B}^\infty(I)$ and assume that

$$\mu\phi(\xi) = K\phi(\xi) \quad \text{ for all } \xi \in I.$$

Set $M = \sup_I |\phi|$. We claim that $|\phi(\xi)| = M$ a.e. $\xi \in I$. In order to check this, we fix $\varepsilon > 0$ and $\gamma > \varepsilon$, and choose $\xi \in I$ so that $|\phi(\xi)| > M - \varepsilon$. Observing that $|\phi(\xi)| \leq \bar{K}|\phi|(\xi)$ and setting $B_\gamma = \{\xi \in I \mid |\phi(\xi)| \leq M - \gamma\}$, we calculate that

$$\begin{aligned} 0 &< \int_I k(\xi, \eta)(|\phi(\eta)| - M + \varepsilon) d\eta \\ &\leq \int_{B_\gamma} k(\xi, \eta)(\varepsilon - \gamma) d\eta + \int_I k(\xi, \eta) \varepsilon d\eta \leq -(\gamma - \varepsilon)\kappa_0 |B_\gamma| + \varepsilon. \end{aligned}$$

Sending $\varepsilon \to 0$, we see that $|B_\gamma| = 0$ for all $\gamma > 0$, which shows that $|\phi(\xi)| = M$ a.e. $\xi \in I$.

By multiplying ϕ by M^{-1} if necessary, we may assume that $M = 1$. We fix $\hat{\xi} \in I$ so that $|\phi(\hat{\xi})| = 1$. We may assume by multiplying ϕ by $\overline{\phi(\hat{\xi})}$, the complex conjugate of $\phi(\hat{\xi})$, that $\phi(\hat{\xi}) = 1$. Define $a \in \mathcal{B}(I)$ by $a(\xi) = \operatorname{Re} \phi(\xi)$. It follows that $a(\hat{\xi}) = 1$ and $|a(\xi)| \leq 1$ for all $\xi \in I$. Setting $B_\varepsilon = \{\xi \in I \mid a(\xi) \leq 1 - \varepsilon\}$ for $\varepsilon > 0$, we argue as before, to get

$$0 \leq -\varepsilon \int_{B_\varepsilon} k(\hat{\xi}, \eta) d\eta \leq -\varepsilon \kappa_0 |B_\varepsilon|,$$

which guarantees that $\phi(\xi) = 1$ a.e. $\xi \in I$. Thus we have

$$\mu\phi(\xi) = K\phi(\xi) = 1 \quad \text{ for } \xi \in I,$$

and conclude that $\mu = 1$ and $\phi(\xi) = 1$ for all $\xi \in I$.

Next, we observe that for $\phi \in \{r\}^\perp$,

$$\int_I \bar{K}\phi(\xi) r(\xi) d\xi = \int_I \phi(\xi) \bar{K}^* r(\xi) d\xi = \int_I \phi(\xi) r(\xi) d\xi = 0. \tag{6.2}$$

This allows us to define the continuous linear operator $\bar{L} : \{r\}^{\perp} \to \{r\}^{\perp}$ by $\bar{L}\phi = \bar{K}\phi$.

Since $\bar{K}$ is a compact operator on $L^2(I)$, we see that $\bar{L}$ is a compact operator on $\{r\}^{\perp}$. By the Fredholm-Riesz-Schauder theory, we know that for each $\varepsilon > 0$, $\sigma(\bar{L}) \cap \{z \in \mathbf{C} \mid |z| > \varepsilon\}$ is a finite set and consists of eigenvalues of $\bar{L}$. Here and henceforth, for any operator L, $\sigma(L)$ denotes the spectrum of L. Since $\mathbf{1} \notin \{r\}^{\perp}$, we see from (6.1) that $\sigma(\bar{L}) \subset \{z \in \mathbf{C} \mid |z| < 1\}$. Since $\sigma(\bar{L})$ is a closed subset of $\mathbf{C}$, we find a constant $\theta \in (0,1)$ such that

$$\sigma(\bar{L}) \subset \{z \in \mathbf{C} \mid |z| \le \theta\}. \tag{6.3}$$

In view of (6.2), we may define the continuous operator $L : \{r\}^{\perp,\infty} \to \{r\}^{\perp,\infty}$ by $L\phi = K\phi$. We claim that

$$\sigma(L) \subset \{z \in \mathbf{C} \mid |z| \le \theta\}. \tag{6.4}$$

To show this, fix $\mu \in \{z \in \mathbf{C} \mid |z| > \theta\}$. For $\phi \in \{r\}^{\perp,\infty}$ choose any

$$\psi \in (\mu I - \bar{L})^{-1}[\phi],$$

and set

$$f(\xi) = \mu^{-1}\big(K\psi(\xi) - \phi(\xi)\big) \quad \text{for } \xi \in I.$$

It is easily seen that $f \in \{r\}^{\perp,\infty}$ and that

$$\mu f(\xi) - Lf(\xi) = \phi(\xi) \quad \text{for all } \xi \in I.$$

Hence, $\mu I - L$ is surjective. Next we fix $\phi \in \{r\}^{\perp,\infty}$. Let $f, g \in \{r\}^{\perp,\infty}$ satisfy

$$(\mu I - L)f(\xi) = \phi(\xi) \quad \text{and} \quad (\mu I - L)g(\xi) = \phi(\xi) \quad \text{for } \xi \in I.$$

Then we see that $[f - g] \in \operatorname{Ker}(\mu I - \bar{L})$, which yields in view of (6.3) that $f(\xi) = g(\xi)$ a.e. $\xi \in I$. Accordingly we have

$$\mu(f - g)(\xi) = L(f - g)(\xi) = 0 \quad \text{for } \xi \in I.$$

Thus $\mu I - L$ is injective. Invoking the open mapping theorem, we conclude that μ is in the resolvent set of L, proving (6.4).

Recall the definition of the spectral radius ρ of the operator L, i.e.,

$$\rho = \lim_{k \to \infty} \|L^k\|^{1/k}.$$

(See [8].) We know that $\rho \le \theta$. Fix any $\lambda \in (\theta, 1)$. Then there is a constant $C \ge 1$ such that

$$\|L^k\| \le C\lambda^k \quad \text{for all } k \in \mathbf{N}.$$

This yields that for $t \geq 0$,

$$\|e^{tL}\| \leq \sum_{k \in \mathbf{Z}_+} \frac{t^k \|L^k\|}{k!} \leq Ce^{\lambda t}.$$

Thus, for $h \in \{r\}^{\perp,\infty}$ and $t \geq 0$ we have

$$\|e^{t(K-I)}h\|_\infty = \|e^{t(L-I)}h\|_\infty \leq Ce^{-(1-\lambda)t}\|h\|_\infty.$$

This completes the proof. $\square$

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