Long-time behavior of solutions of Hamilton-Jacobi equations with convex and coercive Hamiltonians

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Abstract

We investigate the long-time behavior of viscosity solutions of Hamilton-Jacobi equations in $\mathbb{R}^n$ with convex and coercive Hamiltonians and give three general criteria for the convergence of solutions to asymptotic solutions as time goes to infinity. We apply the criteria to obtain more specific sufficient conditions for the convergence to asymptotic solutions and then examine them with examples. We take the dynamical approach to these investigations which is based on tools from weak KAM theory such as extremal curves, Aubry sets and representation formulas for solutions.

1 Introduction.

We are concerned with the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\,\cdot\,, 0) = u_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{1.1}$$

and study the long-time behavior of the solution of (1.1).

We assume the following (A1)–(A5) throughout this paper.

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(A1) $H \in C(\mathbb{R}^n \times \mathbb{R}^n)$,

(A2) $\inf\{H(x,p) \,|\, x \in B(0,r),\ |p| \geq R\} \longrightarrow +\infty$ as $R \to +\infty$ for every $r > 0$,

(A3) $H(x,p)$ is convex with respect to p for every $x \in \mathbb{R}^n$,

(A4) for each $\phi \in \mathcal{S}_H^-$, there exist $C > 0$ and $\psi \in \mathcal{S}_{H-C}^-$ such that

$$\lim_{|x|\to\infty} (\phi - \psi)(x) = \infty,$$

(A5) $u_0 \in C(\mathbb{R}^n)$.

Here and henceforth, we denote by $\mathcal{S}_H$ (resp., $\mathcal{S}_H^-$, $\mathcal{S}_H^+$) the set of all continuous viscosity solutions (resp., subsolutions, supersolutions) of $H(x,\, Du(x)) = 0$ in $\mathbb{R}^n$. Similarly, given a domain Ω, we denote by $\mathcal{S}_H(\Omega)$ (resp., $\mathcal{S}_H^-(\Omega)$, $\mathcal{S}_H^+(\Omega)$) the set of all continuous viscosity solutions (resp., subsolutions, supersolutions) of $H(x,\, Du(x)) = 0$ in Ω. See, for instance, [2, 20] for overviews on Hamilton-Jacobi equations and viscosity solutions theory.

Note that hypotheses (A1)–(A5) are not enough to assure the unique solvability of (1.1) in the sense of viscosity solution. To start our discussion with this generality, we define the (unique) solution of (1.1) as follows. For any $\psi \in C(\mathbb{R}^n)$ and $t \geq 0$, we define the function $T_t\psi$ on $\mathbb{R}^n$ by

$$T_t\psi(x) = \inf\Big\{\int_{-t}^{0} L(\eta(s), \dot\eta(s))\, ds + \psi(\eta(-t)) \ \Big|\ \eta \in \mathcal{C}([-t,\, 0];\, x)\Big\}, \tag{1.2}$$

and refer to the function $u(x,t) := T_t u_0(x)$ as the *solution* of (1.1). Here L is the *Lagrangian* of H defined by $L(x,\xi) = \sup_{p\in\mathbb{R}^n}(\xi \cdot p - H(x,p))$ for $(x,\xi) \in \mathbb{R}^{2n}$ and $\mathcal{C}([a,\, b]; x)$, with $a < b$, denotes the space of all absolutely continuous functions (called curves) $\eta : [a,\, b] \to \mathbb{R}^n$ (i.e. $\eta \in \mathrm{AC}([a,\, b])$) such that $\eta(b) = x$. Also, $\mathcal{C}((-\infty,\, a]; x)$ denotes the space of all functions $\eta \in C((-\infty,\, a])$ such that $\eta \in \mathcal{C}([c,\, a]; x)$ for all $c < a$.

We remark here that $T_t\psi(x)$ is well-defined with the following interpretation and $T_t\psi(x) \in [-\infty,\, \infty)$ for all $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$. To see this, we fix $\psi \in C(\mathbb{R}^n)$. Note that $L(x,\xi)$ is lower semicontinuous on $\mathbb{R}^{2n}$, due to the assumption that $H \in C(\mathbb{R}^{2n})$, and hence that the function $s \mapsto L(\eta(s),\, \dot\eta(s))$ is Lebesgue measurable on $(-\infty,\, 0)$ for any $\eta \in \mathrm{AC}((-\infty,\, 0])$. Noting that $H(x,\, 0) = -\inf_{\xi\in\mathbb{R}^n} L(x,\xi)$ for all $x \in \mathbb{R}^n$, we observe that, for each $t > 0$, $r > 0$ and $\eta \in \mathrm{AC}([-t,\, 0])$ satisfying $\eta([-t,0]) \subset B(0,r)$, we have $L(\eta(s),\, \dot\eta(s)) \geq -\max_{x\in B(0,r)} H(x,\, 0)$. Therefore, for $\eta \in \mathrm{AC}([-t,\, 0])$, it is natural to set

$$\int_{-t}^{0} L(\eta(s),\, \dot\eta(s))\, ds = \infty$$

if $s \mapsto L(\eta(s),\, \dot\eta(s))$ is not integrable on $(-t,\, 0)$. With this interpretation, $T_t\psi(x)$ is well-defined for all $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$. Next, thanks to (A1)–(A3), we may choose a constant $\varepsilon > 0$ for each $r > 0$ (see Lemma 2.3 below or [18, Proposition 2.1]) such that $\sup_{(x,\xi)\in B(0,r)\times B(0,\varepsilon)} L(x,\xi) < \infty$. Then we see that for all $(x,t) \in \mathbb{R}^n \times [0, \infty)$,

$$T_t\psi(x) \le \int_{-t}^{0} L(x,\, 0)\, ds + \psi(x) = L(x,0)t + \psi(x) < \infty.$$

Now, let us consider an example where $T_t u_0(x) = -\infty$ for some (x,t). Let $n = 1$, $H(p) = (1/2)p^2$ and $u_0(x) = -x^2$. Then the Lagrangian L of H is given by $L(\xi) = (1/2)\xi^2$. Consider the curve $\eta \in \mathcal{C}((-\infty, 0]; 0)$ given by $\eta(s) = cs$, with $c > 0$, and observe that for any $t > 0$,

$$T_t u_0(0) \le \int_{-t}^{0} L(\dot\eta(s))\, ds + u_0(\eta(-t)) = \frac{c^2 t}{2} - (ct)^2 = \frac{c^2 t}{2}(1 - 2t),$$

which implies that $T_t u_0(0) = -\infty$ if $t > 1/2$. We recall here (see [18, Theorems A.1, A.2]) that if the function $u(x,t) := T_t u_0(x)$ is continuous on an open set $U \subset \mathbb{R}^n \times (0, \infty)$, then u is a viscosity solution of $u_t + H(x, Du) = 0$ in U.

The objective of this paper is to investigate the long-time behavior of the solution of (1.1). More precisely, we are concerned with the convergence of the form

$$u(x,t) + at - \phi(x) \longrightarrow 0 \qquad \text{in } \; C(\mathbb{R}^n) \qquad \text{as } t \to \infty \tag{1.3}$$

for some $a \in \mathbb{R}$ and $\phi \in C(\mathbb{R}^n)$, where $C(\mathbb{R}^n)$ is equipped with the topology of locally uniform convergence. Note that if u satisfies (1.1) in the viscosity sense, then the function $\phi(x) - at$, which we call an *asymptotic solution* of (1.1), enjoys the following stationary Hamilton-Jacobi equation in the viscosity sense:

$$H(x, D\phi) = a \qquad \text{in } \; \mathbb{R}^n. \tag{1.4}$$

Thus a natural question related to (1.3) is the *additive eigenvalue problem* for H which seeks for a pair $(a, \phi) \in \mathbb{R} \times C(\mathbb{R}^n)$ such that ϕ is a solution of (1.4), i.e., $\phi \in \mathcal{S}_{H-a}$. The additive eigenvalue problem appears in ergodic control, in which it is called the ergodic control problem, and in homogenization, in which it is called the cell problem. A standard approach to (1.3) is first to solve the additive eigenvalue problem for H and then to try to prove the convergence (1.3) for each fixed solution of the additive eigenvalue problem for H. However, to simplify our presentation in this paper, we will deal only with the latter step in the above approach and investigate if (1.3) holds for a fixed $a \in \mathbb{R}$. We assume moreover that $a = 0$. Indeed, convergence (1.3) with general a is equivalent to (1.3) with $a = 0$ once H and u are replaced by $H - a$ and $u(x,t) + at$, respectively. We will show in Theorem 2.9 below that

$$\liminf_{t\to\infty} u(x,t) = u_\infty(x) \quad \text{for all } x \in \mathbb{R}^n,$$

under the additional assumption:

(A6) $-\infty < u_0^-(x) \le u_\infty(x) < \infty$ for all $x \in \mathbb{R}^n$,

where

$$u_0^-(x) := \sup\{\, \phi(x) \,|\, \phi \in \mathcal{S}_H^-, \quad \phi \le u_0 \ \text{ in } \mathbb{R}^n\},$$
$$u_\infty(x) := \inf\{\, \psi(x) \,|\, \psi \in \mathcal{S}_H, \quad \psi \ge u_0^- \ \text{ in } \mathbb{R}^n\}.$$

This condition is equivalent to saying that

$$\{\phi \in \mathcal{S}_H^- \mid \phi \le u_0 \ \text{ in } \mathbb{R}^n\} \ne \emptyset \quad \text{and} \quad \{\phi \in \mathcal{S}_H \mid \phi \ge u_0^- \ \text{ in } \mathbb{R}^n\} \ne \emptyset.$$

Thus our purpose in this paper is to show the following convergence:

$$u(\cdot, t) \longrightarrow u_\infty \quad \text{in } C(\mathbb{R}^n) \ \text{ as } t \to \infty. \tag{1.5}$$

Asymptotic problems of this type has been studied intensively in the last decade. As one of the most typical cases, it was proved that if H satisfies (A1)–(A3) and $H(x,p)$ is $\mathbb{Z}^n$-periodic with respect to x and is strictly convex with respect to p, then for each $\mathbb{Z}^n$-periodic initial function $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$ there exists a solution $(a, \phi) \in \mathbb{R} \times C(\mathbb{R}^n)$ of the additive eigenvalue problem for H such that (1.3) is valid and the constant a is determined independently of u_0. We refer to the literatures [4, 6, 7, 10, 11, 21, 22, 23] and references therein for more details. Remark that [4] deal with non-convex Hamiltonians whereas most of others are concerned only with convex ones.

It has also been of interest in recent years on the long-time behavior of viscosity solutions to (1.1) that are not necessarily spatially periodic. As far as non-periodic solutions are concerned, the above (A1)-(A6) are insufficient to obtain the convergence (1.3) even if we admit strict convexity for H in any sense (see [5, 16]). The papers [3, 14, 16, 18] deal with some situations in which the solution of (1.1) has indeed the desired convergence of the form (1.3) for a suitable (a, ϕ).

Motivated by these earlier results, given a point $z \in \mathbb{R}^n$, we introduce three general criteria, each of which, together mostly with (A1)–(A6) above, guarantees the pointwise convergence of the solution u of (1.1):

$$u(z, t) \longrightarrow u_\infty(z) \quad \text{ as } t \longrightarrow \infty. \tag{1.6}$$

We then apply these criteria to obtain general convergence results of the form (1.5) and apply them to several examples. Our results cover most of existing results, and, on the other hand, involve a few observations which seem to be new.

One of our new observations is concerned with strict convexity for H. As pointed out in several literatures (see e.g. [4]), it is necessary in some situations to require a sort of strict convexity for H, as a genuine nonlinearity for H, so that the solution of (1.1) converges to an asymptotic solution as $t \to \infty$. In the present paper, we use condition $(A7)_+$ or $(A7)_-$ which guarantees, respectively, strict convexity of $H(x,p)$ in p at the zero level-set of H "upward" or "downward" (see Section 4 for the precise requirements). We point out here that some of our convergence results involving $(A7)_-$ are not covered by the previous results. Another important feature in this paper is in the observations of a "switch-back" motion of nearly optimal curves for the variational formula (1.2) for $T_t u_0$ in some situations. In one-dimensional case we have already encountered such a switch-back motion (see [17]), and here we investigate with greater generality the cases where such switch-back motions appear.

We use the dynamical approach in our investigations here which is based on tools from weak KAM theory (see e.g. [10, 13, 8, 9]) such as extremal curves, Aubry sets and representations formulas for solutions. Indeed, we formulate our general criteria for the convergence to asymptotic solutions in terms of extremal curves, which seems inevitable to attain further generality.

This paper is organized as follows. In the next section, we give some preliminaries on tools in our dynamical approach such as dynamic programming principle, extremal curves, Aubry sets, etc. In Section 3, we formulate and prove representation formulas for solutions $u \in \mathcal{S}_H$, which are very close those obtained in [19]. In Section 4, we give three general criteria, called (C1), (C2), (C3), for the pointwise convergence (1.6). Sections 4, 5 and 6 are devoted to establishing some results based on (C1), (C2) and (C3), respectively, and furthermore to applying them to (mostly simple) examples.

Before closing the introduction, we give a few of our notation. We use δ_B to denote the indicator function of the set B, that is, $\delta_B(x) = 0$ if $x \in B$ and $= \infty$ otherwise. For a metric space X, $\mathrm{UC}(X)$ (resp., $\mathrm{BUC}(X)$) denotes the space of uniformly continuous (resp., bounded uniformly continuous) functions on X. For $a, b \in \mathbb{R}$ we write $a \wedge b = \min\{a, b\}$. For real-valued functions f, g, $f \wedge g$ denotes the function given by $f \wedge g(x) = \min\{f(x), g(x)\}$.

2 Preliminaries.

We assume always in this section that (A1)–(A5) hold and will later assume also that (A6) holds. We usually write $u(x,t)$ for $T_t u_0(x)$ for simplicity of notation. We set

$$u_0^-(x) := \sup\{\, \phi(x) \mid \phi \in \mathcal{S}_H^-, \quad \phi \le u_0 \ \text{ in } \mathbb{R}^n\},$$
$$u_\infty(x) := \inf\{\, \psi(x) \mid \psi \in \mathcal{S}_H, \quad \psi \ge u_0^- \ \text{ in } \mathbb{R}^n\},$$

where we use the standard convention: $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

Theorem 2.1. *Assume that $u_0^-(x) = -\infty$ for some $x \in \mathbb{R}^n$. Then*

$$\liminf_{t\to\infty} u(x,t) = -\infty \qquad \textit{for all } x \in \mathbb{R}^n$$

According to this theorem, in order to get convergence (1.3) for $a = 0$, we have to assume that $u_0^-(x) > -\infty$ for all $x \in \mathbb{R}^n$.

We need the following two lemmas to prove the theorem above.

Lemma 2.2. *For any $x \in \mathbb{R}^n$ and $t,\ s > 0$ we have*

$$u(x, t+s) = \inf\left\{ \int_{-s}^{0} L(\eta(r), \dot\eta(r))\, dr + u(\eta(-s),\ t) \ \Bigm|\ \eta \in \mathcal{C}([-s,\ 0];\ x) \right\}$$

The above identity is referred to as the dynamic programming principle in the theory of optimal control and as the semi-group property in terms of PDE. Here we have to be careful that the expression under the infimum symbol may have value $\infty - \infty$, and we regard $\infty - \infty$ as ∞. An explanation on this interpretation is that only those $\eta \in \mathcal{C}([-s,\ 0];\ x)$ which have finite integral

$$\int_{-s}^{0} L(\eta(r), \dot\eta(r))\, dr$$

are considered to be "admissible". The proof of the lemma above is standard and we omit giving it here.

Lemma 2.3. *For each $R > 0$ there are constants $C_R > 0$ and $\delta_R > 0$ such that $L(x, \xi) \le C_R$ for all $(x, \xi) \in B(0,\ R) \times B(0,\ \delta_R)$.*

For a proof of this lemma we refer to [18, Proposition 2.1].

Proof of Theorem 2.1. We argue by contradiction. Thus we suppose that $u_0^-(x) \equiv -\infty$ and that there exists an $x_0 \in \mathbb{R}^n$ such that $\liminf_{t\to\infty} u(x_0, t) > -\infty$. By translation, we may assume that $x_0 = 0$.

We show first that for each $R > 0$ there exists a constant $M_R > 0$ such that $u(x, t) \ge -M_R$ for all $(x, t) \in B(0,\ R) \times [0, \infty)$. For this, we fix $R > 0$ and choose constants $\tau > 0$ and $C_0 > 0$ so that $u(0, t) \ge -C_0$ for all $t \ge \tau$. Let $C_R > 0$ and $\delta_R > 0$ be the constants from Lemma 2.3 and fix any $(x,\ t) \in B(0,\ R) \times [0,\ \infty)$. Let $\eta \in C^1([-t_R,\ 0])$, with $t_R := \tau + R/\delta_R$, be the line segment such that $\eta(0) = 0$, $\eta(-t_R) = x$ and $|\dot\eta(s)| \le \delta_R$ for all $s \in (-t_R,\ 0)$, and observe by the dynamic programming principle that for any $t \ge 0$,

$$u(0, t + t_R) \le \int_{-t_R}^{0} L(\eta(s), \dot\eta(s))\, ds + u(x, t) \le C_R t_R + u(x, t).$$

Hence we get $u(x, t) \ge -M_R$ for all $(x, t) \in \mathbb{R}^n \times (0,\ \infty)$, where $M_R := C_R t_R + C_0$.

Next we observe by (1.2) with $\psi = u_0$ that $u(x, t) \le L(x, 0)t + u_0(x)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Since $L(x, 0) = -\min_{p\in\mathbb{R}^n} H(x, p)$ is a continuous function of x because of (A1)–(A2), we see that u is locally bounded on $\mathbb{R}^n \times [0, \infty)$ and hence by [18, Theorem A.1] for instance that u^* is a viscosity subsolution of $u_t + H(x, Du) = 0$ in $\mathbb{R}^n \times (0,\ \infty)$, where u^* is the upper semicontinuous envelope of u. Set $w(x) = \inf_{t>0} u^*(x,t)$ for $x \in \mathbb{R}$, and observe that w is an upper semicontinuous viscosity solution of $H(x, Dw) = 0$ in $\mathbb{R}^n$. Also, since $u^*(x,t) \le L(x,0)t + u_0(x)$ for all $(x,t) \in \mathbb{R}^n \times (0, \infty)$, we have $w(x) \le u_0(x)$ for all $x \in \mathbb{R}^n$. Since H is coercive, it is a standard observation that $w \in C(\mathbb{R}^n)$. Now we see that $u_0^-(x) \ge w(x) > -\infty$ for all $x \in \mathbb{R}$, which is a contradiction. □

Theorem 2.4. *Suppose that $u_0^-(x) > -\infty$ for $x \in \mathbb{R}^n$ and that $u_\infty(x) \equiv \infty$. Then*

$$\lim_{t\to\infty} u(x,t) \equiv \infty.$$

As a consequence of this theorem, in order to have convergence (1.3) for $a = 0$, we need to assume that $u_\infty(x) < \infty$ for all $x \in \mathbb{R}^n$.

Proof. By assumption, we have $u_0^-(x) > -\infty$ and $u_\infty(x) = \infty$ for all $x \in \mathbb{R}^n$. We suppose that $\liminf_{t\to\infty} u(x_0, t) < \infty$ for some $x_0 \in \mathbb{R}^n$, and will obtain a contradiction.

We define the function u^- on $\mathbb{R}^n \times [0,\infty)$ by $u^-(x,t) = T_t u_0^-(x)$. Since $u_0^- \le u_0$ in $\mathbb{R}^n$, we have $u^-(x,t) \le u(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Recall (see e.g. [18, Proposition 2.5]) that for any $\psi \in \mathcal{S}_H^-$,

$$\psi(\eta(0)) - \psi(\eta(-t)) \le \int_{-t}^{0} L(\eta(s), \dot\eta(s))\, ds \quad \text{for all } \eta \in \mathrm{AC}([-t,\, 0]). \tag{2.1}$$

In particular, since $u_0^- \in \mathcal{S}_H^-$, we get

$$u_0^-(x) \le T_t u_0^-(x) = u^-(x,t) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\infty).$$

This together with the dynamic programming principle yields

$$u^-(x, t+s) \ge \inf\Big\{ \int_{-t}^{0} L(\eta(r), \dot\eta(r))\, dr + u_0^-(\eta(-t)) \;\Big|\; \eta \in \mathcal{C}([-t,\, 0];\, x) \Big\} = u^-(x,t)$$

for all $x \in \mathbb{R}^n$ and $t, s \in [0,\infty)$. Thus we see that the function $u^-(x,t)$ is non-decreasing in t for any $x \in \mathbb{R}^n$.

We may assume without any loss of generality that $x_0 = 0$. We choose a constant $C_1 > 0$ so that $\liminf_{t\to\infty} u(0,t) \le C_1$. By the monotonicity of $u^-(0,t)$, we have

$$u^-(0,t) \le C_1 \qquad \text{for all } t \ge 0.$$

Fix any $R > 0$ and $x \in B(0,\, R)$. Let $C_R > 0$ and $\delta_R > 0$ be the constants from Lemma 2.3. Set $t_R = R/\delta_R$. Let $\eta \in \mathcal{C}([-t_R,\, 0];\, x)$ be the line segment joining x to the origin. Noting that $|\dot\eta(s)| \le \delta_R$ for all $s \in [-t_R,\, 0]$, by the dynamic programming principle we get for all $t \in [0,\, \infty)$,

$$u^-(x, t+t_R) \le \int_{-t_R}^{0} L(\eta(s), \dot\eta(s))\, ds + u^-(0,t) \le C_R t_R + u^-(0,t) \le C_R t_R + C_1.$$

Moreover, since $u(x,t) \le L(x,0)t + u_0(x)$ for all $t \ge 0$, by setting

$$K_R := \max_{y \in B(0,R)} (|L(y,0)|t_R + |u_0(y)|) + C_R t_R + C_1,$$

we obtain $u^-(x,t) \le K_R$ for all $(x,t) \in B(0,\, R) \times [0,\infty)$.

We show next that u^- is locally Lipschitz continuous on $\mathbb{R}^n \times [0,\, \infty)$. Indeed, fixing $R > 0$, $x, y \in B(0,\, R)$ with $x \ne y$ and $t \ge 0$, we observe as above by using the monotonicity of $u^-(x,t)$ in t, the dynamic programming principle and Lemma 2.3 that for any $\tau > 0$, if $|x-y| \le \delta_R \tau$, then

$$u^-(y,t) \le u^-(y,t+\tau) \le u^-(x,t) + C_R\tau.$$

Thus we obtain

$$|u^-(y,t) - u^-(x,t)| \le C_R \delta_R^{-1}|x-y| \quad \text{for all } x, y \in B(0,\, R) \text{ and } t \ge 0.$$

Similarly, we get for $x \in B(0,\, R)$ and $t, s \in [0,\, \infty)$,

$$u^-(x,t) \le u^-(x,t+s) \le u^-(x,t) + C_R s,$$

and hence $|u^-(x,t) - u^-(x,s)| \le C_R|t-s|$ for all $x \in B(0,\, R)$ and $t, s \in [0,\, \infty)$. Thus, the function u^- is Lipschitz continuous on $B(0,\, R) \times [0,\, \infty)$. Now, setting $w(x) := \lim_{t\to\infty} u^-(x,t)$ for $x \in \mathbb{R}^n$, we see that w is locally Lipschitz continuous in $\mathbb{R}^n$ and $w \in \mathcal{S}_H$. The monotonicity of the function $u^-(x,t)$ in t guarantees that $u_0^- \le w$ in $\mathbb{R}^n$. Therefore we see that $u_\infty(x) \le w(x) < \infty$ for all $x \in \mathbb{R}^n$, which is a contradiction. □

In view of Theorems 2.1 and 2.4, we henceforth assume, in addition to (A1)–(A5), that (A6) holds. Of course, we now have $u_0^- \in \mathcal{S}_H^-$ and $u_\infty \in \mathcal{S}_H$.

Theorem 2.5. *Let $\phi \in \mathcal{S}_H$ and $z \in \mathbb{R}^n$. Then there exists a curve $\gamma \in \mathcal{C}((-\infty,\, 0];\, z)$ such that for all $t > 0$,*

$$\int_{-t}^{0} L(\gamma(s),\, \dot\gamma(s))\, ds = \phi(z) - \phi(\gamma(-t)). \tag{2.2}$$

We do not give here the proof of the lemma above. Instead, we give two lemmas which may be useful for the proof. Indeed, with the next two lemmas at hand, one can follow, for instance, the proof of [18, Theorem 1.6].

Recall that a curve $\gamma \in \mathcal{C}((-\infty, 0]; z)$ is said to be *extremal* for ϕ at z if it satisfies (2.2) for all $t > 0$. We denote by $\mathcal{E}_z(\phi)$ the set of all extremal curves for ϕ at z. Also, we often use the notation: $\mathcal{E}(\phi) := \bigcup_{x\in\mathbb{R}^n} \mathcal{E}_x(\phi)$ and $\mathcal{E} := \bigcup_{\phi\in\mathcal{S}_H} \mathcal{E}(\phi)$.

Lemma 2.6. *Let* $\phi \in \mathcal{S}_H$. *Then, for all* $(x,t) \in \mathbb{R}^n \times [0,\,\infty)$,

$$\phi(x) = \inf\Big\{\int_{-t}^{0} L(\eta(s), \dot\eta(s))\,ds + \phi(\eta(-t)) \;\Big|\; \eta \in \mathcal{C}([-t,\,0];\,x)\Big\}.$$

Proof. We write $v(x,t)$ for the right hand side of the above equality. We need to show that $v(x,t) = \phi(x)$ for all $(x,t) \in \mathbb{R}^n \times (0,\,\infty)$.

As before, since $\phi \in \mathcal{S}_H^-$, we have

$$\phi(x) - \phi(\eta(-t)) \le \int_{-t}^{0} L(\eta(s), \dot\eta(s))\,ds \quad \text{ for all } (x,t) \in \mathbb{R}^n,\ \eta \in \mathcal{C}([-t,0];\,x),$$

and consequently

$$\phi(x) \le v(x,t) \quad \text{ for all } (x,t) \in \mathbb{R}^n \times [0,\,\infty). \tag{2.3}$$

We recall that

$$v(x,t) \le \phi(x) + L(x,0)t \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,\,\infty),$$

which shows together with (2.3) that v is bounded locally on $\mathbb{R}^n \times [0,\infty)$. Now, we note (see for instance [18]) that v is a viscosity solution of $v_t + H(x,\,Dv) = 0$ in $\mathbb{R}^n \times (0,\infty)$ in the sense that the upper envelope v^* (resp., lower envelope v_*) of v is a viscosity subsolution (resp., supersolution) of $v_t + H(x, Dv) = 0$ in $\mathbb{R}^n \times (0,\,\infty)$. By the above estimates on v we see that v is continuous for $t = 0$.

It reamins to show that $v(x,t) \le \phi(x)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Since $\phi+1$ and v^* are subsolutions of $u_t + H(x, Du) = 0$ in $\mathbb{R}^n \times (0,\infty)$, we see by the convexity of H that the function $v^*(x,t) \wedge (\phi(x)+1)$ is a subsolution of $u_t + H(x, Du) = 0$ in $\mathbb{R}^n \times (0,\infty)$. Hence, by replacing v^* by the function $v^*(x,t) \wedge (\phi(x)+1)$ if necessary, we may assume that $v^*(x,t) \le \phi(x) + 1$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$. By (A4) there is a pair of a $C > 0$ and a function $\psi \in \mathcal{S}_{H-C}^-$ such that $\lim_{|x|\to\infty}(\phi - \psi)(x) = \infty$. Fix any $\varepsilon \in (0,\,1)$ and define the function w on $\mathbb{R}^n \times [0,\,\infty)$ by $w(x,t) := (1-\varepsilon)v^*(x,t) + \varepsilon(\psi(x) - Ct)$. Observe by the convexity of H that w is a viscosity subsolution of $w_t + H(x, Dw) = 0$ in $\mathbb{R}^n \times (0,\,\infty)$. Here, by adding a constant to ψ we may assume that $\psi \le \phi$ on $\mathbb{R}^n$, and note that $w(x,0) \le \phi(x)$ for all $x \in \mathbb{R}^n$ and

$$w(x,t) - \phi(x) \le \varepsilon(\psi(x) - \phi(x) - 1 - Ct) + 1 \quad \text{ for all } (x,t) \in \mathbb{R}^n \times [0,\,\infty),$$

and therefore that there is a constant $R > 0$ such that $w(x,t) \le \phi(x)$ for all $(x,t) \in (\mathbb{R}^n \setminus B(0,\,R)) \times [0,\infty)$. We apply a standard comparison result for viscosity sub- and supersolutions of $u_t + H(x, Du) = 0$ in $\operatorname{Int} B(0,\,R+1) \times [0,\,\infty)$, to obtain $w \le \phi$ in $B(0,\,R+1) \times [0,\infty)$, which assures that $w \le \phi$ in $\mathbb{R}^n \times [0,\,\infty)$. Sending $\varepsilon \to 0$, we get $v^*(x,t) \le \phi(x)$ for all $(x,t) \in \mathbb{R}^n \times [0,\infty)$, from which we conclude that $v(x,t) = \phi(x)$ for all $(x,t) \in \mathbb{R}^n \times [0,\,\infty)$. □

Remark 2.1. As above, if $\psi \in \mathcal{S}_H^-$, then we have by (2.1)

$$\psi(x) \le \inf\Big\{\int_{-t}^{0} L(\eta(s), \dot\eta(s))\, ds + \psi(\eta(-t)) \;\Big|\; \eta \in \mathcal{C}([-t,\, 0];\, x)\Big\} \quad \text{for } x \in \mathbb{R}^n. \quad (2.4)$$

Moreover, if $\phi \in \mathcal{S}_H$ and $\gamma \in \mathcal{E}_z(\phi)$ for some $z \in \mathbb{R}^n$, then the function $t \mapsto (\phi - \psi)(\gamma(-t))$ is non-increasing on $[0,\, \infty)$. Indeed, by (2.4), we have

$$\psi(\gamma(b)) - \psi(\gamma(a)) \le \int_a^b L(\gamma(s), \dot\gamma(s))\, ds \quad \text{for any } a < b \le 0.$$

Then, by the extremality of γ, we deduce that

$$\int_a^b L(\gamma(s), \dot\gamma(s))\, ds = \phi(\gamma(b)) - \phi(\gamma(a)) \quad \text{for any } a < b \le 0.$$

From these, we get $(\phi - \psi)(\gamma(a)) \le (\phi - \psi)(\gamma(b))$ for any $a < b \le 0$.

The following observation is similar to [18, Lemma 6.5].

Lemma 2.7. *Let $\phi \in \mathcal{S}_H$. Then for each $R > 0$ there is an $M > 0$ such that, for any $y \in B(0,\, R)$, if $\eta \in \mathcal{C}([0,1];\, y)$ satisfies*

$$\int_0^1 L(\eta(s), \dot\eta(s))\, ds < \phi(y) - \phi(\eta(0)) + 1, \quad (2.5)$$

then $\eta(t) \in B(0,\, M)$ for all $t \in [0,\, 1]$.

Proof. Fix $R > 0$, and let $y \in B(0,\, R)$ and $\eta \in \mathcal{C}([0,\, 1];\, y)$ satisfy (2.5). Let $C > 0$ and $\psi \in \mathcal{S}_{H-C}^-$ be those from assumption (A4) corresponding to $\phi \in \mathcal{S}_H$. Then, for all $t \in [0,\, 1)$, we have (see e.g. [18, Proposition 2.5])

$$\phi(\eta(t)) - \phi(\eta(0)) \le \int_0^t L(\eta(s), \dot\eta(s))\, ds$$
$$\psi(y) - \psi(\eta(t)) \le \int_t^1 L(\eta(s), \dot\eta(s))\, ds + C.$$

Adding the these two and (2.5), we obtain

$$(\phi - \psi)(\eta(t)) < (\phi - \psi)(y) + 1 + C,$$

which assures that $\eta(t) \in B(0,\, M)$ for all $t \in [0,\, 1]$ and for some $M > 0$ depending only on $R > 0$, ϕ, ψ and C. □

Assumption (A4) is only needed in our arguments to guarantee the existence of extremal curves as observed in the theorem above.

Example 2.2. Indeed, hypotheses (A1)–(A3) are not enough to guarantee existence of extremal curves. To see this, let $n = 1$ and consider the Hamiltonian $H(x, p) := (x^2 + 1)(|p - 1| - 1)$. Observe that its Lagrangian L is given by $L(x, \xi) = \delta_{[-(x^2+1),\, x^2+1]}(\xi) + \xi + x^2 + 1$. Note that $H(x, 0) = 0$ for any $x \in \mathbb{R}$ and $\phi(x) \equiv 0$ is a solution of $H(x, D\phi(x)) = 0$ in $\mathbb{R}$. Fix any $z \in \mathbb{R}$ and suppose that there is an extremal curve $\gamma \in \mathcal{E}_z(\phi)$. By the extremality of γ, we have

$$0 = \phi(z) - \phi(\gamma(-t)) = \int_{-t}^{0} L(\gamma(s),\, \dot\gamma(s))\, ds \quad \text{for all } t > 0.$$

Consequently, we get $\dot\gamma(s) + \gamma(s)^2 + 1 = 0$ for a.e. $s < 0$ since $L(x, \xi) \geq 0$ for all $(x, \xi) \in \mathbb{R}^2$. Integrating this ODE, we get

$$\arctan z - \arctan \gamma(-t) + t = 0 \quad \text{for all } t > 0.$$

That is, we must have

$$\gamma(-t) = \tan(\arctan z + t) \quad \text{for all } t > 0.$$

This function γ, however, is not continuous on $(-\infty, 0]$, which is a contradiction. Thus we conclude that there is no extremal curve for ϕ.

A class of Hamiltonians H which satisfy (A1)–(A4) is given as follows. Let $H_0 \in C(\mathbb{R}^{2n})$ and $f \in C(\mathbb{R}^n)$. Assume that $H_0 \in \mathrm{BUC}(\mathbb{R}^n \times B(0, R))$ for all $R > 0$ and $\lim_{R\to\infty} \inf\{H_0(x, p) \mid x \in \mathbb{R}^n,\, |p| \geq R\} = \infty$. Then it is easily checked that the function $H(x, p) := H_0(x, p) - f(x)$ satisfies (A1)–(A4).

Proposition 2.8. *The function u is continuous on $\mathbb{R}^n \times [0, \infty)$ and satisfies (1.1) in the viscosity sense.*

Proof. As before, we have $u(x, t) \leq L(x, 0)t + u_0(x)$ and $u(x, t) \geq u_0^-(x)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Hence the function u is locally bounded in $\mathbb{R}^n \times [0, \infty)$. It is known (see e.g. [18, Theorems A.1 and A.2]) that u^* (resp., u_*) is a viscosity subsolution (resp., supersolution) of $u_t + H(x, Du) = 0$ in $\mathbb{R}^n \times (0, \infty)$.

It remains to show that u is continuous in $\mathbb{R}^n \times [0, \infty)$, that is, $u^* \leq u_*$ in $\mathbb{R}^n \times [0, \infty)$. We show first that $\lim_{t\to+0} u(x, t) = u_0(x)$ uniformly on compact subsets of $\mathbb{R}^n$. By assumption (A4), we may choose $C > 0$ and $\psi \in \mathcal{S}^-_{H-C}$ so that $\lim_{|x|\to\infty}(u_0^- - \psi)(x) = \infty$. Let $R > 0$ and $\varepsilon \in (0, 1)$. Fix any $(x, t) \in B(0, R) \times (0, 1]$. We choose a $\gamma \in \mathcal{C}([-t, 0]; x)$ so that

$$u(x, t) + \varepsilon > \int_{-t}^{0} L(\gamma(s), \dot\gamma(s))\, ds + u_0(\gamma(-t)), \tag{2.6}$$

and note that

$$\psi(x) - \psi(\gamma(-t)) \leq \int_{-t}^{0} [L(\gamma(s), \dot\gamma(s)) + C]\, ds. \tag{2.7}$$

Combining these two inequalities, we get

$$u(x,t)+\varepsilon > \psi(x)+(u_0-\psi)(\gamma(-t))-Ct,$$

and hence

$$(u_0^- -\psi)(\gamma(-t)) \le (u_0-\psi)(x)+L(x,0)t+Ct+\varepsilon.$$

From this we find a constant $M_R \ge R$, independent of $(x,t)\in B(0,R)\times(0,1]$, such that $|\gamma(-t)|\le M_R$.

We choose a constant $K_R>0$, depending on R and ε, so that $|u_0(x)-u_0(y)|\le \varepsilon+K_R|x-y|$ for all $x,y\in B(0,\,M_R)$. Now we recall (see e.g. [18, Proof of Lemma 6.4]) that there is a constant $C_1\equiv C_1(R,\varepsilon)>0$ such that $L(y,\xi)\ge (K_R+1)|\xi|-C_1$ for all $(x,\xi)\in B(0,M_R)\times\mathbb{R}^n$. Using this, we get

$$\begin{aligned}
&\int_{-t}^{0}((K_R+1)|\dot\gamma(s)|-C_1)\,ds \le \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds\\
&< \varepsilon+L(x,0)t+u_0(x)-u_0(\gamma(-t)) \le 2\varepsilon+L(x,0)t+K_R|x-\gamma(-t)|\\
&\le 2\varepsilon+L(x,0)t+K_R\int_{-t}^{0}|\dot\gamma(s)|\,ds,
\end{aligned}$$

and therefore

$$|x-\gamma(-t)|\le \int_{-t}^{0}|\dot\gamma(s)|\,ds < 2\varepsilon+(C_1+\max_{B(0,R)}|L(\cdot,0)|)t. \tag{2.8}$$

Let ω_R be the modulus of continuity of $u_0-\psi$ on $B(0,M_R)$. We observe by (2.6) and (2.7) that

$$u(x,t) > -\varepsilon+\psi(x)+(u_0-\psi)(\gamma(-t))-Ct \ge -\varepsilon+u_0(x)-\omega_R(|x-\gamma(-t)|)-Ct.$$

From this and (2.8) we see that there is a modulus ν_R such that $u(x,t)\ge u_0(x)-\nu_R(t)$ for all $(x,t)\in B(0,R)\times[0,\,1]$.

Now, as in the proof of Lemma 2.6, we may apply a comparison result for viscosity sub- and supersolutions, to obtain $(u_0^-+A)\wedge u^*\le u_*$ in $\mathbb{R}^n\times[0,\infty)$ for all $A>0$, which shows that $u^*\le u_*$ in $\mathbb{R}^n\times[0,\infty)$. $\square$

Theorem 2.9. *The following two equalities hold: for all $(x,t)\in\mathbb{R}^n\times[0,\infty)$,*

$$(T_t u_0^-)(x)=\inf_{s\ge t} u(x,s) \quad \textit{and} \quad u_\infty(x)=\liminf_{s\to\infty} u(x,s).$$

We can easily adapt the proof of [16, Lemma 4.1] to prove the above theorem.

As noted in the introduction, because of Theorem 2.9, our problem is now reduced to proving the locally uniform convergence (1.5), that is,

$$T_t u_0 \to u_\infty \quad \text{in } C(\mathbb{R}^n) \ \text{ as } t\to\infty.$$

Proposition 2.10. *The pointwise convergence*

$$u(x,t) \to u_\infty(x) \quad \text{as } t \to \infty \tag{2.9}$$

for every $x \in \mathbb{R}^n$ is equivalent to the locally uniform convergence (1.5).

Proof. We need only to show that the pointwise convergence (2.9) yields the locally uniform convergecne (1.5). Since $u_0^- \in \mathcal{S}_H^-$, the function $t \to T_t u_0^-(x)$ is non-decreasing in $[0, \infty)$ and $T_t u_0^-(x) \to u_\infty(x)$ as $t \to \infty$ for all $x \in \mathbb{R}^n$. Therefore, by Dini's lemma, we see that $T_t u_0^- \to u_\infty$ in $C(\mathbb{R}^n)$ as $t \to \infty$. By comparison, we get $T_t u_0^-(x) \le u(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0, \infty)$. These show that $\max\{0, u_\infty(x) - u(x,t)\} \to 0$ locally uniformly in $x \in \mathbb{R}^n$ as $t \to \infty$.

Next we assume that (2.9) holds for every $x \in \mathbb{R}^n$. Fix any $R > 0$ and let $C_R > 0$ and $\delta_R > 0$ be the constants from Lemma 2.3. Observe by the dynamic programming principle that for any $x, y \in B(0,R)$ and $h > 0$, if $|y - x| \le \delta_R h$, then

$$u(y, t+h) \le C_R h + u(x,t).$$

From this and (2.9), we get

$$\limsup_{t\to\infty} u(y,t) \le u_\infty(x) + C_R h$$

for all $x, y \in B(0,R)$, $h > 0$ such that $|y - x| \le \delta_R h$. Hence, by the continuity of u_∞ and a standard compactness arguemnt, we see that $\max\{0, u(x,t) - u_\infty(x)\} \to 0$ uniformly for $x \in B(0,R)$ as $t \to \infty$. We thus conclude that $u(\cdot,t) \to u_\infty$ in $C(\mathbb{R}^n)$ as $t \to \infty$. □

It is well-known (e.g. [7, 14, 18, 17]) under a little more restrictive hypotheses that u_0^- can be represented as

$$u_0^-(x) = \inf\{u_0(y) + d_H(x,y) \mid y \in \mathbb{R}^n\} \quad \text{for } x \in \mathbb{R}^n, \tag{2.10}$$

where d_H is defined by

$$d_H(x,y) := \sup\{\phi(x) - \phi(y) \,|\, \phi \in \mathcal{S}_H^-\}.$$

Indeed, the formula (2.10) is still valid in our setting. To see this, we observe that

$$v(x) := \inf\{u_0(y) + d_H(x,y) \mid y \in \mathbb{R}^n\} \le u_0(x) + d_H(x,x) = u_0(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Also, since $d_H(\cdot, y)$ is the maximum subsolution of $H(x, Du) = 0$ in $\mathbb{R}^n$ among those vanishing at y, we have

$$u_0^-(x) - u_0(y) \le u_0^-(x) - u_0^-(y) \le d_H(x,y) \quad \text{for all } x, y \in \mathbb{R}^n,$$

from which we get $u_0^-(x) \le v(x)$ for all $x \in \mathbb{R}^n$. We thus conclude that $u_0^- = v$.

Note (see e.g. [18, Proposition 8.2]) that d_H can be written as

$$d_H(x,y) = \inf\Big\{ \int_{-t}^{0} L(\eta(s),\dot\eta(s))\,ds \mid t>0,\ \eta\in\mathcal{C}([-t,0];x),\ \eta(-t)=y \Big\}. \quad (2.11)$$

Now, we introduce the Aubry set $\mathcal{A}_H$ (see e.g. [13]) as the subset of $\mathbb{R}^n$ defined by

$$\mathcal{A}_H = \{y\in\mathbb{R}^n \mid d_H(\cdot,y)\in\mathcal{S}_H\}.$$

A remark here is that $d_H(\cdot,y) \in \mathcal{S}_H(\mathbb{R}^n\setminus\{y\})$ for all $y\in\mathbb{R}^n$, i.e, $d_H(\cdot,y)$ is a viscosity solution of $H(x,Du)=0$ in $\mathbb{R}^n\setminus\{y\}$. Recall (see e.g. [18, 13]) that another characterization for $y\in\mathbb{R}^n$ to be in $\mathcal{A}_H$ is given by the condition

$$\inf\Big\{\int_{-t}^{0} L(\eta(s),\dot\eta(s))\,ds \ \Big|\ t\ge 1,\ \eta\in \mathrm{AC}([-t,\,0]),\ \eta(0)=\eta(-t)=y\Big\}=0. \quad (2.12)$$

In particular, if $L(y,0)\le 0$ for some y, then $y\in\mathcal{A}_H$. Such a point y is called an *equilibrium.* Note here that $L(x,0)\ge 0$ for all $x\in\mathbb{R}^n$ since (A6) is in effect and accordingly $\min_{p\in\mathbb{R}^n} H(x,p)\le 0$ for all $x\in\mathbb{R}^n$. Thus we have: y is an equilibrium if and only if $L(y,0)=0$.

Theorem 2.11. (a) *Let $\gamma\in\mathcal{E}(u_\infty)$. Then $\lim_{t\to\infty}(u_\infty-u_0^-)(\gamma(-t))=0$.* (b) *Let $\phi\in\mathcal{S}_H$ and $\gamma\in\mathcal{E}(\phi)$. Then every ω-limit point of the orbit $\gamma(-t)$, $t\ge 0$, is a point of $\mathcal{A}_H$, that is, if $\{t_j\}\subset(0,\infty)$ is an increasing sequence such that $\lim_{j\to\infty}t_j=\infty$ and $\lim_{j\to\infty}\gamma(-t_j)=y$ for some $y\in\mathbb{R}^n$, then $y\in\mathcal{A}_H$.*

Proof. To see that (a) is valid, we set $\gamma(0)=z$ and observe that

$$T_t u_0^-(z) \le \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds + u_0^-(\gamma(-t)) = u_\infty(z) - u_\infty(\gamma(-t)) + u_0^-(\gamma(-t)).$$

Since $\lim_{t\to\infty} T_t u_0^-(z) = u_\infty(z)$ by Theorem 2.9, from the above inequality, we get $\limsup_{t\to\infty}(u_\infty-u_0^-)(\gamma(-t))\le 0$. But we have $u_\infty\ge u_0^-$ in $\mathbb{R}^n$ and hence $\lim_{t\to\infty}(u_\infty - u_0^-)(\gamma(-t))=0$.

We next show that (b) is valid. Let $y\in\mathbb{R}^n$ be a limit point of a sequence $\{\gamma(-t_j)\}$, where $\{t_j\}\subset(0,\infty)$ is an increasing sequence such that $t_j\to\infty$ and $\gamma(-t_j)\to y$ as $j\to\infty$. Let $\delta>0$ and $C>0$ be, respectively, the constants δ_R and C_R from Lemma 2.3 for $R=|y|+1$. Fix $\varepsilon>0$ and choose $j,k\in\mathbb{N}$ so that $t_j+1<t_k$ and $|\gamma(-t_j)-y|+|\gamma(-t_k)-y|<\min\{\varepsilon\delta/C,\,1\}$ and $|\phi(\gamma(-t_j))-\phi(\gamma(-t_k))|<\varepsilon$. Setting $y_i=\gamma(-t_i)$ and $\tau_i=|y-y_i|/\delta$ for $i=j,k$, $\xi_j=\delta(y_j-y)/|y_j-y|$ if $y_j\ne y$, $\xi_k=\delta(y-y_k)/|y-y_k|$ if $y_k\ne y$, and $r=t_k-t_j$, we define the curve η by

$$\eta(s)=\begin{cases} y-s\xi_j & \text{for } s\in[-\tau_j,\,0],\\ \gamma(s+\tau_j-t_j) & \text{for } s\in[-(\tau_j+r),\,-\tau_j],\\ y_k-(s+\tau_j+r)\xi_k & \text{for } s\in[-(\tau_j+r+\tau_k),\,-(\tau_j+r)]. \end{cases}$$

We observe that $\eta(0) = \eta(-(\tau_j + r + \tau_k)) = y$, $\tau_j + r + \tau_k > 1$ and

$$\begin{aligned}
&\int_{-(\tau_j+r+\tau_k)}^{0} L(\eta(s), \dot\eta(s))\, ds \\
&= \int_{-\tau_j}^{0} L(y - s\xi_j, \xi_j)\, ds + \int_{-t_k}^{-t_j} L(\gamma(s), \dot\gamma(s))\, ds + \int_{-\tau_k}^{0} L(y_k - s\xi_k, \xi_k)\, ds \\
&\le C(\tau_j + \tau_k) + \phi(y_j) - \phi(y_k) < 2\varepsilon,
\end{aligned}$$

and conclude that $y \in \mathcal{A}_H$. □

We recall the following proposition.

Lemma 2.12. *Let $U \subset \mathbb{R}^n$ be an open set. Then, $\mathcal{A}_H \cap U = \emptyset$ if and only if there exist function $f \in C(\mathbb{R}^n)$ and $\psi \in \mathcal{S}^-_{H+f}$ such that $f \ge 0$ in $\mathbb{R}^n$ and $f > 0$ in U.*

Instead of giving a proof of the lemma above, let us recall [18, Lemmas 8.4, 8.5] which states that the same equivalence as above holds for any compact subsets U of $\mathbb{R}^n$. From this, we see immediately that the "if" part is valid. The proof of the "only if" part is as follows. We choose a sequence $\{K_j\}$ of compact subsets of U so that $\bigcup_j K_J = U$. Fix a $\psi_0 \in \mathcal{S}^-_H$ and then select a sequence $\{\psi_j\} \subset \mathcal{S}^-_H$ in view of the proof of [18, Lemma 8.4] so that $H(x, D\psi_j) \le -\delta_j$ in a neighborhood of K_j in the viscosity sense for any $j \in \mathbb{N}$, where $\delta_j > 0$ is a constant, and $\psi_0 \le \psi_j \le \psi_0 + 1$ in $\mathbb{R}^n$ for all $j \in \mathbb{N}$. We next observe that $\psi := \sum_{j\in\mathbb{N}} \psi_j/2^j$ converges in $C(\mathbb{R}^n)$ and that ψ is a subsolution of $H(x, D\psi) \le -f(x)$ in $\mathbb{R}^n$ in the viscosity sense for some function $f \in C(\mathbb{R}^n)$ satisfying $f \ge 0$ in $\mathbb{R}^n$ and $f > 0$ in U. See also Fathi-Siconolfi [12, Proposition 6.1].

3 Representation formula.

We assume throughout this section that (A1)-(A4) are satisfied and that $\mathcal{S}^-_H \ne \emptyset$, and we establish a representation formula for $v \in \mathcal{S}_H$. We do not assume here that (A6) holds, but remark that (A6) implies that $\mathcal{S}^-_H \ne \emptyset$. Our formula is similar to representation formulas for $v \in \mathcal{S}_H(\Omega)$ obtained in [19, Theorems 5.1 and 5.3] for general domain $\Omega \subset \mathbb{R}^n$, which extend the classical representation formula [13, Theorem 6.7] for $v \in \mathcal{S}_H(\Omega)$, in the case where Ω is an n dimensional torus, of the form

$$v(x) = \inf\{d_H(x, y) + v(y) \mid y \in \mathcal{A}_H\},$$

where d_H and $\mathcal{A}_H$ are defined in the same way as in the previous section.

Fix any $v \in \mathcal{S}_H$. Let $\gamma \in \mathcal{E}(v)$. Note that

$$\int_{-r}^{-t} L(\gamma(s), \dot\gamma(s))\, ds = v(\gamma(-t)) - v(\gamma(-r)) \le \int_{-r}^{-t} L(\eta(s), \dot\eta(s))\, ds$$

for $0 \le t < r$ and any $\eta \in \mathrm{AC}([-r,\,-t])$ satisfying $\eta(s) = \gamma(s)$ at $s = -t, -r$. Hence we have

$$v(\gamma(-t)) - v(\gamma(-r)) = \inf \int_{-r}^{-t} L(\eta(s), \dot{\eta}(s))\, ds,$$

where the infimum is taken over all $\eta \in \mathrm{AC}([-r,\,-t])$, with $\eta(s) = \gamma(s)$ at $s = -t, -r$. This together with (2.11) assures that

$$v(\gamma(-t)) = v(\gamma(-r)) + d_H(\gamma(-t),\, \gamma(-r)) \quad \text{for} \;\; 0 \le t \le r. \tag{3.1}$$

Lemma 3.1. *Let $v \in \mathcal{S}_H$ and $\gamma \in \mathcal{E}(v)$. There is a function $\phi \in \mathcal{S}_H$ such that as $t \to \infty$,*

$$v(\gamma(-t)) + d_H(\cdot,\, \gamma(-t)) \longrightarrow \phi \quad \text{in } C(\mathbb{R}^n).$$

Moreover ϕ has the properties: (a) $\phi = v$ on $\gamma((-\infty,\, 0])$ and (b) $\phi \ge v$ in $\mathbb{R}^n$.

The above lemma is similar to [19, Lemma 5.2].

Proof. Set $\psi_t(x) = v(\gamma(-t)) + d_H(x, \gamma(-t))$ for $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$. We have $v(x) \le v(\gamma(-t)) + d_H(x, \gamma(-t))$ for any $(x,t) \in \mathbb{R}^n \times [0,\, \infty)$ and hence $v \le \psi_t$ in $\mathbb{R}^n$ for any $t \ge 0$. Next let $0 \le s \le t$. By (3.1) we see that $v(\gamma(-s)) = \psi_t(\gamma(-s))$. That is, we have $\psi_t = v$ on the set $\gamma([-t,\, 0])$. Since $\psi_t \in \mathcal{S}_H^-$, we get for $x \in \mathbb{R}^n$,

$$\psi_t(x) - \psi_t(\gamma(-s)) \le d_H(x, \gamma(-s)),$$

and hence

$$\psi_t(x) \le \psi_t(\gamma(-s)) + d_H(x, \gamma(-s)) = v(\gamma(-s)) + d_H(x, \gamma(-s)) = \psi_s(x).$$

Now, noting that $\{\psi_t\}_{t \ge 0}$ is locally equi-Lipschitz continuous and locally uniformly bounded in $\mathbb{R}^n$, we conclude by the monotonicity of the sequence $\{\psi_t\}$ that ψ_t converges to a function ϕ in $C(\mathbb{R}^n)$ as $t \to \infty$. It is clear that $v = \phi$ on $\gamma((-\infty,\, 0])$ and $v \le \phi$ on $\mathbb{R}^n$. To check that $\phi \in \mathcal{S}_H$, we consider first the case where $\sup_{t \ge 0} |\gamma(-t)| = \infty$. We choose a sequence $\{t_j\} \subset (0, \infty)$ so that $|\gamma(-t_j)| \to \infty$ (and hence $t_j \to \infty$) as $j \to \infty$. Since $\psi_t \in \mathcal{S}_H(\mathbb{R}^n \setminus \{\gamma(-t)\})$, we see by the stability of the viscosity property that $\phi \in \mathcal{S}_H$. Next assume that $\sup_{t \ge 0} |\gamma(-t)| < \infty$. We may choose a sequence $\{t_j\} \subset (0,\, \infty)$ diverging to infinity such that $\gamma(-t_j) \to y$ as $j \to \infty$ for some $y \in \mathbb{R}^n$, and we have

$$v(\gamma(-t_j)) + d_H(\cdot, \gamma(-t_j)) \longrightarrow v(y) + d_H(\cdot, y) \quad \text{in} \;\; C(\mathbb{R}^n) \;\; \text{as} \;\; j \to \infty,$$

which shows that $\phi = v(y) + d_H(\cdot, y)$. Now, since $y \in \mathcal{A}_H$ due to Theorem 2.11 (b), we conclude that $\phi \in \mathcal{S}_H$. □

Similarly to [19], we write $g(v,\gamma)$ for the function $\phi \in \mathcal{S}_H$ given by the lemma above. That is,

$$g(v,\gamma)(x) = \lim_{t\to\infty}[v(\gamma(-t)) + d_H(x,\gamma(-t))] \quad \text{for } x \in \mathbb{R}^n.$$

We remark in view of Theorem 2.11 (b) that, in the above definition, we have either $\lim_{t\to\infty} |\gamma(-t)| = \infty$ or $g(v,\gamma) = v(y) + d_H(\cdot, y)$ for some $y \in \mathcal{A}_H$.

Theorem 3.2. *Let $v \in \mathcal{S}_H$. Then*

$$v(x) = \inf\{g(v,\gamma)(x) \mid \gamma \in \mathcal{E}(v)\} \quad \textit{for all} \;\; x \in \mathbb{R}^n. \tag{3.2}$$

Proof. We write $w(x)$ for the right hand side of the above identity. Let $z \in \mathbb{R}^n$. In view of Theorem 2.5, there exists a $\gamma \in \mathcal{E}_z(v)$, and, by the definition of $g(v,\gamma)$, we have $v = g(v,\gamma)$ on $\gamma((-\infty, 0])$. In particular, $v(z) = g(v,\gamma)(z) \ge w(z)$. Hence, $w \le v$ in $\mathbb{R}^n$. On the other hand, since $v \le g(v,\gamma)$ in $\mathbb{R}^n$ for any $\gamma \in \mathcal{E}(v)$ by Lemma 3.1, we have $v \le w$ in $\mathbb{R}^n$. Thus we see that $v = w$ in $\mathbb{R}^n$. □

Note that the function $\phi := g(v,\gamma)$, with $v \in \mathcal{S}_H$ and $\gamma \in \mathcal{E}(v)$, has the property that $\phi(y_j) + d_H(\cdot, y_j) \to \phi$ in $C(\mathbb{R}^n)$ as $j \to \infty$ for some sequence $\{y_j\} \subset \mathbb{R}^n$ satisfying either $\lim_{j\to\infty} |y_j| = \infty$ or $\lim_{j\to\infty} y_j = y$ for some $y_0 \in \mathcal{A}_H$. (Take $y_j = \gamma(-t_j)$ with an appropriate sequence $t_j \to \infty$.) We denote the set of the functions $\phi \in \mathcal{S}_H$ having this property by Δ^*, which is similar to the union of Δ_0^* and $\mathcal{A}_H$ in [19].

The above formula (3.2) is more precise than that in [19, Theomre 5.4] due to, roughly speaking, the fact that the choice of the "ideal boundary"

$$\{g(v,\gamma) \mid \gamma \in \mathcal{E}(v)\} \setminus \{v(y) + d_H(\cdot, y) \mid y \in \mathcal{A}_H\}$$

of $\mathbb{R}^n \setminus \mathcal{A}_H$ here depends on v.

We define the functions $g^\pm(v,\gamma)$ on $\mathbb{R}^n$ for $v \in C(\mathbb{R}^n)$ and $\gamma \in \mathcal{E}$ by

$$\begin{aligned} g^+(v,\gamma)(x) &= \limsup_{t\to\infty}[v(\gamma(-t)) + d_H(x,\gamma(-t))], \\ g^-(v,\gamma)(x) &= \liminf_{t\to\infty}[v(\gamma(-t)) + d_H(x,\gamma(-t))]. \end{aligned}$$

We write $g(v,\gamma)$ for $g^+(v,\gamma)$ if $g^+(v,\gamma) = g^-(v,\gamma) \in C(\mathbb{R}^n)$. Of course, we have $-\infty \le g^-(v,\gamma)(x) \le g^+(v,\gamma)(x) \le \infty$ for all $x \in \mathbb{R}^n$. We denote by $\mathcal{B}_H$ the set of all those $\phi \in \mathcal{S}_H$ for which there corresponds a sequence $\{(y_j, c_j)\} \subset \mathbb{R}^{n+1}$ such that $c_j + d_H(\cdot, y_j) \to \phi$ in $C(\mathbb{R}^n)$ as $j \to \infty$.

Proposition 3.3. *Let $(v,\gamma) \in C(\mathbb{R}^n)\times\mathcal{E}$. (a) If $g^+(v,\gamma)(x_0) \in \mathbb{R}$ (resp., $g^-(v,\gamma)(x_0) \in \mathbb{R}$) for some $x_0 \in \mathbb{R}^n$, then $g^+(v,\gamma) \in \mathcal{B}_H$ (resp., $g^-(v,\gamma) \in \mathcal{B}_H$). (b) If $v \in \mathcal{S}_H^-$, then $v \le g^-(v,\gamma)$ in $\mathbb{R}^n$.*

Proof. We begin with assertion (a). Assume that $g^+(v,\gamma)(x_0) \in \mathbb{R}$ for some $x_0 \in \mathbb{R}^n$. Choose a $\phi \in \mathcal{S}_H$ so that $\gamma \in \mathcal{E}(\phi)$. We know that for some $\psi \in \mathcal{S}_H$,

$$\phi(\gamma(-t)) + d_H(\cdot, \gamma(-t)) \to \psi \quad \text{in } C(\mathbb{R}^n) \ \text{ as } t \to \infty.$$

Hence, we get

$$\begin{aligned} g^+(v,\gamma)(x_0) - \psi(x_0) &= \limsup_{t\to\infty}[v(\gamma(-t)) - \phi(\gamma(-t))] \\ &= g^+(v,\gamma)(x) - \psi(x) \quad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

That is, we have $g^+(v,\gamma)(x) = \psi(x) + g^+(v,\gamma)(x_0) - \psi(x_0)$ for all $x \in \mathbb{R}^n$ and therefore $g^+(v,\gamma) \in \mathcal{B}_H$. An argument parallel to the above shows that if $g^-(v,\gamma)(x_0) \in \mathbb{R}$ for some $x_0 \in \mathbb{R}^n$, then $g^-(v,\gamma) \in \mathcal{B}_H$.

Next we turn to assertion (b). Assume that $v \in \mathcal{S}_H^-$. Recalling that $v(x) - v(\gamma(-t)) \le d_H(x,\gamma(-t))$ for all $x \in \mathbb{R}^n$, $t > 0$, we see that $v(x) \le g^-(v,\gamma)(x)$ for all $x \in \mathbb{R}^n$. The proof is complete. □

The following proposition is an obvious consequence of Proposition 3.3 and Theorem 3.2.

Corollary 3.4. *(a) If $\phi \in \mathcal{S}_H^-$, then*

$$\phi(x) \le \inf\{g^-(\phi,\gamma)(x) \mid \gamma \in \mathcal{E}\} \quad \textit{for all } x \in \mathbb{R}^n.$$

(b) If $\phi \in \mathcal{S}_H$, then

$$\phi(x) = \inf\{g^-(\phi,\gamma)(x) \mid \gamma \in \mathcal{E}\} = \inf\{g^+(\phi,\gamma)(x) \mid \gamma \in \mathcal{E}\} \quad \textit{for all } x \in \mathbb{R}^n.$$

4 General criteria for pointwise convergence.

Throughout this section we assume that H and u_0 satisfy (A1)-(A6), and we will later assume additionally either $(A7)_+$ or $(A7)_-$.

As Proposition 2.10 states, in order to show the locally uniform convergence (1.5), we need only to show the pointwise convergence of $T_t u_0(x)$ of $u_\infty(x)$ as $t \to \infty$ for every $x \in \mathbb{R}^n$. Let $z \in \mathbb{R}^n$. In this section we seek for criteria for the pointwise convergence

$$u(z,\, t) := (T_t u_0)(z) \to u_\infty(z) \qquad \text{as } t \to \infty. \tag{4.1}$$

We fix any $\gamma \in \mathcal{E}_z(u_\infty)$, and we introduce the first criterion

(C1) $\lim_{t\to\infty} (u_0 - u_\infty)(\gamma(-t)) = 0.$

Note that $u_0^- \le u_0$ in $\mathbb{R}^n$ and $\lim_{t\to\infty}(u_\infty - u_0^-)(\gamma(-t)) = 0$ by Theorem 2.11 (a). Hence condition (C1) is equivalent to the condition

$$\lim_{t\to\infty} (u_0 - u_0^-)(\gamma(-t)) = 0.$$

Theorem 4.1. *Under condition (C1), the convergence (4.1) holds.*

Proof. By the variational formula (1.2) with $\psi = u_0$ and the definition of extremal curves, we see that

$$\begin{aligned} u(z,t) &\le \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\,ds + u_0(\gamma(-t)) \\ &= u_\infty(z) - u_\infty(\gamma(-t)) + u_0(\gamma(-t)) \qquad \text{for all } \ t>0. \end{aligned}$$

From this together with (C1) and Theorem 2.9, we get

$$\limsup_{t\to\infty} u(z,t) \le u_\infty(z) + \lim_{t\to\infty}(u_0-u_\infty)(\gamma(-t)) = u_\infty(z) = \liminf_{t\to\infty} u(z,t),$$

which implies (4.1). □

Next we introduce our second criterion.

(C2) For each $\varepsilon>0$ there exists a $\tau>0$ such that for any $t>0$ and for some $\eta\in \mathrm{AC}([-t,\,0])$,

$$\eta(-t)=\eta(0)=\gamma(-\tau) \quad \text{ and } \quad \int_{-t}^{0} L(\eta(s),\,\dot\eta(s))\,ds<\varepsilon.$$

Theorem 4.2. *Under condition (C2), the convergence (4.1) holds.*

Proof. Fix any $\varepsilon>0$ and let $\tau>0$ be the constant from assumption (C2). Set $y=\gamma(-\tau)$ and choose a $\sigma>0$ in view of Theorem 2.9 so that $u(y,\,\sigma)<u_\infty(y)+\varepsilon$. Fix any $t>0$. By (C2), we may choose an $\eta\in\mathrm{AC}([-t,\,0])$ such that $\eta(-t)=\eta(0)=y$ and

$$\int_{-t}^{0} L(\eta(s),\,\dot\eta(s))\,ds<\varepsilon.$$

Now, using the dynamic programming principle (Lemma 2.2), we compute that

$$\begin{aligned} u(z,\,\tau+\sigma+t) &\le \int_{-\tau}^{0} L(\gamma(s),\,\dot\gamma(s))\,ds + u(\gamma(-\tau),\,t+\sigma) \\ &\le u_\infty(z)-u_\infty(y)+\int_{-t}^{0} L(\eta(s),\,\dot\eta(s))\,ds + u(\eta(-t),\,\sigma) \\ &< u_\infty(z)-u_\infty(y)+\varepsilon+u(y,\,\sigma) \\ &< u_\infty(z)-u_\infty(y)+u_\infty(y)+2\varepsilon = u_\infty(z)+2\varepsilon. \end{aligned}$$

Consequently we obtain

$$\limsup_{t\to\infty} u(z,\,t) \le u_\infty(z) = \liminf_{t\to\infty} u(z,\,t),$$

which concludes the proof. □

Our third criterion is the following.

(C3) For any $\varepsilon > 0$, there exists a $\tau > 0$ and for each $t \geq \tau$, a $\sigma(t) \in [0, \tau]$ such that

$$u_\infty(\gamma(-t)) + \varepsilon > u(\gamma(-t), \sigma(t)).$$

Note that the above inequality is equivalent to the condition that there is an $\eta \in \mathcal{C}([-\sigma(t), 0]; \gamma(-t))$ such that

$$u_\infty(\gamma(-t)) + \varepsilon > \int_{-\sigma(t)}^{0} L(\eta(s), \dot\eta(s))\, ds + u_0(\eta(-\sigma(t))).$$

In our next theorem, condition (C3) is used together with one of the conditions $(A7)_\pm$ on H, which are certain strict convexity requirements on H. We set $Q := \{(x,p) \in \mathbb{R}^{2n} \mid H(x,p) = 0\}$ and

$$S := \{(x,\xi) \in \mathbb{R}^{2n} \mid (x,p) \in Q, \quad \xi \in D_2^- H(x,p) \ \text{ for some } p \in \mathbb{R}^n\},$$

where $D_2^- H(x,p)$ stands for the subdifferential of H with respect to the p-variable.

$\mathbf{(A7)_+}$ (resp., $\mathbf{(A7)_-}$) There exists a modulus ω satisfying $\omega(r) > 0$ for $r > 0$ such that for all $(x,p) \in Q,\ \xi \in D_2^- H(x,p)$ and $q \in \mathbb{R}^n$,

$$H(x, p+q) \geq \xi \cdot q + \omega((\xi \cdot q)_+) \qquad (\text{resp.,} \quad \geq \xi \cdot q + \omega((\xi \cdot q)_-)),$$

where $r_\pm := \max\{\pm r, 0\}$ for $r \in \mathbb{R}$.

Roughly speaking, $(A7)_+$ (resp., $(A7)_-$) means that $H(x, \cdot)$ is strictly convex "upward" (resp., "downward") at the zero-level set of H uniformly in $x \in \mathbb{R}^n$. We remark here that condition $(A7)_+$ has already been used in [4, 16] to replace the strict convexity of $H(x, \cdot)$ in order to get the convergence (4.1) and also that condition $(A7)_-$ has been discussed in [17] when $n = 1$.

Theorem 4.3. *Assume that (C3) and either $(A7)_+$ or $(A7)_-$ are satisfied. Then (4.1) holds.*

Lemma 4.4. *Assume that H satisfies $(A7)_+$ (resp., $(A7)_-$). Then, there exist a constant $\delta_1 > 0$ and a modulus ω_1 such that for any $\varepsilon \in [0, \delta_1]$ (resp., $\varepsilon \in [-\delta_1, 0]$) and $(x,\xi) \in S$,*

$$L(x, (1+\varepsilon)\xi) \leq (1+\varepsilon) L(x,\xi) + |\varepsilon| \omega_1(|\varepsilon|). \tag{4.2}$$

The estimate of this type was proved first by [7] when $H(x, \cdot)$ is strictly convex.

Proof. The proof of (4.2) under $(A7)_+$ is exactly the same as that of [16, Lemma 3.2]. Moreover, by a careful review of its proof, one sees that (4.2) is also valid under $(A7)_-$. □

Proof of Theorem 4.3. We need only to prove that $\limsup_{t\to\infty} u(z,t) \le u_\infty(z)$. Fix any $\varepsilon > 0$ and choose a $\tau > 0$ and a $\sigma(t) \in [0,\,\tau]$ for each $t > \tau$ as in (C3).

We first consider the case when $(A7)_+$ holds. Let δ_1 and ω_1 be those from Lemma 4.4. Fix $t > \tau$, and set $\theta = \sigma(t)$ and $\delta = \theta/(t-\theta)$. Note that $(1+\delta)(t-\theta) = t$ and $\lim_{t\to\infty} \delta = 0$. We assume that t is sufficiently large, so that $\delta \in [0,\,\delta_1]$. Define $\eta \in \mathcal{C}((-\infty,\,0];\,z)$ by $\eta(s) = \gamma((1+\delta)s)$ and observe that $\eta(-t+\theta) = \gamma(-t)$ and by Lemma 4.4 that for a.e. $s \le 0$,

$$L(\eta(s), \dot\eta(s)) \le (1+\delta)L(\gamma((1+\delta)s), \dot\gamma((1+\delta)s)) + \delta\omega_1(\delta).$$

Hence we get

$$\int_{-t+\theta}^0 L(\eta(s), \dot\eta(s))\, ds \le \int_{-t}^0 L(\gamma(s), \dot\gamma(s))\, ds + (t-\theta)\delta\omega_1(\delta),$$

and furthermore

$$\begin{aligned} u(z,t) &\le \int_{-t+\theta}^0 L(\eta(s), \dot\eta(s))\, ds + u(\eta(-t+\theta), \theta) \\ &\le \int_{-t}^0 L(\gamma(s), \dot\gamma(s))\, ds + \theta\omega_1(\delta) + u(\gamma(-t), \sigma(t)) \\ &< u_\infty(z) - u_\infty(\gamma(-t)) + u_\infty(\gamma(-t)) + \varepsilon + \theta\omega_1(\delta) = u_\infty(z) + \varepsilon + \theta\omega_1(\delta). \end{aligned}$$

From this we get $\limsup_{t\to\infty} u(z,t) \le u_\infty(z) + \varepsilon$.

We next consider the case when $(A7)_-$ holds. As before, let δ_1 and ω_1 be those from Lemma 4.4. Let $t > 2\tau$, and set $\theta = \sigma(t-\tau)$ and $\delta = (\tau-\theta)/(t-\theta)$, so that $(1-\delta)(t-\theta) = t-\tau$ and $\lim_{t\to\infty} \delta = 0$. Assume that t is sufficiently large, so that $\delta \in [0,\,\delta_1]$. Define the curve $\eta \in \mathcal{C}((-\infty,\,0];\,z)$ by $\eta(s) = \gamma((1-\delta)s)$ and observe that $\eta(-t+\theta) = \gamma(-t+\tau)$. As above, using Lemma 4.4, we get

$$\int_{-t+\theta}^0 L(\eta(s), \dot\eta(s))\, ds \le \int_{-t+\tau}^0 L(\gamma(s), \dot\gamma(s))\, ds + (t-\theta)\delta\omega_1(\delta).$$

We then compute that

$$\begin{aligned} u(z,t) &\le \int_{-t+\theta}^0 L(\eta(s), \dot\eta(s))\, ds + u(\eta(-t+\theta), \theta) \\ &\le \int_{-t+\tau}^0 L(\gamma(s), \dot\gamma(s))\, ds + u(\gamma(-t+\tau), \sigma(t-\tau)) \\ &< u_\infty(z) - u_\infty(\gamma(-t+\tau)) + (\tau-\theta)\omega_1(\delta) + u_\infty(\gamma(-t+\tau)) + \varepsilon \\ &= u_\infty(z) + \varepsilon + (\tau-\theta)\omega_1(\delta), \end{aligned}$$

and obtain $\limsup_{t\to\infty} u(z,t) \le u_\infty(z) + \varepsilon$. Now, since $\varepsilon > 0$ is arbitrary, we see that $\lim_{t\to\infty} u(z,t) \le u_\infty(z)$. □

5 Results based on (C1) and examples.

The following result is an easy consequence of Theorem 4.1.

Theorem 5.1. *Assume that (A1)–(A6) hold. Let* $\psi_0 \in \mathcal{S}_H^-$ *and assume that* $u_0 \geq \psi_0$ *in* $\mathbb{R}^n$ *and*

$$\lim_{|x|\to\infty} (u_0 - \psi_0)(x) = 0.$$

Moreover assume that $\mathcal{A}_H = \emptyset$*. Then the convergence (1.5) holds.*

Proof. By the assumption on ψ_0, we see that $\psi_0 \leq u_0^- \leq u_0$ in $\mathbb{R}^n$ and moreover $\lim_{|x|\to\infty}(u_0 - u_0^-)(x) = 0$. Since $\mathcal{A}_H = \emptyset$, by Theorem 2.11 (b), we have $\lim_{t\to\infty} |\gamma(-t)| = \infty$ for all $\gamma \in \mathcal{E}$. It is now clear that (C1) is valid for all $\gamma \in \mathcal{E}(u_\infty)$. Hence, we see from Theorem 4.1 that convergence (1.5) holds. □

A variation of the above theorem is the following.

Theorem 5.2. *Assume that (A1)–(A6) hold. Assume that* $u_0 \geq \psi_0$ *in* $\mathbb{R}^n$ *for some* $\psi_0 \in \mathcal{S}_H^-$*. Assume moreover that for any* $\phi \in \mathcal{S}_H$ *such that* $\phi \geq \psi_0$ *in* $\mathbb{R}^n$*,*

$$\lim_{t\to\infty} (u_0 - \psi_0)(\gamma(-t)) = 0 \quad \textit{for all } \gamma \in \mathcal{E}(\phi).$$

Then the convergence (1.5) holds.

Proof. As above, we have $\psi_0 \leq u_0^- \leq u_0$ and $\psi_0 \leq u_\infty$ in $\mathbb{R}^n$. Hence, by assumption, we get $\lim_{t\to\infty}(u_0 - u_0^-)(\gamma(-t)) = 0$ for all $\gamma \in \mathcal{E}(u_\infty)$. That is, (C1) is valid for all $\gamma \in \mathcal{E}(u_\infty)$. Thus, the convergence (1.5) holds. □

We next generalize [3, Theorem 4.2], a result due to Barles-Roquejoffre, in light of (C1).

Theorem 5.3. *Assume that (A1)–(A5) hold. Assume in addition that there exist functions* $\phi \in \mathcal{S}_H$ *and* $\psi \in \mathcal{S}_{H+\delta}^-$*, with* $\delta > 0$*, such that* $\inf_{\mathbb{R}^n}(u_0 - \phi \wedge \psi) > -\infty$ *and*

$$\lim_{r\to\infty} \sup\{|(u_0 - \phi)(x)| \mid (\psi - \phi)(x) > r\} = 0. \tag{5.1}$$

Then the convergence (1.5) holds, and moreover, $u_\infty = \phi$ *on* $\mathbb{R}^n$*.*

The same convergence assertion as above has been established in [3, Theorem 4.2] under the assumption that

$$\lim_{|x|\to\infty} (u_0 - \phi)(x) = 0, \tag{5.2}$$

which is a stronger requirement than (5.1).

Proof. We intend to apply Theorem 5.2 to prove the above assertion. We choose a constant $C > 0$ so that $u_0 \geq \phi \wedge \psi - C$ in $\mathbb{R}^n$. Fix any $\varepsilon > 0$. By assumption (5.1), we may choose an $R \equiv R(\varepsilon) > 0$ so that for any $x \in \mathbb{R}^n$, if $(\phi - \psi)(x) \leq -R$, then $|\phi(x) - u_0(x)| \leq \varepsilon$. Define the function $w \equiv w_\varepsilon$ by setting $w = \phi \wedge (\psi - R - C) - \varepsilon$. Note that $w \in \mathcal{S}_H^-$. Fix any $x \in \mathbb{R}^n$, and observe that if $\phi(x) \leq \psi(x) - R$, then

$$w(x) \leq \phi \wedge (\psi - R)(x) - \varepsilon = \phi(x) - \varepsilon \leq u_0(x).$$

Next, if $\phi(x) > \psi(x) - R$, then we have

$$w(x) = \psi(x) - R - C - \varepsilon = \phi \wedge (\psi - R)(x) - C - \varepsilon \leq \phi \wedge \psi(x) - C - \varepsilon < u_0(x).$$

Hence, $w \leq u_0$ in $\mathbb{R}^n$. We define the function $\psi_0 \in \mathcal{S}_H^-$ by $\psi_0(x) = \sup_{\varepsilon > 0} w_\varepsilon(x)$. Note that $w_\varepsilon \leq \psi_0 \leq u_0$ in $\mathbb{R}^n$ for any $\varepsilon > 0$ and also by construction that $\psi_0 \leq \phi$ in $\mathbb{R}^n$. Consequently, we have $\psi_0 \leq u_0^- \leq u_0$ in $\mathbb{R}^n$ and, in particular, $u_0^-(x) > -\infty$ for all $x \in \mathbb{R}^n$.

We temporarily denote by V the set of all functions $v \in \mathcal{S}_H$ satisfying $v \geq \phi \wedge \psi - C$ in $\mathbb{R}^n$. We show that $\lim_{t\to\infty}(u_0 - \psi_0)(\gamma(-t)) = 0$ for any $v \in V$ and $\gamma \in \mathcal{E}(v)$. For this let $v \in V$. Fix any $\gamma \in \mathcal{E}_z(v)$, with $z \in \mathbb{R}^n$. We show that

$$\lim_{t\to\infty} (\phi - \psi)(\gamma(-t)) = -\infty. \tag{5.3}$$

Indeed, for any $t > 0$ we get

$$\begin{aligned} \psi(z) - \psi(\gamma(-t)) &\leq \int_{-t}^{0} (L(\gamma(s), \dot\gamma(s)) - \delta)\, ds, \\ &= d_H(z, \gamma(-t)) - \delta t = v(z) - v(\gamma(-t)) - \delta t, \end{aligned}$$

and hence

$$v(\gamma(-t)) - \psi(\gamma(-t)) \leq v(z) - \psi(z) - \delta t.$$

Now, using the inequality $v \geq \phi \wedge \psi - C$, we get

$$\phi \wedge \psi(\gamma(-t)) - \psi(\gamma(-t)) \leq v(z) - \psi(z) - \delta t + C,$$

from which we deduce that (5.3) holds.

Fix any $\varepsilon > 0$. In view of (5.3), we choose a $\tau > 0$ so that $(\phi - \psi)(\gamma(-t)) \leq -R_\varepsilon - C$ for all $t \geq \tau$. Noting that $w_\varepsilon \leq \psi_0 \leq u_0^- \leq u_0$ in $\mathbb{R}^n$ and that for any $x \in \mathbb{R}^n$, if $\phi(x) \leq \psi(x) - R_\varepsilon$, then $|u_0(x) - \phi(x)| \leq \varepsilon$, we observe that for any $t \geq \tau$,

$$\begin{aligned} u_0(\gamma(-t)) \geq u_0^-(\gamma(-t)) \geq \psi_0(\gamma(-t)) &\geq \phi \wedge (\psi - R_\varepsilon - C)(\gamma(-t)) - \varepsilon \\ &= \phi(\gamma(-t)) - \varepsilon \geq u_0(\gamma(-t)) - 2\varepsilon. \end{aligned}$$

From this, we conclude that

$$\lim_{t\to\infty}(u_0-\psi_0)(\gamma(-t))=\lim_{t\to\infty}(u_0^- -\psi_0)(\gamma(-t))=0 \quad \text{for all } \gamma\in\mathcal{E}(v),\ v\in V. \tag{5.4}$$

To check that (A6) is satisfied, we first observe from (5.1) and (5.3) that

$$\lim_{t\to\infty}(\phi-u_0)(\gamma(-t))=0 \quad \text{for all } \gamma\in\mathcal{E}(v),\, v\in V. \tag{5.5}$$

This together with (5.4) shows that $g(\phi,\gamma)=g(u_0,\gamma)=g(u_0^-,\gamma)$ for all $\gamma\in\mathcal{E}(\phi)$. We then observe by Corollary 3.4 and Theorem 3.2 that for any $x\in\mathbb{R}^n$,

$$\begin{aligned}u_0^-(x)&\le\inf\{g^-(u_0^-,\gamma)(x)\mid\gamma\in\mathcal{E}\}\le\inf\{g(u_0^-,\gamma)(x)\mid\gamma\in\mathcal{E}(\phi)\}\\&=\inf\{g(\phi,\gamma)(x)\mid\gamma\in\mathcal{E}(\phi)\}=\phi(x).\end{aligned}$$

We now have $\psi_0\le u_0^-\le u_\infty\le\phi$ in $\mathbb{R}^n$, and we see that (A6) holds. We may now invoke Theorem 5.2, to conclude that the convergence (1.5) holds.

It remains to show that $u_\infty=\phi$. We know already that $u_\infty\le\phi$ in $\mathbb{R}^n$. Since $\phi\wedge\psi\in\mathcal{S}_H^-$ and $\phi\wedge\psi-C\le u_0$ in $\mathbb{R}^n$, we have $u_\infty\ge u_0^-\ge\phi\wedge\psi-C$ in $\mathbb{R}^n$. Hence, $u_\infty\in V$. By (5.4) and (5.5), since $u_0^-\le u_\infty\le\phi$ in $\mathbb{R}^n$, we get $g(u_\infty,\gamma)=g(\phi,\gamma)$ for all $\gamma\in\mathcal{E}(u_\infty)$. Hence we have

$$\begin{aligned}\phi(x)&=\inf\{g(\phi,\gamma)(x)\mid\gamma\in\mathcal{E}\}\le\inf\{g(\phi,\gamma)(x)\mid\gamma\in\mathcal{E}(u_\infty)\}\\&=\inf\{g(u_\infty,\gamma)(x)\mid\gamma\in\mathcal{E}(u_\infty)\}=u_\infty(x)\quad\text{for all } x\in\mathbb{R}^n,\end{aligned}$$

completing the proof. □

We examine the following simple example with Theorem 5.3 and compare it with a result in [3].

Example 5.1. We consider the Hamiltonian H given by $H(p)=|p|-1$. Note that the corresponding Lagrangian is $L(\xi)=\delta_{B(0,1)}(\xi)+1$. It is easy to check that H enjoys (A1)-(A4). We now fix any $p_0\in\partial B(0,1)$, and set $\phi(x):=p_0\cdot x$ and $\psi(x):=0$ for $x\in\mathbb{R}^n$. Note that $\phi\in\mathcal{S}_H$ and $\psi\in\mathcal{S}_{H+1}^-$. Assume that the initial function $u_0\in C(\mathbb{R}^n)$ satisfies

$$\lim_{r\to\infty}\sup\{|(u_0-\phi)(x)|\ \mid\ x\in\mathbb{R}^n,\, p_0\cdot x<-r\}=0$$

as well as the condition that $\inf_{\mathbb{R}^n}(u_0-\phi\wedge\psi)>-\infty$ in $\mathbb{R}^n$. Then, by applying Theorem 5.3, we conclude that $u(\cdot,t)\to\phi$ in $C(\mathbb{R}^n)$ as $t\to\infty$. In this example, the condition (5.2) does not hold, and in this sense, Theorem 5.3 refines [3, Theorem 4.2]. On the other hand, by replacing the previous ϕ by the function $\phi(x):=-|x|$, we still have $\phi\in\mathcal{S}_H$. In order to apply Theorem 5.3 to the present situation, we have to assume that $u_0\in C(\mathbb{R}^n)$ satisfies

$$\lim_{r\to\infty}\sup\{|(u_0-\phi)(x)|\ \mid\ \phi(x)<-r\}=0.$$

But, this assumption is equivalent to the condition (5.2) which is exactly the condition required in [3, Theorem 4.2].

6 Results based on (C2) and examples.

We start by formulating a result based on (C2), which is motivated by the main result in [14].

Theorem 6.1. *Assume that (A1)–(A5) hold and that there are two functions* $\phi_0, \phi_1 \in \mathcal{S}_H^-$ *such that*

$$\lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty \quad \text{and} \quad \inf_{\mathbb{R}^n} (u_0 - \phi_0) > -\infty. \tag{6.1}$$

Assume moreover that

$$\mathcal{A}_H \neq \emptyset \quad \text{and} \quad L(x, 0) = 0 \ \text{ for all } x \in \mathcal{A}_H. \tag{6.2}$$

Then the convergence (1.5) holds.

Note that the second condition of (6.2) says that $\mathcal{A}_H$ consists only of equilibria.

Proof. We may assume by adding a constant to ϕ_0 that $u_0 \geq \phi_0$ in $\mathbb{R}^n$. We then have $\phi_0 \leq u_0^- \leq u_0$ in $\mathbb{R}^n$. Fix a $y \in \mathcal{A}_H$ and observe that $u_\infty \leq u_0^-(y) + d_H(\cdot, y)$ in $\mathbb{R}^n$. Hence, (A6) is valid.

Fix any $\gamma \in \mathcal{E}_z(u_\infty)$, with $z \in \mathbb{R}^n$, and note by Remark 2.1 that the function $t \mapsto (u_\infty - \phi_1)(\gamma(-t))$ is non-increasing on $[0, \infty)$ and hence $(u_\infty - \phi_1)(\gamma(-t)) \leq (u_\infty - \phi_1)(z)$ for all $t > 0$. Since $u_\infty \geq \phi_0$ in $\mathbb{R}^n$, this monotonicity and (6.1) together assure that $\gamma(-t) \in B(0, R)$ for all $t \geq 0$ and some $R > 0$. Theorem 2.11 (b) now assures that $\operatorname{dist}(\gamma(-t), \mathcal{A}_H) \to 0$ as $t \to \infty$. Fix any $t > 0$ and choose a point $y \in \mathcal{A}_H$ so that $|\gamma(-t) - y| = \operatorname{dist}(\gamma(-t), \mathcal{A}_H)$. (Recall that $\mathcal{A}_H$ is a closed subset of $\mathbb{R}^n$.) Let $\delta_R > 0$ and $C_R > 0$ be those constants from Lemma 2.3. Let $r > 0$, set $\xi := \delta_R(y - \gamma(-t))/|y - \gamma(-t)|$ and $\rho := \operatorname{dist}(\gamma(-t), \mathcal{A}_H)$, and define the curve $\eta \in \mathrm{AC}([-r, 0])$ by

$$\eta(s) = \begin{cases} \gamma(-t) - s\xi & \text{for } s \in [-\rho/\delta_R, 0], \\ y & \text{for } s \in [-r + \rho/\delta_R, -\rho/\delta_R], \\ \gamma(-t) + (s + r)\xi & \text{for } s \in [-r, -r + \rho/\delta_R] \end{cases}$$

if $\delta_R r > 2\rho$ and $\eta(s) = \gamma(-t)$ if $\delta_R r \leq 2\rho$. It is easy to see that

$$\int_{-r}^{0} L(\eta(s), \dot{\eta}(s))\, ds \leq \frac{2C_R}{\delta_R}\rho = \frac{2C_R}{\delta_R} \operatorname{dist}(\gamma(-t), \mathcal{A}_H).$$

It is now obvious that (C2) holds for all $\gamma \in \mathcal{E}(u_\infty)$. Thus, applying Theorem 4.2, we conclude that the convergence (1.5) holds. □

Remark 6.1. (a) Under the hypotheses of Theorem 6.1, we have

$$u_\infty(x) = \inf\{u_0^-(y) + d_H(x,y) \mid y \in \mathcal{A}_H\} \quad \text{for all } x \in \mathbb{R}^n.$$

Indeed, in the proof above, we have observed that $\lim_{t\to\infty} \operatorname{dist}(\gamma(-t), \mathcal{A}_H) = 0$, and therefore Theorem 3.2 yields the above representation formula.

(b) We know that for any $x \in \mathbb{R}^n$, $L(x,0) = 0$ if and only if $\inf_{p\in\mathbb{R}^n} H(x,p) = 0$. Therefore, by virtue of Lemma 2.12, we see that the condition (6.2) is stated equivalently that there are functions ψ, $f \in C(\mathbb{R}^n)$ such that ψ is a subsolution of $H(x, D\psi) = f$ in $\mathbb{R}^n$, $f \le 0$ in $\mathbb{R}^n$, and for $x \in \mathbb{R}^n$, $f(x) = 0$ if and only if $\inf_{p\in\mathbb{R}^n} H(x,p) = 0$.

Condition (C2) covers another situation, where nearly optimal curves for (1.2) with $\psi = u_0$ exhibit a "switch-back" motion for large t. We discuss first a simple example.

Let $n = 1$ and consider the case where the Hamiltonian H is given by $H(x,p) := |p| - e^{-|x|}$ and u_0 is given by $u_0(x) = \min\{|x| - 2, 0\}$. It is clear that (A1)–(A5) are satisfied. It is easy to see that $d_H(x,y) = \left|\int_y^x e^{-s}\,ds\right|$ for all $x, y \in \mathbb{R}$. By the formula

$$u_0^-(x) = \inf\{u_0(y) + d_H(x,y) \mid y \in \mathbb{R}\},$$

we see that $u_0^-(x) = -e^{-|x|} - 1$ for $x \in \mathbb{R}$. We define the functions $d_\pm \in \mathcal{S}_H$ by $d_\pm(x) = \lim_{y\to\pm\infty} d_H(x,y)$, and observe that $d_\pm(x) = e^{\mp x}$ for $x \in \mathbb{R}$ and that $u_\infty(x) = \lim_{|y|\to\infty} u_0^-(y) + (d_+ \wedge d_-)(x) = e^{-|x|} - 1$ for $x \in \mathbb{R}$. Note that the Lagrangian L is given by $L(x,\xi) = \delta_{[-1,1]}(\xi) + e^{-|x|}$.

For a given $z \in \mathbb{R}$, we define $\gamma \in \mathcal{C}((-\infty,0]; z)$ by $\gamma(s) = z - \operatorname{sgn}(z)\,s$, where $\operatorname{sgn}(z) = 1$ for $z \ge 0$ and $= -1$ for $z < 0$. Then, it is easy to see that $\gamma \in \mathcal{E}_z(u_\infty)$ and $|\gamma(-t)| \longrightarrow \infty$ as $t \to \infty$. Fix any $\varepsilon > 0$ and choose a $\tau > 0$ so that

$$2\int_{|\gamma(-\tau)|}^{\infty} e^{-s}\,ds < \varepsilon.$$

We define $\eta \in \mathrm{AC}([-t,\,0])$ for any fixed $t > 0$ by

$$\eta(s) := \begin{cases} \gamma(-\tau) - \operatorname{sgn}(z)\,s & \text{for } \ -\dfrac{t}{2} \le s \le 0, \\ \gamma(-\tau) + \operatorname{sgn}(z)\,(s+t) & \text{for } \ -t \le s \le -\dfrac{t}{2}, \end{cases}$$

and observe that $\eta(0) = \eta(-t) = \gamma(-\tau)$ and

$$\int_{-t}^{0} L(\eta(s), \dot\eta(s))\,ds < 2\int_{|\gamma(-\tau)|}^{\infty} e^{-s}\,ds < \varepsilon,$$

so that condition (C2) is valid for the given γ. Now, Theorem 4.2 guarantees that the convergence (1.5) holds.

We remark that the curve $\eta \in \mathrm{AC}([-t,\, 0])$ built here has a switch-back motion in which the point $\eta(-s)$, with $s \in [0,\, t]$, moves toward ∞ or $-\infty$ with a unit speed up to the time $t/2$ and then moves back to the starting point. It is also worth mentioning that condition (C1) does not hold in this case. Indeed, we have $\lim_{t\to\infty}(u_0 - u_\infty)(\gamma(-t)) = 1 > 0$.

We continue by generalizing the above observation, where "switch-back" motions appear in nearly optimal curves for (1.2) with $\psi = u_0$. We introduce the following:

(A8) $H(x,0) \le 0$ for all $x \in \mathbb{R}^n$ and there exists a $\lambda \ge 1$ such that

$$H(x,-p) \le H(x,\lambda p) \quad \text{for all} \ \ (x,p) \in \mathbb{R}^{2n}. \tag{6.3}$$

Note that condition (6.3) is equivalent to the condition

$$L(x,-\xi) \le L(x,\lambda\xi) \quad \text{for all} \ \ (x,\xi) \in \mathbb{R}^{2n}. \tag{6.4}$$

Theorem 6.2. *Assume that (A1)-(A6) and (A8) hold and that u_0 is bounded below on $\mathbb{R}^n$. Then, the convergence (1.5) holds.*

Assumption (A8) can be relaxed in the above assertion as follows. Let $\phi_0 \in \mathcal{S}_H^-$.

(A8)′ There exists a $\lambda \ge 1$ such that for every $(x,p) \in Q$, $\xi \in D_2^- H(x,p)$, $q \in \mathbb{R}^n$ and $q' \in \partial_c \phi_0(x)$,

$$H(x, q' - \lambda q) \ge \xi \cdot (q' + q - p), \tag{6.5}$$

where $\partial_c \phi_0(x)$ denotes the Clarke derivative of ϕ_0 at $x \in \mathbb{R}^n$.

Assumption (A8) is a particular case of (A8)′ where $\phi_0 = 0$. We return to this point and give a generalization of Theorem 6.2 later in this section.

Proof of Theorem 6.2. Since the function $\phi_0 := 0$ is a subsolution of $H(x,Du) = 0$ in $\mathbb{R}^n$ and $u_0(x) \ge -C$ for all $x \in \mathbb{R}^n$ and some $C > 0$, we see that $u_\infty(x) \ge u_0^-(x) \ge -C$ for $x \in \mathbb{R}^n$.

Fix any $\gamma \in \mathcal{E}_z(u_\infty)$, $z \in \mathbb{R}^n$. In view of Theorem 4.2, we only need to show that (C2) holds. By assumption, we have $\min_{\xi\in\mathbb{R}^n} L(x,\xi) = -H(x,0) \ge 0$ for all $x \in \mathbb{R}^n$. Observe that

$$0 \le \int_{-t}^{0} L(\gamma(s),\dot\gamma(s))\, ds = u_\infty(x) - u_\infty(\gamma(-t)) \le u_\infty(x) + C \quad \text{for all} \ \ t \ge 0.$$

Consequently, the function $s \mapsto L(\gamma(s),\dot\gamma(s))$ is integrable on $(-\infty,\, 0)$. Fix an arbitrary $\varepsilon > 0$. Then, there exists a $\tau > 0$ such that

$$\int_{-\infty}^{-\tau} L(\gamma(s),\dot\gamma(s))\, ds < \varepsilon. \tag{6.6}$$

Fix any $t > 0$. Let $\lambda \geq 1$ be the constant from (A8), and set $\theta = t/(\lambda + 1)$, so that $\lambda^{-1}(\theta - t) = -\theta$. We define the curve $\eta \in \mathrm{AC}([-t,\, 0])$ by

$$\eta(s) := \begin{cases} \gamma(s - \tau) & \text{if } \ s \in [-\theta,\, 0], \\ \gamma(-\lambda^{-1}(s + t) - \tau) & \text{if } \ s \in [-t,\, -\theta]. \end{cases}$$

Note that $\eta(-t) = \eta(0) = \gamma(-\tau)$. Then, using (6.4) and (6.6), we compute that

$$\begin{aligned} \int_{-t}^{0} L(\eta(s), \dot\eta(s))\, ds &= \int_{-\theta-\tau}^{-\tau} L(\gamma(s), \dot\gamma(s))\, ds + \lambda \int_{-\theta-\tau}^{-\tau} L(\gamma(s), -\lambda^{-1}\dot\gamma(s))\, ds \\ &\leq (1 + \lambda) \int_{-\theta-\tau}^{-\tau} L(\gamma(s), \dot\gamma(s))\, ds < (1 + \lambda)\varepsilon, \end{aligned}$$

which assures that (C2) holds. □

Next, we give an example of a class of Hamiltonians H which satisfy (A8).

Example 6.2. Let $H_0 \in C(\mathbb{R}^{2n})$ and $f \in C(\mathbb{R}^n)$. Define the function $H \in C(\mathbb{R}^{2n})$ by $H(x, p) = H_0(x, p) - f(x)$. We assume here that H_0 has the convexity property (A3) and that there exist $\alpha \geq 1$, $\beta \geq 1$, $\gamma > 1$ and $C_0 > 0$ such that for all $(x, p) \in \mathbb{R}^{2n}$,

$$\alpha^{-1}|p|^{\beta} \leq H_0(x, p) \leq \alpha|p|^{\beta} \quad \text{and} \quad 0 \leq f(x) \leq C_0(1 + |x|)^{-\beta\gamma}.$$

To prove that H satisfies property (A8), let $\lambda \geq 1$ be a constant which will be specified later and observe that

$$H_0(x, -p) \leq \alpha|p|^{\beta} \leq \alpha^2\lambda^{-\beta} \cdot \alpha^{-1}|\lambda p|^{\beta} \leq \alpha^2\lambda^{-\beta} H_0(x, \lambda p) \quad \text{for all } \ (x, p) \in \mathbb{R}^{2n}.$$

By selecting $\lambda = \alpha^{2/\beta}$, we get $H(x, -p) \leq H(x, \lambda p)$ for all $(x, p) \in \mathbb{R}^{2n}$. Note also that $H_0(x, 0) = 0$ and $H(x, 0) = -f(x) \leq 0$ for all $(x, p) \in \mathbb{R}^{2n}$. Hence, H satisfies (A8). Let $u_0 \in C(\mathbb{R}^n)$ be a bounded function. It is clear that (A1), (A2) and (A5) are satisfied. Also, it is not difficult to check that (A4) is satisfied. We define $\psi_0 \in C(\mathbb{R}^n)$ by $\psi_0(x) = -\alpha C_0 \int_0^{|x|} (1 + r)^{-\gamma} dr$, and observe that for $x \neq 0$,

$$H(x, D\psi_0(x)) \geq \alpha^{-1}|D\psi_0(x)|^{\beta} - f(x) = C_0(1 + |x|)^{-\beta\gamma} - f(x) \geq 0,$$

which implies that $\psi_0 \in \mathcal{S}_H^+$. Note that ψ_0 is a bounded function on $\mathbb{R}^n$ and that $0 \in \mathcal{S}_H^-$. By Perron's method, we see that there is a bounded solution of $H(x, Du) = 0$ in $\mathbb{R}^n$ and moreover that (A6) is satisfied. We conclude that all the hypotheses of Theorem 6.2 are fulfilled with these H and u_0.

The condition (6.4) and the convexity of L implies that $\min_{\xi\in\mathbb{R}^n} L(x, \xi) = L(x, 0)$. Hence, under hypotheses (A3) and (A8), we have $\min_{\xi\in\mathbb{R}^n} L(x, \xi) = L(x, 0) \geq 0$, and, in view of (2.12), we see easily that $x \in \mathbb{R}^n$ is a point in $\mathcal{A}_H$ if and only if x is an equilibrium point. Thus, in general under the hypotheses, nearly optimal curves η for the variational formula (1.2), with $\psi = u_0$, may not have a switch-back motion. But if we suppose in addition to (A8) that $H(x, 0) < 0$ for all $x \in \mathbb{R}^n$. Then, in view of Lemma 2.12 and Theorem 2.11 (b), we have $|\gamma(-t)| \longrightarrow \infty$ as $t \to \infty$ for any $\gamma \in \mathcal{E}(u_\infty)$. Hence, under this additional assumption, the curves η constructed in the proof of Theorem 6.2 have a switch-back motion.

Remark 6.3. Combining proofs of Theorems 6.1 and 6.2 allows us to replace (6.3) in Theorem 6.2 by a weaker condition that $\mathcal{A}_H$ consists only of equilibria and inequality (6.3) is required only for $(x, p) \in (\mathbb{R}^n \setminus B(0, R)) \times \mathbb{R}^n$, with some $R > 0$. This remark applies also to Theorem 6.3 below.

We now give a generalization of Theorem 6.2, in which condition (A8) is replaced by (A8)$'$.

Theorem 6.3. *Assume that (A1)-(A6) hold. Let $\phi_0 \in \mathcal{S}_H^-$ be such that $u_0 \geq \phi_0$ in $\mathbb{R}^n$, and assume that (A8)$'$ holds with this ϕ_0. Then the convergence (1.5) holds.*

We need a lemma for the proof.

Lemma 6.4. *Let H satisfy (A1)-(A3), and let $\phi_0 \in \mathcal{S}_H^-$. Then, (A8)$'$ holds if and only if*

$$L(x, -\lambda^{-1}\xi) + \lambda^{-1}\xi \cdot q' \leq L(x, \xi) - \xi \cdot q' \tag{6.7}$$

for all $(x, \xi) \in S$ and $q' \in \partial_c \phi_0(x)$.

Proof. We assume (A8)$'$. Recalling the definition of Q and S, we observe that

$$\xi \cdot p = H(x, p) + L(x, \xi) = L(x, \xi) \qquad \text{for all } (x, p) \in Q \text{ and } \xi \in D_2^- H(x, p).$$

Fix any $(x, \xi) \in S$ and $q' \in \partial_c \phi_0(x)$. Fix a $p \in \mathbb{R}^n$ so that $\xi \in D_2^- H(x, p)$. Then, in view of (6.5), we have

$$L(x, \xi) = \xi \cdot p \geq \xi \cdot (q' + q) - H(x, q' - \lambda q) \qquad \text{for all } \ q \in \mathbb{R}^n.$$

Hence

$$\begin{aligned} L(x, \xi) &\geq \sup_{q \in \mathbb{R}^n} \{\xi \cdot (q' + q) - H(x, q' - \lambda q)\} \\ &= \sup_{q \in \mathbb{R}^n} \{(-\lambda^{-1}\xi) \cdot q - H(x, q)\} + (1 + \lambda^{-1})\, \xi \cdot q' \\ &= L(x, -\lambda^{-1}\xi) + (1 + \lambda^{-1})\, \xi \cdot q', \end{aligned}$$

from which follows (6.7).

We next assume (6.7) and fix any $(x,p) \in Q$, $\xi \in D_2^- H(x,p)$, $q \in \mathbb{R}^n$ and $q' \in \partial_c \phi_0(x)$. Then

$$
\begin{aligned}
&H(x, q' - \lambda q) \\
&\quad = \sup_{\zeta \in \mathbb{R}^n} \{\zeta \cdot (q' - \lambda q) - L(x,\zeta)\} = \sup_{\zeta \in \mathbb{R}^n} \{\zeta \cdot (-\lambda^{-1} q' + q) - L(x, -\lambda^{-1}\zeta)\} \\
&\quad \geq \xi \cdot q - [L(x, -\lambda^{-1}\xi) + \lambda^{-1}\xi \cdot q'] \geq \xi \cdot q - [L(x,\xi) - \xi \cdot q'] = \xi \cdot (q' + q - p).
\end{aligned}
$$

Hence (A8)$'$ is valid. □

Proof of Theorem 6.3. Fix any $\gamma \in \mathcal{E}_z(u_\infty)$, with $z \in \mathbb{R}^n$. Let q be a measurable function on $(-\infty, 0)$ such that $q(s) \in \partial_c \phi_0(\gamma(s))$ for a.e. $s \in (-\infty, 0)$ and

$$
\phi_0(z) - \phi_0(\gamma(-t)) = \int_{-t}^{0} q(s) \cdot \dot{\gamma}(s)\, ds \qquad \text{for all } \ t \geq 0.
$$

Now, since $u_\infty \geq \phi_0$ in $\mathbb{R}^n$, for $t > 0$ we have

$$
\begin{aligned}
&\int_{-t}^{0} (L(\gamma(s), \dot{\gamma}(s)) - q(s) \cdot \dot{\gamma}(s))\, ds \\
&= (u_\infty - \phi_0)(z) - (u_\infty - \phi_0)(\gamma(-t)) \leq (u_\infty - \phi_0)(z).
\end{aligned}
$$

Note that $q(s) \cdot \dot{\gamma}(s) \leq L(\gamma(s), \dot{\gamma}(s)) + H(\gamma(s), q(s)) \leq L(\gamma(s), \dot{\gamma}(s))$ for a.e. $s \in (-\infty, 0)$. From these, we see that the function $s \mapsto L(\gamma(s), \dot{\gamma}(s)) - q(s) \cdot \dot{\gamma}(s)$ is non-negative a.e. and integrable on $(-\infty, 0)$.

We now follow the proof of Theorem 6.2. Fix any $\varepsilon > 0$ and choose a $\tau > 0$ so that

$$
\int_{-\infty}^{-\tau} (L(\gamma(s), \dot{\gamma}(s)) - q(s) \cdot \dot{\gamma}(s))\, ds < \varepsilon. \tag{6.8}
$$

Fix any $t > 0$. Let $\lambda \geq 1$ be the constant from (A8)$'$, and set $\theta = t/(\lambda + 1)$. We define the curve $\eta \in \mathrm{AC}([-t,\, 0])$ by

$$
\eta(s) := \begin{cases} \gamma(s - \tau) & \text{if } \ s \in [-\theta,\, 0], \\ \gamma(-\lambda^{-1}(s + t) - \tau) & \text{if } \ s \in [-t,\, -\theta]. \end{cases}
$$

Then we have $\eta(-t) = \eta(0) = \gamma(-\tau)$. Next, noting that $(\gamma(s), \dot{\gamma}(s)) \in S$ for a.e $s \in (-\infty, 0)$ and using (6.7) and (6.8), we compute that

$$
\begin{aligned}
&\int_{-t}^{0} L(\eta(s), \dot{\eta}(s))\, ds = \int_{-\theta-\tau}^{-\tau} L(\gamma(s), \dot{\gamma}(s))\, ds + \lambda \int_{-\theta-\tau}^{-\tau} L(\gamma(s), -\lambda^{-1}\dot{\gamma}(s))\, ds \\
&= \int_{-\theta-\tau}^{-\tau} [L(\gamma(s), \dot{\gamma}(s)) - q(s) \cdot \dot{\gamma}(s)]\, ds \\
&+ \lambda \int_{-\theta-\tau}^{-\tau} [L(\gamma(s), -\lambda^{-1}\dot{\gamma}(s)) + \lambda^{-1} q(s) \cdot \dot{\gamma}(s)]\, ds \\
&\leq (1 + \lambda) \int_{-\theta-\tau}^{-\tau} [L(\gamma(s), \dot{\gamma}(s)) - q(s) \cdot \dot{\gamma}(s)]\, ds < (1 + \lambda)\varepsilon,
\end{aligned}
$$

and conclude that (C2) holds. □

7 Results based on (C3) and examples.

A variant of Theorem 6.1 is given by the next theorem which can be also regarded as a version of [18, Theorem 1.3]

Theorem 7.1. *Assume that (A1)–(A5) and either $(A7)_+$ or $(A7)_-$ hold and that there are two functions ϕ_0, $\phi_1 \in \mathcal{S}_H^-$ such that*

$$\lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty \quad \text{and} \quad \inf_{\mathbb{R}^n}(u_0 - \phi_0) > -\infty.$$

Assume moreover that $\mathcal{A}_H \neq \emptyset$. Then the convergence (1.5) holds.

Proof. As in the proof of Theorem 6.1, we see that (A6) holds. It remains to show that (C3) holds for any $\gamma \in \mathcal{E}(u_\infty)$. Fix $\gamma \in \mathcal{E}_z$, with $z \in \mathbb{R}^n$, and observe as in Theorem 6.1 that there is a constant $R > 0$ such that $\gamma(s) \in B(0, R)$ for all $s \leq 0$. Now we fix any $\varepsilon > 0$ and choose, in view of Theorem 2.9, a $\tau_y > 0$ for each $y \in B(0, R)$ so that $u_\infty(y) + \varepsilon > u(y, \tau_y)$. Next, using the compactness of $B(0, R)$ and the continuity of u_∞, u, we deduce that there exists a $\tau > 0$ such that $u_\infty(x) + \varepsilon > u(x, \tau_x)$ for any $x \in B(0,\, R)$ and some $\tau_x \in [0,\, \tau]$. That is, (C3) is valid for any $\gamma \in \mathcal{E}(u_\infty)$. □

It is easily seen by the compactness argument in the proof above that condition (C3) holds for any $\gamma \in \mathcal{E}(u_\infty)$ in the case where $H(\cdot, p)$ and u_0 are $\mathbb{Z}^n$-periodic. The criterion (C3) applies to the case where H and u_0 are upper semi-periodic and obliquely lower semi-almost periodic, respectively. The convergence result in this case has been established in [16] (see also [15]). We recall that a function H on $\mathbb{R}^{2n}$ is said to be *upper (resp., lower) semi-periodic* if for any $\{y_j\} \subset \mathbb{R}^n$ there exist a subsequence $\{y_j'\} \subset \{y_j\}$, a sequence $\{\xi_j\} \subset \mathbb{R}^n$ converging to zero and a function $G \in C(\mathbb{R}^{2n})$ such that $H(x + y_j', p) \to G(x, p)$ in $C(\mathbb{R}^{2n})$ as $j \to \infty$ and $H(x + y_j' + \xi_j, p) \leq G(x, p)$ (resp., $H(x + y_j' + \xi_j, p) \geq G(x, p)$) for all $(x, p, j) \in \mathbb{R}^{2n} \times \mathbb{N}$. Also, a function f on $\mathbb{R}^n$ is said to be *obliquely lower (resp., upper) semi-almost periodic* if for any $\{y_j\} \subset \mathbb{R}^n$ and any $\varepsilon > 0$ there exist a subsequence $\{y_j'\} \subset \{y_j\}$ and a function $g \in C(\mathbb{R}^n)$ such that $f(x + y_j') \to g(x)$ in $C(\mathbb{R}^n)$ as $j \to \infty$ and $f(x + y_j') + \varepsilon \geq g(x)$ (resp., $f(x + y_j') - \varepsilon \leq g(x)$) for all $(x, j) \in \mathbb{R}^n \times \mathbb{N}$. It is easily checked that if H is upper (or lower) semi-periodic, then it is bounded, uniformly continuous on $\mathbb{R}^n \times B(0, R)$, i.e. $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0, R))$, for any $R > 0$ and that if f is obliquely upper (or lower) semi-almost periodic, then it is uniformly continuous on $\mathbb{R}^n$, i.e., $f \in \mathrm{UC}(\mathbb{R}^n)$.

For the later references, we introduce two conditions on H:

(A1)′ $H \in \mathrm{BUC}(\mathbb{R}^n \times B(0, R))$ for all $R > 0$.

(A2)′ $\inf\{H(x, p) \,|\, x \in \mathbb{R}^n,\ |p| \geq R\} \longrightarrow +\infty$ as $R \to +\infty$.

We state [16, Theorem 2.2] as follows and prove it in a way based on (C3).

Theorem 7.2. *Assume that H and u_0 are upper semi-periodic and obliquely lower semi-almost periodic, respectively, and that (A2)′, (A3), (A6) and either $(A7)_+$ or $(A7)_-$ hold. Then the convergence (1.5) holds.*

Proof. Note that (A1) (A2) and (A5) are satisfied. As is well-known (see e.g. [16]), doe to (A2)′ and the fact that H satisfies (A1)′ and $u_0 \in \mathrm{UC}(\mathbb{R}^n)$, the solution u of (1.1) is uniformly continuous on $\mathbb{R}^n \times [0, \infty)$. Moreover, because of assumption (A6), we have real-valued functions u_0^- and u_∞. To prove the convergence (1.5), it is enough to show that the function $u(\cdot, t) \wedge (u_\infty + 1)$, which is also a solution of (1.1) with u_0 replaced by $u_0 \wedge (u_\infty + 1)$, converges to u_∞ in $C(\mathbb{R}^n)$ as $t \to \infty$. Therefore, we may assume by replacing the function u by the function $u \wedge (u_\infty + 1)$ if necessary that $u_0^-(x) \leq u(x, t) \leq u_\infty(x) + 1$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$.

We need only to show that (C3) holds for any $\gamma \in \mathcal{E}(u_\infty)$. Fix any $\gamma \in \mathcal{E}(u_\infty)$. We argue by contradiction, and thus suppose that (C3) does not hold with this γ and therefore there exists an $\varepsilon > 0$ and for each $j \in \mathbb{N}$ a $t_j \geq j$ such that

$$\min_{0 \leq s \leq j} u(\gamma(-t_j), s) \geq u_\infty(\gamma(-t_j)) + \varepsilon \quad \text{for all } j \in \mathbb{N}. \tag{7.1}$$

Set $y_j = \gamma(-t_j)$ for $j \in \mathbb{N}$. Noting that the function $t \mapsto (u_\infty - u_0^-)(\gamma(-t))$ is non-increasing and non-negative on $[0, \infty)$ and that $u_0^-(x) \leq u(x, t) \leq u_\infty(x) + 1$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, we observe that

$$\sup\{|u_\infty(\gamma(s)) - u(\gamma(s), t)| \mid t \geq 0,\ s \leq 0\} < \infty.$$

Now, we may assume that there are functions $G \in C(\mathbb{R}^{2n})$, $v \in C(\mathbb{R}^n \times [0, \infty))$ and a sequence $\{\xi_j\} \subset \mathbb{R}^n$ converging to zero such that as $j \to \infty$,

$$\begin{aligned} &H(\cdot + y_j + \xi_j, \cdot) \to G \quad \text{in } C(\mathbb{R}^{2n}), \\ &u(\cdot + y_j + \xi_j, \cdot) - u_\infty(y_j + \xi_j) \to v \quad \text{in } C(\mathbb{R}^n \times [0, \infty)), \end{aligned} \tag{7.2}$$

$H(\cdot + y_j + \xi_j, \cdot) \leq G$ in $\mathbb{R}^{2n}$ and $u(\cdot + y_j + \xi_j, 0) - u_\infty(y_j + \xi_j) \geq v(\cdot, 0) - \varepsilon/4$ in $\mathbb{R}^n \times [0, \infty)$ for all $j \in \mathbb{N}$. Note that v is the solution of (1.1), with H and u_0 replaced by G and $v_0 := v(\cdot, 0)$, respectively. Note also that $w := u(\cdot + y_j + \xi_j, \cdot) - u_\infty(y_j + \xi_j)$ is a supersolution of $w_t + G(x, Dw) = 0$ in $\mathbb{R}^n \times (0, \infty)$ and that $w(\cdot, 0) \geq v_0 - \varepsilon/4$ in $\mathbb{R}^n$. Now we obtain by comparison,

$$u(\cdot + y_j + \xi_j, \cdot) - u_\infty(y_j + \xi_j) \geq v - \varepsilon/4 \quad \text{in } \mathbb{R}^n \times [0, \infty) \text{ for all } j \in \mathbb{N}.$$

In particular, we see that $0 \geq v_\infty(0) - \varepsilon/4$ for all $j \in \mathbb{N}$, where $v_\infty(x) := \liminf_{t \to \infty} v(x, t)$. We choose a $\tau > 0$ so that $v(0, \tau) - \varepsilon/4 < v_\infty(0)$. Since u_∞ is (globally) Lipschitz continuous in $\mathbb{R}^n$ by (A2)′, we may assume that $u_\infty(y_j + \xi_j) < u_\infty(y_j) + \varepsilon/4$ for all $j \in \mathbb{N}$. Also we may assume in view of (7.2) that $u(y_j, \tau) - u_\infty(y_j + \xi_j) < v(0, \tau) + \varepsilon/4$ for all $j \in \mathbb{N}$. Combining these together, we get $u(y_j, \tau) < u_\infty(y_j) + \varepsilon$, which contradicts (7.1). The proof is complete. □

The proof above suggests formulating the following theorem.

Theorem 7.3. *Let H satisfy (A1)$'$, (A2)$'$, (A3), (A6) and either (A7)$_+$ or (A7)$_-$. Let $u_0 \in \mathrm{UC}(\mathbb{R}^n)$, and assume that $u_0 \le u_\infty + C$ in $\mathbb{R}^n$ for some $C > 0$. Then, the convergence (1.5) is valid provided that for any sequence $\{y_j\} \subset \mathbb{R}^n$ satisfying*

$$u(\cdot + y_j, \cdot) - u_\infty(y_j) \longrightarrow v \quad \text{in } C(\mathbb{R}^n \times [0, \infty)) \tag{7.3}$$

for some $v \in C(\mathbb{R}^n \times [0, \infty))$, the inequality $v_\infty(0) \le 0$ holds, where v_∞ is the function on $\mathbb{R}^n$ defined by $v_\infty(x) := \liminf_{t\to\infty} v(x,t)$.

We remark that, as the following proof shows, we need to assume the condition (7.3) only for those sequences $\{y_j\}$ given by $y_j = \gamma(-t_j)$, with sequences $\{t_j\} \subset (0, \infty)$ diverging to infinity and $\gamma \in \mathcal{E}(u_\infty)$, in the above theorem.

Proof. Note that condition (A4) is a consequence of (A1)$'$ and (A2)$'$. Hence, (A1)-(A6) are fulfilled. It is enough to show that (C3) is valid for all $\gamma \in \mathcal{E}(u_\infty)$.

Fix any $\gamma \in \mathcal{E}(u_\infty)$, and we suppose that (C3) does not hold with this γ and therefore there is an $\varepsilon > 0$ and for each $j \in \mathbb{N}$ a positive number $t_j \ge j$ such that

$$\min_{0\le s\le j} u(\gamma(-t_j), s) \ge u_\infty(y_j) + \varepsilon, \tag{7.4}$$

and will get a contradiction.

We set $y_j = \gamma(-t_j)$ for $j \in \mathbb{N}$, and observe that $u_\infty(y_j) < u_0(y_j) \le u_\infty(y_j) + C$ for all $j \in \mathbb{N}$. It is a standard observation, due to (A1)$'$, (A2)$'$ and the uniform continuity of u_0, that $u \in \mathrm{UC}(\mathbb{R}^n \times [0, T])$ for all $T > 0$. By taking a subsequence of $\{t_j\}$ if necessary, we may assume that the convergnece (7.3) holds. Also, since u_0^- is Lipschitz continuous on $\mathbb{R}^n$, we may assume that $u_0^-(\cdot + y_j) - u_\infty(y_j) \to w$ in $C(\mathbb{R}^n)$ as $j \to \infty$ for some $w \in \mathcal{S}_H^-$. Recalling that $\lim_{t\to\infty}(u_\infty - u_0^-)(\gamma(-t)) = 0$ and that $u(x,t) \ge u_0^-(x)$ for all $(x,t) \in \mathbb{R}^n \times [0, \infty)$, we see that $w(0) = 0$ and $v(x,t) \ge w(x)$ for all $(x,t) \in \mathbb{R}^n \times [0, \infty)$. Hence we get $v_0^-(x) := \inf_{t\ge0} v(x,t) \ge w(x)$ for all $x \in \mathbb{R}^n$. In particular, $v_\infty(0) \ge v_0^-(0) \ge w(0) = 0$. On the other hand, by assumption, we have $v_\infty(0) \le 0$ and therefore $v_\infty(0) = 0$. By the definition of $v_\infty(0)$, we may choose a $\tau > 0$ so that $v_\infty(0) + \varepsilon/2 > v(0,\tau)$. Moreover, by definition, we may assume by taking a subsequence if necessary that $v(0,\tau) + \varepsilon/2 > u(y_j,\tau) - u_\infty(y_j)$ for all $j \in \mathbb{N}$. Thus we obtain

$$u(y_j,\tau) < v(0,\tau) + u_\infty(y_j) + \frac{\varepsilon}{2} < v_\infty(0) + u_\infty(y_j) + \varepsilon = u_\infty(y_j) + \varepsilon,$$

which contradicts (7.4). □

We give here two examples to which we may apply Theorem 7.3.

Example 7.1. Let $n = 1$, and let $f \in \mathrm{BUC}(\mathbb{R})$ be any function such that $f \geq 0$ in $\mathbb{R}$. We set $F(x) = \int_0^x f(y)\,dy$ for $x \in \mathbb{R}$ and define $H \in C(\mathbb{R}^2)$ and $\phi \in \mathrm{UC}(\mathbb{R})$ by $H(x,p) = p^2 - f(x)^2$ and $\phi(x) := \min\{F(x), -F(x)\} = -|F(x)|$. Note that H satisfies (A1)$'$, (A2)$'$, (A3) and (A7)$_\pm$. Since F, $-F \in \mathcal{S}_H$, we see in view of convexity (A3) that $\phi \in \mathcal{S}_H$. Moreover, it is easily seen that $d_H(x,y) = |F(x) - F(y)|$ for all $x, y \in \mathbb{R}$.

Now, let $p_0 \in \mathrm{BUC}(\mathbb{R})$ be any function satisfying the following property: for any $\varepsilon > 0$, there exists an $l > 0$ such that

$$\min_{|y|\leq l} p_0(x+y) < \inf_{\mathbb{R}} p_0 + \varepsilon \qquad \text{for all } \ x \in \mathbb{R}. \tag{7.5}$$

Remark that (7.5) is valid for any almost periodic function. But, we do not assume any kind of periodicities on neither H nor p_0.

We set $u_0 = \phi + p_0 \in \mathrm{UC}(\mathbb{R})$ and let u be the solution of the Cauchy problem (1.1) with H and u_0 defined above. What we prove is the following convergence:

$$u(\,\cdot\,,t) \longrightarrow \phi + \inf_{\mathbb{R}}(u_0 - \phi) \qquad \text{in } \ C(\mathbb{R}) \quad \text{as } \ t \to \infty. \tag{7.6}$$

In what follows, we assume that $\inf_{\mathbb{R}}(u_0 - \phi) = \inf_{\mathbb{R}} p_0 = 0$, which does not lose any generality. In this case, we have $u_0^- = u_\infty = \phi$ in $\mathbb{R}$. To see this, we may just apply, for instance, formula (2.10) for u_0^-. We observe that $D\phi(x) = \pm f(x)$ if $\pm x < 0$ and $D_2H(x,p) = 2p$ for all $(x,p) \in \mathbb{R}^2$ and therefore that for any $\gamma \in \mathcal{E}(\phi)$, if $\pm\gamma(0) < 0$, then $\dot{\gamma}(s) = \pm 2f(\gamma(s))$ for all $s \leq 0$. It is then easy to see that neither condition (C1) nor (C2) holds in general.

To show the convergence (7.6), we intend to use Theorem 7.3. Fix any $\{y_j\} \subset \mathbb{R}$. By passing to a subsequence if necessary, we may assume that there are only three cases: (i) $\lim_{j\to\infty} y_j = y_0 \in \mathbb{R}$ for some $y_0 \in \mathbb{R}$, (ii) $\lim_{j\to\infty} y_j = \infty$, and (iii) $\lim_{j\to\infty} y_j = -\infty$. As in Theorem 7.3, we assume that $u(\cdot + y_j, \cdot) - \phi(y_j) \to v$ in $C(\mathbb{R}^n \times [0,\infty))$ as $j \to \infty$. In the case when (i) $\lim_{j\to\infty} y_j = y_0$ for some $y_0 \in \mathbb{R}$, we have $v = u(\cdot + y_0, \cdot) - \phi(y_0)$ and hence $v_\infty(0) = \liminf_{t\to\infty} v(0,t) = u_\infty(y_0) - \phi(y_0) = 0$.

Next consider the case when (ii) $\lim_{j\to\infty} y_j = \infty$. We may assume that $H(\cdot + y_j, \cdot) \to G$ in $C(\mathbb{R}^2)$ and $f(\cdot + y_j) \to e$, $p_0(\cdot + y_j) \to q_0$ in $C(\mathbb{R}^n)$ as $j \to \infty$ for some $G \in C(\mathbb{R}^2)$, e, $q_0 \in \mathrm{BUC}(\mathbb{R}^n)$. Set $E(x) = \int_0^x e(y)\,dy$. Observe that $G(x,p) = p^2 - e(x)^2$, that q_0 satisfies property (7.5) with p_0 replaced by q_0, and that $v_0(x) := v(x,0) = -E(x) + q_0(x)$ for all $x \in \mathbb{R}$. Now the function v solves problem (1.1) with G and v_0 in place of H and u_0, respectively. As before, we see that $v_0^- = v_\infty = -E$ in $\mathbb{R}$. We now conclude that $v_\infty(0) = -E(0) = 0$. An argument similar to the above applies to the case when (iii) $\lim_{j\to\infty} y_j = -\infty$, to yield $v_\infty(0) = 0$. Theorem 7.3 now guarantees that (7.6) is valid.

Finally, the condition (7.5) can be relaxed and it is indeed replaced by the following:

for each $\varepsilon > 0$ there exists an $l > 0$ such that

$$\limsup_{|x|\to\infty} \min_{|y|\le l} p_0(x+y) < \inf_{\mathbb{R}} p_0 + \varepsilon.$$

The example above leads us to formulate the following proposition.

Theorem 7.4. *Let H satisfy (A1)$'$, (A2)$'$, (A3) and either (A7)$_+$ or (A7)$_-$. Let $\phi_0 \in \mathcal{S}_H^-$ and $p_0 \in \mathrm{BUC}(\mathbb{R}^n)$. Assume that $u_0 = \phi_0 + p_0$ and that for any $\varepsilon > 0$ and $\gamma \in \mathcal{E}(u_\infty)$ such that $\lim_{t\to\infty} |\gamma(-t)| = \infty$ there exists an $l > 0$ for which*

$$\limsup_{t\to\infty} \min_{|s|\le l} p_0(\gamma(-t+s)) < \inf_{\mathbb{R}^n} p_0 + \varepsilon. \tag{7.7}$$

Then, the convergence (1.5) is valid.

Proof. We may assume without loss of generality that $\inf_{\mathbb{R}^n} p_0 = 0$. Since $\phi_0 \le u_0 \le \phi_0 + \sup_{\mathbb{R}^n} p_0$ in $\mathbb{R}^n$, we have $\phi_0 \le u_0^- \le u_\infty \le \phi_0 + \sup_{\mathbb{R}^n} p_0$ in $\mathbb{R}^n$. In particular, (A6) holds.

Let $\gamma \in \mathcal{E}(u_\infty)$ and let $\{t_j\} \subset (0,\infty)$ be an increasing sequence diverging to infinity. We set $y_j = \gamma(-t_j)$ for $j \in \mathbb{N}$. We note that

$$\dot\gamma(s) \in D_2H(\gamma(s), b(s)) \quad \text{and} \quad H(\gamma(s), b(s)) = 0 \quad \text{for a.e. } s \le 0$$

for some measurable function b on $(-\infty, 0)$, and in view of (A1)$'$ and (A2)$'$ we see that γ is (globally) Lipschitz continuous on $(-\infty, 0]$. Now, by replacing $\{y_j\}$ by a subsequence if necessary, we may assume that as $j \to \infty$,

$$\begin{gathered} H(\cdot + y_j, \cdot) \to G \quad \text{in } C(\mathbb{R}^{2n}), \\ u_\infty(\cdot + y_j) - u_\infty(y_j) \to w, \quad p_0(\cdot + y_j) \to q_0 \quad \text{in } C(\mathbb{R}^n), \\ u(\cdot + y_j, \cdot) - u_\infty(y_j) \to v \quad \text{in } C(\mathbb{R}^n \times [0,\infty)), \\ \gamma(\cdot - t_j) - \gamma(-t_j) \to \eta \quad \text{in } H^1(-k, k),\ k \in \mathbb{N}, \end{gathered}$$

where $H^1(a,b)$ indicates the usual topology of the Sobolev space consisting of functions f on (a, b) such that $\int_a^b (f(s)^2 + \dot f(s)^2)\,ds < \infty$.

We consider the case when $\liminf_{j\to\infty} |y_j| < \infty$. We may assume by selecting again a subsequence if needed that $\lim_{j\to\infty} y_j = y_0$ for some $y_0 \in \mathbb{R}^n$. Then we have $u(\cdot + y_0, \cdot) - u_\infty(y_0) = v$ in $\mathbb{R}^n \times [0,\infty)$ and hence

$$v_\infty(0) := \liminf_{t\to\infty} v(0,t) = \liminf_{t\to\infty} (u(0,t) - u_\infty(0)) = 0.$$

Next we consider the case when $\lim_{t\to\infty} |y_j| = \infty$. Fix any $a < b$ and take j to be large enough so that $b - t_j \le 0$. Note by the extremality of γ that

$$u_\infty(\gamma(-t_j + b)) - u_\infty(\gamma(-t_j + a)) = \int_a^b L(\gamma(-t_j + s), \dot\gamma(-t_j + s))\,ds. \tag{7.8}$$

Fix any $\delta > 0$ and define the function $K_\delta \in C(\mathbb{R}^{2n})$ by

$$K_\delta(x,\xi) = \max_{p\in\mathbb{R}^n}\{\xi\cdot p - G(x,p) - \delta|p|^2\}.$$

Here we note that G satisfies (A1)$'$, (A2)$'$ and (A3). Setting

$$F(x,\xi) = \{p \in \mathbb{R}^n \mid K_\delta(x,\xi) + G(x,p) + \delta|p|^2 - \xi\cdot p \le 0\},$$

we observe that $F(x,\xi) \subset \mathbb{R}^n$ is a non-empty, compact, convex subset for any $(x,\xi) \in \mathbb{R}^{2n}$ and the multi-function F is continuous in $\mathbb{R}^{2n}$. By a selection theorem (see e.g. [1, Theorem 1.8.1]), there is a function $f \in C(\mathbb{R}^{2n})$ such that $f(x,\xi) \in F(x,\xi)$ for all $(x,\xi) \in \mathbb{R}^{2n}$. Set $g(s) = f(\eta(s),\dot\eta(s))$ for $s \in \mathbb{R}$ and observe that

$$\int_a^b L(\gamma(-t_j+s),\dot\gamma(-t_j+s))\,ds \ge \int_a^b [\dot\gamma(-t_j+s)\cdot g(s) - H(\gamma(-t_j+s),g(s))]\,ds.$$

Combining this with (7.8) and sending $j \to \infty$, we get

$$\begin{aligned} w(\eta(b)) - w(\eta(a)) &\ge \int_a^b (\dot\eta(s)\cdot g(s) - G(\eta(s),g(s)))\,ds \\ &\ge \int_a^b (K_\delta(\eta(s),\dot\eta(s)) + \delta|g(s)|^2)\,ds \ge \int_a^b K_\delta(\eta(s),\dot\eta(s))\,ds. \end{aligned}$$

By applying the monotone convergence theorem, we get

$$w(\eta(b)) - w(\eta(a)) \ge \int_a^b K(\eta(s),\dot\eta(s))\,ds,$$

where K denotes the Lagrangian corresponding to G. From this we infer that

$$w(\eta(b)) - w(\eta(a)) = d_G(\eta(b),\eta(a)) \quad \text{for any } a < b.$$

Hence, η is an extremal curve for w. It is not difficult to deduce from (7.7) that for each $\varepsilon > 0$ there is an $l > 0$ such that $\min_{|s|\le l} q_0(\eta(t+s)) \le \varepsilon$ for all $t \in \mathbb{R}$. In particular, we have $\inf_{s\le 0} q_0(\eta(s)) = 0$.

We set $v_0 = v(\cdot,0)$ and define the function $v_0^- \in C(\mathbb{R}^n)$ by

$$v_0^-(x) = \inf\{v_0(y) + d_G(x,y) \mid y \in \mathbb{R}^n\}.$$

Observe that for any $x \in \mathbb{R}^n$,

$$\begin{aligned} v_0(x) &= \lim_{j\to\infty}(\phi_0(x+y_j) + p_0(x+y_j) - u_\infty(y_j)) \\ &\le \lim_{j\to\infty}(u_\infty(x+y_j) - u_\infty(y_j) + p_0(x+y_j)) = w(x) + q_0(x). \end{aligned}$$

Moreover, noting that $\eta(0) = 0$ and $\inf_{s\le 0} q_0(\eta(s)) = 0$ and using the extremality of η, we observe that

$$v_0^-(0) \le \inf\{w(\eta(s)) + q_0(\eta(s)) + d_G(0,\eta(s)) \mid s \le 0\} = w(0) + \inf_{s\le 0} q_0(\eta(s)) = 0.$$

Theorem 7.3 (with the remark next to it) now guarantees that (1.5) holds. $\square$

The following example indicates another direction to generalize Example 7.1 to multi-dimensional cases.

Example 7.2. For each $i = 1, \ldots, n$, let $f_i \in \mathrm{BUC}(\mathbb{R}^n)$, $i = 1, \ldots, n$, be such that $\inf_{\mathbb{R}^n} f_i \geq 0$ or $\sup_{\mathbb{R}^n} f_i \leq 0$. We set

$$H(x,p) = \max_{1 \leq i \leq n} \left\{ p_i^2 - f_i(x) p_i \right\} \quad \text{for } x \in \mathbb{R}^n,\ p = (p_1, \ldots, p_n) \in \mathbb{R}^n.$$

Clearly, $H(x, 0) = 0$ for all $x \in \mathbb{R}^n$ and hence $\phi_0 := 0$ is a solution of $H(x, Du) = 0$ in $\mathbb{R}^n$. Moreover, H satisfies (A1)$'$, (A2)$'$, (A3), (A6) and (A7)$_\pm$. Let $u_0 \in \mathrm{BUC}(\mathbb{R}^n)$ be such that for any $\varepsilon > 0$ and some $l > 0$,

$$\min_{|y| \leq l} u_0(x+y) < \inf_{\mathbb{R}^n} u_0 + \varepsilon \qquad \text{for all } \ x \in \mathbb{R}^n. \tag{7.9}$$

As usual let u be the solution of the Cauchy problem (1.1) with H and u_0 defined above. We claim here that (7.6) holds with $\phi = 0$, that is,

$$u(\,\cdot\,, t) \longrightarrow \inf_{\mathbb{R}^n} u_0 \qquad \text{in } \ C(\mathbb{R}^n) \quad \text{as } \ t \to \infty.$$

To prove this, we check that the hypotheses of Theorem 7.3 are valid. For this purpose, we may assume without loss of generality that $\inf_{\mathbb{R}^n} u_0 = 0$. Then, $u_0 \geq u_0^- \geq 0$ in $\mathbb{R}^n$. We also observe from the assumption on f_i that, for any $\phi \in \mathcal{S}_H^-$, $\phi(x)$ is non-increasing or non-decreasing with respect to the k-th component of x for every $1 \leq k \leq n$. This and (7.9) imply that $u_0^- = 0$. Let $\{y_j\} \subset \mathbb{R}^n$ be any sequence such that

$$u(\,\cdot\, + y_j, \cdot) \longrightarrow v \qquad \text{in } \ C(\mathbb{R}^n \times [0, \infty)) \quad \text{as } j \to \infty$$

for some $v \in \mathrm{BUC}(\mathbb{R}^n \times [0, \infty))$. Set $v_0 := v(\cdot, 0)$ and remark that $\inf_{\mathbb{R}^n} v_0 = 0$ and v_0 inherits property (7.9). By taking a subsequence of $\{y_j\}$ if necessary, we may assume that

$$f_i(\,\cdot\, + y_j) \longrightarrow g_i \qquad \text{in } \ C(\mathbb{R}^n) \quad \text{as } j \to \infty \quad \text{for each } \ i = 1, \ldots, n$$

for some $g_i \in \mathrm{BUC}(\mathbb{R}^n)$, $i = 1, \ldots, n$. Then, we have $\inf_{\mathbb{R}^n} g_i \geq 0$ or $\sup_{\mathbb{R}^n} g_i \leq 0$ according to the sign of f_i for each $i = 1, \ldots, n$. Now, we set

$$G(x,p) = \max_{1 \leq i \leq n} \left\{ p_i^2 - g_i(x) p_i \right\} \quad \text{for } x \in \mathbb{R}^n,\ p = (p_1, \ldots, p_n) \in \mathbb{R}^n.$$

Then, for any $\phi \in \mathcal{S}_G^-$, $\phi(x)$ is non-increasing or non-decreasing with respect to the k-th component of x for every $1 \leq k \leq n$. This fact together with property (7.9) for v_0 ensures that $v_0^-(x) := \sup\{\psi(x) \mid \psi \in \mathcal{S}_G^-,\ \psi \leq v_0 \text{ in } \mathbb{R}^n\} = 0$ for all $x \in \mathbb{R}^n$. Hence, we have $v_\infty(0) = 0$ and conclude that (1.5) holds.

An important feature of H in Example 7.2 is the following: the polars of $K_H(x) := \{p \in \mathbb{R}^n \mid H(x,p) \le 0\}$, defined as $K_H(x)^* := \{\xi \in \mathbb{R}^n \mid \xi \cdot p \le 0 \text{ for all } p \in K_H(x)\}$, and $K_G(x)^*$ of $K_G(x) := \{p \in \mathbb{R}^n \mid G(x,p) \le 0\}$, with $x \in \mathbb{R}^n$, contain a convex cone K_0 with non-empty interior and with vertex at the origin. More explicitly, if $f_j \ge 0$ in $\mathbb{R}^n$ for all j, then $K_H(x) \cup K_G(x) \subset [0,\, \infty)^n$ and $(-\infty,\, 0]^n \subset K_H(x)^* \cap K_G(x)^*$. That is, in this case, we can select $(-\infty,\, 0]^n$ as K_0. In general, if $H(x,0) = 0$ for all $x \in \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ is a closed convex cone with vertex at the origin such that $K \subset K_H(x)^*$ for all $x \in \mathbb{R}^n$, then we have

$$d_H(x,y) = 0 \quad \text{for any } x, y \in \mathbb{R}^n \text{ such that } x - y \in K.$$

Indeed, since $0 \in \mathcal{S}_H$, we have $d_H(x,y) \ge 0$ for all $x, y \in \mathbb{R}^n$. For the function $v := d_H(\cdot, y)$, with $y \in \mathbb{R}^n$, we have

$$v(y + t\xi) - v(y) \le \int_0^t q(s) \cdot \xi \, ds \le 0 \quad \text{for any } \xi \in K,\ t > 0,$$

where $q \in L^\infty(0,\, t)$ is a function satisfying $H(y + s\xi,\, q(s)) \le 0$ for a.e. $s \in (0,t)$. Accordingly, we see that $d_H(x,y) = 0$ if $x - y \in K$. Moroeover, we see from this that if K has a nonempty interior and u_0 has the the property that for each $\varepsilon > 0$ there is an $l > 0$ such that

$$\inf_{|y| \le l} u_0(x + y) < \inf_{\mathbb{R}^n} u_0 + \varepsilon,$$

then $u_\infty(x) \equiv \inf_{\mathbb{R}^n} u_0$. Using this observation, we may extend the convergence assertion of Example 7.2 to some extent, but we do not give here the details.

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