Asymptotic Solutions for Large Time of Hamilton-Jacobi Equations

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Plan:

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• Introduction.

• Problem: The asymptotic behavior, as $t \to \infty$, of solutions $u = u(x,t)$ of the Cauchy problem

$$(\mathrm{CP}) \qquad \begin{cases} u_t + H(x, Du) = 0 & \text{in } \Omega \times (0,\infty), \\ u|_{t=0} = u_0, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$, $H : \Omega \times \mathbf{R}^n \to \mathbf{R}$, $u : \Omega \times [0,\infty) \to \mathbf{R}$ is the unknown, $u_t = \partial u/\partial t$, $Du = (\partial u/\partial x_1, ..., \partial u/\partial x_n)$, and $u_0 : \Omega \to \mathbf{R}$.

• It is a basic question on evolution PDE. Such investigations concerning Hamilton-Jacobi equations go back to S. N. Kruzkov ('67), P.-L. Lions ('83), and G. Barles ('85).

• An interesting feature of the recent developments is the interaction with weak KAM theory introduced by A. Fathi ('97).

• The large-time behavior of solution of (CP) is related to the "stationary" equation:

$$H(x, Dv) = c \quad \text{in } \Omega, \quad \text{where } c \text{ is a constant.}$$

The structure of solutions of this "stationary" equation can be studied with help of weak KAM theory.

• I call the function $H = H(x, p)$ a Hamiltonian and use the notation

$$H[u] := H(x, Du(x)).$$

• Hamilton-Jacobi equations arise in calculus of variations (mechanics, geometric optics, geometry), optimal control, differential games, etc. They are called Bellman equations in optimal control and Isaacs equations in differential games, where they appear as dynamic programming equations. Basic references are books by W. Fleming-H. M. Soner ('91) and M. Bardi–I. Capuzzo Dolcetta ('97).

• Additive eigenvalue problem.

• From the formal expansion of the solution u of (CP)

$$u(x,t) = a_0(x)t + a_1(x) + a_2(x)t^{-1} + \cdots \qquad \text{as } t \to \infty,$$

one gets

$$a_0(x) + \frac{-a_1(x)}{t^2} + \cdots + H(x, Da_0(x)t + Da_1(x) + Da_2(x)t^{-1} + \cdots) = 0,$$

which suggests

$$\begin{cases} a_0(x) \equiv a_0 \text{ for a constant } a_0, \\ a_0 + H(x, Da_1(x)) = 0. \end{cases}$$

• We are led to the additive eigenvalue problem for H: to find $(c, v) \in \mathbf{R} \times C(\Omega)$ such that

$$H[v] = c \quad \text{in } \Omega.$$

• c is called an (additive) eigenvalue for H, v an (additive) eigenfunction for H.

• If (c, v) is a solution of the additive eigenvalue problem for H, then

$$u(x,t) := -ct + v(x)$$

is a solution of $u_t + H[u] = 0$. The function $-ct + v(x)$ is called an asymptotic solution for $u_t + H[u] = 0$.

• The right notion of weak solution for Hamilton-Jacobi equations is that of viscosity solution introduced by M. G. Crandall–P.-L. Lions ('81). It is based on the maximum principle.

◇ Additive eigenvalue problem arises in ergodic control problems, where one seeks to minimize the long-time average of cost

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t),\alpha(t))\,\mathrm{d}t,$$

$$\begin{cases} \alpha : [0,\infty) \to A \quad \text{(control)}, \quad A \quad \text{(control region)}, \\ \dot{X}(t) = g(X(t),\alpha(t)) \text{ (state equation)}, \quad X(0) = x. \end{cases}$$

• Such an ergodic control problem is closely related to the problem of finding the limit

$$\lim_{t\to\infty} \frac{1}{t} u(x,t)$$

for the solution of $u_t + H[u] = 0$ in $\Omega \times (0,\infty)$, $u|_{t=0} = 0$, where

$$H(x,p) = \sup_{a\in A}(-g(x,a)\cdot p - f(x,a)).$$

◇ Homogenization for Hamilton-Jacobi equations

• Additive eigenvalue problems play an important role in homogenization for Hamilton-Jacobi equations, where they are referred to as cell problems. In this theory one is concerned with the macroscopic effects of small scale oscillating phenomena.

• A standard problem is

$$\lambda u^{\varepsilon}(x) + H(x, x/\varepsilon, Du^{\varepsilon}(x)) = 0 \quad \text{ in } \Omega,$$

where

$$\begin{cases} \lambda > 0 & \text{is a given constant,} \\ \varepsilon > 0 & \text{is the small scale parameter to be sent to zero.} \end{cases}$$

The basic scheme in periodic homogenization:

(i) solve the additive eigenvalue problem for fixed (x, p),

$$H(x, y, p + D_y v(y)) = c \quad \text{for } y \in \mathbf{T}^n := \mathbf{R}^n/\mathbf{Z}^n,$$

$$(\ G(y, q) := H(x, y, p + q)\ )$$

(ii) define the so-called effective Hamiltonian $\bar{H}$ by $\bar{H}(x, p) = c$,

(iii) the limit function $\bar{u}(x) := \lim_{\varepsilon \to 0+} u^\varepsilon(x)$ then satisfies

$$\lambda \bar{u} + \bar{H}(x, D\bar{u}(x)) = 0 \quad \text{in } \Omega.$$

P.-L. Lions–G. Papanicolaou–S. R. S. Varadhan ('87),
L. C. Evans ('89) (the perturbed test functions method)

Almost periodic homogenization: HI ('00) and P.-L. Lions–P. E. Souganidis ('04).

Random homogenization: P. E. Souganidis ('99), P.-L. Lions–P. E. Souganidis ('03), H. Kosygina–F. Rezakhanlou–S. R. S. Varadhan ('06), L. Caffarelli–P. E. Souganidis–L.-H. Wang ('05).

• A remark on Hamilton-Jacobi equations with convex Hamiltonian.

Always assume that H is convex. “solution” instead of “viscosity solution”.

• Notation:

$$\mathcal{S}_H^- \equiv \mathcal{S}_H^-(\Omega) := \{u \text{ solution of } H[u] \le 0 \text{ in } \Omega\},$$

$$\mathcal{S}_H^+ \equiv \mathcal{S}_H^+(\Omega) := \{u \text{ solution of } H[u] \ge 0 \text{ in } \Omega\},$$

$$\mathcal{S}_H \equiv \mathcal{S}_H(\Omega) := \mathcal{S}_H^- \cap \mathcal{S}_H^+.$$

◇ The theory of semicontinuous viscosity solutions due to E. N. Barron–R. Jensen ('90) says: if $H(x,p)$ is convex in $p \in \mathbf{R}^n$, then

$$S \subset \mathcal{S}_H, \quad u(x) := \inf\{v(x) \mid v \in S\} \Rightarrow u \in \mathcal{S}_H.$$

• A result in $\mathbf{T}^n$.

A typical result, regarding the large-time asymptotic behavior of the solution u of (CP), from those obtained by G. Namah and J.-M. Roquejoffre ('97–), A. Fathi ('98), G. Barles–P. E. Souganidis ('00), A. Davini–A. Siconolfi ('06) is stated as follows:

- $\Omega = \mathbf{T}^n$, $u_0 \in C(\mathbf{T}^n)$, $H \in C(\mathbf{T}^n \times \mathbf{R}^n)$.
- H is coercive:

$$\lim_{|p|\to\infty} H(x,p) = \infty \quad \text{uniformly in } x \in \mathbf{T}^n.$$

- H is convex: $p \mapsto H(x,p)$ is convex $\forall x \in \mathbf{T}^n$.

Theorem 1. (i) The additive eigenvalue problem $H[v] = c$ in $\mathbf{T}^n$ has a solution $(c, v) \in \mathbf{R} \times C(\mathbf{T}^n)$. Moreover the constant c is uniquely determined.

(ii) The Cauchy problem $u_t + H[u] = 0$ in $\mathbf{T}^n \times (0, \infty)$, $\quad u|_{t=0} = u_0$ has a unique solution $u \in C(\mathbf{T}^n \times [0, \infty))$.

(iii) Assume that $H = H(x, p)$ is strictly convex in p. Then there exists an additive eigenfunction $u_\infty \in C(\mathbf{T}^n)$ for H such that

$$\lim_{t\to\infty} \max_{x\in\mathbf{T}^n} |u(x, t) + ct - u_\infty(x)| = 0.$$

(iv) The function $u_\infty \in C(\mathbf{T}^n)$ is characterized by

$$u_\infty(x) = \inf\{\phi(x) \mid \phi \in \mathcal{S}_{H-c},\ \phi \geq u_0^- \ \text{ in } \mathbf{T}^n\},$$

where

$$u_0^-(x) := \sup\{\psi(x) \mid \psi \in \mathcal{S}^-_{H-c},\ \psi \leq u_0 \ \text{ in } \mathbf{T}^n\}.$$

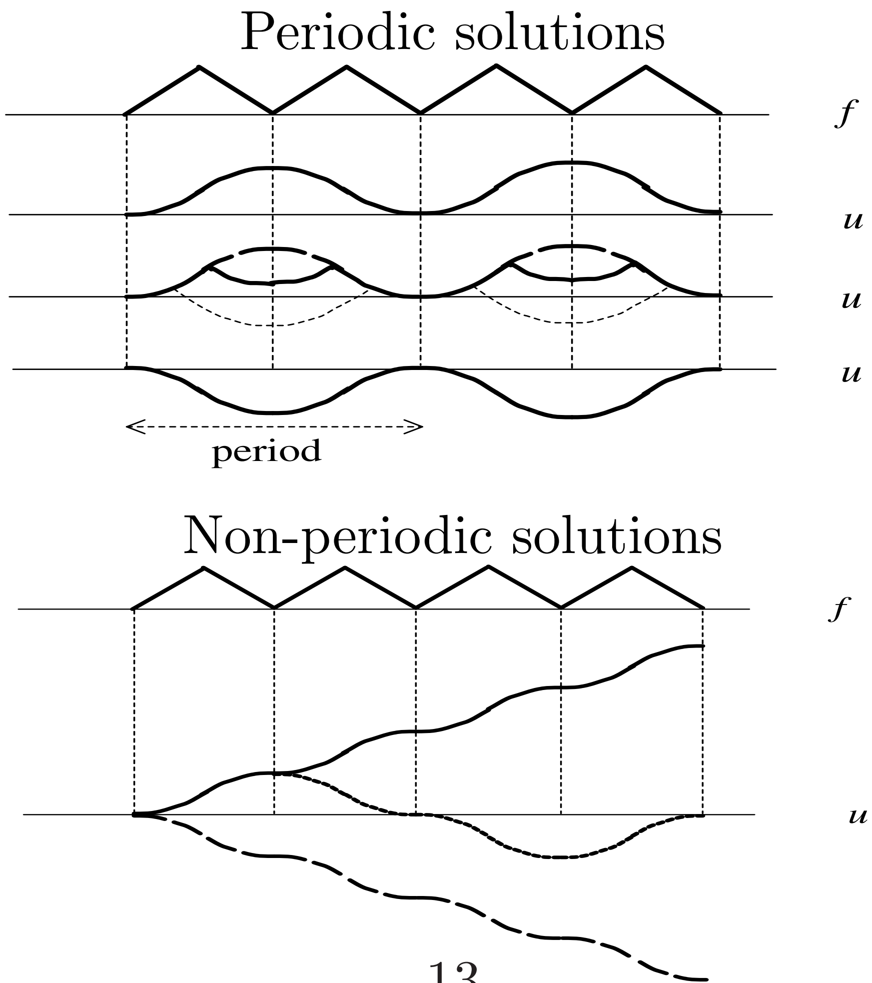
Assertion (i) is due to Lions-Papanicolaou-Varadhan ('87). Assertion (ii) is a more classical result due to M. G. Crandall–P.-L. Lions ('83), M. G. Crandall–L. C. Evans–P.-L. Lions, G. Barles, P. E. Souganidis, HI,...

• A remark is the complex structure of eigenfunctions for H in $\mathbf{T}^n$:

$$v \;\text{ an eigenfunction } \quad \Rightarrow \quad v + a \text{ an eigenfunction for any } a \in \mathbf{R}.$$

The complexity is more than this.

Example. Consider $|Du| = f(x)$ in $\mathbf{R}$, where f is a periodic function and $\min f = 0$. $(c = 0.)$



• A review of weak KAM theory and a formula for asymptotic solutions.

◇ Why is the strict convexity of H needed in Theorem 1?

• It can be replaced by a weaker assumption (G. Barles and P. E. Souganidis ('00)).

• The following example shows that some condition is needed more than the coercivity and convexity of H.

Example (Barles–Souganidis ('00)). Consider the Cauchy problem

$$u_t + |Du + 1| = 1 \quad \text{in } \mathbf{R} \times (0, \infty) \ \text{ and } \ u(x, 0) = \sin x.$$

Then $u(x, t) := \sin(x - t)$ is a classical solution and $u(0, t) = -\sin t$. Hence as $t \to \infty$,

$$u(x, t) \not\to u_\infty(x) - ct.$$

◇ A short review of weak KAM theory:

- $c = 0$ will be assumed: otherwise, replace H by $H - c$.
- Let $\Omega = \mathbf{T}^n$. Let H be coercive and convex. Define

$$d_H(x,y) := \sup\{w(x) - w(y) \mid w \in \mathcal{S}_H^-(\Omega)\}.$$

- The function $d_H(\cdot,y)$ is the maximum subsolution of $H[u] = 0$ in Ω among those satisfying $u(y) = 0$.
- Basic properties:

$$\begin{aligned} &d_H(y,y) = 0, \\ &d_H(\cdot,y) \in \mathcal{S}_H^-(\Omega), \\ &d_H(\cdot,y) \in \mathcal{S}_H(\Omega \setminus \{y\}), \\ &d_H(x,y) \le d_H(x,z) + d_H(z,y). \end{aligned}$$

$$\begin{pmatrix} \mathcal{S}_H^-(\Omega)=\{w \mid\ H[w]\le 0\}, \\ \mathcal{S}_H^+(\Omega)=\{w \mid\ H[w]\ge 0\}, \\ \mathcal{S}_H=\mathcal{S}_H^-(\Omega)\cap\mathcal{S}_H^+(\Omega). \end{pmatrix}$$

Definition: (Projected) Aubry set $\mathcal{A}_H \subset \Omega$ is defined by

$$\mathcal{A}_H := \{y \in \Omega \mid d_H(\cdot, y) \in \mathcal{S}_H(\Omega)\}.$$

$$(\text{for general } y \in \Omega, \quad d_H(\cdot, y) \in \mathcal{S}_H(\Omega \setminus \{y\}).)$$

- Aubry sets play an important role in weak KAM theory.
  The above definition is from A. Fathi and A. Siconolfi ('04).

- In the PDE viewpoint, one of main observations regarding Aubry sets is:

Theorem 2 (representation). If u is a solution of $H[u] = 0$ in $\mathbf{T}^n$, then

$$u(x) = \inf\{u(y) + d_H(x, y) \mid y \in \mathcal{A}_H\} \quad \forall x \in \mathbf{T}^n.$$

The above representation formula and (iv) of Theorem 1 yield the formula:

- $$u_\infty(x) = \inf\{d_H(x,y) + d_H(y,z) + u_0(z) \mid y \in \mathcal{A}_H,\ z \in \mathbf{T}^n\}.$$

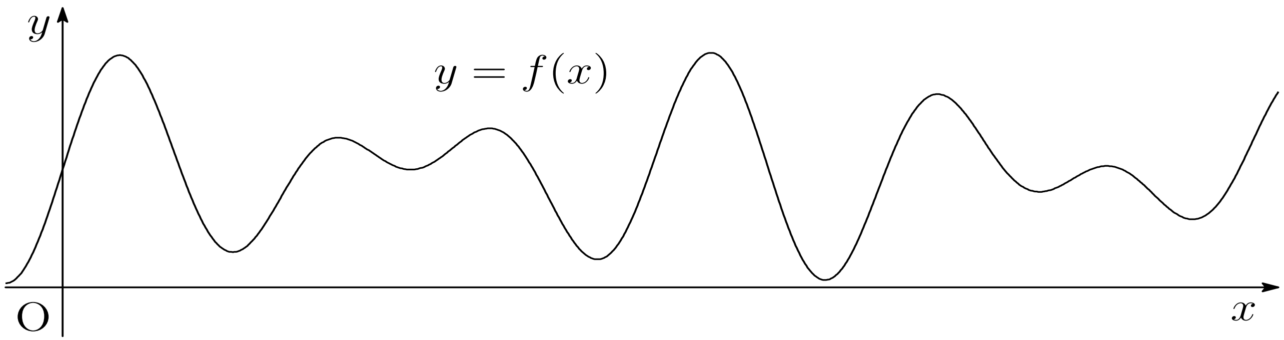
This formula, together with another formulation of Aubry sets and variational formulas for d_H and the solution u, gives a nice insight how the solution u of (CP) converges to the asymptotic solution u_∞ and why the strict convexity of H is useful for the convergence, but let me skip this point.

- **Asymptotic solutions in $\mathbf{R}^n$.**

$\mathbf{T}^n$ compact $\cdots\cdots$ $\mathbf{R}^n$ not compact.

Example 1 (Lions–Souganidis ('03)). Let $n = 1$ and $f(x) = 2 + \sin x + \sin\sqrt{2}x$. Set $H(x,p) = |p|^2 - f(x)^2$.

- f is quasi-periodic,
- $\inf f = 0$ and $f(x) > 0$ for all $x \in \mathbf{R}$.



Consider the Cauchy problem

$$u_t + H[u] = 0 \quad \text{in } \mathbf{R} \times [0,\infty) \qquad \text{and} \qquad u|_{t=0} = 0.$$

- $\exists!$ solution $u \in \mathrm{BUC}(\mathbf{R} \times [0,T]) \quad \forall T > 0,$
- $u \geq 0$ in $\mathbf{R} \times [0,\infty)$.

If u converges to an asymptotic solution $-ct + v(x)$ in $C(\mathbf{R})$ as $t \to \infty$, then $c = 0$ and $v \in \mathcal{S}_H$. Also, $v \geq 0$.

On the other hand, equation $H[v] = 0$ does not have any solution which is bounded below. Therefore, u does not converge to any asymptotic solution.

H quasi-periodic + u_0 periodic $\not\Rightarrow$ convergence to an asymptotic solution

Example 2 (Barles-Souganidis ('00)). Let $n = 1$. Consider the Cauchy problem

$$u_t - Du + \frac{1}{2}|Du|^2 = 0 \quad \text{in } \mathbf{R} \times (0, \infty), \qquad u|_{t=0} = u_0.$$

Lax-Oleinik formula for u:

$$u(x,t) = \inf_{y \in \mathbf{R}} \left(u_0(y) + \frac{|x+t-y|^2}{2t} \right) \qquad \left(u(0,t) = \inf_{y \in \mathbf{R}} \left(u_0(y) + \frac{|t-y|^2}{2t} \right) \right).$$

Assume that $0 \le u_0(x) \le 1$ for all $x \in \mathbf{R}$. Then we have $0 \le u(0,t) \le 1$ for all $t \ge 0$.

If $u_0(t) = 0$, then

$$u(0,t) = 0.$$

If $u_0(x) = 1$ for $x \in [t - \sqrt{2t},\, t + \sqrt{2t}]$, then

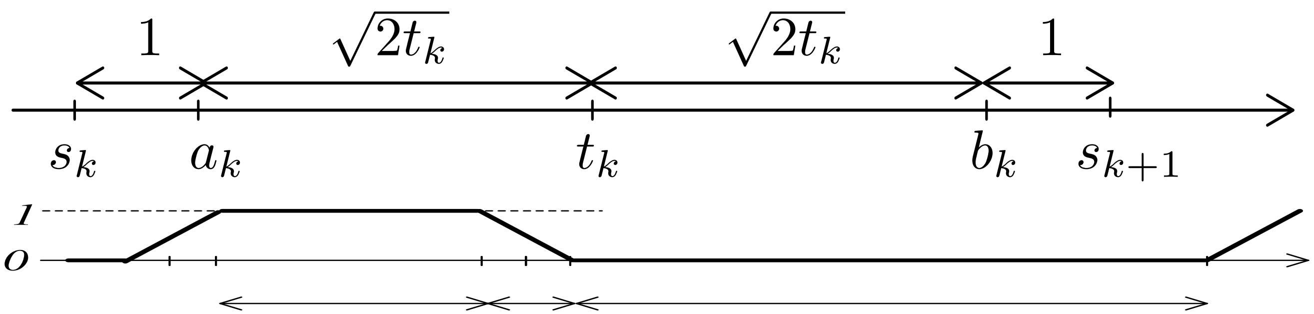
$$u(0,t) = 1.$$

Define the increasing non-negative sequences $\{s_k\}$ and $\{t_k\}$ by

$$
\begin{aligned}
&s_1 = 0, \quad s_1 + 1 = t_1 - \sqrt{2t_1}, \\
&t_1 + \sqrt{2t_1} + 1 = s_2, \quad s_2 + 1 = t_2 - \sqrt{2t_2}, \\
&t_2 + \sqrt{2t_2} + 1 = s_3, \quad s_3 + 1 = t_3 - \sqrt{2t_3}, \\
&\vdots
\end{aligned}
$$

Then set $a_k = t_k - \sqrt{2t_k}$ and $b_k = t_k + \sqrt{2t_k}$ for $k \in \mathbf{N}$, and define the piecewise linear function u_0 by

$$
u_0(x) = \begin{cases} 1 & \text{for } \ a_k \le x \le b_k, \\ 0 & \text{for } x = s_k, \\ 0 & \text{for } x \le s_1. \end{cases}
$$



It is clear that

$$
\begin{aligned}
& s_k < a_k < t_k < b_k < s_{k+1} \quad \text{for all } k \in \mathbf{N}, \\
& \lim_{k\to\infty} s_k = \infty, \\
& u(0, s_k) = 0 \quad \text{and} \quad u(0, t_k) = 1 \quad \forall k \in \mathbf{N}.
\end{aligned}
$$

The conclusion is:

Slowly oscillating initial data $\not\Rightarrow$ convergence to an asymptotic solution

In view of the above examples the following theorem is interesting.

H periodic + u_0 almost periodic $\Rightarrow$ convergence to an asymptotic solution

Let $\Omega = \mathbf{R}^n$.

- H is coercive,
- H is strictly convex,
- $H(x,p)$ is $\mathbf{Z}^n$-periodic in x for all p.

- H satisfies all the assumptions of Theorem 1.
- Define $c :=$ the additive eigenvalue given by Theorem 1. That is, c is the unique constant such that

$$\exists v \in C(\mathbf{R}^n) \text{ such that } H[v] = c \text{ in } \mathbf{R}^n, \quad v \quad \text{is } \mathbf{Z}^n\text{-periodic.}$$

- u_0 is almost periodic in $\mathbf{R}^n$.

Theorem 3. (i) There exists a unique solution u of the Cauchy problem

$$u_t + H[u] = 0 \;\text{ in }\; \mathbf{R}^n \times (0, \infty) \quad \text{and} \quad u|_{t=0} = u_0.$$

$$(u \in \mathrm{BUC}(\mathbf{R}^n \times [0, T]) \quad \forall T > 0)$$

(ii) There exists an almost periodic solution u_∞ of $H[u_\infty] = c$ in Ω for which

$$u(\cdot, t) - u_\infty + ct \to 0 \quad \text{in } C(\Omega) \quad as\ t \to \infty.$$

N. Ichihara–HI.

- Generalizations in terms of "semi-periodic".

◇ A result with compactness of Aubry sets.

- $u_0 \in C(\mathbf{R}^n)$, $H \in C(\mathbf{R}^n \times \mathbf{R}^n)$.
- H is coercive in the sense that

$$\lim_{|p|\to\infty} H(x,p) = \infty \quad \text{uniformly for } x \text{ in each compact subsets of } \mathbf{R}^n.$$

- $H(x,p)$ is strictly convex in p.
- There exist functions $\phi_i \in C(\mathbf{R}^n)$ and $\sigma_i \in C(\mathbf{R}^n)$, with $i = 0, 1$, such that

$$\begin{aligned} H[\phi_i] &\le -\sigma_i \quad \text{in } \mathbf{R}^n, \\ \lim_{|x|\to\infty} \sigma_i(x) &= \infty, \\ \lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) &= \infty. \end{aligned}$$

◇ The function spaces Φ_0, Ψ_0 are convenient:

$$\Phi_0 = \{f \in C(\mathbf{R}^n) \mid \inf_{\mathbf{R}^n}(f - \phi_0) > -\infty\},$$

$$\Psi_0 = \{g \in C(\mathbf{R}^n \times [0,\infty)) \mid \inf_{\mathbf{R}^n\times[0,T]}(g - \phi_0) > -\infty \ \text{ for all } T > 0\}.$$

Theorem 4. (i) The additive eigenvalue problem

$$H[v] = c \quad \text{in } \mathbf{R}^n$$

has a solution $(c, v) \in \mathbf{R} \times \Phi_0$. The additive eigenvalue c is unique.
(ii) There exists a unique solution $u \in \Psi_0$ of the Cauchy problem

$$u_t + H[u] = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty) \quad \text{and} \quad u|_{t=0} = u_0.$$

(iii) There exists a solution $u_\infty \in \Phi_0$ of $H = c$ in $\mathbf{R}^n$ for which

$$u(\cdot, t) + ct - u_\infty \to 0 \quad \text{in } C(\mathbf{R}^n) \text{ as } t \to \infty.$$

Moreover

$$u_\infty(x) = \inf\{d_{H-c}(x, y) + d_{H-c}(y, z) + u_0(z) \mid z \in \mathbf{R}^n,\ y \in \mathcal{A}_{H-c}\}\ \forall x \in \mathbf{R}^n.$$

HI, Y. Fujita–HI–P. Loreti ('06): $u_t + \alpha x \cdot Du + H(Du) = f(x)$, where $\alpha > 0$ and H has the superlinear growth, $\lim_{|p|\to\infty} H(p)/|p| = \infty$.

• A simple example: $u_t + |Du|^2 = |x|$. Then $H(x,p) = |p|^2 - |x|$ and a choice of (ϕ_i, σ_i) is:

$$\phi_1(x) = -|x|, \qquad \sigma_1(x) = |x| - 1, \qquad \phi_0(x) = -\frac{1}{2}|x|, \qquad \sigma_0(x) = |x| - \frac{1}{4}.$$

• Existence of the pairs (ϕ_i, σ_i), $i = 0, 1 \Rightarrow$ the compactness of the Aubry set $\mathcal{A}_{H-c}$.

◇ In unbounded domains the uniqueness of additive eigenvalue does not hold: for unbounded Ω, if

$$c_H = \inf\{a \in \mathbf{R} \mid \exists v \in S^-_{H-a}\},$$

then for any $b \geq c_H$ there exists a solution v of $H[v] = b$ in Ω.

• Restriction of the eigenfunctions to $\Phi_0 \Rightarrow$ uniqueness of the additive eigenvalue. (Theorem 4)

• A related remark is that, under standard assumptions on H (e.g., the periodic case), one can show that if $H[v] = a$ in $\mathbf{R}^n$, $H[w] = b$ in $\mathbf{R}^n$, and

$$\lim_{|x|\to\infty} \frac{|v(x)| + |w(x)|}{|x|} = 0,$$

then $a = b$. (Uniqueness of additive eigenvalue under sublinear growth!)

On the other hand, if $H(x, Dv) = a$ in $\mathbf{R}^n$, $p \neq 0$, $H(x, p + Dw) = b$ in $\mathbf{R}^n$, and

$$\lim_{|x|\to\infty} \frac{|v(x)| + |w(x)|}{|x|} = 0,$$

then $a \neq b$ in general, but $z(x) := p \cdot x + w(x)$ is a solution of $H(x, Dz(x)) = b$ in $\mathbf{R}^n$. (Non-uniqueness of additive eigenvalue under linear growth condition!)

• It is important to understand well the structure of additive eigenfunctions. In this regard, representation formulas of solutions of the stationary problem $H[u] = 0$ like Theorem 2 are useful.

Theorem 2 is due to A. Fathi. There are generalizations to general domains due to HI–H. Mitake, C. Walsh.

Non-uniqueness of additive eigenvalues is created by "ideal boundary points" sitting at infinity. Φ_0 in Theorem 4 kills any contributions from ideal boundary points at infinity.

◇ The final remarks are: (i) Boundary value problems: the state constraint problem by H. Mitake; (ii) Viscous Hamilton-Jacobi equations. (iii) Time-periodic or almost periodic solutions for HJ equations, (iv) Rate of convergence: G. Barles, Y. Fujita,... And my conclusion is that there are a lot to be done even in the case when $\Omega = \mathbf{R}^n$.

Thank you!