Nonlinear oblique derivative problems for singular degenerate parabolic equations on a general domain

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Abstract

We establish comparison and existence theorems of viscosity solutions of the initial-boundary value problem for some singular degenerate parabolic partial differential equations with nonlinear oblique derivative boundary conditions. The theorems cover the capillary problem for the mean curvature flow equation and apply to more general Neumann type boundary problems for parabolic equations in the level set approach to motion of hypersurfaces with velocity depending on the normal direction and curvature.

1 Introduction

In this paper we are concerned with the following boundary value problem

(1.1) $$u_t + F(t, x, u, Du, D^2u) = 0 \qquad \text{in } Q = (0,T) \times \Omega,$$

(1.2) $$B(x, Du) = 0 \qquad \text{in } S = (0,T) \times \partial\Omega,$$

where Ω is a bounded domain in $\mathbf{R}^n$ and $T > 0$. Here $u_t = \partial u/\partial t$, and Du and D^2u denote, respectively, the gradient and Hessian of u. We *assume throughout this paper* that Ω is a bounded domain in $\mathbf{R}^n$ with C^1 boundary.

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We deal with equations (1.1) in a class of singular degenerate parabolic equations which includes the mean curvature flow equation.

The boundary condition (1.2) is a type of the fully nonlinear oblique derivative boundary condition, which will be made precise later on. A typical example we have in mind is the capillary boundary condition

$$\frac{\partial u}{\partial \nu} = a(x)|Du|,$$

where $a(x)$ is a smooth function on $\overline{\Omega}$ with values in $(0,1)$. This boundary condition appears, in fact, as the interpretation of the standard capillary condition in the level set approach to motion of hypersurfaces by curvatures. See, for instance, [S2, ESY]. See also [OS].

In the case when F is continuous in its variables, there are already a lot of comparison and existence results for viscosity solutions of second order degenerate parabolic PDE with boundary condition (1.2). We refer for this to [B1, I] and references therein.

In the case of singular PDE like the mean curvature flow equation, Giga and the second author [GS] have established comparison and existence results for viscosity solutions under the Neumann condition. When Ω is a half space the second author has established comparison and existence theorems under the capillary condition in [S2].

Our aim in this paper is to establish comparison and existence theorems concerning viscosity solutions of (1.1)-(1.2). Our results in this paper extends the results obtained in [GS, S2] to general C^1 bounded domains Ω and fully nonlinear oblique boundary conditions. A variant of these results has been already announced and utilized in studying the stability of motion of hypersurface driven by curvature in the work [ESY] by Ei, Sato, and Yanagida, where this paper is referred as "Capillary problem for singular degenerate parabolic equations". (We hope this change of title does not cause any confusion.)

After we had almost completed this work we noticed the work by Barles [B2] which treated (1.1)-(1.2). (We would like to thank Mariko Arisawa who brought [B2] to our attention.) Compared with our results here, his results [B2] cover more general F and B, but less general domains. This difference is similar to that between [B1] and [I].

This paper is organized as follows. In Section 2 we state and prove our comparison result. In Section 3 we establish our existence result. In Section 4 we explain how to build test functions which are needed in the proof of the comparison and existence theorems. In Section 5 we discuss about a few examples including the Capillary problem for the mean curvature flow equation.

2 A comparison theorem

We start by listing our assumptions on F and B. Henceforth, for $p,q \in \mathbf{R}^n \setminus \{0\}$ we write $\overline{p} = \frac{p}{|p|}$ and $\rho(p,q) = [(|p| \wedge |q|)^{-1}|p-q|] \wedge 1$. Here and henceforth we use the notation: $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$.

(F1) $$F \in C([0,T] \times \overline{\Omega} \times \mathbf{R} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n),$$

where $\mathcal{S}^n$ denotes the space of $n \times n$ real matrices equipped with the usual ordering.

(F2) There exists a constant $\gamma \in \mathbf{R}$ such that for each $(t,x,p,X) \in [0,T] \times \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n$ the function $u \mapsto F(t,x,u,p,X) - \gamma u$ is non-decreasing on $\mathbf{R}$.

(F3) For each $R > 0$ there exists a continuous function $\omega_R : [0,\infty) \to [0,\infty)$ satisfying $\omega_R(0) = 0$ such that if $X, Y \in \mathcal{S}^n$ and $\mu_1, \mu_2 \in [0,\infty)$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then

$$\begin{aligned} &F(t,x,u,p,X) - F(t,y,u,q,-Y) \\ &\qquad \ge -\omega_R(\mu_1(|x-y|^2 + \rho(p,q)^2) + \mu_2 + |p-q| + |x-y|(|p| \vee |q| + 1)). \end{aligned}$$

for all $t \in [0,T],\ x,y \in \overline{\Omega},\ u \in \mathbf{R}$, with $|u| \le R$, and $p,q \in \mathbf{R}^n \setminus \{0\}$.

(B1) $$B \in C(\mathbf{R}^n \times \mathbf{R}^n) \cap C^{1,1}(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})).$$

(B2) For each $x \in \mathbf{R}^n$ the function $p \mapsto B(x,p)$ is positively homogeneous of degree one in p, i.e., $B(x,\lambda p) = \lambda B(x,p)$ for all $\lambda \ge 0$ and $p \in \mathbf{R}^n \setminus \{0\}$.

(B3) There exists a positive constant θ such that $\langle \nu(z), D_pB(z,p) \rangle \ge \theta$ for all $z \in \partial\Omega$ and $p \in \mathbf{R}^n \setminus \{0\}$. Here $\nu(z)$ denotes the unit outer normal vector of Ω at $z \in \partial\Omega$.

We may and do assume that the function ω_R from (F3) is non-decreasing on $[0,\infty)$.

Theorem 2.1. *Suppose that (F1)–(F3) and (B1)–(B3) hold. Let $u \in \mathrm{USC}([0,T) \times \overline{\Omega})$ and $v \in \mathrm{LSC}([0,T) \times \overline{\Omega})$ be, respectively, viscosity sub- and supersolutions of (1.1)–(1.2). If $u(0,x) \le v(0,x)$ for $x \in \overline{\Omega}$, then $u \le v$ on $(0,T) \times \overline{\Omega}$.*

Remark 2.2. Assumptions (F1) and (F3) imply that

$$-\infty < F_*(t,x,u,0,0) = F^*(t,x,u,0,0) < \infty \tag{2.1}$$

holds for all $(t,x,u) \in [0,T] \times \overline{\Omega} \times \mathbf{R}$ and the degenerate ellipticity of F. Here F^* and F_* denote the upper and lower semi-continuous envelopes, which are defined on $[0,T] \times \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$, of F, respectively. Indeed, for any $X, Y \in \mathcal{S}^n$ and $\varepsilon > 0$ we have

$$\begin{aligned} \langle X\xi, \xi \rangle - \langle Y\eta, \eta \rangle &= \langle X(\xi - \eta), \xi - \eta \rangle + 2\langle X(\xi - \eta), \eta \rangle + \langle (X - Y)\eta, \eta \rangle \\ &\le \|X\| |\xi - \eta|^2 + 2\|X\| |\xi - \eta| |\eta| + \|X - Y\| |\eta|^2 \\ &\le \left(\|X\| + \frac{1}{\varepsilon} \|x\| \right) |\xi - \eta|^2 + (\|X - Y\| + \varepsilon \|X\|) |\eta|^2 \quad \text{for } \xi, \eta \in \mathbf{R}^n, \end{aligned}$$

that is,

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \left(1+\frac{1}{\varepsilon}\right)\|X\|\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + (\|X-Y\|+\varepsilon\|X\|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Hence, from (F3) we get

$$\begin{aligned} F(t,y,u,q,Y) \le F(t,x,u,p,X) + \omega_R\Big(&\|X\|\left(1+\frac{1}{\varepsilon}\right)\left(\rho(p,q)^2+|x-y|^2\right) \\ &+\|X-Y\|+\varepsilon\|X\|+|p-q|+|x-y|(|p|\vee|q|+1)\Big) \end{aligned}$$

for all $t\in[0,T]$, $x,y\in\overline{\Omega}$, $u\in\mathbf{R}$, with $|u|\le R$, $p,q\in(\mathbf{R}^n\setminus\{0\})$, and $R>0$. From this it is easily seen that (2.1) holds.

In what follows we use the notation: for any $p,q\in\mathbf{R}^n$,

$$\rho^*(p,q)=\begin{cases}\rho(p,q) & \text{if } \ p,q\ne 0,\\ 1 & \text{if either } \ p=0 \ \text{ or } \ q=0.\end{cases}$$

Note that the function ρ^* is upper semi-continuous on $\mathbf{R}^n\times\mathbf{R}^n$.

Remark 2.3. The last inequality in (F3) can be stated as

$$\begin{aligned} \text{(2.2)}\quad & F_*(t,x,u,p,X)-F^*(t,y,u,q,-Y) \\ &\ge\ -\omega_R(\mu_1(|x-y|^2+\rho^*(p,q)^2)+\mu_2+|p-q|+|x-y|(|p|\vee|q|+1)) \end{aligned}$$

for all $t\in[0,T]$, $x,y\in\overline{\Omega}$, $u\in\mathbf{R}$, with $|u|\le R$, and $p,q\in\mathbf{R}^n$. Notice that this is valid for $p=0$ or $q=0$.

We need the next lemma for the proof of Theorem 2.1. We refer the reader to [I,Lemma 3.4] for a proof of the lemma.

Lemma 2.4 *Assume that (B1) and (B3) hold. For any $\varepsilon\in(0,1)$ there exists a function $\psi\in C^\infty(\overline{\Omega})$ satisfying the properties: $D\psi(x)\ne 0$ for all $x\in\partial\Omega$, $\psi(x)\ge 0$ for all $x\in\overline{\Omega}$, $\langle\nu(x),D\psi(x)\rangle\ge(1-\varepsilon)|D\psi(x)|$ for all $x\in\partial\Omega$, and $\langle D_pB(x,p),D\psi(x)\rangle\ge 1$ for all $(x,p)\in\partial\Omega\times(\mathbf{R}^n\setminus\{0\})$.*

Proof of Theorem 2.1. We may assume by replacing $T>0$ by a smaller number if necessary that u and $-v$ is bounded above on $[0,T)\times\overline{\Omega}$. For any constant $A\ge\max_{x\in\overline{\Omega}}u(0,x)\vee(-v(0,x))$, if we choose a constant $B>0$ large enough, then the functions

$$f(t,x)=-A-Bt \quad \text{and} \quad g(t,x)=A+Bt$$

are, respectively, (viscosity) sub- and supersolutions of (1.1)–(1.2). For such functions f and g, we set

$$\tilde{u}(t,x)=u(t,x)\vee f(t,x) \qquad \text{and} \qquad \tilde{v}(t,x)=v(t,x)\wedge g(t,x),$$

and observe that $\tilde{u}$ and $\tilde{v}$ are, respectively, sub- and supersolutions of (1.1)–(1.2) and that $\tilde{u}(0,x) \le \tilde{v}(0,x)$ for $x \in \overline{\Omega}$. If we can show that $\tilde{u} \le \tilde{v}$ on $[0,T) \times \overline{\Omega}$ for any such f and g, then we see that $u \le v$ on $[0,T) \times \overline{\Omega}$. This observation reduces the proof to the case where u and v are bounded.

Also, the standard technique reduces the proof to the case when $\gamma = 0$ in (F2). Indeed, if $\gamma < 0$, then the functions $\hat{u}(t,x) = e^{\gamma t}u(t,x)$ and $\hat{v}(t,x) = e^{\gamma t}v(t,x)$ are, respectively, sub- and supersolutions of (1.1)–(1.2) with $F(t,x,r,p,X)$ replaced by the function

$$e^{\gamma t}(-\gamma r + F(t,x,e^{-\gamma t}r, e^{-\gamma t}p, e^{-\gamma t}X)).$$

Thus we may assume that u and v are bounded on $[0,T) \times \overline{\Omega}$ and that the function $r \mapsto F(t,x,r,p,X)$ is non-decreasing in $\mathbf{R}$ for each $(t,x,p,X) \in [0,T] \times \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n$.

In view of Lemma 2.4, we may choose a function $\psi \in C^2(\overline{\Omega})$ so that $\psi(x) \ge 0$ for all $x \in \overline{\Omega}$ and $\langle D_p B(x,p), D\psi(x) \rangle \ge 1$ for all $(x,p) \in \partial\Omega \times (\mathbf{R}^n \setminus \{0\})$.

By virtue of Theorem 4.4 in Section 4, there are a function $w \in C^{1,1}(\overline{\Omega} \times \overline{\Omega})$ and a positive constant C such that for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

$$\begin{aligned} &|x-y|^4 \le w(x,y) \le C|x-y|^4, \\ &|D_x w(x,y)| \vee |D_y w(x,y)| \le C|x-y|^3, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &B(x, D_x w(x,y)) \ge 0 \quad \text{if } x \in \partial\Omega, \\ &B(y, -D_y w(x,y)) \ge 0 \quad \text{if } y \in \partial\Omega, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &|D_x w(x,y) + D_y w(x,y)| \le C|x-y|^4, \\ &\rho^*(D_x w(x,y), -D_y w(x,y)) \le C|x-y|, \end{aligned} \tag{2.5}$$

and for a. e. $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

$$D^2 w(x,y) \le C\Big\{|x-y|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + |x-y|^4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\Big\}. \tag{2.6}$$

We argue by contradiction. So we suppose that

$$m_0 := \sup\{u(t,x) - v(t,x) \ : \ (t,x) \in [0,T) \times \overline{\Omega}\} > 0. \tag{2.7}$$

For $\alpha > 0$, $\varepsilon > 0$, $\delta > 0$ we define

$$\begin{aligned} \Psi(t,x,y) &= \frac{\varepsilon}{T-t} + \alpha w(x,y) + \delta(\psi(x) + \psi(y)), \\ \Phi(t,x,y) &= u(t,x) - v(t,y) - \Psi(t,x,y) \end{aligned}$$

for $(t,x,y) \in [0,T) \times \overline{\Omega} \times \overline{\Omega}$. From (2.7) we infer that for sufficiently small $\varepsilon > 0$ and $\delta > 0$, the function Φ attains a maximum greater that $m_0/2$. Fix such δ and ε, and choose a maximum point $(\hat{t},\hat{x},\hat{y})$ of Φ. Note that Φ and $(\hat{t},\hat{x},\hat{y})$ depend on α, ε, δ.

It is now well-known (see, e.g., [CIL]) that

$$\lim_{\varepsilon \searrow 0} \lim_{\alpha \to \infty} \lim_{\delta \searrow 0} \Phi(\hat{t},\hat{x},\hat{y}) = m_0, \tag{2.8}$$

$$(2.9) \qquad \lim_{\alpha\to\infty} \sup\{\alpha w(\hat{x},\hat{y}) \ : \ 0<\delta<1,\ 0<\varepsilon<1\} = 0.$$

We will pass to the limit as $\delta \searrow 0$, $\alpha \to \infty$ in this order. Thus, in view of (2.8), we may assume that $\hat{t} > 0$ and that $u(\hat{t},\hat{x}) > v(\hat{t},\hat{y})$.

Note that

$$(2.10) \qquad B(\hat{x},\hat{p}) > 0 \qquad \text{if } \hat{x} \in \partial\Omega,$$

$$(2.11) \qquad B(\hat{y},\hat{q}) < 0 \qquad \text{if } \hat{x} \in \partial\Omega.$$

To check (2.10), let us assume that $\hat{x} \in \partial\Omega$. If $D_x w(\hat{x},\hat{y}) = 0$, then we have for some $\hat{\xi} \in \mathbf{R}^n \setminus \{0\}$,

$$B(\hat{x},\hat{p}) = B(\hat{x},0) + \delta\langle D_p B(\hat{x},\hat{\xi}), D\psi(\hat{x})\rangle \geq \delta.$$

Consider the case where $D_x w(\hat{x},\hat{y}) \neq 0$. Suppose for the moment that $\alpha D_x w(\hat{x},\hat{y}) + sD\psi(\hat{x}) = 0$ for some $s > 0$. Let $r > 0$ be the smallest among such s. Then we have for some $\hat{\eta} \in \mathbf{R}^n \setminus \{0\}$,

$$B(\hat{x}, \alpha D_x w(\hat{x},\hat{y}) + rD\psi(\hat{x})) = B(\hat{x}, \alpha D_x w(\hat{x},\hat{y})) + r\langle D_p B(\hat{x},\hat{\eta}), D\psi(\hat{x})\rangle \geq r.$$

This is a contradiction, which shows that $\alpha D_x w(\hat{x},\hat{y}) + sD\psi(\hat{x}) = 0$ for all $s > 0$. Moreover, the above computation shows that $B(\hat{x},\hat{p}) \geq \delta$. Thus (2.10) is valid. Similarly we can show that (2.11) holds, the details of which we leave to the reader.

We apply the maximum principle for semi-continuous functions (see [CIL]), to find matrices $X, Y \in \mathcal{S}^n$ such that

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3C\alpha|\hat{x}-\hat{y}|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_1\left(\alpha|\hat{x}-\hat{y}|^4 + \delta\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

where C is the constant from (2.6) and $C_1 = C \vee \sup_{x\in\Omega} \|D^2\psi(x)\|$, and such that

$$\frac{\varepsilon}{(T-\hat{t})^2} + F_*(\hat{t},\hat{x},\hat{u},\hat{p},X) - F^*(\hat{t},\hat{y},\hat{v},\hat{q},-Y) \leq 0,$$

where

$$\hat{u} = u(\hat{t},\hat{x}), \qquad \hat{v} = v(\hat{t},\hat{y}),$$
$$\hat{p} = \alpha D_x w(\hat{t},\hat{x}) + \delta D\psi(\hat{x}), \qquad \hat{q} = -\alpha D_y w(\hat{t},\hat{y}) - \delta D\psi(\hat{y}).$$

Using (2.2) and writing $\omega = \omega_R$, where $R = \sup_{[0,T)\times\overline{\Omega}}(|u|+|v|)$, we get

$$0 \geq \frac{\varepsilon}{T^2} + F_*(\hat{t},\hat{x},\hat{u},\hat{p},X) - F^*(\hat{t},\hat{y},\hat{u},\hat{q},-Y) \geq \frac{\varepsilon}{T^2} - \omega(r_1+r_2+r_3),$$

where

$$r_1 = 3C\alpha|\hat{x}-\hat{y}|^2(|\hat{x}-\hat{y}|^2 + \rho^*(\hat{p},\hat{q})^2),$$
$$r_2 = C_1(\alpha|\hat{x}-\hat{y}|^4 + \delta),$$
$$r_3 = |\hat{p}-\hat{q}| + |\hat{x}-\hat{y}|(|\hat{p}| \vee |\hat{q}| + 1).$$

Sending $\delta \searrow 0$ along a sequence, we may assume that $\hat{t} \to \bar{t}$, $\hat{x} \to \bar{x}$, $\hat{y} \to \bar{y}$, $\hat{p} \to \bar{p}$, $\hat{q} \to \bar{q}$, and $r_i \to s_i$ for $i = 1, 2, 3$. We then get

$$0 \geq \frac{\varepsilon}{(T - \bar{t})^2} - \omega(s_1 + s_2 + s_3), \tag{2.12}$$

$$\bar{p} = \alpha D_x w(\bar{x}, \bar{y}), \qquad \bar{q} = -\alpha D_y w(\bar{x}, \bar{y}),$$
$$s_1 \leq 3C\alpha(1 + C)|\bar{x} - \bar{y}|^4 \leq 3C(1 + C)\alpha w(\hat{x}, \hat{y}),$$
$$s_2 \leq C_1 \alpha w(\bar{x}, \bar{y}),$$

$$\begin{aligned} s_3 &\leq |\bar{p} - \bar{q}| + |\bar{x} - \bar{y}|(|\bar{p}| \vee |\bar{q}| + 1) \\ &\leq +C\alpha|\hat{x} - \hat{y}|^4 + |\hat{x} - \hat{y}|(C\alpha|\hat{x} - \hat{y}|^3 + 1) \\ &\leq 2C\alpha w(\bar{x}, \bar{y}) + |\bar{x} - \bar{y}|. \end{aligned}$$

Sending $\alpha \to \infty$ in (2.12), we get a contradiction, which proves that $\sup_{[0,T)\times\overline{\Omega}}(u-v) \leq 0$. □

3 An existence theorem

We next show the existence of a viscosity solution of the initial-boundary value problem

$$u_t + F(t, x, u, Du, D^2u) = 0 \qquad \text{in } Q, \tag{3.1}$$

$$B(x, Du) = 0 \qquad \text{on } S, \tag{3.2}$$

$$u(0, x) = g(x) \qquad \text{for } x \in \overline{\Omega}, \tag{3.3}$$

where $g \in C(\overline{\Omega})$ is a given function.

The main result in this section is the following.

Theorem 3.1. *Assume that (F1)–(F3) and (B1)–(B3) hold. Then for each $g \in C(\overline{\Omega})$ there is a (unique) viscosity solution $u \in C([0, T) \times \overline{\Omega})$ of (3.1)–(3.2) satisfying (3.3).*

The uniqueness assertion above is an immediate consequence of Theorem 2.1.

Proof. We use the Perron method (see [CIL]) to show the existence of a continuous viscosity solution.

Thus the first step is to build sub- and supersolutions of (3.1)-(3.2) satisfying (3.3).

If we introduce the new unknown $\hat{u}(t, x) = e^{\gamma t}u(t, x)$, where $\gamma \in \mathbf{R}$ is the constant form (F2), then the problem (3.1)–(3.3) is reduced to the case when $\gamma = 0$. Hence, we may assume that $\gamma = 0$.

According to Theorem 4.4 in Section 4 below, there is a function $w \in C(\overline{\Omega} \times \overline{\Omega})$ having the following properties:

(3.4) $B(x, D_x w(x,y)) \geq 0$ for $x \in \partial\Omega, y \in \overline{\Omega}$,

(3.5) $B(y, -D_y w(x,y)) \leq 0$ for $y \in \partial\Omega, x \in \overline{\Omega}$,

(3.6) $|x-y|^4 \leq w(x,y) \leq C|x-y|^4$ for all $x, y \in \overline{\Omega}$ and for some constant $C > 0$.

Choose a constant $C_0 > 0$ so that $|g(x)| \leq C_0$ for all $x \in \overline{\Omega}$.

As we observed in Remarks 2.2 and 2.3, we have

$$F^*(t,y,u,q,Y) \leq F_*(t,x,u,p,X) \\ +\omega_R\Big(2\|X\|(|x-y|^2 + \rho^*(p,q)^2) + \|X-Y\| + \|X\| + |x-y|(|p| \vee |q| + 1)\Big)$$

for all $t \in [0,T)$, $x, y \in \overline{\Omega}$, $u \in \mathbf{R}$, with $|u| \leq R$, $p, q \in \mathbf{R}^n$, $X, Y \in \mathcal{S}^n$, and $R > 0$.

Fix any $\bar{p} \in \mathbf{R}^n \setminus \{0\}$. Using the above observation, for each $\alpha > 0$ we have

$$F_*(t,x,-C_0,\alpha D_x w(x,y), \alpha D_y^2 w(x,y)) \geq F_*(t,x,-C_0,\bar{p},0) - C_1(\alpha) \geq -C_2(\alpha)$$

for some constants $C_i(\alpha) > 0$, with $i = 1, 2$. Similarly, for any $\alpha > 0$ and for some constant $C_3(\alpha) > 0$, we have

$$F^*(t,x,C_0,-\alpha D_y w(x,y), -\alpha D_y^2 w(x,y)) \leq C_3(\alpha).$$

For each $0 < \varepsilon < 1$ we can choose constants $\alpha(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ such that for all $x, y \in \overline{\Omega}$,

$$|g(x) - g(y)| \leq \varepsilon + \alpha(\varepsilon) w(x,y),$$
$$C_2(\alpha(\varepsilon)) \vee C_3(\alpha(\varepsilon)) \leq \beta(\varepsilon).$$

We define functions $V^\pm$ on $[0,T) \times \overline{\Omega}$ parametrized by $\varepsilon, y \in \overline{\Omega}$, respectively, by

$$V^+(t,x;\varepsilon,y) = g(y) + \varepsilon + \alpha(\varepsilon) w(x,y) + \beta(\varepsilon) t,$$

$$V^-(t,x;\varepsilon,y) = g(y) - \varepsilon - \alpha(\varepsilon) w(y,x) - \beta(\varepsilon) t.$$

It is easily seen that functions $V^+(t,x;\varepsilon,y)$ and $V^-(t,x;\varepsilon,y)$ are classical sub- and supersolutions of (3.1)–(3.2), respectively. Moreover, for all $x, y \in \overline{\Omega}$, $0 < t < T$, and $0 < \varepsilon < 1$, we have

$$V^-(t,x;\varepsilon,y) \leq g(x) \leq V^+(t,x;\varepsilon,y),$$
$$V^-(0,x;\varepsilon,x) = g(x) = V^+(0,x;\varepsilon,x).$$

Next we define functions $f^\pm$ on $[0,T) \times \overline{\Omega}$ by

$$f^+(t,x) = \inf\{V^+(t,x;\varepsilon,y) \ : \ 0 < \varepsilon < 1, \ y \in \overline{\Omega}\},$$

$$f^-(t,x) = \sup\{V^-(t,x;\varepsilon,y) \ : \ 0 < \varepsilon < 1, \ y \in \overline{\Omega}\}.$$

Then we easily deduce that $f^\pm$ are continuous on $[0,T) \times \overline{\Omega}$ and $f^\pm(0,x) = g(x)$ for all $x \in \overline{\Omega}$ and that f^+ and f^- are viscosity sub- and supersolutions of (3.1)–(3.2).

Now we conclude by the Perron method together with Theorem 2.1 that if we define the function u on $[0,T)\times\overline{\Omega}$ by

$$u(t,x)=\sup\{v(t,x)\ :\ v\in S^-\},$$

where S^- denotes the set of functions v on $[0,T)\times\overline{\Omega}$ such that v is a viscosity subsolution of (3.1)–(3.2) and such that $f^-\le v\le f^+$ on $[0,T)\times\overline{\Omega}$, then u is a continuous function on $\overline{\Omega}\times[0,T)$ and a viscosity solution of (3.1)–(3.2). Noting that u satisfies (3.3), we conclude the proof. □

4 Construction of a test function

Since Ω is a bounded C^1 domain, there are a compact neighborhood N of $\partial\Omega$ and a vector field $\tilde\nu\in C(N,\mathbf{R}^n)$ such that

$$|\tilde\nu|=1\quad\text{on}\ \ N\quad\text{and}\quad\tilde\nu=\nu\quad\text{on}\ \ \partial\Omega.$$

We fix such a pair N and $\tilde\nu$, and we write ν for $\tilde\nu$.

For any vector field $\mu\in C(N,\mathbf{R}^n)$ and constant $\sigma>0$ we write

$$W(\mu,\sigma)=\{(x,\xi)\in N\times(\mathbf{R}^n\setminus\{0\})\ :\ |\langle\mu(x),\xi\rangle|\le\sigma|\xi|\}.$$

Lemma 4.1. *Assume (B1)–(B3). Then there are positive constants σ, C and a function $u\in C^{1,1}(W(\nu,\sigma))$ such that for all $(x,\xi)\in W(\nu,\sigma)$,*

$$|\xi|^2\le u(x,\xi)\le C|\xi|^2, \tag{4.1}$$

$$u(x,\lambda\xi)=\lambda^2u(x,\xi)\qquad\textit{for all}\ \lambda\in[0,\infty), \tag{4.2}$$

$$B(x,D_\xi u(x,\xi))=0. \tag{4.3}$$

Remark 4.2. If we assume (B1)–(B2), then there is a constant $M>0$ such that

$$|p|^{-1}|B(x,p)|\vee|D_pB(x,p)|\vee|p|\|D_p^2B(x,p)\|\le M\qquad\text{a. e. }(x,\xi)\in N\times\mathbf{R}^n. \tag{4.4}$$

As the proof below shows, the constants σ and C can be chosen to depend only on θ from(B3) and M.

Proof. Our proof parallels that of [I, Lemma 4.3].

We prove this lemma in the case when $B\in C^2(N\times\mathbf{R}^n)$. The general case can be treated by a limiting argument based on smooth approximations of B, but the details will be left to the reader.

Let $\sigma > 0$ be a constant to be fixed later on. Choose a C^2 vector field $\mu : N \to \mathbf{R}^n$ so that $|\mu| = 1$ and $|\mu - \nu| \le \sigma/2$ on N.

We define the function u on $N \times \mathbf{R}^n$ by

$$u(x, \xi) = \sup\{\langle p, \xi\rangle - \tfrac{1}{2}|p - \langle p, \mu(x)\rangle \mu(x)|^2 \ : \ p \in \mathbf{R}^n,\ B(x, p) = 0\}.$$

It is easy to check that u is positively homogeneous of degree two, i.e., $u(x\lambda\xi) = \lambda^2 u(x, \xi)$ for all $(x, \xi, \lambda) \in N \times \mathbf{R}^n \times [0, \infty)$.

We fix any $z \in N$ and examine the function $u(x, \xi)$ for x in a neighborhood of z.

There are a neighborhood $V \subset N$ of z and a family $\{e_1(x), ..., e_n(x)\}_{x\in V}$ of orthonormal bases of $\mathbf{R}^n$ such that $e_n(x) = \mu(x)$ for $x \in V$ and $e_i \in C^2(V)$ for $i = 1, ..., n$.

We set

$$\tilde{B}(x, p) = B(x, p_1 e_1(x) + \cdots + p_n e_n(x)).$$

Note that

$$\begin{aligned} &\tilde{B} \in C(V \times \mathbf{R}^n), \\ &\tilde{B} \in C^{1,1}(V \times (\mathbf{R}^n \setminus \{0\})). \end{aligned}$$

Setting

$$E(x) = \begin{pmatrix} e_1(x) \\ \vdots \\ e_n(x) \end{pmatrix},$$

we have

$$\begin{aligned} D_p \tilde{B}(x, p) &= E(x) D_p B\Big(x, \sum_{i=1}^{n} p_i e_i(x)\Big), \\ D_p^2 \tilde{B}(x, p) &= E(x) D_p^2 B\Big(x, \sum_{i=1}^{n} p_i e_i(x)\Big) E(x)^*, \end{aligned}$$

where $E(x)^*$ denotes the transposed matrix of $E(x)$. According to (4.4), we thus have

$$|p|^{-1}|\tilde{B}(x, p)| \vee |D_p \tilde{B}(x, p)| \vee |p| \|D_p^2 \tilde{B}(x, p)\|) \le M \qquad \text{for all } (x, p) \in V \times (\mathbf{R}^n \setminus \{0\}).$$

Note as well that the function $p \mapsto \tilde{B}(x, p)$ is positively homogeneous of degree one for each $x \in V$.

We now need to assume that $M\sigma \le \theta$. Since

$$\begin{aligned} \frac{\partial}{\partial p_n} \tilde{B}(x, p) \ &= \ \Big\langle \mu(x), D_p B\Big(x, \sum_{i=1}^{n} p_i e_i(x)\Big)\Big\rangle \\ &\ge \ \theta - \frac{M\sigma}{2} \ge \frac{\theta}{2} \quad \text{for all } (x, p) \in V \times (\mathbf{R}^n \setminus \{0\}). \end{aligned} \tag{4.5}$$

by (B3), there is a unique continuous function H on $V \times \mathbf{R}^{n-1}$ such that

$$\tilde{B}(x, p) = 0 \quad \text{if and only if} \ \ p_n + H(x, p_1, ..., p_{n-1}) = 0. \tag{4.6}$$

It follows from the homogeneity of $\tilde{B}$ and the uniqueness of H that $q \mapsto H(x,q)$ is positively homogeneous of degree one. Also, it is easily seen that there is a constant $A > 0$ depending only on θ and M such that

$$(4.7) \quad |q|^{-1}|H(x,q)| \vee |D_q H(x,q)| \vee |q| \|D_q^2 B(x,q)\| \le A \quad \text{for all } (x,\xi) \in V \times (\mathbf{R}^{n-1} \setminus \{0\}).$$

For $(x,\xi) \in V \times \mathbf{R}^n$, we have

$$(4.8) \qquad \begin{aligned} u(x,\xi) &= \sup\Big\{\Big\langle \sum_{i=1}^{n-1} q_i e_i(x) - H(x,q), \xi \Big\rangle - \frac{1}{2}|q|^2 \ : \ q \in \mathbf{R}^{n-1}\Big\} \\ &= \sup\Big\{\sum_{i=1}^{n-1} q_i \langle e_i(x), \xi\rangle - tH(x,q)\langle e_n(x), \xi\rangle - \frac{1}{2}|q|^2 \ : \ q \in \mathbf{R}^{n-1}\Big\}. \end{aligned}$$

For $(x,\xi) \in V \times \mathbf{R}^n$, if $\xi = \sum_{i=1}^{n-1} y_i e_i(x) + t e_n(x)$, then we have

$$(x,\xi) \in W(\mu,\sigma) \quad \text{if and only if} \quad |t| \le \sigma |y| \ \text{ and } \ t \ne 0.$$

We set

$$\widetilde{W} = \{(x,y,t) \in V \times \mathbf{R}^{n-1} \times (\mathbf{R} \setminus \{0\}) \ : \ |t| \le \sigma |y|\},$$

$$\tilde{u}(x,y,t) = u\Big(x, \sum_{i=1}^{n-1} y_i e_i(x) + t e_n(x)\Big) \quad \text{for } (x,y,t) \in V \times \mathbf{R}^{n-1} \times \mathbf{R}.$$

From (4.8) we see that

$$(4.9) \qquad \tilde{u}(x,y,t) = \sup\left\{\langle q, y\rangle - tH(x,q) - \frac{1}{2}|q|^2 \ : \ q \in \mathbf{R}^{n-1}\right\}.$$

It is now obvious that for each $(x,y,t) \in V \times \mathbf{R}^{n-1} \times \mathbf{R}$, the above optimization problem has a maximizer q and it satisfies the relation

$$(4.10) \qquad q = y - tD_q H(x,q).$$

Let $(x,y,t) \in \widetilde{W}$ and $q \in \mathbf{R}^{n-1}$ satisfy (4.10). As in [I, the proof of Lemma 4.3], we easily see that if $\sigma \le 1/(3A)$, then

$$|q - y| \le A|t| \le \frac{1}{3}|y|, \qquad \frac{2}{3}|y| \le |q| \le \frac{4}{3}|y|,$$

$$\|tD_q^2 H(x,q)\| \le \frac{1}{2}, \qquad \|(I + tD_q^2 H(x,q))^{-1}\| \le 2,$$

and that the solution q of (4.10) is unique. We write $p(x,y,t)$ for the unique solution q of (4.10).

Now, as in [I], we see that $\tilde{u} \in C^2(\widetilde{W})$, $D_y \tilde{u}(x,y,t) = p(x,y,t)$ in $\widetilde{W}$, and

$$\tilde{u}_t(x,y,t) + H(x, D_y \tilde{u}(x,y,t)) = 0 \qquad \text{in } \widetilde{W}.$$

Furthermore fixing $\sigma = \left(\frac{1}{4A}\right) \wedge \left(\frac{M}{\theta}\right)$, we get

$$\tilde{u}(x,y,t) \geq \frac{1}{2}|y|^2 - tH(x,y) \geq \frac{1}{4}|y|^2,$$

$$\tilde{u}(x,y,t) \leq \frac{1}{2}|y|^2 - tH(x,p(x,y,t)) \leq \left(\frac{1}{2} + A|t||p(x,y,t)|\right)|y|^2 \leq 2|y|^2.$$

Hence, noting that

$$|y|^2 + t^2 \leq (1+\sigma^2)|y|^2 \qquad \text{for } (x,y,t) \in \widetilde{W},$$

we get

$$\frac{1}{4(1+\sigma^2)}(|y|^2+t^2) \leq \tilde{u}(x,y,t) \leq 2(|y|^2+t^2) \quad \text{for } (x,y,t) \in \widetilde{W}.$$

These readily imply that $u \in C^2(W(\mu,\sigma))$,

$$B(x, D_\xi u(x,\xi)) = 0 \qquad \text{in } W(\mu,\sigma),$$

$$\frac{1}{4(1+\sigma^2)}|\xi|^2 \leq u(x,\xi) \leq 2|\xi|^2 \quad \text{for all } (x,\xi) \in W(\mu,\sigma).$$

What remains is to observe that $W(\nu,\sigma/2) \subset W(\mu,\sigma)$ and that the function $4(1+\sigma^2)u(x,\xi)$ satisfies (4.1)–(4.3) with $\sigma/2$ replacing σ. □

Lemma 4.3. *Assume (B1)–(B3). Then there are a positive constant σ and a function $v \in C^{1,1}(\overline{\Omega} \times \mathbf{R}^n)$ such that for all $(x,\xi) \in \overline{\Omega} \times \mathbf{R}^n$,*

$$v(x,\xi) \geq |\xi|^2, \tag{4.11}$$

$$v(x,\lambda\xi) = \lambda^2 v(x,\xi) \qquad \text{for all } \lambda \in [0,\infty), \tag{4.12}$$

$$B(x, D_\xi v(x,\xi)) \geq 0 \quad \text{if } x \in \partial\Omega \text{ and } \langle \nu(x), \xi\rangle \geq -\sigma|\xi|, \tag{4.13}$$

$$B(x, D_\xi v(x,\xi)) \leq 0 \quad \text{if } x \in \partial\Omega \text{ and } \langle \nu(x), \xi\rangle \leq \sigma|\xi|, \tag{4.14}$$

Proof. Again, our proof is similar to that of [I, Lemma 4.2].

We divide our arguments into two steps.

Step 1. Choose $\sigma > 0$ and $u \in C^{1,1}(W(\nu,\sigma))$ as in Lemma 4.1. As before we choose a vector field $\mu \in C(N)$ so that $|\mu| = 1$ and $|\nu - \mu| \leq \sigma/24$ on N. Note that, since $|\nu - \mu| \leq \sigma/2$ on N, we have

$$W(\mu,\sigma/2) \subset W(\nu,\sigma).$$

Replacing $\sigma > 0$ by a smaller number, we may assume that $4M\sigma \leq \theta$, where M is a positive constant such that (4.4) holds, and that

$$\langle D_p B(x,p), \mu(x)\rangle \geq \frac{\theta}{2} \qquad \text{for all } (x,p) \in N \times (\mathbf{R}^n \setminus \{0\}).$$

We choose a C^∞ function f on $\mathbf{R}$ such that $0 \leq f \leq 1$ on $\mathbf{R}$, $f(r) = 1$ if $|r| \leq 1$ $f(r) = 0$ if $|r| \geq 2$, $f'(r) \leq 0$ for $r \geq 0$ and $f'(r) \geq 0$ for $r \leq 0$. Also we choose a C^∞ function g on $\mathbf{R}$ such that $0 \leq g \leq 1$ on $\mathbf{R}$, $g(r) = 0$ if $|r| \leq \frac{1}{2}$, $g(r) = 1$ if $|r| \geq 1$, $g'(r) \geq 0$ for $r \geq 0$ and $g'(r) \leq 0$ for $r \leq 0$. We may assume that $|f'(r)| \vee |g'(r)| \leq 3$ for all $r \in \mathbf{R}$.

Setting $a = \sigma/6$, we define functions f_a, g_a on $N \times \mathbf{R}^n$ by

$$f_a(x,\xi) = \begin{cases} f\left(\dfrac{\langle \xi, \mu(x)\rangle}{a|\xi|}\right) & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases}$$

$$g_a(x,\xi) = \begin{cases} g\left(\dfrac{\langle \xi, \mu(x)\rangle}{a|\xi|}\right) & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Note that if $(x,\xi) \in (N \times \mathbf{R}^n) \setminus W(\mu, \sigma/2)$, then $|\langle \mu(x), \xi\rangle| \geq \frac{\sigma}{2}|\xi| = 3a|\xi|$ and hence $f_a(x,\xi) = 0$. Let $b > 0$ be a constant to be fixed later on. We set

$$v(x,\xi) = f_a(x,\xi)u(x,\xi) + g_a(x,\xi)\frac{\langle \xi, \nu(x)\rangle^2}{b}.$$

It is easy to see that

$$v(x,\lambda\xi) = \lambda^2 v(x,\xi) \qquad \text{for all } (x,\xi,\lambda) \in N \times \mathbf{R}^n \times [0,\infty),$$

$$v \in C^{1,1}(N \times \mathbf{R}^n).$$

We are to show that v satisfies (4.13)–(4.14) for some σ.

Fix $(x,\xi) \in N \times (\mathbf{R}^n \setminus \{0\})$. If $D_\xi v(x,\xi) = 0$, then $B(x, D_\xi v(x,\xi)) = 0$. Therefore, we may assume that $D_\xi v(x,\xi) \neq 0$. We compute that

$$\begin{aligned} B(x, D_\xi v(x,\xi)) &= B\Big(x, uD_\xi f_a + f_a D_\xi u + \frac{\langle b,\mu\rangle}{b} D_\xi g_a + g_a \frac{2\langle \xi,\mu\rangle\mu}{b}\Big) \\ &= f_a B(x, D_\xi u) \\ &\quad + \Big\langle D_p B(x,\hat{p}), uD_\xi f_a + \frac{\langle \xi,\mu\rangle^2}{b} D_\xi g_a + g_a \frac{2\langle \xi,\mu\rangle\mu}{b}\Big\rangle \end{aligned} \tag{4.15}$$

for some $\hat{p} \in (\mathbf{R}^n \setminus \{0\})$.

Consider the case when $|\langle \xi, \mu(x)\rangle| \leq \frac{a}{2}|\xi|$. We have $g_a(x,\xi) = 0$, $f_\alpha(x,\xi) = 1$ and $D_\xi g_a(x,\xi) = D_\xi f_a(x,\xi) = 0$. Hence, noting that $(x,\xi) \in W(\nu,\sigma)$, we have

$$B(x, D_\xi v(x,\xi)) = B(x, D_\xi u(x,\xi)) = 0.$$

Next consider the case when $\langle \xi, \mu(x)\rangle > 2a|\xi|$. Since $f_a(x,\xi) = 0$, $g_a(x,\xi) = 1$, and $D_\xi f_a(x,\xi) = D_\xi g_a(x,\xi) = 0$, from (4.15) we have

$$B(x, D_\xi v(x,\xi)) = B(x,0) + \Big\langle D_p B(x,\hat p), \frac{2g_a}{b}\langle \xi,\mu\rangle\mu\Big\rangle \geq \frac{\theta}{b}\langle \xi,\mu\rangle > 0.$$

The case when $\langle \xi, \mu(x)\rangle < -2a|\xi|$ can be treated similarly to the previous case, and in this case we have

$$B(x, D_\xi v(x,\xi)) < 0.$$

Now, we consider the case when $\frac{a}{2}|\xi| < |\langle \xi,\mu(x)\rangle| \leq 2a|\xi|$. We want to show that

$$B(x, D_\xi v(x,\xi)) \geq 0 \quad \text{if } \langle \xi,\mu(x)\rangle > 0,$$

$$B(x, D_\xi v(x,\xi)) \leq 0 \quad \text{if } \langle \xi,\mu(x)\rangle < 0.$$

We only deal with the case when $\langle \xi,\mu(x)\rangle > 0$, i.e, the case when $\frac{a}{2}|\xi| < \langle \xi,\mu(x)\rangle \leq 2a|\xi|$. The other case can be treated in a parallel way and we do not give the details in this case.

Note that

$$D_\xi f_a = f_a'\left(\frac{\nu}{a|\xi|} - \frac{\langle \xi,\nu\rangle}{a|\xi|^3}\xi\right),$$
$$D_\xi g_a = g_a'\left(\frac{\nu}{a|\xi|} - \frac{\langle \xi,\nu\rangle}{a|\xi|^3}\xi\right)$$

where $f_a' = f'(\frac{\langle \xi,\nu\rangle}{a|\xi|})$ and $g_a' = g'(\frac{\langle \xi,\nu\rangle}{a|\xi|})$. Therefore,

$$\begin{aligned}\langle D_p B(x,\hat p), u D_\xi f_a\rangle &\geq -|u|\,|D_p B(x,\hat p)|\frac{|f_a'|}{a|\xi|}\\ &\geq \begin{cases} -\dfrac{3CM|\xi|g_a}{a} & \text{if } a|\xi| \leq \langle \xi,\mu(x)\rangle \leq 2a|\xi|,\\ 0 & \text{if } \dfrac{1}{2}a|\xi| \leq \langle \xi,\mu(x)\rangle \leq a|\xi|.\end{cases}\end{aligned}$$

Here and henceforth $\hat p$ and C are from (4.15) and and (4.1), respectively. Also, we have

$$\begin{aligned}\left\langle D_p B(x,\hat p), \frac{\langle \xi,\mu\rangle^2}{b} D_\xi g_a\right\rangle &= \frac{\langle \xi,\mu\rangle^2 g_a'}{ab|\xi|}\left\{\langle D_p B(x,\hat p),\mu\rangle - M\frac{|\langle \xi,\mu\rangle|}{|\xi|}\right\}\\ &\geq \frac{\langle \xi,\mu\rangle^2 g_a'}{ab|\xi|}\Big(\frac{\theta}{2} - 2aM\Big) \geq 0,\end{aligned}$$

and

$$\left\langle D_p B(x,\hat p),\ \frac{2g_a}{b}\langle \xi,\mu\rangle\mu\right\rangle \geq \frac{g_a\langle \xi,\mu\rangle\theta}{b} \geq \frac{a|\xi|\theta g_a}{2b}.$$

Fixing $b = \theta a^2/(6CM)$, we observe that

$$B(x, D_\xi v(x,\xi)) \geq |\xi| g_a \left(\frac{\theta a}{2b} - \frac{3CM}{a}\right) = 0.$$

Note that, since $|\nu - \mu| \leq \frac{a}{4}$ on N, for $(x,\xi) \in N \times \mathbf{R}^n$, if

$$\langle \xi, \nu(x)\rangle \geq -\frac{a}{4}|\xi| \quad (\text{ resp., } \langle \xi, \nu(x)\rangle \leq \frac{a}{4}|\xi| \text{ }),$$

then we have

$$\langle \xi, \mu(x)\rangle \geq -\frac{a}{2}|\xi| \quad (\text{ resp., } \langle \xi, \mu(x)\rangle \leq \frac{a}{2}|\xi| \text{ }).$$

Thus we conclude that the function $v \in C^{1,1}(N \times \mathbf{R}^n)$ satisfies (4.11)–(4.14), with σ replaced by a, for all $(x,\xi) \in N \times \mathbf{R}^n$.

Step 2. We choose a function $\zeta \in C^2(\mathbf{R}^n)$ so that $\zeta(x) = 1$ in a neighborhood of $\partial\Omega$, the support $\operatorname{spt}\zeta$ is contained in the interior of N, and $0 \leq \zeta \leq 1$ in $\mathbf{R}^n$. Set

$$\tilde{v}(x,\xi) = \zeta(x)v(x,\xi) + (1 - \zeta(x))|\xi|^2 \quad \text{for } (x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n.$$

It is now obvious that the function $\tilde{v}$ has all the required properties. □

Theorem 4.4. *Assume that (B1)–(B3) hold. There are a function $w \in C^{1,1}(\overline{\Omega} \times \overline{\Omega})$ and a positive constant C such that for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,*

(i) $|x-y|^4 \leq w(x,y) \leq C|x-y|^4,$
$|D_x w(x,y)| \vee |D_y w(x,y)| \leq C|x-y|^3,$

(ii) $B(x, D_x w(x,y)) \geq 0$ *if* $x \in \partial\Omega,$
$B(y, -D_y w(x,y)) \geq 0$ *if* $y \in \partial\Omega,$

(iii) $|D_x w(x,y) + D_y w(x,y)| \leq C|x-y|^4,$
$\rho^*(D_x w(x,y), -D_y w(x,y)) \leq C|x-y|,$

and for a. e. $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

(iv) $$D^2 w(x,y) \leq C\left\{|x-y|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + |x-y|^4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right\}.$$

Proof. Let $\varepsilon \in (0,1)$, and, according to Lemma 2.4, we choose a function $\psi \in C^\infty(\overline{\Omega})$ having the properties: $D\psi(x) \neq 0$ for all $x \in \partial\Omega$,

$$\langle \nu(x), D\psi(x)\rangle \geq (1-\varepsilon)|D\psi(x)| \quad \text{for all } x \in \partial\Omega,$$

and $\psi \geq 0$ on $\overline{\Omega}$. Let $(x,p) \in \partial\Omega \times (\mathbf{R}^n \setminus \{0\})$. Setting $\mu = D\psi(x)/|D\psi(x)|$, we observe that

$$|\mu - \langle \nu(x), \mu\rangle \nu(x)|^2 = 1 - \langle \nu(x), \mu\rangle^2 < 2\varepsilon.$$

Hence, using (B3), we have

$$\begin{aligned}\langle D_pB(x,p), D\psi(x)\rangle &= |D\psi(x)|\langle D_pB(x,p), \mu\rangle \\ &= |D\psi(x)|(\langle D_pB(x,p), \langle \nu(x), \mu\rangle \nu(x)\rangle + \langle D_pB(x,p), \mu - \langle \nu(x), \mu\rangle \nu(x)\rangle) \\ &\geq |D\psi(x)|(\theta\langle \nu(x), \mu\rangle - M\sqrt{2\varepsilon}) \geq |D\psi(x)|(\theta(1-\varepsilon) - M\sqrt{2\varepsilon}),\end{aligned}$$

where M is the positive constant from (4.4). Accordingly, we may assume by selecting ε small enough that

$$\langle D_pB(x,p), D\psi(x)\rangle \geq \frac{\theta}{2}|D\psi(x)| \qquad \text{for } (x,p) \in \partial\Omega \times (\mathbf{R}^n \setminus \{0\}).$$

Let $A > 0$ be a constant to be chosen later. We may assume by replacing ψ by a constant multiple of ψ that

$$\langle D_pB(x,p), D\psi(x)\rangle \geq A \qquad \text{for } (x,p) \in \partial\Omega \times (\mathbf{R}^n \setminus \{0\}). \tag{4.16}$$

Define

$$w(x,y) = e^{\psi(x)+\psi(y)} g(x, x-y) \quad \text{for } (x,y) \in \overline{\Omega} \times \overline{\Omega},$$

where $g = v^4$ and v is the function from Lemma 4.3.

We intend to show that if A is large enough, then w satisfies conditions (i)–(iv).

It is clear from our choice of v and ψ that w satisfies (i). It is a standard observation (see, e.g., [I, the proof of Theorem 4.1]) that (iv) is satisfied.

Let $\sigma > 0$ be the constant from Lemma 4.3. Since Ω is a bounded C^1 domain, we can find $d > 0$ so that if $x \in \partial\Omega$, $y \in \overline{\Omega}$, and $|x-y| \leq d$, then

$$\langle \nu(x), y - x\rangle \leq \sigma|y-x|.$$

Calculating that

$$D_x w(x,y) = w(x,y)D\psi(x) + e^{\psi(x)+\psi(y)}\left(D_x g(x, x-y) + D_\xi g(x, x-y)\right),$$

$$D_y w(x,y) = w(x,y)D\psi(y) - e^{\psi(x)+\psi(y)} D_\xi g(x, x-y),$$

we see that

$$\begin{aligned}|D_x w(x,y) + D_y w(x,y)| \leq\ & e^{\psi(x)+\psi(y)}(|D\psi(x) + D\psi(y)|g(x,x-y) \\ & +|D_x g(x,x-y)|) \leq C_1|x-y|^4\end{aligned} \tag{4.17}$$

for some constant $C_1 > 0$. Fix $x \in \overline{\Omega}$ and consider the function

$$\xi \mapsto g(x,\xi)D\psi(x) + D_x g(x,\xi) + D_\xi g(x,\xi)$$

on $\mathbf{R}^n$. Observe that the third term $D_\xi g(x,\xi)$ is homogeneous of degree 3 and the other two terms are homogeneous of degree 4 in the variable ξ and that $4g(x,\xi) = \langle D_\xi g(x,\xi), \xi\rangle \leq |D_\xi g(x,\xi)||\xi|$, by Euler's identity. These together yield that

$$|g(x,\xi)D\psi(x) + D_x g(x,\xi) + D_\xi g(x,\xi)| \geq 4|\xi|^3 - C_2|\xi|^4 = |\xi|^3(4 - C_2|\xi|)$$

for some constant $C_2 > 0$. Similarly, we have

$$|g(x,\xi)D\psi(y) - D_\xi g(x,\xi)| \geq |\xi|^3(4 - C_3|\xi|)$$

for some constant $C_3 > 0$. We may assume that $(C_2 \vee C_3)d \leq 3$. Then we have

$$|g(x,\xi)D\psi(x) + D_x g(x,\xi) + D_\xi g(x,\xi)| \geq |\xi|^3,$$

$$|g(x,\xi)D\psi(y) - D_\xi g(x,\xi)| \geq |\xi|^3.$$

Thus we see that if $x, y \in \overline{\Omega}$ and $0 < |x-y| \leq d$ then

$$\frac{|D_x w(x,y) + D_y w(x,y)|}{|D_x w(x,y)| \wedge |D_y w(x,y)|} \leq C_1|x-y|,$$

and conclude from this and (4.17) that (iii) holds for some constant C.

We want to show that (ii) holds if we choose A large enough. Let $x \in \partial\Omega$ and $y \in \overline{\Omega}$. It suffices to prove that if A is large enough, then

$$B(x, D_1 w(x,y)) \geq 0 \quad \text{and} \quad B(x, -D_2 w(y,x)) \leq 0, \tag{4.18}$$

where D_1 and D_2 denote, respectively, the differentiation of any function $f(x,y)$ on $\mathbf{R}^n \times \mathbf{R}^n$ with respect to the first variable $x \in \mathbf{R}^n$ and the second variable $y \in \mathbf{R}^n$.

If $x = y$, then $D_x w(x,y) = D_y w(x,y) = 0$ and hence,

$$B(x, D_1 w(x,y)) = B(x, -D_2 w(y,x)) = 0.$$

Henceforth we may assume that $x \neq y$.

First we consider the case when $|x-y| \leq d$. Since $\langle \nu(x), x-y\rangle \geq -\sigma|x-y|$, by (4.14) we have

$$B(x, D_\xi g(x, x-y)) \geq 0.$$

Using this and writing $\hat{e} = e^{\psi(x)+\psi(y)}$, we calculate that

$$\begin{aligned} B(x, D_1 w(x,y)) &\geq \hat{e}B(x, D_\xi g(x, x-y)) + \langle D_p B(x,\hat{p}), w(x,y)D\psi(x) + \hat{e}D_x g(x, x-y)\rangle \\ &\geq Aw(x,y) - C_4 w(x,y) \end{aligned}$$

for some $\hat{p} \in \mathbf{R}^n \setminus \{0\}$ and some constant $C_4 > 0$, independent of A, and that

$$\begin{aligned} B(x, -D_2 w(y,x)) &= B(x, -w(y,x)D\psi(x) + \hat{e}D_\xi g(y, y-x)) \\ &\leq \hat{e}B(x, D_\xi g(y, y-x)) - \langle D_p B(x,\hat{q}), w(y,x)D\psi(x)\rangle \\ &\leq \hat{e}B(x, D_\xi g(x, y-x)) + C_5 w(y,x) - Aw(y,x) \\ &\leq -(A - C_5)w(y,x) \end{aligned}$$

for some $\hat{q} \in \mathbf{R}^n \setminus \{0\}$ and some constant $C_5 > 0$ independent of A.

Next consider the case when $|x-y|>d$. Assuming that $D_1w(x,y)\neq 0$, we compute that

$$B(x,D_1w(x,y))\geq\langle D_pB(x,\hat p),D_1w(x,y)\rangle\geq Aw(x,y)-C_6\hat e\geq\hat e(Ad^4-C_6)$$

for some $\hat p\in\mathbf{R}^n\setminus\{0\}$ and some constant $C_6>0$ independent of A. Now, assuming that $D_2w(y,x)\neq 0$, we obtain

$$B(x,-D_2w(y,x))\leq\hat e(Ad^4-C_7)$$

for some constant $C_7>0$ independent of A. We now fix A so that

$$C_4\vee C_5(C_6/d^4)\vee(C_7/d^4)\leq A,$$

and conclude that (4.18) holds. Thus the function w has all the required properties. □

5 Examples

In this section we discuss typical examples of F and B to which Theorems 2.1 and 3.1 apply.

First we treat examples of F. Let $A:\overline{\Omega}\times(\mathbf{R}^n\setminus\{0\})\to M^{n\times m}$, where $M^{n\times m}$ denotes the space of real $n\times m$ matrices, be a function which is homogeneous of degree zero, i.e.,

$$A(x,\lambda p)=A(x,p)\quad\text{for all }(x,p,\lambda)\in\overline{\Omega}\times(\mathbf{R}^n\setminus\{0\})\times(0,\infty)\tag{5.1}$$

and which satisfies

$$\|A(x,p)-A(y,q)\|\leq C_1(|x-y|+|p-q|)\quad\text{for all }x,y\in\overline{\Omega}\text{ and }p,q\in S^{n-1},\tag{5.2}$$

where $C_1>0$ is a constant and S^{n-1} denotes the unit sphere $\{\xi\in\mathbf{R}^n\ :\ |\xi|=1\}$. It follows that for all $x,y\in\overline{\Omega}$ and $p,q\in\mathbf{R}^n\setminus\{0\}$,

$$\begin{aligned}\|A(x,p)-A(y,q)\|&\leq C_1\Big(|x-y|+\Big|\frac{p}{|p|}-\frac{q}{|q|}\Big|\Big)\\&\leq C_1\Big(|x-y|+\frac{|p-q|}{|p|\vee|q|}\Big)\leq C_1(|x-y|+2\rho(p,q)).\end{aligned}$$

Let $b\in C(\overline{\Omega},\mathbf{R}^n)$ satisfy

$$|b(x)-b(y)|\leq C_2|x-y|\quad\text{for all }x,y\in\overline{\Omega}.\tag{5.3}$$

Furthermore let $c,f\in C(\overline{\Omega},\mathbf{R})$ be given. Define the function $F\in C(\overline{\Omega}\times\mathbf{R}\times(\mathbf{R}^n\setminus\{0\})\times\mathcal{S}^n)$ by

$$F(x,u,p,X)=-\operatorname{tr}[A(x,p)^*A(x,p)X]+\langle b(x),p\rangle+c(x)u+f(x).\tag{5.4}$$

As is observed in [CIL], if $X, Y \in \mathcal{S}^n$ and $\mu_1, \mu_2 \in [0, \infty)$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then

$$\begin{aligned} -\operatorname{tr}[A(x,p)^*A(x,p)X] - \operatorname{tr}[A(y,q)^*A(y,q)Y] &\le C_3 \|A(x,p) - A(y,q)\|^2 \\ &\le 4C_3C_1(|x-y|^2 + \rho(p,q)^2). \end{aligned}$$

It is now easy to see that F satisfies condition (F3). Also, it is immediate to see that condition (F2) is satisfied with $\gamma \le \min_{\overline{\Omega}} c$.

If $A(x,p) = I - |p|^{-2}(p \otimes p)$, $b = 0$, and $c = f = 0$, then it is the case of the mean curvature flow equation and the above conditions on A, b, c, and f are valid.

A standard way of generalization of the above example is to take the max-min of a family of functions F having the form (5.1). More precisely, let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty index sets, and let $A_{\alpha\beta} \in C(\overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}), M^{n\times m})$, $b_{\alpha\beta} \in C(\overline{\Omega}, \mathbf{R}^n)$, $c_{\alpha\beta} \in C(\overline{\Omega})$, and $f_{\alpha\beta} \in C(\overline{\Omega})$, with $(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}$, be given. Assume that these sets of functions are uniformly bounded, that $\{c_{\alpha\beta}\}$ and $\{f_{\alpha\beta}\}$ are equi-continuous, that $\{A_{\alpha\beta}\}$ satisfies (5.1) and (5.2) with a uniform constant C_1, and that $\{b_{\alpha\beta}\}$ is equi-Lipschitz continuous (i.e., satisfies (5.3) with a uniform constant C_2. Define

$$\begin{aligned} F_{\alpha\beta}(x,u,p,X) = &- \operatorname{tr}[A_{\alpha\beta}(x,p)^*A_{\alpha\beta}(x,p)X] \\ &+ \langle b_{\alpha\beta}(x), p\rangle + c_{\alpha\beta}(x)u + f_{\alpha\beta}(x), \end{aligned}$$

and

$$F(x,u,p,X) = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} F_{\alpha\beta}(x,u,p,X).$$

Then the function F satisfies (F1)–(F3).

Next we deal with the boundary condition. Consider the function B of the form

$$B(x,p) = \langle \mu(x), p\rangle - |C(x)p|,$$

where $\mu : \mathbf{R}^n \to \mathbf{R}^n$ is a $C^{1,1}$ vector field over $\mathbf{R}^n$ and $C : \mathbf{R}^n \to M^{n\times n}$ is a $C^{1,1}$ function satisfying $\det C(x) \ne 0$ in a neighborhood of $\partial\Omega$. It is clear that (B2) is satisfied. We can modify the definition of B so that the resulting function $\widetilde{B}$ satisfies (B1) and $\widetilde{B}(x,\cdot) = B(x,\cdot)$ for all x in a neighborhood of $\partial\Omega$.

As before let $\nu(x)$ denote the unit outer normal of Ω at $x \in \partial\Omega$. By calculation, we have

$$D_pB(x,p) = \mu(x) - \frac{C(x)^*C(x)p}{|C(x)p|} \quad \text{if } p \ne 0,$$

and we see that (B3) is equivalent to the condition

$$\langle \mu(x), \nu(x)\rangle > \langle \xi, C(x)\nu(x)\rangle \quad \text{for all } (x,\xi) \in \partial\Omega \times S^{n-1}.$$

A particular case is when $\mu = \nu$ and $C(x) = a(x)I$ for some $a \in C^{1,1}(\mathbf{R}^n)$ such that $0 < a(x) < 1$ for $x \in \partial\Omega$, which corresponds to the Capillary condition. In this case the boundary regularity of Ω should be of class $C^{2,1}$ in order that $\mu = \nu \in C^{1,1}(\mathbf{R}^n)$ is satisfied, which is one of requirements of Theorems 2.1 and 3.1. It is interesting to find that the results in [B2] need the same $C^{2,1}$ regularity of the boundary.

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