JOURNAL
Vol. 0, No. 0, March 2004

LIMITS OF SOLUTIONS OF P-LAPLACE EQUATIONS AS P GOES TO INFINITY AND RELATED VARIATIONAL PROBLEMS *

HITOSHI ISHII* AND PAOLA LORETI*

Abstract. We show that the convergence, as $p \to \infty$, of the solution u_p of the Dirichlet problem for $-\Delta_p u(x) = f(x)$ in a bounded domain $\Omega \subset \mathbf{R}^n$ with zero-Dirichlet boundary condition and with continuous f in the following cases: (i) one dimensional case, radial cases, (ii) the case of no balanced family, and (iii) two cases with vanishing integral. We also give some properties of the maximizers for the functional $\int_\Omega f(x)v(x)\,\mathrm{d}x$ in the space of functions $v \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ satisfying $v|_{\partial\Omega} = 0$ and $\|Dv\|_{L^\infty(\Omega)} \le 1$.

Key words. p-Laplace equation, asymptotic behavior, variational problem, L^∞ variational problem, eikonal equation, ∞-Laplace equation

AMS subject classifications. 35B40, 35J60, 35J20, 35F30

1. Introduction. We study the asymptotic behavior, as $p \to \infty$, of the solution u_p of the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here and henceforth Δ_p denotes the p-Laplacian, i.e.,

$$\Delta_p u(x) := \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(|Du|^{p-2}\frac{\partial u}{\partial x_i}\right),$$

$\Omega \subset \mathbf{R}^n$ is a bounded open set, the exponent p satisfies $p > 1$, and $f \in C(\overline{\Omega})$.

The PDE in (1.1) is the Euler-Lagrange equation of the maximization problem for the functional

$$I_p(u) := \int_\Omega \left(f(x)u(x) - \frac{1}{p}|Du(x)|^p\right) \mathrm{d}x \quad \text{over } W_0^{1,p}(\Omega). \tag{1.2}$$

As is well known, the two problems (1.1) and (1.2) are equivalent. The problem (1.1) has a unique solution $u \in W_0^{1,p}(\Omega)$ and so does (1.2). For the existence and uniqueness of a solution of (1.1), we refer to [L]. According to the regularity results for (1.1), the solution u_p has Hölder continuous derivatives in Ω. That is, $u_p \in C^{1,\gamma}(\Omega)$ for some constant $\gamma \in (0, 1)$ which depends on p. Moreover, if the boundary $\partial\Omega$ is smooth, then $u_p \in C^{1,\gamma}(\overline{\Omega})$. See [U, D, Lb, T] for these regularity properties.

The asymptotic problem for (1.1) as $p \to \infty$ appears in modelling of a torsional creep phenomenon for a prismatic elastoplastic rod. This corresponds to the case

*Received by the editors March x, 2004; accepted by the editors Month, x, xxxx..

*Department of Mathematics, School of Education, Waseda University, Nishi-Waseda 1-6-1, Shinjuku, Tokyo 169-8050, Japan, supported in part by Grant-in-Aid for Scientific Research, No. 15340051 and No. 14654032 and by Waseda University Grant for Special Research Projects, No. 2001A-838 (ishii@edu.waseda.ac.jp).

*Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma "La Sapienza", Via Scarpa n. 16, 00161 Roma, Italy (loreti@dmmm.uniroma1.it).

where $n = 2$ and f is a positive constant (see, for instance, [BDM, K, PP]). In fact, if $f > 0$, then the limit of u_p in $C(\overline{\Omega})$ exists and is the distance function from the boundary $\partial\Omega$, i.e., the function $d(x) := \operatorname{dist}(x, \partial\Omega)$. See [BDM] for this result and [IK, IL1, IL2, FIN, BK, JLM] for some of related topics.

This convergence result is then generalized to the case of general non-negative functions f, by using ∞-Laplace equation in the region ω where f vanishes, i.e, solving the problem

$$\begin{cases} -\Delta_\infty w(x) = 0 & \text{in } \omega, \\ \quad w(x) = d(x) & \text{on } \partial\omega, \end{cases} \tag{1.3}$$

where

$$\Delta_\infty w(x) := \sum_{i,j=1}^n \frac{\partial w(x)}{\partial x_i} \frac{\partial w(x)}{\partial x_j} \frac{\partial^2 w(x)}{\partial x_i \partial x_j} \quad \text{and} \quad \omega := \operatorname{int}\{x \in \Omega \mid f(x) = 0\}.$$

Due to [J] (see also [BB]), the problem (1.3) has a unique viscosity solution $w \in C(\overline{\omega})$ which is Lipschitz continuous in ω and satisfies $\|Dw\|_{L^\infty(\omega)} \le 1$. If we assume that $f \ge 0$ in Ω and define $U \in C(\overline{\Omega})$ by

$$U(x) = \begin{cases} d(x) & \text{for } x \in \overline{\Omega} \setminus \omega, \\ w(x) & \text{for } x \in \omega, \end{cases}$$

where w is the unique viscosity solution of (1.3), then U gives the limit of u_p in $C(\overline{\Omega})$ as $p \to \infty$. See Remark 5.2 in [BDM], where the above idea of finding the limit function appears. See also [CIL] for an introduction to viscosity solutions.

In 1967 G. Aronsson initiated the study of the ∞-Laplace equation in his study of absolutely minimal Lipschitz extensions (AMLE for short), also called as canonical Lipschitz extensions, to a domain ω of a function given on $\partial\omega$. The AMLE and ∞-Laplace equation are subjects which have received intensive research activities recently. For these recent developments, we refer to [ACJ].

As we will recall in Section 5, the family $\{u_p\}_{p>q}$, with $q > n$, is precompact in $C(\overline{\Omega})$. Therefore, $\{u_p\}_{p>1}$ has a sequence $\{u_{p_j}\}_{j\in\mathbf{N}}$ convergent in $C(\overline{\Omega})$, where $p_j \to \infty$ as $j \to \infty$. However, it is not clear if the whole family $\{u_p\}_{p>1}$ is convergent in $C(\overline{\Omega})$ or not, except in the case where $f \ge 0$.

In this paper we address ourselves to the question whether the whole family $\{u_p\}_{p>1}$ is convergent in $C(\overline{\Omega})$ as $p \to \infty$ or not. We present only partial positive answers to this question in this paper.

In the cases where $n = 1$ or when Ω is an open ball and f is a radial function, we show the convergence of u_p in $C(\overline{\Omega})$ and identify the limit function. In these cases, our proof relies heavily on an explicit formula for u_p.

In the general situation we do not know any convenient formula for u_p and in our approach we make a careful study (especially the structure of its maximizers) of the variational problem for the functional

$$I_\infty(u) := \int_\Omega f(x)u(x)\,\mathrm{d}x \tag{1.4}$$

over the set $X := \{v \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega) \mid v|_{\partial\Omega} = 0,\ \|Dv\|_{L^\infty(\Omega)} \le 1\}$. This variational problem appears as the limit problem of (1.2). (See Proposition 5.3 below.) This problem may be conceived of an L^∞ variational problem because of the L^∞ bound on the gradient Du and because it appears as the limit problem for the variational problem (1.2) as $p \to \infty$.

As a generalization of the case where $f \geq 0$, we show the convergence of u_p in $C(\overline{\Omega})$ under the condition of *no balanced family*, i.e., under the assumption that for any nonempty family $\mathcal{C}$ of Lipschitz-connected components of $\{x \in \Omega \mid f(x) \neq 0\}$ that stay away from $\partial\Omega$ and $\omega := \bigcup\{U \in \mathcal{C}\}$ (union of the sets U, where U ranges over all $U \in \mathcal{C}$),

$$\int_\omega f(x)\,\mathrm{d}x \neq 0.$$

Here the standard definition of connected components is not appropriate and we have used the notion of Lipschitz-connected (L-connected for short) component. See Section 2 for the precise assumption, (2.4), and for the definition of L-connected components.

We also consider the case when

$$\int_\Omega f(x)\,\mathrm{d}x = 0 \quad \text{and} \quad f \neq 0. \tag{1.5}$$

This is the case when the above assumption (the assumption of no balanced family) is not satisfied. Also, this is the case related to the Monge-Kantorovich mass transfer problem. The Monge-Kantorovich mass transfer problem has received much attention in the last decade. We refer to [EG, BBD, ACBBV] for the recent developments of the Monge-Kantorovich mass transfer problem and the role of the asymptotic problem for (1.1) as $p \to \infty$ in the mass transfer problem.

In the Monge-Kantorovich case, i.e., the case where (1.5) holds, we only have two special results besides those in the cases when $n = 1$ or when Ω is an open ball and f is radial. One of them says that if Ω is symmetric with respect to the origin and f is an odd function, u_p converges in $C(\overline{\Omega})$, and the other roughly says that if the distance between two sets $\Omega_+ := \{x \in \Omega \mid f(x) > 0\}$ and $\Omega_- := \{x \in \Omega \mid f(x) < 0\}$ is greater than or equal to the sum of the supremum and infimum of the distances between $x \in \Omega_+ \cup \Omega_-$ and $\partial\Omega$, then the convergence of u_p in $C(\overline{\Omega})$ is valid.

The main results of this paper, concerned with convergence of u_p, are precisely stated in Section 2. The proof of convergence in one-dimensional case and radial case are presented in Sections 3 and 4, respectively. Section 5 is devoted to general properties of $\{u_p\}$, the set $\mathcal{M}$ of maximizers of the variational problem (1.4), the set $\mathcal{A}$ of the limits of u_p, i.e.,

$$\mathcal{A} = \{U \in C(\overline{\Omega}) \mid \text{there is a sequence } p_j \to \infty \text{ such that } u_{p_j} \to U \text{ in } C(\overline{\Omega})\}. \tag{1.6}$$

Section 6 is devoted to further properties of the set $\mathcal{M}$ which are useful in our study of convergence of u_p. These observations on $\mathcal{M}$ comprise main results of this paper together with our results on the convergence of u_p.

We prove our convergence results in the case of no balanced family and in the vanishing integral case (the case of (1.5)), respectively, in Sections 7 and 8.

We explain the notation in this paper. For $a, b \in \mathbf{R}$ we write $a \vee b = \max\{a,\, b\}$, $a \wedge b = \min\{a,\, b\}$, $a^+ = a \vee 0$, and $a^- = a \wedge 0$. We use the same notation for functions. We denote by $\mu(A)$ the Lebesgue measure of measurable set $A \subset \mathbf{R}^n$. If needed, we denote by $\mu_n(A)$ in order to specify the dimension of the space where A lives. We denote by $B(x, a)$ the closed ball of radius a and with x as its center.

Finally, we remark that most of the results in this paper are already announced in [IL4].

2. Main results on the convergence. In this section we state our results concerning the limit, as $p \to \infty$, of the solution u_p of the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{2.1}$$

Here, as before, $\Omega \subset \mathbf{R}^n$ is a bounded, open subset of $\mathbf{R}^n$ and $f \in C(\overline{\Omega})$.

To begin with, let us recall that the problem (2.1) has a unique solution $u_p \in W_0^{1,p}(\Omega)$ for any $p \in (1, \infty)$. See, e.g., [L].

Let $X = \{v \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega) \mid v|_{\partial\Omega} = 0,\ \|Dv\|_{L^\infty(\Omega)} \le 1\}$. We will recall that the family $\{u_p\}_{p>r}$ is bounded in $W^{1,q}(\Omega)$ for any $q > 1$ and $r > 1$, which guarantees that for any sequence $p_j \to \infty$, there is a subsequence $\{p_{j_k}\}_{k\in\mathbf{N}}$ such that $u_{p_{j_k}}$ converges to a function $U \in X$ uniformly in $\overline{\Omega}$ as $k \to \infty$.

We are interested in the question whether the following claim (C) is true or not:

(C) *The solution u_p converges uniformly to a function $U \in X$ as $p \to \infty$.*

We are not yet able to determine if the claim (C) is always true or not, and in what follows we present a couple of sufficient conditions for (C) to hold as our main results in this paper.

The first of all we treat the case when $n = 1$. In this case we can show not only that (C) holds but also identify the limit as the next theorem states.

Let $n = 1$ and $\Omega = (0, a)$, where $a > 0$ is a constant. We define the function $F \in C^1([0,\ a])$ by

$$F(x) = \int_0^x f(t)\,\mathrm{d}t.$$

We define

$$h(r) = \mu(\{x \in \Omega \mid F(x) < r\}), \qquad \beta^* = \sup\{r \in \mathbf{R} \mid h(r) \le \frac{a}{2}\},$$

$$O_- = \{x \in \Omega \mid F(x) < \beta^*\}, \quad O_+ = \{x \in \Omega \mid F(x) > \beta^*\}, \quad O_0 = \{x \in \Omega \mid F(x) = \beta^*\},$$

$$k = \begin{cases} 0 & \text{if } \mu(O_0) = 0, \\ \dfrac{\mu(O_+) - \mu(O_-)}{\mu(O_0)} & \text{if } \mu(O_0) > 0. \end{cases}$$

Then we introduce the function $U \in C([0,\ a])$ by

$$U(x) = \int_0^x (\mathbf{1}_{O_-}(t) - \mathbf{1}_{O_+}(t) + k\mathbf{1}_{O_0}(t))\,\mathrm{d}t. \tag{2.2}$$

Here and henceforth $\mathbf{1}_A$ denotes the characteristic function of the set A. We will see in the next section (Lemma 3.5) that $|k| \le 1$, which assures that $U \in X$.

Theorem 2.1. *If $n = 1$ and $\Omega = (0,\ a)$, then* (C) *holds and moreover the limit function U is given by* (2.2).

As above in the radial case we can show that (C) is valid and give an explicit formula for the limit function.

Let $a > 0$ be a constant and we assume that $\Omega = \operatorname{int} B(0, a)$ and $f(x) = g(|x|)$ for some $g \in C([0, a])$.

We define $O_\pm \subset \mathbf{R}^n$ by

$$O_+ = \{t \in (0, a) \mid \int_{B(0,t)} f(x)\,\mathrm{d}x > 0\}, \qquad O_- = \{t \in (0, a) \mid \int_{B(0,t)} f(x)\,\mathrm{d}x < 0\},$$

and $U \in X$ by

$$U(x) = \int_{|x|}^{a} (\mathbf{1}_{O_+}(t) - \mathbf{1}_{O_-}(t))\,\mathrm{d}t. \tag{2.3}$$

THEOREM 2.2. *If* $\Omega = \operatorname{int} B(0,a)$ *and* $f(x) = g(|x|)$ *is a radial function, then* (C) *holds and the limit function* U *is given by* (2.3).

The next condition under which (C) holds is a generalization of the well-known observation due to [BDM] and [J] (see Remark 5.2 of [BDM] and the uniqueness result of [J]) that if $f \geq 0$ in Ω, then (C) holds.

In order to make a precise statement, we need to introduce the notation.

We write

$$\Omega_+ = \{x \in \Omega \mid f(x) > 0\}, \quad \Omega_- = \{x \in \Omega \mid f(x) < 0\}, \quad \text{and} \quad \Omega_* = \Omega_+ \cup \Omega_-.$$

Note that $\operatorname{spt} f = \overline{\Omega}_*$, where $\operatorname{spt} f$ denotes the support of the function f. Let $\mathcal{O}_*$ denote the sets of all connected components of Ω_*.

We modify the notion of "connectedness" for a better formulation as follows. Let $A, B \subset \mathbf{R}^n$. Define $\rho(A,B) \in [0,\infty]$ by setting

$$\rho(A,B) = \inf\{d(A,U_1) + d(U_1,U_2) + \cdots + d(U_m,B) \mid U_1,...,U_m \in \mathcal{O}_*\},$$

where $d(U,V) = \inf\{|x-y| \mid x \in U, y \in V\}$. Notice that $\rho(A,B) = \infty$ if and only if either $A = \emptyset$ or $B = \emptyset$. Since, as is easily checked,

$$\rho(A,B) = \rho(B,A) \geq 0, \qquad \rho(A,B) \leq \rho(A,C) + \rho(C,B)$$

for any $A, B, C \subset \mathbf{R}^n$, if we write $A \sim B$ for $A, B \subset \mathbf{R}^N$ when $\rho(A,B) = 0$, then this relation $\sim$ defines an equivalence relation in $\mathcal{O}_*$.

Using the above equivalence relation, we classify $\mathcal{O}_*$ as

$$\mathcal{O}_* = \bigcup\{\mathcal{O}_\lambda \mid \lambda \in \Lambda\},$$

where
(i) for each $\lambda \in \Lambda$, $\mathcal{O}_\lambda \neq \emptyset$,
(ii) for each $\lambda \in \Lambda$, if $U \in \mathcal{O}_\lambda$, then $\mathcal{O}_\lambda = \{V \in \mathcal{O}_* \mid V \sim U\}$,
and
(iii) if $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1 \neq \lambda_2$, then $\mathcal{O}_{\lambda_1} \cap \mathcal{O}_{\lambda_2} = \emptyset$.

We set

$$G_\lambda = \bigcup\{U \mid U \in \mathcal{O}_\lambda\} \quad \text{for } \lambda \in \Lambda.$$

and define

$$\Lambda_0 = \{\lambda \in \Lambda \mid \rho(G_\lambda, \partial\Omega) = 0\}.$$

We note that $\{G_\lambda \mid \lambda \in \Lambda\}$ classifies the set Ω_*. Each G_λ, with $\lambda \in \Lambda$, is called an L-connected component of Ω_*.

As the proof of Lemma 7.1 below shows, if w is a Lipschitz continuous function on Ω and $Dw(x) = 0$ a.e. $x \in \Omega_*$, then w is constant on each G_λ, with $\lambda \in \Lambda$. Conversely, one can show the following: let U, V be connected components of Ω_* having the property that if w is Lipschitz continuous on Ω and $Dw(x) = 0$ a.e. $x \in \Omega_*$, then w is constant on $U \cup V$. Then $U \sim V$, i.e., U, V are subsets of an

L-connected component G_λ. In light of these observations, we have chosen the term "Lipschitz-connected".

Our assumption on (f, Ω) is:

(2.4) For any nonempty $\Gamma \subset \Lambda \setminus \Lambda_0$ and $\omega := \bigcup\{G_\lambda \mid \lambda \in \Gamma\}$,

$$\int_\omega f(x)\,\mathrm{d}x \neq 0.$$

We call this condition that of *no balanced family* (of L-connected components).

Fig. 1 below gives pictorially an example of function f which satisfies condition (2.4). Here $\int_{\alpha_1}^{\alpha_2} f(x)\,\mathrm{d}x = \int_{\alpha_2}^{\alpha_3} f(x)\,\mathrm{d}x = -\int_{\alpha_4}^{\alpha_5} f(x)\,\mathrm{d}x$ is assumed. In this example, $\Omega_+ = (\alpha_1, \alpha_2) \cup (\alpha_2, \alpha_3)$, $\Omega_- = (\alpha_4, \alpha_5)$, and the L-connected components are Ω_+ and Ω_-. The integral of f over $\omega = \Omega_+$, Ω_-, or $\Omega_+ \cup \Omega_-$ does not vanish. On the other hand, the connected components of Ω_* are (α_1, α_2), (α_2, α_3), and (α_4, α_5), and the integral of f over $\omega = (\alpha_1, \alpha_2) \cup (\alpha_4, \alpha_5)$ vanishes. For this f, the condition similar to (2.4) but with the usual notion of connectedness in place of that of L-connectedness does not hold.

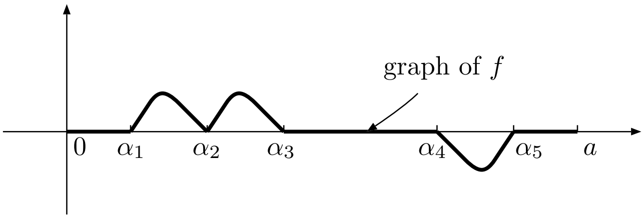

Fig. 1.

Next, we examine the function f given pictorially by Fig. 2, where $\int_{\alpha_1}^{\alpha_2} f(x)\,\mathrm{d}x = 0$ is assumed. For this function f, $\Omega_* = (\alpha_1, \beta) \cup (\beta, \alpha_2)$ is the only L-connected component of Ω_*, and condition (2.4) does not hold. This function f will appear in Example 3.2 in Section 3.

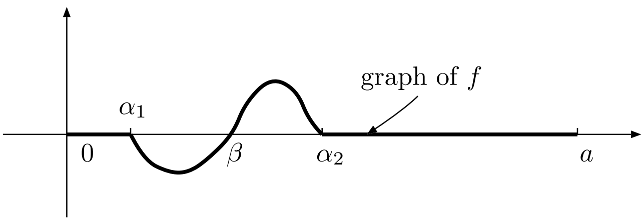

Fig. 2.

Theorem 2.3. *Under the assumption* (2.4)*,* (C) *holds.*

Regarding the cases when (2.4) is violated, we restrict ourselves to the case where

$$\int_\Omega f(x)\,\mathrm{d}x = 0 \quad \text{and} \quad f \neq 0, \tag{2.5}$$

and

(2.6) Ω_+ and Ω_- are connected.

We give two sufficient conditions for (C) to hold. One is a symmetry requirement on (f, Ω). That is, we assume that

(2.7) Ω is symmetric with respect to the origin, i.e., $-\Omega = \Omega$,

and

(2.8) f is an odd function, i.e., $f(-x) = -f(x)$ for all $x \in \Omega$.

The asymptotic problem, as $p \to \infty$, for (1.1) has applications to the Monge-Kantorovich mass transfer problem. In the mass transfer problem, the condition of vanishing integral, (2.5), is a natural compatibility condition, which means conservation of the total mass in the process of mass transfer.

The second one is the condition that

$$\min\{\inf_{x\in\Omega_+}\sup_{y\in\Omega_-}[d(x)+d(y)-|x-y|],\ \inf_{y\in\Omega_-}\sup_{x\in\Omega_+}[d(x)+d(y)-|x-y|]\} \le 0. \tag{2.9}$$

holds. Here and henceforth $d(x)$ denotes the distance between x and $\partial\Omega$, i.e., $d(x) = \operatorname{dist}(x, \partial\Omega)$. A sufficient condition for (2.9) to hold is that

$$\operatorname{dist}(\Omega_+, \Omega_-) \ge \inf_{\Omega_*} d + \sup_{\Omega_*} d.$$

See Fig. 3 below.

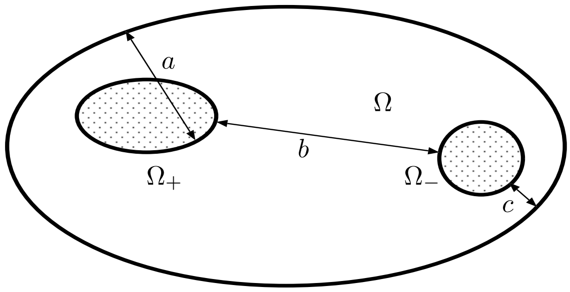

FIG. 3. A case where $b \ge a + c$.

THEOREM 2.4. *Under the assumptions* (2.5) *and* (2.6)*, if either* (2.7) *and* (2.8) *or* (2.9) *are satisfied, then* (C) *holds.*

3. One dimensional case. In this section we prove Theorem 2.1.

Let $\Omega = (0, a)$, where $a > 0$ is a constant, and $f \in C([0, a])$. Fix $p > 1$ and we consider the $(p+1)$-Laplace equation with the inhomogeneous term f

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(|u'(x)|^{p-1}u'(x)\right) = -f(x) \quad \text{in } \Omega, \tag{3.1}$$

together with the Dirichlet condition

$$u(0) = u(a) = 0. \tag{3.2}$$

Here u' denotes the derivative of u. The unique solution in $W_0^{1,p+1}(\Omega)$ of (3.1) (i.e., the solution of (3.1)-(3.2)) is denoted by u_{p+1} as in the previous sections.

We seek for an explicit formula for u_{p+1}. For this, noting that $u := u_{p+1} \in C^1([0,\, a])$ and integrating both sides of (3.1), we get

$$|u'(x)|^{p-1}u'(x) = |u'(0)|^{p-1}u'(0) - F(x) \quad \text{ for } x \in \Omega,$$

where $F(x) := \int_0^x f(t)\,\mathrm{d}t$.

Let ψ_p be the inverse function of $r \mapsto |r|^{p-1}r$. That is, $\psi_p(s) = |s|^{\frac{1}{p}-1}s$ for $s \in \mathbf{R}$. Note that as $p \to \infty$,

$$\psi_p(r) \to \begin{cases} 1 & \text{for } r > 0, \\ -1 & \text{for } r < 0. \end{cases}$$

Observe moreover that for any $\varepsilon \in (0, 1)$, the above convergence is uniform for $|r| \in [\varepsilon, \varepsilon^{-1}]$.

Writing $\beta = |u'(0)|^{p-1}u'(0)$ and integrating the equality $u'(x) = \psi_p(\beta - F(x))$, we get

$$u(x) = \int_0^x \psi_p(\beta - F(t))\,\mathrm{d}t \quad \text{ for } x \in \overline{\Omega}. \tag{3.3}$$

Conversely, if we can choose $\beta \in \mathbf{R}$ so that

$$\int_0^a \psi_p(\beta - F(t))\,\mathrm{d}t = 0,$$

then the function u defined by (3.3) is in $C^1([0,\, a])$ and the unique solution of (3.1)-(3.2).

We show directly that there is a unique constant $\beta_p \in \mathbf{R}$ such that

$$\int_0^a \psi_p(\beta_p - F(t))\,\mathrm{d}t = 0, \tag{3.4}$$

although this can be deduced from the general existence and uniqueness result for solutions of (1.1).

Set

$$G_p(r) = \int_0^a \psi_p(r - F(t))\,\mathrm{d}t$$

for $r \in \mathbf{R}$. Since the function $\psi_p(r)$ is strictly increasing, the function G_p is strictly increasing on $\mathbf{R}$. In view of the monotone convergence theorem, we see that the function G_p is continuous on $\mathbf{R}$. If $f = 0$, then it is clear that $\beta_p = 0$ gives the unique solution of (3.4).

We may thus assume in what follows that $f \neq 0$. We set

$$F_- = \min_{[0,a]} F, \qquad F_+ = \max_{[0,a]} F, \qquad \delta(F) = F_+ - F_-. \tag{3.5}$$

Note that $F_- \le 0 \le F_+$ and $\delta(F) > 0$. Since $F_- - F(x) \le 0$ for all $x \in \Omega$ and $\delta(F) > 0$, we have $G_p(F_-) < 0$. Similarly, we have $G_p(F_+) > 0$. Thus we see that there is a unique constant $\beta_p \in (F_-, F_+)$ such that $G_p(\beta_p) = 0$, and we find an explicit formula

$$u_{p+1}(x) = \int_0^x \psi_p(\beta_p - F(t))\,\mathrm{d}t \quad \text{ for } x \in \overline{\Omega}. \tag{3.6}$$

Next, we study the asymptotic behavior of the function u_{p+1} given by (3.6) as $p \to \infty$. Recall that

$$h(r) = \mu(\{x \in \Omega \mid F(x) < r\}), \qquad \beta^* = \sup\{r \in \mathbf{R} \mid h(r) \le \frac{a}{2}\},$$
$$O_- = \{x \in \Omega \mid F(x) < \beta^*\}, \quad O_+ = \{x \in \Omega \mid F(x) > \beta^*\}, \quad O_0 = \{x \in \Omega \mid F(x) = \beta^*\}.$$

We define open sets $O(r) \subset \mathbf{R}$ for $r \in \mathbf{R}$ by

$$O(r) = \{x \in \Omega \mid F(x) < r\}.$$

We have: (i) for $r \le s$, $O(r) \subset O(s)$, (ii) $O(F_-) = \emptyset$, (iii) $O(r) = (0, a)$ for all $r > F_+$, and (iv)

$$\bigcup_{t<r} O(t) = O(r), \qquad \bigcap_{t>r} O(t) = \{x \in \Omega \mid F(x) \le r\}.$$

Consequently, we have: (i) h is non-decreasing in $\mathbf{R}$, (ii) $h(r) = \mu(\emptyset) = 0$ for $r \in (-\infty, F_-]$, (iii) $h(r) = \mu(\Omega) = a$ for $r \in (F_+, \infty)$, and (iv)

$$\lim_{t \nearrow r} h(t) = h(r) \le \mu(\{x \in \Omega \mid F(x) \le r\}) = \lim_{t \searrow r} h(t).$$

Now, the property (iv) for h and the definition of β^* implies that

$$h(\beta^*) \le \frac{a}{2} \le h(\beta^* + 0) := \lim_{t \searrow \beta^*} h(t).$$

A key step in the proof of Theorem 2.1 is in the following lemma.

Lemma 3.1. *We have*

$$\lim_{p\to\infty} \beta_p = \beta^*.$$

We prepare with three lemmas for the proof of Lemma 3.1.

Lemma 3.2. *Let $r \in (F_-, F_+)$. Then, if $s > r$ (resp., $s < r$), then $h(s) > h(r)$ (resp., $h(s) < h(r)$).*

Proof. We consider the case when $s > r$. We may assume that $s \in (F_-, F_+)$. By the intermediate value theorem, we have

$$F(y) = \frac{s + r}{2}$$

for some $y \in \overline{\Omega}$. By the continuity of F, we can choose $\delta > 0$ so that

$$F(x) \in (r, s) \quad \text{for all } x \in \omega := (y - \delta, y + \delta) \cap \Omega.$$

It is clear that $\omega \subset O(s)$, $\mu(\omega) > 0$, and $\omega \cap O(r) = \emptyset$. Hence we have

$$h(s) = \mu(O(s)) = \mu(O(r)) + \mu(O(s) \setminus O(r)) \ge \mu(O(r)) + \mu(\omega) > \mu(O(r)) = h(r).$$

The proof for the case when $s < r$ is similar and will be omitted. ☐

Lemma 3.3. *Let $\beta \in [F_-, F_+]$. We have*

$$|\psi_p(\beta - F(x))| \le \psi_p(\max\{\delta(F), 1\}) \le \psi_1(\max\{\delta(F), 1\}) \quad \text{for } x \in \Omega.$$

Proof. For $x \in \Omega$ we have

$$-\delta(F) = F_- - F_+ \le \beta - F(x) \le F_+ - F_- = \delta(F),$$

and hence, $|\psi_p(\beta - F(x))| \le \psi_p(\max\{\delta(F), 1\}) \le \psi_1(\max\{\delta(F), 1\})$. $\square$

LEMMA 3.4. *Let $\{\alpha_j\} \subset [F_-, F_+]$ be a sequence converging to some $r \in \mathbf{R}$, $\{p_j\} \subset (1,\infty)$ a sequence such that $p_j \to \infty$ as $j \to \infty$, and $\phi \in L^1(\Omega)$. Set*

$$O_-(r) = \{x \in \Omega \mid F(x) < r\} \quad \text{and} \quad O_+(r) = \{x \in \Omega \mid F(x) > r\}.$$

Then

$$\int_{O_-(r)} \psi_{p_j}(\alpha_j - F(x))\phi(x)\,\mathrm{d}x \to \int_{O_-(r)} \phi(x)\,\mathrm{d}x,$$

$$\int_{O_+(r)} \psi_{p_j}(\alpha_j - F(x))\phi(x)\,\mathrm{d}x \to -\int_{O_+(r)} \phi(x)\,\mathrm{d}x.$$

Proof. Fix $x \in O_-(r)$. Since $r - F(x) > \delta$ for some constant $\delta > 0$, there is a $J \in \mathbf{N}$ such that for all $j \ge J$,

$$\delta < \alpha_j - F(x) \le \delta(F),$$

which implies that

$$\lim_{j\to\infty} \psi_{p_j}(\alpha_j - F(x)) = 1.$$

Now, in view of Lemma 3.3 and the Lebesgue convergence theorem, we conclude that

$$\lim_{j\to\infty} \int_{O_-(r)} \psi_{p_j}(\alpha_j - F(x))\phi(x)\,\mathrm{d}x = \int_{O_-(r)} \phi(x)\,\mathrm{d}x.$$

In the same way we see that

$$\lim_{j\to\infty} \int_{O_+(r)} \psi_{p_j}(\alpha_j - F(x))\phi(x)\,\mathrm{d}x = -\int_{O_+(r)} \phi(x)\,\mathrm{d}x. \qquad \square$$

Proof of Lemma 3.1. First of all we show that $\liminf_{p\to\infty} \beta_p \ge \beta^*$. For this, we argue by contradiction and thus suppose that $r := \liminf_{p\to\infty} \beta_p < \beta^*$. There is a sequence $\{p_j\} \subset (1,\infty)$ such that $\lim_{j\to\infty} p_j = \infty$ and $\lim_{j\to\infty} \beta_{p_j} = r$. By Lemma 3.2, we see that $h(r + 0) < h(\beta^*) \le a/2$. Therefore, setting $A = \{x \in \Omega \mid F(x) \le r\}$ and $B = \{x \in \Omega \mid F(x) > r\}$, we have

$$h(r+0) = \mu(A) < \frac{a}{2}, \qquad \mu(B) = a - \mu(A) > \frac{a}{2}.$$

Since $|\psi_p(\beta_p - F(x))| \le \psi_p(\max\{\delta(F), 1\})$ for $x \in \Omega$ by Lemma 3.3, we have

$$\begin{aligned} &\limsup_{j\to\infty} \left| \int_A \psi_{p_j}(\beta_{p_j} - F(x))\,\mathrm{d}x \right| \\ &\le \limsup_{j\to\infty} \int_A \psi_{p_j}(\max\{\delta(F), 1\})\,\mathrm{d}x = \int_A \mathrm{d}x = \mu(A) < \frac{a}{2}. \end{aligned}$$

Observe next by Lemma 3.4 that

$$\lim_{j\to\infty} \int_B \psi_{p_j}(\beta_{p_j} - F(x))\,\mathrm{d}x = -\mu(B) < -\frac{a}{2}.$$

Since

$$0 = \int_A \psi_{p_j}(\beta_{p_j} - F(x))\,\mathrm{d}x + \int_B \psi_{p_j}(\beta_{p_j} - F(x))\,\mathrm{d}x,$$

we find that $0 < \frac{a}{2} - \frac{a}{2} = 0$, which is a contradiction. This shows that $\liminf_{p\to\infty} \beta_p \geq \beta^*$.

An argument similar to the above shows that $\limsup_{p\to\infty} \beta_p \leq \beta^*$, and we conclude that $\lim_{p\to\infty} \beta_p = \beta^*$. $\square$

Recall that $k = 0$ if $\mu(O_0) = 0$ and otherwise,

$$k = \frac{\mu(O_+) - \mu(O_-)}{\mu(O_0)}.$$

LEMMA 3.5. *We have* $|k| \leq 1$.

Proof. Since

$$\mu(O_-) = h(\beta^*) \leq \frac{a}{2} \leq h(\beta^* + 0) = \mu(O_-) + \mu(O_0),$$

we have

$$0 \geq 2\mu(O_-) - a = \mu(O_-) - \mu(O_+) - \mu(O_0),$$
$$0 \leq 2\mu(O_-) + 2\mu(O_0) - a = \mu(O_-) + \mu(O_0) - \mu(O_+).$$

Hence, if $k \neq 0$, then

$$-1 \leq k = \frac{\mu(O_+) - \mu(O_-)}{\mu(O_0)} \leq 1. \qquad \square$$

Proof of Theorem 2.1. We show first that the family $\{u_p\}_{p>2}$ is uniformly bounded and equi-continuous on $\overline{\Omega}$.

To see this, fix $x \in \Omega$ and $p > 1$. By Lemma 3.3, we have

$$|u_{p+1}(x)| \leq \int_0^x \psi_1(\max\{\delta(F), 1\})\,\mathrm{d}x \leq a\psi_1(\max\{\delta(F), 1\})$$

and $|u'_{p+1}(x)| \leq \psi_1(\max\{\delta(F), 1\})$. These show that the family $\{u_p\}_{p>2}$ is uniformly bounded and equi-continuous on $\overline{\Omega}$.

Next we show that

$$\lim_{p\to\infty} \psi_p(\beta_p - \beta^*) = k \quad \text{ if } \mu(O_0) > 0. \tag{3.7}$$

In fact, we have

$$0 = \int_0^a \psi_p(\beta_p - F(x))\,\mathrm{d}x = \int_{O_-} \psi_p(\beta_p - F(x))\,\mathrm{d}x$$
$$+\psi_p(\beta_p - \beta^*)\mu(O_0) + \int_{O_+} \psi_p(\beta_p - F(x))\,\mathrm{d}x,$$

and then Lemma 3.4 yields

$$0 = \mu(O_-) - \mu(O_+) + \lim_{p\to\infty} \psi_p(\beta_p - \beta^*)\mu(O_0),$$

which shows (3.7).

Since $\{u_p\}_{p>2}$ is precompact in $C([0,\, a])$, we only need to show that for each fixed $x \in \Omega$,

$$u_{p+1}(x) \to U(x) := \int_0^x (\mathbf{1}_{O_-}(t) - \mathbf{1}_{O_+}(t) + k\mathbf{1}_{O_0}(t))\,\mathrm{d}t \qquad \text{as } p \to \infty.$$

Fix $x \in \Omega$ and note that

$$\begin{aligned} u_{p+1}(x) = &\int_{(0,x)\cap O_-} \psi_p(\beta_p - F(t))\,\mathrm{d}t + \int_{(0,x)\cap O_+} \psi_p(\beta_p - F(t))\,\mathrm{d}t \\ &+ \psi_p(\beta_p - \beta^*) \int_{(0,x)\cap O_0} \mathrm{d}t. \end{aligned}$$

Sending $p \to \infty$ and using Lemma 3.4, we get

$$\lim_{p\to\infty} u_p(x) = \int_{(0,x)\cap\Omega_-} \mathrm{d}t - \int_{(0,x)\cap\Omega_+} \mathrm{d}t + k\int_{(0,x)\cap\Omega_0} \mathrm{d}t = U(x). \qquad \square$$

Next we examine the limit function U in a few cases.

Example 3.1. We consider the case when $f(x) > 0$ for all $x \in \Omega$. Then the function F is strictly increasing in $\overline{\Omega}$. Therefore we have $O_- = (0,\, a/2)$ and $O_+ = (a/2,\, a)$, and hence

$$U(x) = \int_0^x \big(\mathbf{1}_{O_-}(t) - \mathbf{1}_{O_+}(t)\big)\, dt = \begin{cases} x & \text{for } 0 \le x \le a/2, \\ a - x & \text{for } a/2 \le x \le a. \end{cases}$$

This is the distance function from $\partial\Omega = \{0,\, a\}$ and, as is well-known, it is the unique viscosity solution of $|U'(x)| = 1$ in Ω and $U(0) = U(a) = 0$.

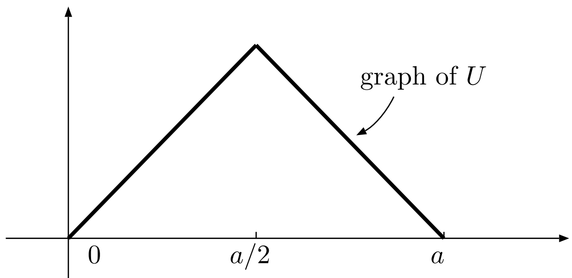

FIG. 4. *A case where $f > 0$.*

Example 3.2. Let $0 < \alpha_1 < \alpha_2 < a$ satisfy $\alpha_2 - \alpha_1 < \dfrac{a}{2}$. Let F satisfy $F(x) = 0$ for $x \in [0, \alpha_1] \cup [\alpha_2, a]$ and $F(x) < 0$ for $x \in (\alpha_1, \alpha_2)$.

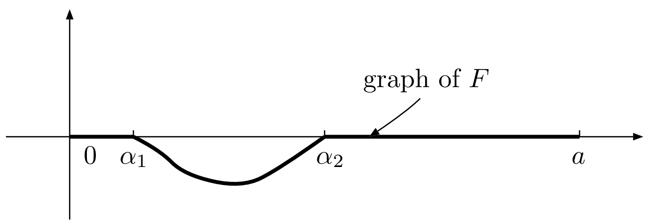

FIG. 5.

Then we have $\beta^* = 0$, $O_- = (\alpha_1, \alpha_2)$, $O_+ = \emptyset$, and $O_0 = (0, \alpha_1) \cup (\alpha_2, a)$. Furthermore, we have $k = -(\alpha_2 - \alpha_1)/(a - (\alpha_2 - \alpha_1))$, and

$$U(x) = \int_0^x \big(\mathbf{1}_{O_-}(t) + k\mathbf{1}_{O_0}(t)\big)\, dt = \begin{cases} kx & \text{for } 0 \le x \le \alpha_1, \\ k\alpha_1 + x - \alpha_1 & \text{for } \alpha_1 \le x \le \alpha_2, \\ k(x-a) & \text{for } \alpha_2 \le x \le a. \end{cases}$$

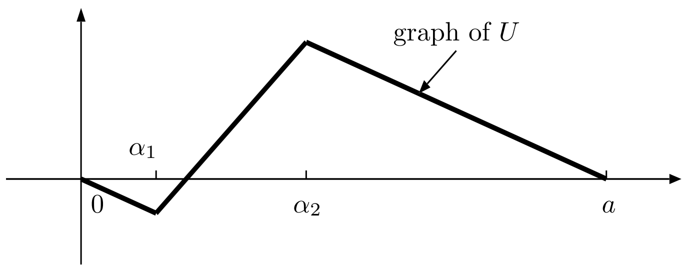

FIG. 6. *A case where* $|k| < 1$.

Example 3.3. Let $0 < \alpha_1^- < \alpha_2^- < \alpha_1^+ < \alpha_2^+ < a$ satisfy $\alpha_2^- - \alpha_1^- = \alpha_2^+ - \alpha_1^+ < \dfrac{a}{2}$. Let F satisfy: $F(x) = 0$ for $x \in [0, \alpha_1^-] \cup [\alpha_2^-, \alpha_1^+] \cup [\alpha_2^+, a]$, $F(x) < 0$ for $x \in (\alpha_1^-, \alpha_2^-)$, and $F(x) > 0$ for $x \in (\alpha_1^+, \alpha_2^+)$.

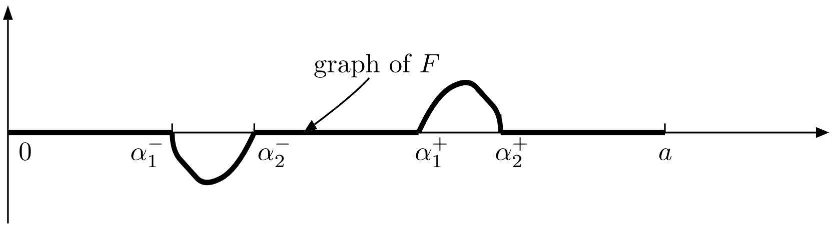

FIG. 7.

Then we have $\beta^* = 0$ and $k = (\alpha_2^+ - \alpha_1^+ - (\alpha_2^- - \alpha_1^-))/(a - (\alpha_2^+ - \alpha_1^+) - (\alpha_2^- - \alpha_1^-)) = 0$, and the limit function U is given by

$$U(x) = \begin{cases} 0 & \text{for } x \in [0, \alpha_1^-] \cup [\alpha_2^+, a], \\ x - \alpha_1^+ & \text{for } x \in (\alpha_1^-, \alpha_2^-), \\ \alpha_2^- - \alpha_1^- & \text{for } x \in [\alpha_2^-, \alpha_1^+], \\ -x + \alpha_2^+ & \text{for } x \in (\alpha_1^+, \alpha_2^+). \end{cases}$$

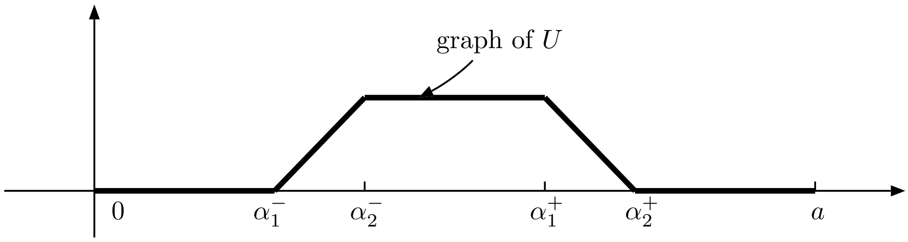

FIG. 8. *A case where* $k = 0$.

4. Radial case. In this section we give a proof of Theorem 2.2, which is rather close to that of Theorem 2.1 presented in the previous section.

Let $a > 0$ and $g \in C([0,\, a])$, and define $f \in C(B(0,a))$ by $f(x) = g(|x|)$. Set $\Omega = \operatorname{int} B(0,a)$.

We consider the Dirichlet problem for u_{p+1} as in the previous section:

$$\begin{cases} -\Delta_{p+1} u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

By the uniqueness of the solution of (4.1), we see that the function u_{p+1} is a radial function, i.e., $u_{p+1}(x) = v_p(|x|)$ for some $v_p \in C([0,a])$. By the regularity results for (4.1), we know that $u_{p+1} \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$. In particular, we have

$$v_p \in C^1([0,a]), \qquad v_p'(0) = 0.$$

The PDE (4.1) is now reduced to the following ODE for v_p

$$(r^{n-1}|v'(r)|^{p-1}v'(r))' = -r^{n-1}g(r) \quad \text{in } (0,a), \tag{4.2}$$

and the boundary condition for v_p is: $v'(0) = v(a) = 0$. Integrating twice yields

$$v(r) = \alpha - \int_0^r \psi_p\Big(t^{1-n}\int_0^t s^{n-1}g(s)\,\mathrm{d}s\Big)\,\mathrm{d}t \quad \text{for all } r \in [0,\, a]$$

and for some $\alpha \in \mathbf{R}$, where $\psi_p \in C(\mathbf{R})$ is the function given by $\psi_p(s) = |s|^{\frac{1}{p}-1}s$ as in the previous section. Here the constant α for $v = v_p$ should be determined by

$$\alpha = \int_0^a \psi_p\Big(t^{1-n}\int_0^t s^{n-1}g(s)\,\mathrm{d}s\Big)\,\mathrm{d}t.$$

Setting

$$\alpha_p = \int_0^a \psi_p\Big(t^{1-n}\int_0^t s^{n-1}g(s)\,\mathrm{d}s\Big)\,\mathrm{d}t,$$

we have

$$v_p(r) = \alpha_p - \int_0^r \psi_p\Big(t^{1-n}\int_0^t s^{n-1}g(s)\,\mathrm{d}s\Big)\,\mathrm{d}t \quad \text{for } r \in [0,r].$$

At this point one can check directly and easily that $v_p \in C^1([0\,a])$ and it satisfies (4.2) and the boundary condition $v_p'(0) = v_p(a) = 0$.

Completion of proof of Theorem 2.2. It is easy to see that as $p \to \infty$,

$$\alpha_p \to \alpha^* := \int_0^a \big(\mathbf{1}_{O_+}(r) - \mathbf{1}_{O_-}(r)\big)\,\mathrm{d}r,$$

and

$$v_p(r) \to V(r) := \alpha^* + \int_0^r \big(\mathbf{1}_{O_-}(t) - \mathbf{1}_{O_+}(t)\big)\,\mathrm{d}t \quad \text{for each } r \in [0,a],$$

where

$$O_+ = \{t \in (0,a) \mid \int_0^t s^{n-1}g(s)\,\mathrm{d}s > 0\} = \{t \in (0,a) \mid \int_{B(0,t)} f(x)\,\mathrm{d}x > 0\},$$

$$O_- = \{t \in (0,a) \mid \int_0^t s^{n-1}g(s)\,\mathrm{d}s < 0\} = \{t \in (0,a) \mid \int_{B(0,t)} f(x)\,\mathrm{d}x < 0\}.$$

As in the previous section, it is easy to show that the collection of functions $v_p(|x|)$, with $p > 1$, is precompact in $C(\overline{\Omega})$. Thus the above pointwise convergence is enough for us to conclude that $u_p(x)$ converges to $U(x) := V(|x|)$ uniformly for $x \in \overline{\Omega}$ as $p \to \infty$. ☐

Remark. Contrary to the general one dimensional case, the limit function V has the property that $V'(r) \in \{-1, 0, 1\}$ for all $r \in [0, a]$.

Remark. We also have a convergence result in the case when Ω is an annulus and f is radial. Indeed, let $0 < r_1 < r_2$, $\Omega = \{x \in \mathbf{R}^n \mid r_1 < |x| < r_2\}$, and $f(x) = g(|x|)$ for some $g \in C([r_1, r_2])$. Let u_p be the solution of (1.1). We define $G(r) = \int_{r_1}^{r} t^{n-1} g(t)\,\mathrm{d}t$ for $r \in [r_1, r_2]$, $h(r) = \mu_1(\{t \in (r_1, r_2) \mid G(t) < r\})$ for $r \in \mathbf{R}$, $\beta^* = \sup\{t \in (r_1, r_2) \mid h(r) \le \frac{r_2 - r_1}{2}\}$, $O_+ = \{r \in (r_1, r_2) \mid G(r) > \beta^*\}$, $O_- = \{r \in (r_1, r_2) \mid G(r) < \beta^*\}$, $O_0 = \{r \in (r_1, r_2) \mid G(r) = \beta^*\}$, $k = 0$ if $\mu_1(O_0) = 0$, $k = \frac{\mu_1(O_+) - \mu_1(O_-)}{\mu_1(O_0)}$ otherwise, and

$$U(x) = \int_{r_1}^{|x|} \big(\mathbf{1}_{O_-}(t) - \mathbf{1}_{O_+}(t) + k\mathbf{1}_{O_0}(t)\big)\,\mathrm{d}t \quad \text{for } x \in \overline{\Omega}.$$

Then we have

$$u_p \to U \quad \text{in } C(\overline{\Omega}) \quad \text{as } p \to \infty.$$

We do not give here the proof of this result since it is a simple combination of the proofs of Theorems 2.1 and 2.2.

5. General observations. Here we study a few of general properties of the solution u_p of (1.1), the set $\mathcal{A}$ of the limits of u_p defined by (1.6), and the set $\mathcal{M}$ of the maximizers of the variational problem (1.4), i.e.,

$$\mathcal{M} = \{v \in X \mid I_\infty(v) = \sup_{u \in X} I_\infty(u)\}.$$

We start by observing that the following estimate

$$\|Du_p\|_{L^p(\Omega)} \le C \tag{5.1}$$

holds, where the constant C can be chosen independently of p for $p > 2$. Indeed, using the test function $u = u_p$ in the weak formulation of (1.1), we get

$$\int_\Omega |Du|^p\,\mathrm{d}x = \int_\Omega fu\,\mathrm{d}x,$$

and hence by the Poincaré inequality for functions in $W_0^{1,1}(\Omega)$,

$$\begin{aligned}
&\int_\Omega |Du|^p\,\mathrm{d}x \le \|f\|_{L^\infty(\Omega)} \|u\|_{L^1(\Omega)} \le C_1 \|f\|_{L^\infty(\Omega)} \|Du\|_{L^1(\Omega)} \\
&\le C_1 \|f\|_{L^\infty(\Omega)} \mu(\Omega) \int_\Omega |Du| \frac{\mathrm{d}x}{\mu(\Omega)} \le C_1 \|f\|_{L^\infty(\Omega)} \mu(\Omega) \left(\int_\Omega |Du|^p \frac{\mathrm{d}x}{\mu(\Omega)}\right)^{\frac{1}{p}} \\
&\le C_1 \|f\|_{L^\infty(\Omega)} \mu(\Omega)^{1-\frac{1}{p}} \|Du\|_{L^p(\Omega)},
\end{aligned}$$

where C_1 is a positive constant independent of p. Hence, we obtain

$$\|Du\|_{L^p(\Omega)} \le \big(C_1 \|f\|_{L^\infty(\Omega)}\big)^{\frac{1}{p-1}} \mu(\Omega)^{\frac{1}{p}},$$

which shows (5.1).

From the above estimate (5.1), we have the following well-known observations (see [BDM] for instance).

PROPOSITION 5.1. (i) *For any $q > n$, the collection $\{u_p\}_{p\geq q}$ is precompact in $C(\overline{\Omega})$. In particular, for any sequence $1 < p_k \to \infty$ there is a subsequence p_{k_j} such that $u_{p_{k_j}}(x) \to U(x)$ uniformly on $\overline{\Omega}$ for some $U \in C(\overline{\Omega})$.*

(ii) *Let $U \in C(\overline{\Omega})$ be as above. Then $U \in W^{1,\infty}(\Omega)$ and $|DU(x)| \leq 1$ a.e. $x \in \Omega$.*

Proof. We show first (i). For $p \geq q$, we have

$$\|Du_p\|_{L^q(\Omega)} \leq \mu(\Omega)^{\frac{1}{q}-\frac{1}{p}} \|Du_p\|_{L^p(\Omega)} \tag{5.2}$$

For $q > n$, by the Sobolev embedding theorem (see, e.g., [GT]), we have

$$\|u_p\|_{C^{0,\gamma}(\overline{\Omega})} \leq C_q \|Du\|_{L^q(\Omega)}$$

for some constants $\gamma \in (0,1)$ and $C_q > 0$. These together with (5.1) imply that for any $q > n$, the collection $\{u_p\}_{p\geq q}$ is precompact.

Next, we prove (ii). The estimates (5.1) and (5.2) and the weak compactness of the balls in $W_0^{1,q}(\Omega)$, with $1 < q < \infty$, guarantee that $U \in W_0^{1,q}(\Omega)$ for any $q \in (1,\infty)$. This weak compactness, (5.1), and (5.2) yield

$$\|DU\|_{L^q(\Omega)} \leq \mu(\Omega)^{\frac{1}{q}} \qquad \text{for any } q > 1,$$

which implies that $|DU(x)| \leq 1$ a.e. in Ω. $\square$

Recalling the definition (1.6) of the set $\mathcal{A}$, from Proposition 5.1 we immediately have:

PROPOSITION 5.2. (i) *$\mathcal{A} \neq \emptyset$ and $\mathcal{A} \subset X$.* (ii) *$u_p \to U$ in $C(\overline{\Omega})$ as $p \to \infty$ if and only if $\mathcal{A} = \{U\}$.*

Next, we consider the functional $I_\infty(u)$ for $u \in X$ defined by (1.4) and study the set $\mathcal{M}$ of maximizers of this functional.

The following Proposition states a basic relation between $\mathcal{A}$ and $\mathcal{M}$.

PROPOSITION 5.3. (i) *$\mathcal{A} \subset \mathcal{M}$.* (ii) *As $p \to \infty$, $I_p(u_p) \to \sup_{u\in X} I_\infty(u)$.*

Proof. Let $U \in \mathcal{A}$ and $p_j \to \infty$ be such that $u_{p_j} \to U$ in $C(\overline{\Omega})$ as $j \to \infty$. As $p = p_j \to \infty$, we have

$$I_p(u_p) = I_\infty(u_p) - \frac{1}{p}\int_\Omega |Du_p|^p \,\mathrm{d}x \leq I_\infty(u_p) \to I_\infty(U).$$

Fix any $V \in X$ and observe that as $p \to \infty$,

$$I_p(u_p) \geq I_p(V) = I_\infty(V) - \frac{1}{p}\int_\Omega |DV(x)|^p \,\mathrm{d}x \to I_\infty(V).$$

Hence we get

$$I_\infty(U) \geq \limsup_{j\to\infty} I_{p_j}(u_{p_j}) \geq \liminf_{p\to\infty} I_p(u_p) \geq I_\infty(V).$$

Since $U \in X$ by Proposition 5.2, we thus conclude that

$$I_\infty(U) = \sup_{u\in X} I_\infty(u), \qquad \lim_{j\to\infty} I_{p_j}(u_{p_j}) = \sup_{u\in X} I_\infty(u),$$

and $\mathcal{A} \subset \mathcal{M}$. Using (i) of Proposition 5.1, we deduce that

$$I_p(u_p) \to \sup_{u\in X} I_\infty(u) \quad \text{as } p \to \infty. \qquad \square$$

PROPOSITION 5.4. *If $u \in \mathcal{A}$, then u satisfies*

$$-\Delta_\infty u(x) \le 0 \quad \text{in } \Omega \setminus \overline{\Omega}_+ \quad \text{and} \quad -\Delta_\infty u(x) \ge 0 \quad \text{in } \Omega \setminus \overline{\Omega}_-$$

in the viscosity sense.

Proof. We prove only the first inequality as the proof of the other inequality is similar. We set $W = \Omega \setminus \overline{\Omega}_+$. Let $\varphi \in C^2(W)$ and $\hat{x} \in W$. We assume that $u - \varphi$ attains a strict maximum at $\hat{x}$ and will show that $-\Delta_\infty\varphi(\hat{x}) \le 0$. For this, we argue by contradiction and hence we assume that $-\Delta_\infty\varphi(\hat{x}) > 0$. Here we may assume that $u(\hat{x}) = \varphi(\hat{x})$.

Since $u \in \mathcal{A}$, there is a sequence $1 < p_j \to \infty$ such that $u_{p_j} \to u$ in $C(\overline{\Omega})$ as $j \to \infty$. Fix an $r > 0$ so that $B(\hat{x}, r) \subset W$ and $\Delta_\infty\varphi(x) < 0$ for all $x \in B(\hat{x}, r)$. Since

$$\Delta_p\varphi(x) = |D\varphi(x)|^{p-4}(|D\varphi(x)|^2\Delta\varphi(x) + (p-2)\Delta_\infty\varphi(x)) \quad \text{for } x \in B(\hat{x}, r),$$

and

$$\min_{B(\hat{x},r)} |D\varphi| > 0 \quad \text{and} \quad \max_{B(\hat{x},r)} \Delta_\infty\varphi < 0,$$

we see that if p is large enough, then $\Delta_p\varphi(x) < 0$ for all $x \in B(\hat{x}, r)$.

Set $\omega = \operatorname{int} B(\hat{x}, r)$. Choose an $\varepsilon > 0$ so that $(u - \varphi)|_{\partial\omega} < -3\varepsilon$. If we choose $j \in \mathbf{N}$ large enough, then we have $(u_{p_j} - \varphi)|_{\partial\omega} < -2\varepsilon$ and $(u_{p_j} - \varphi)(\hat{x}) > -\varepsilon$. Fix such a j and set $v = u_{p_j} + \varepsilon$ and $q = p_j$ for notational simplicity. We may assume as well that $\Delta_q\varphi(x) < 0$ for all $x \in \omega$.

Since $f \le 0$ in ω and $(v - \varphi)^+ \in W_0^{1,q}(\omega)$, we have

$$\int_\omega |Dv|^{q-2}Dv \cdot D(v-\varphi)^+ \,\mathrm{d}x = \int_\omega f(v-\varphi)^+ \,\mathrm{d}x \le 0,$$
$$\int_\omega |D\varphi|^{q-2}D\varphi \cdot D(v-\varphi)^+ \,\mathrm{d}x = -\int_\omega \Delta_q\varphi(v-\varphi)^+ \,\mathrm{d}x > 0,$$

and hence

$$\int_{\omega_+} (|Dv|^{q-2}Dv - |D\varphi|^{q-2}D\varphi) \cdot D(v-\varphi) \,\mathrm{d}x < 0,$$

where $\omega_+ = \{x \in \omega \mid v(x) > \varphi(x)\}$. On the other hand, because of the convexity of the function: $\xi \mapsto |\xi|^q$, we know that

$$\int_{\omega_+} (|Dv|^{q-2}Dv - |D\varphi|^{q-2}D\varphi) \cdot D(v-\varphi) \,\mathrm{d}x \ge 0,$$

which contradicts to the above inequality. $\square$

Remark. Let $u \in \mathcal{A}$. By an argument similar to the above proof, we can prove that $\max\{|Du(x)| - 1, -\Delta_\infty u(x)\} \le 0$ in Ω in the viscosity sense. However, we have a stronger conclusion that

$$|Du(x)| \le 1 \quad \text{in } \Omega \text{ in the viscosity sense.} \tag{5.3}$$

Indeed, if $u \in \mathcal{A}$, then $u \in X$ and hence $|Du(x)| \le 1$ a.e., which implies (see, for instance, Proposition 3.4 in [Ln]) that u satisfies (5.3).

Definition. *Let $Y \subset X$. We call Y essentially single if for any u, $v \in Y$, $u = v$ on* $\operatorname{spt} f$.

Proposition 5.5. *Let $Y \subset X$ be such that $\mathcal{A} \subset Y$. If Y is essentially single, then $\mathcal{A}$ is a singleton. In particular, the whole family $\{u_p\}_{p>1}$ converges in $C(\overline{\Omega})$.*

The following proof has been already explained in Introduction.

Proof. Let u, $v \in \mathcal{A}$. By assumption, we have $u = v$ on $\operatorname{spt} f$. By Proposition 5.4, we see that u and v are both viscosity solutions of

$$-\Delta_\infty w(x) = 0 \quad \text{in } \Omega \setminus \operatorname{spt} f.$$

By the uniqueness result for this PDE due to [J], we conclude that $u = v$ in $\Omega \setminus \operatorname{spt} f$, which guarantees that $u = v$ in $\overline{\Omega}$. ∎

6. Properties of the set $\mathcal{M}$. In this section we collect some properties of the set $\mathcal{M}$ of the maximizers of the functional I_∞.

Proposition 6.1. *Let $u \in \mathcal{M}$. Then*

$$u(x) = \inf\{u(y) + |x - y| \mid y \in \Omega_- \cup \partial\Omega\} \quad \textit{for all } x \in \Omega_+ \tag{6.1}$$

and

$$u(x) = \sup\{u(y) - |x - y| \mid y \in \Omega_+ \cup \partial\Omega\} \quad \textit{for all } x \in \Omega_-. \tag{6.2}$$

A proposition similar to this can be found in [EG] (Lemma 3.1 of [EG]), the proof of which can be easily adapted to our case, but we give a proof here for completeness.

Proof. We prove only (6.1), since the proof of (6.2) is similar. Let $u \in X$. Then

$$|u(x) - u(y)| \le |x - y| \quad \text{for all } x, y \in \overline{\Omega},$$

from which we have

$$u(x) \le \inf\{u(y) + |x - y| \mid y \in A\}$$

for all $x \in \overline{\Omega}$ and any $A \subset \overline{\Omega}$. In particular, we have

$$u(x) \le \inf\{u(y) + |x - y| \mid y \in \Omega_- \cup \partial\Omega\} \quad \text{for all } x \in \overline{\Omega}. \tag{6.3}$$

Now, let $u \in \mathcal{M}$. Since $\mathcal{M} \subset X$, the inequality (6.3) holds with this u. Setting

$$v(x) = \inf\{u(y) + |x - y| \mid y \in \Omega_- \cup \partial\Omega\} \quad \text{for } x \in \overline{\Omega},$$

we see immediately from the definition of v that

$$v(x) - v(y) \le |x - y| \quad \text{for all } x, y \in \overline{\Omega},$$

which implies that $|Dv(x)| \le 1$ a.e. in Ω. Also, we have

$$u(x) \le v(x) \quad \text{for all } x \in \overline{\Omega}, \quad \text{by (6.3)},$$

and

$$v(x) \le u(x) \quad \text{for all } x \in \Omega_- \cup \partial\Omega, \quad \text{by the definition of } v.$$

Combining these we find that $u(x) = v(x)$ for all $x \in \Omega_- \cup \partial\Omega$. In particular, $v(x) = 0$ for all $x \in \partial\Omega$. Thus we see that $v \in X$.

Next note that $I_\infty(u) = \max_{w\in X} I_\infty(w) \ge I_\infty(v)$. On the other hand, since $u = v$ on Ω_- and $v \ge u$ on Ω_+, we get $I_\infty(u) \le I_\infty(v)$. Hence, we see that $I_\infty(v) = I_\infty(u)$. Now, since

$$\int_{\Omega_+} f(x)u(x)\,\mathrm{d}x = \int_{\Omega_+} f(x)v(x)\,\mathrm{d}x$$

and $v \ge u$ on Ω_+, we conclude that $u = v$ on Ω_+, which completes the proof. ∎

Remark. As one can see from the above proof, the set $\Omega_- \cup \partial\Omega$ in (6.1) can be replaced by any set $A \subset \overline{\Omega}$ satisfying $\Omega_- \cup \partial\Omega \subset A$. Similarly, the set $\Omega_+ \cup \partial\Omega$ in (6.2) can be replaced by any set $A \subset \overline{\Omega}$ satisfying $\Omega_+ \cup \partial\Omega \subset A$.

PROPOSITION 6.2. *Let $u, v \in \mathcal{M}$ and $k \ge 0$. Then $u \wedge (v + k)$, $(u - k) \vee v \in \mathcal{M}$.*

Proof. It is easy to see that $u \wedge (v + k)$, $(u - k) \vee v \in X$. In particular, we have

$$\max\{I_\infty(u \wedge (v + k)),\ I_\infty((u - k) \vee v)\} \le I_\infty(u) = I_\infty(v).$$

Noting that $u \wedge (v + k) = u - (u - v - k)^+$ and $(u - k) \vee v = v + (u - v - k)^+$, we see that

$$I_\infty(u\wedge(v+k)) = I_\infty(u) - I_\infty((u-v-k)^+), \quad I_\infty((u-k)\vee v) = I_\infty(v) + I_\infty((u-v-k)^+),$$

and hence

$$\begin{aligned} 0 &\le I_\infty(u) - I_\infty(u \wedge (v + k)) \\ &= I_\infty((u - v - k)^+) = I_\infty((u - k) \vee v) - I_\infty(v) \le 0. \end{aligned}$$

Consequently, we have

$$I_\infty(u \wedge (v + k)) = I_\infty((u - k) \vee v) = I_\infty(u),$$

Thus we conclude that $u \wedge (v + k)$, $(u - k) \vee v \in \mathcal{M}$. ∎

The following proposition establishes the existence of the maximal and minimal elements of $\mathcal{M}$.

PROPOSITION 6.3. *Define $V, W : \overline{\Omega} \to \mathbf{R}$ by*

$$V(x) = \sup\{v(x) \mid v \in \mathcal{M}\} \quad \text{and} \quad W(x) = \inf\{v(x) \mid v \in \mathcal{M}\}.$$

Then $V, W \in \mathcal{M}$.

Proof. We only prove the identity for V, since the proof of the other one is similar.

First of all, note that $V \in X$. Choose a dense subset $\{y_k\}_{k\in\mathbf{N}}$ of Ω. For each $k \in \mathbf{N}$ we choose a sequence $\{v_{kj}\}_{j\in\mathbf{N}} \subset \mathcal{M}$ such that $\lim_{j\to\infty} v_{kj}(y_k) = V(y_k)$. By the definition of V, we have $v(x) \le V(x)$ for all $x \in \Omega$ and $v \in \mathcal{M}$. Therefore, we find that

$$V(y_l) = \sup\{v_{kj}(y_l) \mid k, j \in \mathbf{N}\} \quad \text{for } l \in \mathbf{N}.$$

Define $w \in X$ by setting

$$w(x) = \sup\{v_{kj}(x) \mid k, j \in \mathbf{N}\} \quad \text{for } x \in \overline{\Omega}.$$

It is immediate to see that $V = w$ on $\overline{\Omega}$.

We intend to show that $V \in \mathcal{M}$. Relabeling the countable set $\{v_{kj}\}_{k,j\in\mathbf{N}}$, we find a sequence $\{v_m\}_{m\in\mathbf{N}} \subset \mathcal{M}$ such that

$$V(x) = \sup\{v_m(x) \mid m \in \mathbf{N}\} \quad \text{for all } x \in \overline{\Omega}.$$

We define the non-decreasing sequence $\{w_j\}_{j\in\mathbf{N}}$ by induction as follows:

$$w_1 = v_1, \quad w_{j+1} = w_j \vee v_{j+1} \text{ for } j \in \mathbf{N}.$$

By Proposition 6.2, we see that $w_j \in \mathcal{M}$ for all $j \in \mathbf{N}$. It is clear that

$$\lim_{j\to\infty} w_j(x) = V(x) \quad \text{for all } x \in \overline{\Omega}.$$

Therefore we see by the monotone convergence theorem that

$$I_\infty(V) = \lim_{j\to\infty} I_\infty(w_j) = \max_{v\in X} I_\infty(v),$$

and conclude that $V \in \mathcal{M}$. □

Proposition 6.4. *For any $u, v \in \mathcal{M}$, we have*

$$\sup_{\Omega_+}(u-v)^+ = \sup_{\Omega_-}(u-v)^+.$$

Proof. Set $k = \sup_{\Omega_-}(u-v)^+$, and observe that $u(y) \le v(y) + k$ for $y \in \Omega_- \cup \partial\Omega$. Using Proposition 6.1, we see that for $x \in \Omega_+$,

$$u(x) = \inf\{u(y) + |x-y| \mid y \in \Omega_- \cup \partial\Omega\} \le \inf\{v(y) + |x-y| \mid y \in \Omega_- \cup \partial\Omega\} + k.$$

Hence, we have $u(x) \le v(x) + k$ for all $x \in \Omega_+$, and therefore

$$\sup_{\Omega_+}(u-v)^+ \le \sup_{\Omega_-}(u-v)^+.$$

Exchanging the role of Ω_+ and Ω_- in the above argument, we get

$$\sup_{\Omega_-}(u-v)^+ \le \sup_{\Omega_+}(u-v)^+,$$

and finish the proof. □

Proposition 6.5. *If $u \in \mathcal{M}$, then u satisfies in the viscosity sense*

$$|Du(x)| = 1 \quad \text{in } \Omega_+ \quad \text{and} \quad -|Du(x)| = -1 \quad \text{in } \Omega_-. \tag{6.4}$$

This proposition is an easy consequence of Proposition 6.1. For completeness we give a proof here.

Proof. Fix $u \in \mathcal{M}$. Let $\varphi \in C^1(\Omega)$ and $\hat{x} \in \Omega_+$. Assume that $u - \varphi$ attains a maximum at $\hat{x}$. Then, since $|u(x) - u(\hat{x})| \le |x - \hat{x}|$ for all $x \in \Omega$, we have as $x \to \hat{x}$

$$-|x-\hat{x}| \le u(x) - u(\hat{x}) \le \varphi(x) - \varphi(\hat{x}) = D\varphi(\hat{x}) \cdot (x - \hat{x}) + o(|x-\hat{x}|).$$

Substituting $\hat{x} - tD\varphi(\hat{x})$ for x and sending $t \searrow 0$, we see that $|D\varphi(\hat{x})| \le 1$.

Now, we assume that $u - \varphi$ attains a minimum at $\hat{x}$. In view of Proposition 6.1, we choose a point $y \in \overline{\Omega}_- \cup \partial\Omega$ so that $u(\hat{x}) = u(y) + |\hat{x} - y|$ holds. As before, we have as $x \to \hat{x}$

$$|x-y| - |\hat{x}-y| \ge u(x) - u(\hat{x}) \ge \varphi(x) - \varphi(\hat{x}) \ge -|D\varphi(\hat{x})||x-\hat{x}| + o(|x-\hat{x}|).$$

Substituting $\hat{x} + t(y - \hat{x})$ for x and sending $t \searrow 0$, we see that $|D\varphi(\hat{x})| \ge 1$.

Thus we see that u is a viscosity solution of $|Du(x)| = 1$ in Ω_+. A parallel argument shows that u is a viscosity solution of $-|Du(x)| = -1$ in Ω_-. ☐

PROPOSITION 6.6. *$\mathcal{M}$ is a convex set as a subset of $C(\overline{\Omega})$.*

Proof. Note that $X \subset C(\overline{\Omega})$ is a convex set. Since I_∞ is a linear functional on $C(\overline{\Omega})$, we conclude that $\mathcal{M}$ is convex. ☐

PROPOSITION 6.7. *Let $u, v \in \mathcal{M}$. Then*

$$Du(x) = Dv(x) \qquad a.e.\ x \in \Omega_*.$$

Proof. Let u and $v \in \mathcal{M}$. Define $w \in C(\overline{\Omega})$ by

$$w = \frac{1}{2}(u + v).$$

According to Proposition 6.6, we have $w \in \mathcal{M}$. By Rademacher's theorem, we see that functions u, v, w are almost everywhere differentiable in Ω. Now, Proposition 6.5 yields

$$|Du(x)| = |Dv(x)| = |Dw(x)| = 1 \qquad \text{a.e. } x \in \Omega_*,$$

and therefore the strict convexity of the Euclidean norm in $\mathbf{R}^n$ implies that

$$Du(x) = Dv(x) = Dw(x) \qquad \text{a.e. } x \in \Omega_*. \qquad ☐$$

7. Case of no balanced family. In this section we first prove Theorem 2.3 and then examine a case where the hypothesis (2.4) is satisfied.

We begin with a lemma. Let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ be the classification of $\mathcal{O}_*$,

$$G_\lambda := \bigcup\{U \mid U \in \mathcal{O}_\lambda\} \qquad \text{for } \lambda \in \Lambda$$

as in Section 2. Also, let $\Lambda_0 \subset \Lambda$ be as in Section 2.

LEMMA 7.1. *If u, $v \in \mathcal{M}$, then $u - v$ is constant on any G_λ, with $\lambda \in \Lambda$.*

Proof. Let u, $v \in \mathcal{M}$. First of all, we observe that for any A, $B \subset \Omega$,

$$\begin{aligned} \inf_{(x,y)\in A\times B} |(u-v)(x) - (u-v)(y)| &\le \inf_{(x,y)\in A\times B} (|u(x) - u(y)| + |v(x) - v(y)|) \\ &\le 2d(A, B). \end{aligned} \tag{7.1}$$

Fix $\lambda \in \Lambda$ and $U, V \in \mathcal{O}_\lambda$. By Proposition 6.7, we have

$$(u - v)(x) = \begin{cases} k_U & \text{for } x \in U, \\ k_V & \text{for } x \in V \end{cases}$$

for some constants k_U, k_V. Fix any $\varepsilon > 0$. Since $\rho(U, V) = 0$, there is a finite family $W_1, ..., W_m \in \mathcal{O}_*$ such that $d(U, W_1)+d(W_1, W_2)+\cdots+d(W_m, V) < \varepsilon$. By Proposition 6.7, for each $i \in \{1, ..., m\}$ there is a constant k_i such that $(u-v)(x) = k_i$ for all $x \in U_i$.

Now, using (7.1), we get

$$\begin{aligned} |k_U - k_V| &\le |k_U - k_1| + |k_1 - k_2| + \cdots + |k_m - k_V| \\ &\le 2(d(U, W_1) + d(W_1, W_2) + \cdots + d(W_m, V)) < 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $k_U = k_V$. This shows that $u - v$ is constant on G_λ. ☐

Lemma 7.2. *Let u, $v \in \mathcal{M}$ and $\lambda \in \Lambda_0$. Then $u = v$ on G_λ.*

Proof. In view of Lemma 7.1, let $k \in \mathbf{R}$ be a constant such that $u = v + k$ on G_λ. Fix any $U \in \mathcal{O}_\lambda$ and $\varepsilon > 0$. There is a finite sequence $U_1, ..., U_m \in \Omega$ such that $d(U, U_1) + d(U_1, U_2) + \cdots + d(U_m, \partial\Omega) < \varepsilon$. As in the proof of Lemma 7.1, since $u = v = 0$ on $\partial\Omega$, we find that $|k| \leq 2(d(U, U_1) + d(U_1, U_2) + \cdots + d(U_m, \partial\Omega)) < 2\varepsilon$. This is enough for us to conclude that $u = v$ on G_λ. □

Proof of Theorem 2.3. In view of Proposition 5.5, it is enough to show that $\mathcal{M}$ is essentially single.

For this we argue by contradiction. Thus we let $u, v \in \mathcal{M}$ and assume that $u \neq v$ on $\operatorname{spt} f$. We may assume that u and v are, respectively, the maximal and minimal elements of $\mathcal{M}$, i.e.,

$$u(x) \geq w(x) \geq v(x) \quad \text{for all } x \in \Omega \text{ and } w \in \mathcal{M}.$$

Fix any $k > 0$ so that $k < \sup_{\Omega_*}(u - v)$. For $t \in (0, k]$ we set $w_t = u \wedge (v + t)$. Note that for $x \in \Omega$ and $0 \leq t < s \leq k$,

$$v(x) \leq w_t(x) \leq w_s(x) \leq w_k(x).$$

Also, since $0 < k < \sup_{\Omega_*}(u - v)$, we see that $w_k - v$ attains the maximum value k at some point of Ω_*.

By Proposition 6.2, we have $w_t \in \mathcal{M}$ for all $t \in (0, k]$. Hence we have $I_\infty(w_k) = I_\infty(w_t)$ for all $t \in (0, k)$, which reads

$$0 = \int_\Omega \frac{(w_k - w_t)(x)}{k - t} f(x)\,\mathrm{d}x \quad \text{for all } t \in (0, k).$$

For $0 \leq t < k$, we set

$$A_t = \{x \in \Omega_* \mid w_t(x) < w_k(x)\} \quad \text{and} \quad B = \bigcap_{0<t<k} A_t.$$

Note that $A_t \supset A_s$ for $0 < t < s < k$.

We claim here that

$$B = \{x \in \Omega_* \mid (w_k - v)(x) = k\}.$$

To see this, we write C for the right hand side of the above identity. Let $x \in B$. By definition, we have $w_t(x) < w_k(x)$ for all $t \in (0, k)$. This implies that $w_t(x) = v(x) + t$ for all $t \in (0, k)$, and hence that $v(x) + t < u(x)$ for all $t \in (0, k)$. Therefore, we have $v(x) + k \leq u(x)$, and moreover, $w_k(x) = v(x) + k$. Thus, we see that $B \subset C$.

Next, let $x \in C$. We then have $w_k(x) = v(x) + k$, which yields that $u(x) \geq v(x) + k > v(x) + t$ for all $t \in [0, k)$. Hence we have $w_t(x) < w_k(x)$ for all $t \in (0, k)$. That is, we have $x \in B$, which concludes that $C \subset B$ and moreover $B = C$.

Since $w_k - v$ takes the value k at some point of Ω_*, we have $B \neq \emptyset$.

Now we go back the equality

$$0 = \int_\Omega \frac{(w_k - w_t)(x)}{k - t} f(x)\,\mathrm{d}x = \int_{A_t} \frac{(w_k - w_t)(x)}{k - t} f(x)\,\mathrm{d}x \quad \text{for } t \in (0, k).$$

We are going to apply the Lebesgue convergence theorem. Since $A_t \supset A_s$ for $0 < t < s < k$ and $\bigcap_{0<t<k} A_t = B$, we see that as $t \nearrow k$,

$$\mathbf{1}_{A_t}(x) \to \mathbf{1}_B(x) \quad \text{for all } x \in \Omega.$$

Note that $|(w_k - w_t)(x)| \le |k-t|$ for all $t \in (0,k)$ and $x \in A_t$, and hence,

$$\mathbf{1}_{A_t}(x)\left|\frac{(w_k - w_t)(x)}{k-t}f(x)\right| \le |f(x)| \quad \text{for all } t \in (0,k) \text{ and } x \in \Omega.$$

For $x \in B$, we have $w_k(x) = v(x) + k$ and $w_t(x) = v(x) + t$, and therefore,

$$\frac{(w_k - w_t)(x)}{k-t} = 1.$$

Therefore, as $t \nearrow k$,

$$\mathbf{1}_{A_t}(x)\frac{(w_k - w_t)(x)}{k-t}f(x) \to \mathbf{1}_B(x)f(x).$$

We apply the Lebesgue convergence theorem along any sequence $t_k \nearrow k$, to conclude that $\int_B f(x)\,\mathrm{d}x = 0$.

Finally, noting by Lemma 7.1 that the function $w_k - v$ is constant on any G_λ, with $\lambda \in \Lambda$, and setting

$$\Gamma = \{\lambda \in \Lambda \mid (w_k - v)(x) = k \ \text{ on } G_\lambda\},$$

we have

$$B = \bigcup\{G_\lambda \mid \lambda \in \Gamma\}.$$

We see from Lemma 7.2 that $\Gamma \subset \Lambda \setminus \Lambda_0$. Recalling that $B \neq \emptyset$, by the assumption (2.4) we have $\int_B f(x)\,\mathrm{d}x \neq 0$. This is a contradiction. ∎

Let us examine the case where

$$\mu(\{x \in \Omega \mid f(x) = 0\}) = 0. \tag{7.2}$$

We have the following proposition as a corollary of Proposition 6.7.

THEOREM 7.3. *Under the assumption* (7.2), $\mathcal{M}$ *is a singleton. As a consequence, the whole family* $\{u_p\}_{p>1}$ *converges in* $C(\overline{\Omega})$.

Proof. Let $u, v \in \mathcal{M}$. By Proposition 6.7, we have $Du(x) = Dv(x)$ a.e. in Ω_*. By (7.2), we have $Du(x) = Dv(x)$ a.e. in Ω. Hence, $u = v$ in Ω. ∎

We wish here to explain that the convergence result in Theorem 7.3 can be shown as a consequence of Theorem 2.3.

PROPOSITION 7.4. *If* (7.2) *holds, then* $\Lambda = \Lambda_0$ *and hence* (2.4) *is satisfied.*

Proof. We only need to show that $\rho(U, \partial\Omega) = 0$ for all $U \in \mathcal{O}_*$.

To do this, we fix $U \in \mathcal{O}_*$ and $x \in U$. Choose a closest point y in $\partial\Omega$ to the point x. Set $R = |y - x|$. Choose a constant $r \in (0, R)$ so that $B(x, r) \subset U$. Let H be the hyperplane normal to the vector $y - x$ and passing through the point x, i.e., $H = \{\xi \in \mathbf{R}^n \mid (\xi - x)\cdot(y - x) = 0\}$. Let C be the truncated open cone generated by the point y and the $(n-1)$-dimensional sphere $H \cap B(x, r)$. That is, we write

$$C = \{ty + (1-t)\xi \mid (t, \xi) \in (0,\,1) \times (H \cap B(x,\,r))\}.$$

Note that $C \subset \operatorname{int} B(x, R) \subset \Omega$.

By the assumption (7.2), we have

$$\mu(C) = \mu(C \cap \Omega_*).$$

Using the Fubini theorem, from this we deduce that for μ_{n-1}-almost all $\xi \in H \cap B(x, r)$, we have $\mu_1(\{t \in (0,1) \mid ty + (1-t)\xi \in \Omega_*\}) = 1$.

Fix a point $\xi \in H \cap B(x, r)$ so that $\mu_1(\{t \in (0,1) \mid ty + (1-t)\xi \in \Omega_*\}) = 1$. Define $I \subset (0, 1)$ by setting $I = \{t \in (0,1) \mid ty + (1-t)\xi \in \Omega_*\}$. Since I is an open subset of (0, 1), there is a sequence $\{I_j\}_{j\in J}$, with $J \subset \mathbf{N}$, of non-empty open intervals $I_j \subset (0, 1)$ such that $I = \bigcup\{I_j \mid j \in J\}$. We may assume as well that if $i, j \in J$ and $i \neq j$, then $I_i \cap I_j = \emptyset$. Since $\mu_1(I) = 1$, we have $\sum_{j\in J} \mu_1(I_j) = 1$.

Fix any $\varepsilon > 0$. There is a finite subset $J_\varepsilon \subset J$ such that $\sum_{j\in J_\varepsilon} \mu_1(I_j) > 1 - \varepsilon$. We may assume that $J_\varepsilon = \{1, ..., m\}$, where m is a positive integer which depends on ε. For each $j \in J_\varepsilon$, we choose a_j, $b_j \in [0, 1]$ so that $I_j = (a_j, b_j)$. We may further assume that $a_1 < b_1 \le a_2 < b_2 \le ... \le a_m < b_m$. Then we have $\sum_{j\in J_\varepsilon} \mu_1(I_j) = \sum_{j\in J_\varepsilon}(b_j - a_j) > 1 - \varepsilon$.

For each $j \in J_\varepsilon$, since $ty+(1-t)\xi \in \Omega_*$ for $t \in I_j$, and the set $\{ty+(1-t)\xi \mid t \in I_j\}$ is connected, we see that there is a $U_j \in \mathcal{O}_*$ such that

$$ty + (1-t)\xi \in U_j \text{ for } t \in I_j.$$

Observe that

$$\begin{aligned}
d(U, U_1) &\le |\xi - (a_1 y + (1-a_1)\xi)| \le a_1|y - \xi|, \\
d(U_{j-1}, U_j) &\le |(b_{j-1}y + (1-b_{j-1})\xi) - (a_j y + (1-a_j)\xi)| \\
&\le (a_j - b_{j-1})|y - \xi| \quad \text{ for all } j \in \{2, ..., m\}, \\
d(U_m, \partial\Omega) &\le |y - (b_m y + (1-b_m)\xi)| \le (1-b_m)|y - \xi|.
\end{aligned}$$

Adding all of these, we get

$$\begin{aligned}
d(U, U_1) + d(U_1, U_2) + \cdots + d(U_m, \partial\Omega) &\le (a_1 + (a_2 - b_1) + \cdots + (1 - b_m))|y - \xi| \\
&= (1 - \sum_{j\in J_\varepsilon}(b_j - a_j))|y - \xi| < \varepsilon|y - \xi|.
\end{aligned}$$

Thus we see that $\rho(U, \partial\Omega) = 0$ and finish the proof. □

8. Cases with vanishing integral. In this section we prove Theorem 2.4.

Proof of Theorem 2.4. Assume that (2.5) and (2.6) are satisfied.

First, we assume that (2.7) and (2.8) are satisfied and show in view of Proposition 5.5 that $\mathcal{A}$ is essentially single.

We observe that every $u \in \mathcal{A}$ is an odd function. This follows from the uniqueness of solutions of the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.1}$$

Indeed, if u is a solution of (8.1), then the function $-u(-x)$ is a solution of (8.1) as well and, by the uniqueness, $u(x) = -u(-x)$ for all $x \in \overline{\Omega}$. This shows that every function $u \in \mathcal{A}$ is an odd function.

Now, let $u, v \in \mathcal{A}$. Since Ω_+ and Ω_- are connected and $Du(x) = Dv(x)$ a.e. $x \in \Omega_*$, there is a constant $k \in \mathbf{R}$ such that $u(x) = v(x) + k$ for all $x \in \Omega_+$. By symmetry in u and v, we may assume that $k \ge 0$.

Since u and v are odd functions, we have

$$-u(-x) = -v(-x) + k \quad \text{ for all } x \in \Omega_+.$$

That is, $u(x) = v(x) - k$ for all $x \in \Omega_-$. Using Proposition 6.4, we thus get

$$k = \max_{\Omega_+}(u - v)^+ = \max_{\Omega_-}(u - v)^+ = 0,$$

which shows that $u(x) = v(x)$ for all $x \in \Omega_*$ and hence $\mathcal{A}$ is essentially single.

Next we turn to the case where (2.9) is satisfied. It is enough to show that $\mathcal{M}$ is essentially single. Let $u, v \in \mathcal{M}$ and we will show that $u = v$ on Ω_*.

By Proposition 6.4, we have

$$\sup_{\Omega_+}(u - v)^+ = \sup_{\Omega_-}(u - v)^+. \tag{8.2}$$

We argue by contradiction, and hence suppose that $u(z) \neq v(z)$ for some $z \in \Omega_*$. We may assume that $u(z) > v(z)$. In view of (8.2), there is a constant $k > 0$ such that $u(x) - v(x) = k$ for all $x \in \Omega_*$.

From (2.9) we have either

$$\inf_{x\in\Omega_+} \sup_{y\in\Omega_-} [d(x) + d(y) - |x - y|] \leq 0, \tag{8.3}$$

or

$$\inf_{y\in\Omega_-} \sup_{x\in\Omega_+} [d(x) + d(y) - |x - y|] \leq 0. \tag{8.4}$$

We consider only the case where (8.3) holds since the other case can be treated similarly.

For each $\varepsilon > 0$ there exists a point $x_\varepsilon \in \Omega_+$ such that $d(x_\varepsilon) + d(y) - |x_\varepsilon - y| \leq \varepsilon$ for all $y \in \Omega_-$. Since $u, v \in X$, we have $|u(x)| \vee |v(x)| \leq d(x)$ for all $x \in \Omega$. From these we get

$$d(x_\varepsilon) \leq \inf\{v(y) + |x_\varepsilon - y| \mid y \in \Omega_-\} + \varepsilon,$$

Since $v = 0$ on $\partial\Omega$, we have

$$d(x_\varepsilon) = \inf\{v(y) + |x_\varepsilon - y| \mid y \in \partial\Omega\}.$$

Therefore, using Proposition 6.1, we have

$$d(x_\varepsilon) \leq \inf\{v(y) + |x_\varepsilon - y| \mid y \in \Omega_- \cup \partial\Omega\} + \varepsilon = v(x_\varepsilon) + \varepsilon.$$

Thus we obtain $u(x_\varepsilon) \leq d(x_\varepsilon) \leq v(x_\varepsilon) + \varepsilon$, which yields a contradiction by choosing $\varepsilon \in (0, k)$. This completes the proof. □

Remark. In Theorem 2.4 we may replace the condition (2.6) by the condition that $\#(\Lambda \setminus \Lambda_0) = 2$, where Λ and Λ_0 are as in the assumption (2.4), and the condition (2.5) by

$$\int_\omega f(x)\,\mathrm{d}x = 0,$$

with $\omega := \bigcup\{G_\lambda \mid \lambda \in \Lambda \setminus \Lambda_0\}$.

REFERENCES

[ACBBV] L. Ambrosio, L. A. Caffarelli, Y. Brenier, G. Buttazzo, C. Villani, *Optimal transportation and applications*, L. A. Caffarelli, S. Salsa, eds., Lecture Notes in

Mathematics, 1813, Springer-Verlag, Berlin; Centro Internazionale Matematico Estivo, Florence, 2003.

[A1] G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Ark. Mat., 6 (1967), pp. 551–561.

[A2] G. Aronsson, *On the partial differential equation* ${u_x}^2 u_{xx} + 2u_x u_y u_{xy} + {u_y}^2 u_{yy} = 0$, Ark. Mat., 7 (1968), 395–425.

[ACJ] G. Aronsson, M. G. Crandall, and P. Juutinen, *A tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 4, 439–505.

[BB] G. Barles and J. Busca, *Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term*, Comm. Partial Differential Equations, 26 (2001), no. 11-12, pp. 2323–2337.

[BBD] G. Bouchitté, G. Buttazzo, and L. De Pascale, *A p-Laplacian approximation of some mass optimization problems*, J. Optim. Theory Appl., 118 (2003) no. 1, pp. 1–25.

[BDM] T. Bhattacharya, E. DiBenedetto, and J. Manfredi, *Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems*, Some topics in nonlinear PDEs (Turin, 1989). Rend. Sem. Mat. Univ. Politec. Torino 1989, Special Issue, pp. 15–68 (1991).

[BK] M. Belloni and B. Kawohl, *The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$*, ESAIM Control Optim. Calc. Var. 10 (2004), no. 1, 28–52 .

[CEG] M. G. Crandall, L. C. Evans, and R. F. Gariepy, *Optimal Lipschitz extensions and the infinity Laplacian*, Calc. Var. Partial Differential Equations, 13 (2001), no. 2, pp. 123–139.

[CIL] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), no. 1, pp. 1–67.

[D] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., 7 (1983), no. 8, pp. 827–850.

[EG] L. C. Evans and W. Gangbo, *Differential equations methods for the Monge-Kantorovich mass transfer problem*, Mem. Amer. Math. Soc., 137 (1999), no. 653.

[FIN] N. Fukagai, M. Ito, and K. Narukawa, *Limit as $p \to \infty$ of p-Laplace eigenvalue problems and L^∞-inequality of the Poincaré type*, Differential Integral Equations, 12 (1999), no. 2, pp. 183–206.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order. Second edition*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224, Springer-Verlag, Berlin, 1983.

[IK] T. Ishibashi and S. Koike, On fully nonlinear PDEs derived from variational problems of L^p norms, SIAM J. Math. Anal., 33 (2001), no. 3, pp. 545–569.

[IL1] H. Ishii and P. Loreti, *On relaxation in an L^∞ optimization problem*, Proc. Roy. Soc. Edinburgh Sect. A, 133 (2003), no. 3, pp. 599–615.

[IL2] H. Ishii and P. Loreti, *Relaxation of Hamilton-Jacobi equations*, Arch. Rational Mech. Anal., 169 (2003), no. 4, pp. 265–304.

[IL3] H. Ishii and P. Loreti, *Asymptotic behavior of the solution of the p-Laplace equation on a finite interval*, unpublished.

[IL4] H. Ishii and P. Loreti, *Asymptotic behavior of solutions of p-Laplace type equations as p goes to infinity*, *Proceedings of the twelfth Tokyo conference on nonlinear PDE 2003*, pp. 32–50, Tokyo, Japan, 2004.

[J] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*, Arch. Rational Mech. Anal., 123 (1993), no. 1, pp. 51–74.

[JPM] P. Juutinen, P. Lindqvist, J.J. Manfredi, *The ∞-eigenvalue problem*, Arch. Ration. Mech. Anal. 148 (1999), no. 2, pp. 89–105.

[K] B. Kawohl, *On a family of torsional creep problems*, J. Reine Angew. Math., 410 (1990), pp. 1–22.

[L] J.-L. Lions, *Quelques méhodes de réolution des problèmes aux limites non linéires*, (French) Dunod; Gauthier-Villars, Paris 1969.

[Lb] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. 12 (1988), no. 11, pp. 1203–1219.

[Ln] P.-L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Mathematics, 69, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[PP] L. E. Payne and G. A. Philippin, *Some applications of the maximum principle in the problem of torsional creep*, SIAM J. Math. Anal., 33 (1977), pp. 446–455.

[T] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, 51 (1984), no. 1, pp. 126–150.

[U] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math., 138 (1977), no. 3-4, pp. 219–240.