A LEVEL SET APPROACH TO THE WEARING PROCESS OF A NONCONVEX STONE

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Abstract: We study the geometric evolution of a nonconvex stone by the wearing process via the partial differential equation methods. We use the so-called level set approach to this geometric evolution of a set. We establish a comparison theorem, an existence theorem, and some stability properties of solutions of the partial differential equation arising in this level set approach, and define the flow of a set by the wearing process via the level set approach.

Keywords: geometric evolution, wearing process, nonconvex stone, level set approach

Subject classification: 53C44, 35K65

1. Introduction

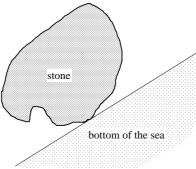
In this paper we study a mathematical model of the wearing process of a stone rolling on beach via the PDE methods.

In [F] Firey proposed and studied a mathematical model of the wearing process of such a stone in the case when it has a convex shape. In his model the motion of a stone is described by the Gauss curvature flow. See, for instance, [A, C, CEI, H, T] for the mathematical developments regarding the Gauss curvature flow.

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In [IM2, IM3] we extended Firey's arguments to the case when the stone does not necessarily have a convex shape. The idea of our extension was very simple, which is explained as follows. Let $V_t \subset \mathbf{R}^{n+1}$ be a stone at time t being worn due to hits of the bottom of the sea (or beach). In the model the bottom of the sea is supposed to be a hyperplane.



We look at the stone in a moving coordinates system for which the stone does not rotate and translate and, for each unit vector $p \in \mathbf{R}^{n+1}$, we associate the hyperplane

$$\pi(p) = \{ z \in \mathbf{R}^{n+1} \mid z \cdot p = 0 \},$$

where $z \cdot p$ denotes the Euclidean inner product of z and p.

The set V_t evolves by loosing its volume near the point where it is hit by the hyperplane $\pi(p)$ coming from the direction p. Here the sentence "a point $X \in V_t$ is hit by $\pi(p)$ coming from the direction p" means that the half space

$$X + \{z \in \mathbf{R}^{n+1} \mid z \cdot p > 0\}$$

does not intersects with V_t . In other words, part of the surface of a stone in our model which locates in a cavity of the stone does not evolve until it is exposed so that it can be hit by a hyperplane.

Three hypotheses in this model of the wearing process are imposed: (i) the probability of $\pi(p)$ hitting the stone V_t is uniform with respect to the direction p, (ii) the volume loss near a point $X \in \partial V_t$ is proportional to how often the point X is hit by hyperplanes, and (iii) the total volume loss of the stone in a time period is proportional to the length of the time period. Moreover, it is imposed that, once V_t becomes empty at a time t_0 then $V_t = \emptyset$ for all $t > t_0$.

In [IM2, IM3] we restricted ourselves to the case when the boundary of the stone V_t at time t is given by the graph of an evolving function, the case which corresponds to stones with infinite volume.

Our main purpose here is to remove this non-realistic (in applications) hypothesis that the boundary of stones is given by the graph of functions. This is carried out by adapting the level set approach (see [OS, CGG, ES]) to our model of the wearing process. One of main difficulties was in establishing the comparison assertion (see Theorem 2.6) for viscosity solutions of the partial differential equation arising in the level set approach, which has been resolved by reducing the problem to the case of evolving graphs. This idea has been employed successfully in [GG] in the study of a general planar anisotropic curvature flow. The PDE has a non-local factor which describes the "cavity" effect. This non-local effect resembles in its character which one encounters in a mathematical model of etching (see [AEI]).

The paper is organized as follows: In section 2, we recall the level set approach to Gauss curvature flow, then introduce the PDE in the level set approach to our model of the evolution of a set (stone) by wearing process, and state main results in this paper. The rest of the paper is devoted to the proof of the main results: in section 3, we establish our comparison theorem for (viscosity) solutions of the PDE in the level set approach and in section 4 we study stability properties of (viscosity) solutions of the PDE and then establish our existence theorem for the PDE. An elementary lemma is presented in the appendix which gives a representation formula for Gauss curvature of level sets of a function.

2. Level set approach and main results

We first recall the level set approach to the Gauss curvature flow.

Let $\{V_t\}_{t\geq 0}$ be a collection of closed subsets V_t of \mathbf{R}^{n+1} parametrized by $t\geq 0$. The collection $\{V_t\}_{t\geq 0}$ is called the generalized Gauss curvature flow issued from V_0 if there exists a function $u\in C(\mathbf{R}^{n+1}\times[0,\infty))$ which satisfies

(2.1)
$$u_t(z,t) = G(Du(z,t), D^2u(z,t)) \quad \text{for } (z,t) \in \mathbf{R}^{n+1} \times (0,\infty)$$

in the viscosity sense (see the definition in [IS], which is recalled below),

$$V_t = \{(z, t) \in \mathbf{R}^{n+1} \times [0, \infty) \mid u(z, t) \le 0\}$$
 for $t \ge 0$,

and for any $\lambda \in \mathbf{R}$,

(2.2) the set
$$\{(z,t) \mid u(z,t) \leq \lambda\} \subset \mathbf{R}^{n+2}$$
 is compact.

Here $G: (\mathbf{R}^{n+1} \setminus \{0\}) \times \mathcal{S}^{n+1} \to \mathbf{R}$ is defined by

$$G(p,X) = |p| \det_+ \left(|p|^{-1} (I - \overline{p} \otimes \overline{p}) X (I - \overline{p} \otimes \overline{p}) + \overline{p} \otimes \overline{p} \right)$$

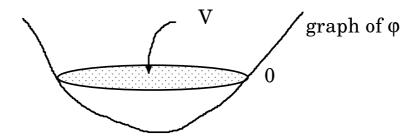
where $\overline{p} = p/|p|$ and

$$\det_{+} A = \prod_{i=1}^{n+1} \max\{\lambda_i, 0\} \quad \text{for } A \in \mathcal{S}^{n+1},$$

with λ_i (i = 1, ..., n + 1) denoting the eigenvalues of the matrix A.

Note (see Lemma A in the appendix) that for any $\varphi \in C^2(\mathbf{R}^{n+1})$, if $V := \{z \in \mathbf{R}^{n+1} \mid \varphi(z) \leq 0\}$ is convex, $0 \in \partial V$, and $D\varphi(0) \neq 0$, then

the Gauss curvature of ∂V at $0 = |D\varphi(0)|^{-1}G(D\varphi(0), D^2\varphi(0))$.



The choice of condition (2.2) is somewhat optional and it is put here in order to make sure the uniqueness of the solution of the Cauchy problem for (2.1). See Theorem 2.2 below.

In what follows, for convenience of notation, we write N = n + 1.

We introduce a class of functions u defined on $\Omega \subset \mathbf{R}^N \times [0, \infty)$ by imposing a condition at infinity. The condition for u is stated as follows:

(A) for each $T \in (0, \infty)$ and $(z,t) \in \Omega \cap (\mathbf{R}^N \times [0,T])$ there is a constant R > 0 such that for all $(y,s) \in \Omega \cap (\mathbf{R}^N \times [0,T])$, if $|y| \geq R$, then $u(y,s) \geq u(z,t)$.

Similarly, we introduce condition $(A)_0$ for functions $u:\Omega\subset\mathbf{R}^N\to\mathbf{R}$:

(A)₀ for each $z \in \Omega$ there is a constant R > 0 such that $u(y) \ge u(z)$ for all $y \in \Omega \setminus B(0,R)$.

For $\Omega_0 \subset \mathbf{R}^N$ and $\Omega \subset \mathbf{R}^N \times [0, \infty)$ we set

$$\mathcal{U}_0(\Omega_0) = \{ u \in C(\Omega_0) \mid u \text{ satisfies condition } (A)_0 \},$$

$$\mathcal{U}(\Omega) = \{u \in C(\Omega) \mid u \text{ satisfies condition (A)}\}.$$

Since G(p, X) is singular for p = 0, the standard definition of viscosity solution (see [CIL]) is not good enough for (2.1). Following [IS], we recall a definition of viscosity solution which is appropriate to (2.1). We first introduce the set \mathcal{A} of admissible test functions. We define \mathcal{F} as the set of those functions $f \in C^2([0,\infty))$ which satisfy

$$f(0) = f'(0) = f''(0) = 0, f''(r) > 0 \text{for all } r > 0,$$
$$\lim_{p \to 0} |p|^{-1} f'(|p|) \max\{|G(p, I)|, |G(p, -I)|\} = 0,$$

where the prime ' denotes the differentiation. This last condition can be phrased as

$$\lim_{r \searrow 0} f'(r)r^{-N+1} = 0.$$

Let Ω be an open subset of $\mathbf{R}^N \times (0, \infty)$. A function $\varphi \in C^2(\Omega)$ is called admissible in Ω if for each $(\hat{z}, \hat{t}) \in \Omega$ where $D\varphi$ vanishes, there is an $f \in \mathcal{F}$ such that

$$|\varphi(z,t) - \varphi(\hat{z},\hat{t}) - \varphi_t(\hat{z},\hat{t})(t-\hat{t})| \le f(|z-\hat{z}|) + o(|t-\hat{t}|)$$
 as $(z,t) \to (\hat{z},\hat{t})$.

We denote by $\mathcal{A}(\Omega)$ the set of all admissible functions in Ω .

We remark that the function $f \in C^{\infty}([0,\infty))$ defined by $f(r) = r^{N+1}$ belongs to \mathcal{F} and the function $\varphi \in C^2(\mathbf{R}^N \times (0,\infty))$ defined by $\varphi(z,t) = f(|z-\hat{z}|)$ is an admissible function in $\mathbf{R}^N \times (0,\infty)$ for any $\hat{z} \in \mathbf{R}^N$.

It is convenient to define

$$G(D\varphi(z,t), D^2\varphi(z,t)) = 0$$
 if $D\varphi(z,t) = 0$

for all $\varphi \in \mathcal{A}(\Omega)$. With this notation, we consider the set of all those functions $\varphi \in \mathcal{A}(\Omega)$ for which the function: $(z,t) \mapsto G(D\varphi(z,t), D^2\varphi(z,t))$ is continuous on Ω . We denote this set by $\mathcal{A}_0(\Omega)$. Note that for any $f \in \mathcal{F}$, $\hat{z} \in \mathbf{R}^N$, $\psi \in C^2(\mathbf{R})$, and $a \in \mathbf{R}$, if we set $\varphi(z,t) = af(|z-\hat{z}|) + \psi(t)$, then $\varphi \in \mathcal{A}_0(\Omega)$.

A function $u \in \mathrm{USC}(\Omega)$ is called a viscosity subsolution of (2.1) in Ω if whenever $\varphi \in \mathcal{A}(\Omega), (y,s) \in \Omega$, and $u - \varphi$ attains a local maximum at (y,s), then

$$\varphi_t(y,s) \le G(D\varphi(y,s), D^2\varphi(y,s)).$$

Similarly, a function $u \in \mathrm{USC}(\Omega)$ is called a viscosity supersolution of (2.1) in Ω if whenever $\varphi \in \mathcal{A}(\Omega)$, $u - \varphi$ attains a local minimum at $(y, s) \in \Omega$, then

$$\varphi_t(y,s) \ge G(D\varphi(y,s), D^2\varphi(y,s)).$$

A function $u \in C(\Omega)$ is called a viscosity solution of (2.1) in Ω if it is both a viscosity subsolution and a viscosity supersolution of (2.1) in Ω .

It is an easy exercise to check that in the above definition of viscosity sub- and supersolutions we may replace $\mathcal{A}(\Omega)$ by $\mathcal{A}_0(\Omega)$.

A few of basic observations on the viscosity solutions of (2.1) are stated in the following theorem.

Theorem 2.1. (a) Let $\theta \in C(\mathbf{R})$ be a non-decreasing function and Ω an open subset of $\mathbf{R}^N \times (0, \infty)$. Then, if $u \in \mathrm{USC}(\Omega)$ (resp., $v \in \mathrm{LSC}(\Omega)$) is a viscosity subsolution (resp., supersolution) of (2.1) in Ω , then so is the function $\theta \circ u$ (resp., $\theta \circ v$).

- (b) Let $u, v \in BUC(\mathbf{R}^N \times [0, \infty))$ be a viscosity subsolution and a viscosity supersolution of (2.1), respectively. Assume that $u(z, 0) \leq v(z, 0)$ for all $z \in \mathbf{R}^N$. Then $u \leq v$ in $\mathbf{R}^N \times [0, \infty)$.
- (c) Let $h \in BUC(\mathbf{R}^N)$. Then there is a viscosity solution $u \in BUC(\mathbf{R}^N \times [0, \infty))$ of (2.1) satisfying the initial condition

(2.3)
$$u(z,0) = h(z) \quad \text{for all } z \in \mathbf{R}^N.$$

For a proof of the above theorem, see [IS].

The following result justifies this definition of the generalized Gauss curvature flow.

Theorem 2.2. (a) For any $h \in \mathcal{U}_0(\mathbf{R}^N)$ there is a unique viscosity solution $u \in \mathcal{U}(\mathbf{R}^N \times [0,\infty))$ of (2.1) satisfying the initial condition (2.3).

(b) Let $\lambda \in \mathbf{R}$ and $u, v \in \mathcal{U}(\mathbf{R}^N \times [0, \infty))$ be two viscosity solutions of (2.1). If

$$\{z \in \mathbf{R}^N \mid u(z,0) \le \lambda\} \subset \{z \in \mathbf{R}^N \mid v(z,0) \le \lambda\},\$$

then

$$(2.5) \qquad \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid u(z,t) \le \lambda\} \subset \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid v(z,t) \le \lambda\}.$$

It follows from the above theorem that for any compact set $S \subset \mathbf{R}^N$, there is a unique generalized Gauss curvature flow $\{V_t\}_{t\geq 0}$ such that $V_0 = S$. To see the existence, for a given compact set $S \subset \mathbf{R}^N$ choose $h \in \mathcal{U}_0$ so that $S = \{z \in \mathbf{R}^N \mid h(z) \leq 0\}$, apply Theorem 2.2, (a), to find a viscosity solution $u \in \mathcal{U}$ satisfying the initial condition (2.3), and set $V_t = \{z \in \mathbf{R} \mid u(z,t) \leq 0\}$ for $t \geq 0$. The uniqueness is an immediate consequence of Theorem 2.2, (b).

We remark that in [IS] the generalized Gauss curvature flow is defined even for noncompact initial sets S. However, we restrict ourselves to the case when initial sets are compact in order to make presentation simple.

Theorems 1.8 and 1.9 of [IS] contain an assertion similar to Theorem 2.2. However, there is a small difference in their formulations, and so we give below an outline how to get it from Theorem 2.1.

In the following arguments we use many ideas developed in [CGG, ES] without mentioning.

Outline of proof of Theorem 2.2. Observe that if $\theta \in C(\mathbf{R})$ is a non-decreasing function such that $\sup_{\mathbf{R}} \theta < \infty$, $h \in \mathcal{U}_0$, and $u \in \mathcal{U}$, then $\theta \circ h \in \mathrm{BUC}(\mathbf{R}^N)$ and $\theta \circ u \in \mathrm{BUC}(\mathbf{R}^N \times [0, \infty))$.

Proof of (b): Fix $\lambda \in \mathbf{R}$ and $u, v \in \mathcal{U}$ so that (2.4) holds. It is not hard to see that there are increasing functions $\theta_1, \theta_2 \in C(\mathbf{R})$ such that $\sup_{\mathbf{R}} (\theta_1 + \theta_2) < \infty$, $\theta_1(\lambda) = \theta_2(\lambda)$, and $\theta_1 \circ u(z, 0) \ge \theta_2 \circ v(z, 0)$ for all $z \in \mathbf{R}^N$. Noting that $\theta_1 \circ u, \theta_2 \circ v \in \mathrm{BUC}(\mathbf{R}^N \times [0, \infty))$, we conclude from Theorem 2.1, (b) that $\theta_1 \circ u \ge \theta_2 \circ v$ in $\mathbf{R}^N \times [0, \infty)$, from which follows (2.5).

Proof of (a): The uniqueness assertion is an immediate consequence of (b).

Fix any $h \in \mathcal{U}_0$. Fix an increasing function $\theta \in C(\mathbf{R})$ so that $\theta(\mathbf{R}) = (-\infty, 0)$. By virtue of Theorem 2.1, (c), there exists a viscosity solution $w \in \mathrm{BUC}(\mathbf{R}^N \times [0, \infty))$ satisfying (2.3) with h replaced by $\theta \circ h$.

Note that θ^{-1} is a continuous function on $(-\infty,0)$. We intend to show that w(z,t) < 0 for all $(z,t) \in \mathbf{R}^N \times [0,\infty)$.

For this purpose, we set

$$\varphi(z,t) = |z|^N + Nt$$
 for $(z,t) \in \mathbf{R}^N \times [0,\infty)$,

and observe that φ is a viscosity solution of (2.1).

Fix any R > 0 and $\varepsilon > 0$ and observe as well that the function

$$\psi(z,t) := \varepsilon \min\{\varphi(z,t) - R, 0\}$$

is a viscosity solution of (2.1). If $\varepsilon > 0$ is sufficiently small, then we have

$$w(z,0) \le \psi(z,0)$$
 for all $z \in \mathbf{R}^N$,

and hence, thanks to Theorem 2.1, (b), we get

$$w(z,t) \le \psi(z,t)$$
 for all $(z,t) \in \mathbf{R}^N \times [0,\infty)$.

That is, for any R > 0, if $|z|^N + Nt < R$, then $\psi(z, t) < 0$ and w(z, t) < 0, which implies that w < 0 in $\mathbf{R}^N \times [0, \infty)$.

Set $u = \theta^{-1} \circ w$. Then $u \in C(\mathbf{R}^N \times [0, \infty))$ and u is a viscosity solution of (2.1).

We wish to show that $u \in \mathcal{U}$. Fix any $\lambda \in \mathbf{R}$. Let φ be the function defined above. Note that

$$\{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid u(z,t) \le \lambda\} = \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid w(z,t) \le \theta(\lambda)\}.$$

We choose R > 0 so that

$${z \in \mathbf{R}^N \mid h(z) \le \lambda} \subset B(0, R),$$

and define $\psi \in C(\mathbf{R}^N \times [0, \infty))$ by

$$\psi(z,t) = A \min\{\varphi(z,t) - (R+1)^N, 0\} + \theta(\lambda), \text{ with } A > 0.$$

If A is sufficiently large, then

$$w(z,0) \ge \psi(z,0)$$
 for all $z \in \mathbf{R}^N$,

and then, by comparison, we have

$$w \ge \psi$$
 in $\mathbf{R}^N \times [0, \infty)$.

This shows that

$$\{(z,t) \in \mathbf{R}^{N} \times [0,\infty) \mid w(z,t) < \theta(\lambda)\} \subset \{(z,t) \in \mathbf{R}^{N} \times [0,\infty) \mid \psi(z,t) < \theta(\lambda)\}$$

= \{(z,t) \in \mathbf{R}^{N} \times [0,\infty) \preceq \preceq \preceq \preceq (R+1)^N \}.

Hence, we see that $\{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid u(z,t) < \lambda\}$ is bounded, and conclude that u satisfies condition (A).

Remark. Each sub-level set of the function $\varphi(z,t)=|z|^N+Nt$ corresponds to the Gauss curvature flow of a ball.

Now, we introduce the partial differential equation in the level set approach to our mathematical model of the wearing process of a rolling stone.

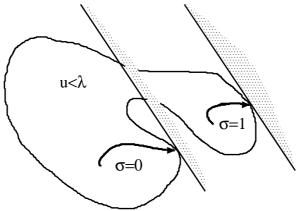
It is a simple modification of (2.1) and is given by

(L)
$$u_t(z,t) = \sigma(u, Du(z,t), z, t)G(Du(z,t), D^2u(z,t))$$
 for $(z,t) \in \mathbf{R}^N \times (0,\infty)$.

The only difference of this PDE from (2.1) consists in the new factor " $\sigma(u, Du(z, t), z, t)$ ". The precise interpretation of this factor σ will be explained soon, but, roughly speaking, $\sigma = 1$ if the sub-level set

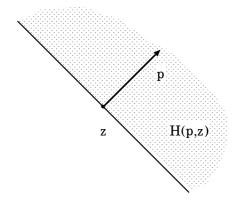
$$\{y \in \mathbf{R}^N \mid u(y,t) \le u(z,t)\}$$

is hit by the hyperplane $\pi(Du(z,t))$ coming from the direction of Du(z,t) and $\sigma=0$ otherwise.



We henceforth use the notation: for $p \in \mathbf{R}^N \setminus \{0\}$ and $z \in \mathbf{R}^N$,

$$H(p, z) = \{ y \in \mathbf{R}^N \mid (y - z) \cdot p \ge 0, \ y \ne z \}.$$



The precise definition of solution of (L) is given in the following.

Definition 2.3. Let $0 < T \le \infty$ and set $\Omega = \mathbb{R}^N \times (0,T)$. A function $u \in \mathrm{USC}(\Omega)$ is called a viscosity subsolution of (L) in Ω if whenever $\varphi \in \mathcal{A}(\Omega)$, $(z,t) \in \Omega$, and $u - \varphi$ attains a local maximum at (z,t), then

$$\varphi_t(z,t) \le \sigma^+(u, D\varphi(z,t), z, t)G(D\varphi(z,t), D^2\varphi(z,t)),$$

where for $p \in \mathbf{R}^N \setminus \{0\}$,

$$\sigma^{+}(u, p, z, t) = \begin{cases} 1 & \text{if } u(\cdot, t) \ge u(z, t) \text{ on } H(p, z), \\ 0 & \text{otherwise.} \end{cases}$$

A function $u \in LSC(\Omega)$ is called a viscosity supersolution of (L) in Ω if whenever $\varphi \in \mathcal{A}(\Omega), (z,t) \in \Omega$, and $u - \varphi$ attains a local minimum at (z,t), then

$$\varphi_t(z,t) \ge \sigma^-(u, D\varphi(z,t), z, t)G(D\varphi(z,t), D^2\varphi(z,t)),$$

where for $p \in \mathbf{R}^N \setminus \{0\}$,

$$\sigma^{-}(u, p, z, t) = \begin{cases} 1 & \text{if } u(\cdot, t) > u(z, t) \text{ on } H(p, z), \\ 0 & \text{otherwise.} \end{cases}$$

A function $u \in C(\Omega)$ is called a viscosity solution of (L) in Ω if it is both a viscosity subsolution and a viscosity supersolution of (L) in Ω . Here and henceforth we use the convention: for any $u : \Omega \subset \mathbf{R}^N \times (0, \infty) \to \mathbf{R}$, $\varphi \in \mathcal{A}(\Omega)$, and $(z,t) \in \Omega$,

$$\sigma^{\pm}(u, D\varphi(z, t), z, t)G(D\varphi(z, t), D^{2}\varphi(z, t)) = 0$$
 if $D\varphi(z, t) = 0$.

Note that $\sigma^+(u, p, z, t) \geq \sigma^-(u, p, z, t)$ for all $u : \Omega \subset \mathbf{R}^N \times (0, \infty) \to \mathbf{R}$ and all $(p, z, t) \in (\mathbf{R}^N \setminus \{0\}) \times \Omega$. Note as well that in the above definition we may replace $\mathcal{A}(\Omega)$ by $\mathcal{A}_0(\Omega)$.

In order to define the flow of a stone by the wearing process in the level set approach, we will establish theorems corresponding to Theorem 2.2 for (L).

We state our main results below in this section without proof. Their proofs will be presented in the following sections.

We start with stability properties of solutions of (L).

For any $S \subset \mathbf{R}^m$ and $f: S \to \mathbf{R}$ we define the upper (resp., lower) semicontinuous envelope $f^*: \overline{S} \to \mathbf{R} \cup \{\infty\}$ (resp., $f_*: \overline{S} \to \mathbf{R} \cup \{-\infty\}$) by

$$f^*(x) = \lim_{r \to 0} \sup \{ f(y) \mid y \in S, |y - x| < r \}$$
 and $f_* = -(-f)^*$.

Proposition 2.4. Let $T \in (0, \infty]$ and set $\Omega = \mathbb{R}^N \times (0, T)$.

(a) Let S be a non-empty collection of viscosity subsolutions of (L) in Ω . Set

$$u(z,t) = \sup\{v(z,t) \mid v \in \mathcal{S}\} \quad \text{for } (z,t) \in \Omega.$$

Assume that $u^*(z,t) < \infty$ for all $(z,t) \in \Omega$. Then u^* is a viscosity subsolution of (L) in Ω .

(b) Let S be a non-empty collection of viscosity supersolutions of (L) in Ω . Set

$$u(z,t) = \inf\{v(z,t) \mid v \in \mathcal{S}\} \quad \text{for } (z,t) \in \Omega.$$

Assume that $u_*(z,t) > -\infty$ for all $(z,t) \in \Omega$ and that u_* satisfies condition (A). Then u_* is a viscosity subsolution of (L) in Ω .

(c) Let $f_1, f_2 \in C(\Omega)$ be a viscosity subsolution and a viscosity supersolution of (L) in Ω , respectively. Assume that $f_1 \leq f_2$ in Ω . Set

$$u(z,t)=\sup\{v(z,t)\mid v\ \ is\ \ a\ \ viscosity\ \ subsolution\ \ of\ (\mathrm{L})\ \ in\ \Omega,\ \ f_1\leq v\leq f_2\ \ in\ \Omega\}.$$

Assume that u_* satisfies (A). Then u_* is a viscosity supersolution of (L) in Ω .

The next proposition is similar to Theorem 2.1, (a).

Proposition 2.5. Let Ω be as in the previous theorem. Let $\theta \in USC(\mathbf{R})$ (resp., $\theta \in LSC(\mathbf{R})$) be a non-decreasing function and u a viscosity subsolution (resp., supersolution) of (L) in Ω . Then $\theta \circ u$ is a viscosity subsolution (resp., supersolution) of (L) in Ω .

One of our main results is the following comparison theorem.

Theorem 2.6. Let $T \in (0, \infty]$. Let $u \in \mathrm{USC}(\mathbf{R}^N \times [0, T))$ and $v \in \mathrm{LSC}(\mathbf{R}^N \times [0, T))$ be a viscosity subsolution and a viscosity supersolution of (L) in $\mathbf{R}^N \times (0, T)$, respectively. Assume that v satisfies condition (A) with $\Omega = \mathbf{R}^N \times [0, T)$ and that $u(z, 0) \leq v(z, 0)$ for all $z \in \mathbf{R}^N$. Then $u \leq v$ in $\mathbf{R}^N \times (0, T)$.

Our existence result for the Cauchy problem for (L) is stated as follows.

Theorem 2.7. Let $h \in \mathcal{U}_0$. Then there exists a (unique) viscosity solution $u \in \mathcal{U}$ of (L) satisfying the initial condition

(2.6)
$$u(z,0) = h(z) \quad \text{for all } z \in \mathbf{R}^N.$$

We are now in a position to define the flow of a set by the wearing process. Let $S \in \mathbf{R}^N$ be a compact set. We choose an $h \in \mathcal{U}_0$ so that

(2.7)
$$S = \{ z \in \mathbf{R}^N \mid h(z) \le 0 \}.$$

By virtue of Theorem 2.7, there exists a viscosity solution $u \in \mathcal{U}$ of (L) satisfying (2.6). We define

$$(2.8) V = \{(z,t) \mid u(z,t) \le 0\},$$

(2.9)
$$V_t = \{ z \in \mathbf{R}^N \mid (z, t) \in V \} \quad \text{for } t \ge 0.$$

The collection of mappings : $S \mapsto V_t$, with $t \geq 0$, of compact subsets of \mathbf{R}^N to compact subsets of \mathbf{R}^N is well-defined as the following theorem ensures.

Theorem 2.8. For fixed compact $S \subset \mathbf{R}^N$, as far as $h \in \mathcal{U}_0$ satisfies (2.7), the set V defined by (2.8) is determined independently of the choice of h.

We give here a proof of this theorem, which is based on previous results.

Proof. Let $S \subset \mathbf{R}^N$ be a compact set. Let $h_1, h_2 \in \mathcal{U}_0$ satisfy (2.7) and $u_1, u_2 \in \mathcal{U}$ be the viscosity solutions of (L) satisfying (2.6) with $h = h_1$ and with $h = h_2$, respectively. Define $V^1, V^2 \subset \mathbf{R}^N \times [0, \infty)$ by (2.8) with $u = u_1$ and with $u = u_2$, respectively.

Define
$$v_1, v_2 \in C(\mathbf{R}^N \times [0, \infty))$$
 by

$$v_1(z,t) = \min\{u_1(z,t), 1\}$$
 and $v_2(z,t) = \max\{u_2(z,t), 0\}.$

By virtue of Proposition 2.5, v_1 and v_2 are viscosity solutions of (L). Note as well that

$$V^{1} = \{(z,t) \in \mathbf{R}^{N} \times [0,\infty) \mid v_{1}(z,t) \leq 0\},\$$

$$V^{2} = \{(z,t) \in \mathbf{R}^{N} \times [0,\infty) \mid v_{2}(z,t) \leq 0\}.$$

Since $v_1(\cdot,0)$ is uniformly continuous in \mathbf{R}^N , we may choose an increasing function $\theta \in C(\mathbf{R})$ such that $\theta(0) = 0$ and

$$v_1(z,0) \le \theta \circ v_2(z,0)$$
 for all $z \in \mathbf{R}^N$.

Now, thanks to Theorem 2.6, we have

$$v_1 \le \theta \circ v_2 \quad \text{in } \mathbf{R}^N \times [0, \infty).$$

Observing that $V^2 = \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid \theta \circ v_2(z,t) \leq 0\}$, we see that $V^2 \subset V^1$. By symmetry, we have as well $V^1 \subset V^2$. Thus, we conclude that $V^1 = V^2$.

Finally, we may define the flow of a set starting from S by the wearing process as the collection $\{V_t\}_{t>0}$, where V_t are defined by (2.9).

3. Comparison principle for PDE (L)

In this section we prove Theorem 2.6. The proof is divided into four steps, each of which is described in subsections from 3.1 to 3.4.

3.1. Graphs representing part of the boundary of an evolving set

In this subsection we take a step to reduce the proof of the comparison principle for (L) to that for PDE for functions describing part of the boundary, as their graphs, of an evolving set by the wearing process.

Let $\Omega \subset \mathbf{R}^n \times (0, \infty)$. Consider the case when part of the boundary of an evolving set by the wearing process is represented as the graph of the function v in Ω . Following [IM2], the PDE for v is

(G)
$$v_t(x,t) = \chi(v, Dv(x,t), x, t)g(Dv(x,t), D^2v(x,t)) \quad \text{for } (x,t) \in \Omega,$$

where the functions g and χ are defined respectively by

$$\begin{split} g(p,X) &= \frac{\det_+ X}{(1+|p|)^{(n+1)/2}}, \\ \chi(v,p,x,t) &= \begin{cases} 1 & \text{if } v(y,t) \geq v(x,t) + p \cdot (y-x) & \text{for all } y \in \Omega_t, \\ 0 & \text{otherwise,} \end{cases} \end{split}$$

and Ω_t denotes the set $\{x \in \mathbf{R}^n \mid (x,t) \in \Omega\}$ for t > 0.

We call $v \in \mathrm{USC}(\Omega)$ a viscosity subsolution of (G) if whenever $\varphi \in C^2(\Omega)$ and $v - \varphi$ attains a local maximum at $(\hat{x}, \hat{t}) \in \Omega$ then

$$\varphi_t(\hat{x},\hat{t}) \leq \chi^+(v, D\varphi(\hat{x},\hat{t}), \hat{x},\hat{t})g(D\varphi(\hat{x},\hat{t}), D^2\varphi(\hat{x},\hat{t})),$$

where $\chi^{+}(v, p, x, t) = \chi(v, p, x, t)$.

We call a function $v \in LSC(\Omega, \mathbf{R} \cup \{\infty\})$ a viscosity supersolution of (G) if whenever $\varphi \in C^2(\Omega)$ and $v - \varphi$ attains a finite local minimum at $(\hat{x}, \hat{t}) \in \Omega$ then

$$\varphi_t(\hat{x},\hat{t}) \ge \chi^-(v, D\varphi(\hat{x},\hat{t}), \hat{x},\hat{t})g(D\varphi(\hat{x},\hat{t}), D^2\varphi(\hat{x},\hat{t})),$$

where $\chi^-(v, p, x, t) = 1$ if

$$v(y,t) > v(x,t) + p \cdot (y-x) \quad \text{for } y \in \Omega_t \setminus \{x\},$$

and there is an $\varepsilon > 0$ such that for any $(y, s) \in \Omega$ satisfying $|y| > \varepsilon^{-1}$ and $|s - t| < \varepsilon$,

$$v(y,s) > p \cdot y + \varepsilon |y|,$$

and otherwise, $\chi^{-}(v, p, x, t) = 0$.

Because of the factor χ in (G), the PDE (G) has a nonlocal character. However, it is clear that if $v \in USC(\Omega)$ is a viscosity subsolution of (G) in Ω and Ω_1 is an open subset of Ω , then v is a viscosity subsolution of (G) in Ω_1 . On the other hand, this local property does not hold for supersolutions.

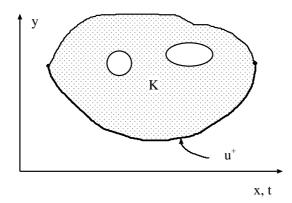
It is convenient for us to allow viscosity supersolutions of (G) to take the value $+\infty$ as we do here.

The above PDE was the starting point of the mathematical analysis in [IM2], but here we show that, speaking loosely, certain part of the boundary of any of sub-level sets of viscosity solutions of (L) is represented as a viscosity solution of (G). This is precisely stated in the following two theorems.

Let $u \in LSC(\mathbf{R}^N \times [0,\infty))$ be a viscosity supersolution of (L). Fix $\lambda \in \mathbf{R}$, and consider the set

$$K = \{(z, t) \in \mathbf{R}^N \times [0, \infty) \mid u(z, t) \le \lambda\}.$$

Since u is lower semicontinuous, K is a closed subset of $\mathbf{R}^N \times [0, \infty)$.



Assume that

(3.1) K is a bounded subset of $\mathbf{R}^N \times [0, \infty)$.

We define $u^+: \mathbf{R}^n \times [0, \infty) \to \mathbf{R} \cup \{\infty\}$ by

$$u^+(x,t) = \inf\{y \in \mathbf{R} \mid u(x,y,t) \le \lambda\} \equiv \inf\{y \in \mathbf{R} \mid (x,y,t) \in K\}.$$

Theorem 3.1. The function u^+ belongs to $LSC(\mathbf{R}^n \times [0,\infty), \mathbf{R} \cup \{\infty\})$ and it is a viscosity supersolution of (G) in $\mathbf{R}^n \times (0,\infty)$.

Proof. In view of Proposition 2.5 (whose proof will be given below), we may assume by replacing u by $\theta \circ u$, where $\theta(s) = 0$ for $s \leq \lambda$ and $\theta(s) = 1$ for $s > \lambda$, that $\lambda = 0$ and u has only two values 0 and 1. Then, by the definition of u^+ , we have

$$u(x, y, t) = 1$$
 for all $(x, t) \in \mathbf{R}^n \times [0, \infty)$ and $y < u^+(x, t)$.

Since u is lower semicontinuous, by the definition of u^+ we have for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$,

$$u(x, u^{+}(x, t), t) = 0$$
 if $u^{+}(x, t) \in \mathbf{R}$.

We next check the lower semicontinuity of u^+ . Note first that for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$,

$$(x, u^+(x,t), t) \in \partial K$$
 if $u^+(x,t) \neq \infty$.

In particular, since K is a bounded subset of \mathbf{R}^{n+2} , we see that the function u^+ is bounded from below on $\mathbf{R}^n \times [0, \infty)$.

Let
$$\mathbf{R}^n \times [0, \infty) \ni (x_k, t_k) \to (x, t) \in \mathbf{R}^n \times [0, \infty)$$
 as $k \to \infty$ and

$$\gamma := \liminf_{k \to \infty} u^+(x_k, t_k).$$

We need to show that $u^+(x,t) \leq \gamma$. There are two possibilities: $\gamma \in \mathbf{R}$ or $\gamma = \infty$. If $\gamma = \infty$, then there is nothing to prove.

Suppose that $\gamma \in \mathbf{R}$. By replacing $\{(x_k, t_k)\}$ by a subsequence if necessary, we may assume that $u^+(x_k, t_k) \in \mathbf{R}$ for all $k \in \mathbf{N}$ and that the sequence $\{u^+(x_k, t_k)\}$ converges to γ . Since $u(x_k, u^+(x_k, t_k), t_k) = 0$ for all $k \in \mathbf{N}$ and u is lower semicontinuous, we see that $u(x, \gamma, t) \leq 0$. From this we see that

$$u^+(x,t) \le \gamma.$$

Let $(\hat{x}, \hat{t}) \in \mathbf{R}^n \times (0, \infty)$ and $\varphi \in C^2(\mathbf{R}^n \times (0, \infty))$. Assume that $u^+(\hat{x}, \hat{t}) \in \mathbf{R}$ and that $u^+ - \varphi$ has a minimum at $(\hat{x}, \hat{t}) \in \mathbf{R}^n \times (0, \infty)$. We may assume that

$$u^+(\hat{x},\hat{t}) = \varphi(\hat{x},\hat{t}),$$

$$u^+(x,t) \ge \varphi(x,t)$$
 for $(x,t) \in \mathbf{R}^n \times (0,\infty)$.

We write $\hat{y} = u^+(\hat{x}, \hat{t})$ and $\hat{p} = D\varphi(\hat{x}, \hat{t})$.

Now we claim that the function $\Phi(x, y, t) := u(x, y, t) - \varphi(x, t) + y$ attains a local minimum at $(\hat{x}, \hat{y}, \hat{t})$.

To show this, we first note that

$$\Phi(\hat{x}, \hat{y}, \hat{t}) = 0.$$

Fix any $(x, y, t) \in \mathbf{R}^N \times (0, \infty)$. Observe that if $y \geq u^+(x, t)$, then

$$\Phi(x, y, t) \ge -\varphi(x, t) + y \ge -u^+(x, t) + y \ge 0.$$

If $y < u^+(x,t)$, then

$$\Phi(x, y, t) \ge 1 - \varphi(x, t) + y.$$

Since $1 - \varphi(\hat{x}, \hat{t}) + \hat{y} = 1$, there exists a constant $\delta > 0$ such that for all $(x, y, t) \in B(\hat{x}, \delta) \times [\hat{y} - \delta, \hat{y} + \delta] \times [\hat{t} - \delta, \hat{t} + \delta]$,

$$1 - \varphi(x, t) + y > 0.$$

Then we have

$$\Phi(x, y, t) \ge 0$$
 for $(x, y, t) \in B(\hat{x}, \delta) \times [\hat{y} - \delta, \hat{y} + \delta] \times [\hat{t} - \delta, \hat{t} + \delta].$

Thus we conclude that Φ has a local minimum at $(\hat{x}, \hat{y}, \hat{t})$.

Define $\psi \in C^2(\mathbf{R}^N \times (0,\infty))$ by $\psi(x,y,t) = \varphi(x,t) - y$. Since u is a viscosity supersolution of (L), we get

(3.2)
$$\psi_t(\hat{x}, \hat{y}, \hat{t}) \ge \sigma^-(u, D\psi(\hat{x}, \hat{y}, \hat{t}), \hat{x}, \hat{y}, \hat{t})G(D\psi(\hat{x}, \hat{y}, \hat{t}), D^2\psi(\hat{x}, \hat{y}, \hat{t})).$$

We need to show

$$\varphi_t(\hat{x},\hat{t}) \ge \chi^-(u^+,\hat{p},\hat{x},\hat{t})g(\hat{p},D^2\varphi(\hat{x},\hat{t})).$$

Consider the case when $\chi^{-}(u^{+}, \hat{p}, \hat{x}, \hat{t}) = 0$. From (3.2) we have

$$\varphi_t(\hat{x},\hat{t}) = \psi_t(\hat{x},\hat{y},\hat{t}) \ge 0 = \chi^-(u^+,\hat{p},\hat{x},\hat{t})g(\hat{p},D^2\varphi(\hat{x},\hat{t})).$$

Consider next the case when $\chi^-(u^+, \hat{p}, \hat{x}, \hat{t}) = 1$. Observe that if

$$\sigma^{-}(u, D\psi(\hat{x}, \hat{y}, \hat{t}), \hat{x}, \hat{y}, \hat{t}) = 1,$$

then (3.2) reads

$$\psi_t(\hat{x}, \hat{y}, \hat{t}) \ge G(D\psi(\hat{x}, \hat{y}, \hat{t}), D^2\psi(\hat{x}, \hat{y}, \hat{t})),$$

from which we get

$$\varphi_t(\hat{x},\hat{t}) \ge g(\hat{p}, D^2 \varphi(\hat{x},\hat{t})) = \chi^-(u^+, \hat{p}, \hat{x}, \hat{t}) g(\hat{p}, D^2 \varphi(\hat{x}, \hat{t})).$$

(See, e.g., Lemma A in the appendix.) Therefore we need only to show that

$$\sigma^{-}(u, D\psi(\hat{x}, \hat{y}, \hat{t}), \hat{x}, \hat{y}, \hat{t}) = 1.$$

From the assumption that $\chi^-(u^+, \hat{p}, \hat{x}, \hat{t}) = 1$, we see that

$$u^+(x,\hat{t}) > \hat{y} + \hat{p} \cdot (x - \hat{x})$$
 for all $x \in \mathbf{R}^n \setminus \{\hat{x}\},$

which guarantees that for all $(x, y) \in \mathbf{R}^N$, if

$$y \le \hat{y} + \hat{p} \cdot (x - \hat{x})$$
 and $(x, y) \ne (\hat{x}, \hat{y}),$

then $y < u^+(x, \hat{t})$ and hence

$$u(x, y, \hat{t}) = 1.$$

That is,

$$u(x, y, \hat{t}) > 0$$
 for all $(x, y) \in H(D\psi(\hat{x}, \hat{y}, \hat{t}), \hat{x}, \hat{y}),$

which shows that

$$\sigma^{-}(u, D\psi(\hat{x}, \hat{y}, \hat{t}), \hat{x}, \hat{y}, \hat{t}) = 1.$$

Now let $u \in \mathrm{USC}(\mathbf{R}^N \times [0,\infty))$ be a viscosity subsolution of (L). Fix $\lambda \in \mathbf{R}$, and consider the sub-level set

$$W = \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid u(z,t) < \lambda\}.$$

Note that W is an open subset of $\mathbf{R}^N \times [0, \infty)$ in the relative topology.

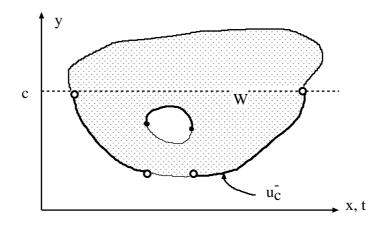
We assume that

(3.3) W is a bounded set.

Fix any $c \in \mathbf{R}$. We consider the intersection of W and the hyperplane y = c of \mathbf{R}^{N+1} , and its projection Ω onto the hyperplane y = 0, which we may identify as a subset of \mathbf{R}^{n+1} . In other words, we define Ω as the set

$$\Omega_c = \{ (x, t) \in \mathbf{R}^{n+1} \mid (x, c, t) \in W \}.$$

Observe that Ω_c is an open subset of $\mathbf{R}^n \times [0, \infty)$ in the relative topology.



We define $u_c^-:\Omega_c\to\mathbf{R}$ by

$$u_c^-(x,t) = \inf\{a \in \mathbf{R} \mid \max_{a \leq y \leq c} u < \lambda\} \equiv \inf\{a \in \mathbf{R} \mid \{x\} \times [a,c] \times \{t\} \subset W\}.$$

Note that $u_c^-(x,t) < c$ for all $(x,t) \in \Omega_c$.

Theorem 3.2. $u_c^- \in USC(\Omega_c)$ and u_c^- is a viscosity subsolution of (G) in Ω_c .

Proof. We may assume by replacing u by $\theta \circ u$, where $\theta(s) = 0$ for $s \geq \lambda$ and $\theta(s) = -1$ for $s < \lambda$, that $\lambda = 0$ and u has only two values 0 and -1. Then, by the definition of u_c^- , we have

(3.4)
$$u(x,y,t) = -1 \quad \text{for all } (x,t) \in \Omega_c \text{ and } y \in (u_c^-(x,t), c].$$

Since u is upper semicontinuous, by the definition of u_c^- we have

$$u(x, u_c^-(x,t), t) = 0$$
 for all $(x,t) \in \Omega_c$.

We next show that $u_c^- \in \mathrm{USC}(\Omega_c)$. Note first that

$$(x, u_c^-(x, t), t) \in \partial W$$
 for all $(x, t) \in \Omega_c$.

Hence, by (3.3), we see that the function u_c^- is bounded on Ω_c .

Fix $(x,t) \in \Omega_c$ and $a \in (u_c^-(x,t),c)$. By (3.4), we have

$$\{x\} \times [a,c] \times \{t\} \subset W$$
,

and, since W is an open set, there is a positive number ε such that $B(x,\varepsilon) \times [a,c] \times [\max\{t-\varepsilon,0\},t+\varepsilon] \subset W$, which shows that

$$u_c^-(y,s) \le a$$
 for all $(y,s) \in B(x,\varepsilon) \times [\max\{t-\varepsilon,0\},t+\varepsilon]$.

This guarantees that $u_c^- \in USC(\Omega_c)$.

Let $(\hat{x}, \hat{t}) \in \Omega_c$ and $\varphi \in C^2(\Omega_c)$. Assume $u_c^- - \varphi$ has a maximum at $(\hat{x}, \hat{t}) \in \Omega_c$. We may assume that

$$u_c^-(\hat{x},\hat{t}) = \varphi(\hat{x},\hat{t}),$$

$$u_c^-(x,t) \leq \varphi(x,t)$$
 for $(x,t) \in \Omega_c$.

We write $\hat{y} = u_c^-(\hat{x}, \hat{t})$ and $\hat{p} = D\varphi(\hat{x}, \hat{t})$.

Now we claim that the function $u(x,y,t) - \varphi(x,t) + y$ attains a local maximum at $(\hat{x},\hat{y},\hat{t})$.

To see this, let $(x,t) \in \Omega_c$ and $y \in (-\infty,c]$, and observe that if $\varphi(x,t) \geq y$, then

$$u(x, y, t) \le 0 \le \varphi(x, t) - y,$$

and if $-1 \le \varphi(x,t) - y < 0$, then $u_c^-(x,t) < y$ and hence

$$u(x, y, t) = -1 \le \varphi(x, t) - y.$$

Since

$$u(\hat{x}, \hat{y}, \hat{t}) - \varphi(\hat{x}, \hat{t}) + \hat{y} = 0,$$

and the set

$$\{(x, y, t) \mid (x, t) \in \Omega_c, y \in (-\infty, c], \varphi(x, t) - y > -1\}$$

contains a neighborhood of $(\hat{x}, \hat{y}, \hat{t})$, we conclude that $u(x, y, t) - \varphi(x, t) + y$ attains a local maximum at $(\hat{x}, \hat{y}, \hat{t})$.

Define $\psi(x,y,t) = \varphi(x,t) - y$ for $(x,y,t) \in W$. Since u is a viscosity subsolution of (L), we get

(3.5)
$$\psi_t(\hat{x}, \hat{y}, \hat{t}) \le \sigma^+(u, D\psi(\hat{x}, \hat{y}, \hat{t}), \hat{x}, \hat{y}, \hat{t})G(D\psi(\hat{x}, \hat{y}, \hat{t}), D^2\psi(\hat{x}, \hat{y}, \hat{t})).$$

Consider the case when $\sigma^+(u, D\psi(\hat{x}, \hat{y}, \hat{t}), \hat{x}, \hat{y}, \hat{t}) = 0$. In this case, from (3.5) we get

$$\varphi_t(\hat{x},\hat{t}) = \psi_t(\hat{x},\hat{y},\hat{t}) \le 0 \le \chi^+(u_c^-,\hat{p},\hat{x},\hat{t})g(\hat{p},D^2\varphi(\hat{x},\hat{t})).$$

Consider next the case when $\sigma^+(u, D\psi(\hat{x}, \hat{y}, \hat{t}), \hat{x}, \hat{y}, \hat{t}) = 1$. In this case, we see easily that

$$\chi^{+}(u_{c}^{-}, \hat{p}, \hat{x}, \hat{t}) = 1.$$

Hence, from (3.5) we get

$$\psi_t(\hat{x}, \hat{y}, \hat{t}) \le G(D\psi(\hat{x}, \hat{y}, \hat{t}), D^2\psi(\hat{x}, \hat{y}, \hat{t})).$$

It is a standard fact (see, e.g., Lemma A in the appendix) that the above inequality yields

$$\varphi_t(\hat{x},\hat{t}) \leq g(D\varphi(\hat{x},\hat{t}), D^2\varphi(\hat{x},\hat{t})).$$

Thus in both cases we have

$$\varphi_t(\hat{x},\hat{t}) \le \chi^+(u_c^-, D\varphi(\hat{x},\hat{t}), \hat{x}, \hat{t})g(D\varphi(\hat{x},\hat{t}), D^2\varphi(\hat{x},\hat{t})).$$

3.2. Comparison lemma for PDE (G)

We establish a comparison lemma for (G), which is a key observation in our proof of Theorem 2.6.

Let Ω be a bounded, relatively open subset of $\mathbf{R}^N \times [0, \infty)$. As before, we use the notation: $\Omega_t = \{x \in \mathbf{R}^n \mid (x,t) \in \Omega\}$ for $t \geq 0$. Let E be a compact subset of $\mathbf{R}^n \times [0, \infty)$.

Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\mathbf{R}^n \times [0, \infty), \mathbf{R} \cup \{\infty\})$ be a viscosity subsolution of

$$(G)_{\delta} \qquad u_t(x,t) = \chi(u,Du(x,t),x,t)g(Du(x,t),D^2u(x,t)) - \delta \quad \text{in } \Omega,$$

where $\delta > 0$ is a constant, and a viscosity supersolution of (G) in $\mathbf{R}^n \times (0, \infty)$, respectively. Here the closure $\overline{\Omega}$ of Ω is taken as a subset of \mathbf{R}^{n+1} and viscosity solutions for (G)_{δ} are defined in the same way as those for (G).

Lemma 3.3. In addition to the above hypotheses, let $c \in \mathbf{R}$ and assume that $u \leq v$ on $\partial \Omega$, that if $(x,t) \in \Omega$ is a maximum point over $\overline{\Omega}$ of u-v, then

- (1) v(x,t) = c,
- (2) $0 \in D_x^+ u(x,t), i.e.,$

$$u(x+\xi,t) \le u(x,t) + o(|\xi|)$$
 as $\xi \to 0$,

and that

$$v(x,t) \begin{cases} > c & for (x,t) \in E \setminus \Omega, \\ = \infty & for (x,t) \in (\mathbf{R}^n \times [0,\infty)) \setminus E. \end{cases}$$

Then $u \leq v$ in Ω .

We adapt the arguments of [IM, Theorem 1] to the following proof.

Proof. We may assume by replacing u and v respectively by u-c and v-c if necessary, that c=0. We argue by contradiction, and thus suppose that

$$\theta := \sup_{\Omega} (u - v) > 0.$$

For $k \in \mathbb{N}$, we define functions (sup- and infconvolutions) u_k , v_k on \mathbb{R}^{n+1} by

$$u_k(x,t) = \max\{u(y,s) - \frac{k}{2}(|y-x|^2 + |s-t|^2) \mid (y,s) \in \overline{\Omega}\},$$

 $v_k(x,t) = \min\{v(y,s) + \frac{k}{2}(|y-x|^2 + |s-t|^2) \mid (y,s) \in \overline{\Omega}\}.$

Since $u \leq u_k$ and $v \geq v_k$ on $\overline{\Omega}$, we have

$$\max_{\overline{\Omega}}(u_k - v_k) \ge \theta.$$

Since $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega}, \mathbf{R} \cup \{\infty\})$, u and -v are bounded above on $\overline{\Omega}$. We may thus choose a constant M > 0 so that

$$\max_{\overline{\Omega}} u \leq M \quad \text{ and } \quad \max_{\overline{\Omega}} (-v) \leq M.$$

For $(x,t) \in \overline{\Omega}$, if $u_k(x,t) \geq v_k(x,t)$, then we have

$$-M \le v_k(x,t) \le u_k(x,t) \le M.$$

If, in addition, $(y, s) \in \overline{\Omega}$ satisfies

$$u_k(x,t) = u(y,s) - \frac{k}{2}(|x-y|^2 + |t-s|^2),$$

then we have

$$(3.6) |x - y|^2 + |t - s|^2 \le 4k^{-1}M.$$

Similarly, if $(x,t) \in \overline{\Omega}$ and and $(y,s) \in \overline{\Omega}$ satisfy

$$u_k(x,t) \ge v_k(x,t)$$
 and $v_k(x,t) = v(y,s) + \frac{k}{2}(|x-y|^2 + |t-s|^2),$

then

$$(3.7) |x - y|^2 + |t - s|^2 \le 4k^{-1}M.$$

Define

$$P = \{(x,t) \in \overline{\Omega} \mid (u-v)(x,t) = \theta\},\$$

which is a closed subset of $\overline{\Omega}$.

For $\varepsilon > 0$ we write

$$U_{\varepsilon} = \{(x,t) \in \Omega \mid t > \varepsilon, \operatorname{dist}((x,t), \overline{\Omega} \setminus \Omega) > \varepsilon\}.$$

Since $u \leq v$ on $\partial\Omega$, $u - v \in \mathrm{USC}(\overline{\Omega}, \mathbf{R} \cup \{-\infty\})$, and

$$\overline{\Omega} \setminus U_{\varepsilon} \subset \{(x,t) \in \overline{\Omega} \mid \operatorname{dist}((x,t),\partial\Omega) \leq \varepsilon\},\$$

we may choose a constant $\varepsilon > 0$ so that

$$(3.8) P \subset U_{2\varepsilon}.$$

Since v > 0 on $E \setminus \Omega$ and $v \in LSC(E, \mathbf{R} \cup \{\infty\})$, we may choose a constant $\nu > 0$ so that

$$v > \nu$$
 on $[E \cap (\mathbf{R}^n \times (\varepsilon, \infty))] \setminus U_{\varepsilon}$.

We fix a constant $\gamma > 0$ such that for any $a \in (-\infty, \gamma], p \in B(0, \gamma),$ and $(y, s) \in U_{\varepsilon}$,

$$a + p \cdot (x - y) < \nu$$
 for all $x \in \mathbf{R}^n$ such that $(x, s) \in E$.

Accordingly, if $a \in (-\infty, \gamma]$, $p \in B(0, \gamma)$, and $(y, s) \in U_{\varepsilon}$, then we have

(3.9)
$$a + p \cdot (x - y) < v(x, s)$$
 for all $x \in \mathbf{R}^n$ such that $(x, s) \notin U_{\varepsilon}$.

We recall here a standard fact that if $(x,t) \in \mathbf{R}^{n+1}$ and $(p,q,X) \in J^{2,+}u_k(x,t)$, then for $y = x + \frac{1}{k}p$ and $s = t + \frac{1}{k}q$,

$$u_k(x,t) = u(y,s) - \frac{k}{2} (|x-y|^2 + |t-s|^2)$$

and

$$(p,q,X) \in J^{2,+}_{\overline{\Omega}}u(y,s).$$

We refer to [CIL] for the definition of $J^{2,\pm}$ and $J^{2,\pm}_{\overline{\Omega}}$, and basic properties of inf- and supconvolutions u_k and v_k . Similarly, if $(x,t) \in \mathbf{R}^{n+1}$ and $(p,q,X) \in J^{2,-}v_k(x,t)$, then for $y = x - \frac{1}{k}p$ and $s = t - \frac{1}{k}q$,

$$v_k(x,t) = v(y,s) + \frac{k}{2} (|x-y|^2 + |t-s|^2)$$

and

$$(p,q,X)\in J^{2,-}_{\overline{\Omega}}v(y,s).$$

We remark that, as is well-known, u_k and $-v_k$ are locally Lipschitz continuous and semiconvex in \mathbf{R}^{n+1} . In fact, regarding the semiconvexity, the functions

$$u_k(x,t) + \frac{k}{2}(|x|^2 + t^2)$$
 and $-v_k(x,t) + \frac{k}{2}(|x|^2 + t^2)$

are convex on \mathbf{R}^{n+1} .

Next, for each $k \in \mathbf{N}$ let $(\hat{x}_k, \hat{t}_k) \in \overline{\Omega}$ satisfy

$$(u_k - v_k)(\hat{x}_k, \hat{t}_k) \ge \theta - \frac{1}{k}.$$

Choose points $(\hat{y}_k, \hat{s}_k) \in \overline{\Omega}$ and $(\hat{z}_k, \hat{r}_k) \in \overline{\Omega}$, with $k \in \mathbb{N}$, so that

(3.10)
$$u_k(\hat{x}_k, \hat{t}_k) = u(\hat{y}_k, \hat{s}_k) - \frac{k}{2} (|\hat{x}_k - \hat{y}_k|^2 + |\hat{t}_k - \hat{s}_k|^2),$$

(3.11)
$$v_k(\hat{x}_k, \hat{t}_k) = v(\hat{z}_k, \hat{r}_k) + \frac{k}{2} (|\hat{x}_k - \hat{z}_k|^2 + |\hat{t}_k - \hat{r}_k|^2).$$

Then we have

$$u(\hat{y}_k, \hat{s}_k) - v(\hat{z}_k, \hat{r}_k) \ge (u_k - v_k)(\hat{x}_k, \hat{t}_k) \ge \theta - \frac{1}{k}$$
 for all $k \in \mathbf{N}$.

We may assume that $(\hat{x}_k, \hat{t}_k) \to (\hat{x}, \hat{t})$ along a subsequence for some $(\hat{x}, \hat{t}) \in \overline{\Omega}$ as $k \to \infty$. In view of (3.6) and (3.7), we see that, along the same subsequence, $(\hat{y}_k, \hat{s}_k) \to (\hat{x}, \hat{t})$ and $(\hat{z}_k, \hat{r}_k) \to (\hat{x}, \hat{t})$ as $k \to \infty$. Since $u, -v \in \mathrm{USC}(\overline{\Omega}, \mathbf{R} \cup \{-\infty\})$, we see from the above inequality that

$$(3.12) (u-v)(\hat{x},\hat{t}) \ge \theta,$$

i.e., $(\hat{x}, \hat{t}) \in P$. This together with assumption (1) implies that

(3.13)
$$\lim_{k \to \infty} v(\hat{z}_k, \hat{r}_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} u(\hat{y}_k, \hat{s}_k) = \theta.$$

Now we want to "strictly" maximize functions which are small perturbations of the function $u_k - v_k$ over $\overline{\Omega}$.

We introduce a sequence $\{\delta_k\}_{k\in\mathbb{N}}\subset (0,\delta/2)$ such that for all $p,q\in B(0,1)$ and $X\in\mathcal{S}^n$, if $-kI\leq X\leq kI$ and $|p-q|<\delta_k$, then

$$|g(p,X) - g(q,X)| < \frac{\delta}{2}.$$

By perturbed optimization techniques (see, e.g., [EL, Corollary 3.8]), for each $k \in \mathbf{N}$ there is a point $(a_k, b_k) \in \mathbf{R}^n \times \mathbf{R}$ satisfying $|a_k| + |b_k| < \delta_k$ such that the function

$$\Phi_k(x,t) := u_k(x,t) - v_k(x,t) - a_k \cdot x - b_k t$$

attains a strict maximum over $\overline{\Omega}$ at a point $(x_k, t_k) \in \overline{\Omega}$.

By choosing $|a_k| + |b_k|$ small enough, we may assume that $(u_k - v_k)(x_k, t_k) > \theta - \frac{1}{k}$ for all $k \in \mathbb{N}$. Recalling (3.12) and (3.8), we may further assume that $(x_k, t_k) \in U_{2\varepsilon}$ for all $k \in \mathbb{N}$.

Now, for each $k \in \mathbf{N}$ there is a function $\psi_k \in C^2(\mathbf{R}^n)$ such that

$$\psi_k(x) > 0$$
 for all $x \neq x_k$, $\psi_k(x_k) = 0$,

 $\psi_k(x)$ is strictly convex on \mathbb{R}^n ,

 $\Phi_k(x,t) + \psi_k(x)$ attains a strict maximum over $\overline{\Omega}$ at (x_k,t_k) ,

$$\|\psi_k\|_{\infty} + \|D\psi_k\|_{\infty} + \|D^2\psi_k\|_{\infty} < \delta_k,$$

where the sup-norms in the above inequality are taken over $\overline{\Omega}$.

By Jensen's maximum principle (see, e.g., [CIL, Lemmas A.2 and A.3]), there are $(\bar{a}_k, \bar{b}_k) \in \mathbf{R}^n \times \mathbf{R}$ and $(\bar{x}_k, \bar{t}_k) \in \overline{\Omega}$ such that

$$u_k(x,t) - v_k(x,t) + \psi_k(x) - (a_k + \bar{a}_k) \cdot x - (b_k + \bar{b}_k)t$$

attains a maximum over $\overline{\Omega}$ at (\bar{x}_k, \bar{t}_k) , u_k and v_k are twice differentiable at (\bar{x}_k, \bar{t}_k) , and $|\bar{a}_k| + |\bar{b}_k| < \delta_k$.

Here we may choose $(\bar{a}_k, \bar{b}_k, \bar{x}_k, \bar{t}_k)$ as close to $(0, 0, x_k, t_k)$ as we wish. Thus we may assume that $(\bar{x}_k, \bar{t}_k) \in U_{2\varepsilon}$ and $(u_k - v_k)(\bar{x}_k, \bar{t}_k) > \theta - \frac{1}{k}$ for all $k \in \mathbb{N}$.

By the elementary maximum principle, we have

$$\begin{split} p_k := Du_k(\bar{x}_k, \bar{t}_k) &= Dv_k(\bar{x}_k, \bar{t}_k) - D\psi_k(\bar{x}_k) + a_k + \bar{a}_k, \\ q_k := u_{k,t}(\bar{x}_k, \bar{t}_k) &= v_{k,t}(\bar{x}_k, \bar{t}_k) + b_k + \bar{b}_k, \\ X_k := D^2u_k(\bar{x}_k, \bar{t}_k) &\leq D^2v_k(\bar{x}_k, \bar{t}_k) - D^2\psi_k(\bar{x}_k). \end{split}$$

Note that $-kI \leq X_k \leq D^2 v(\bar{x}_k, \bar{t}_k) \leq kI$.

For each $k \in \mathbb{N}$ select $(y_k, s_k) \in \overline{\Omega}$ and $(z_k, r_k) \in \overline{\Omega}$ so that (3.10) and (3.11) hold with (\hat{x}_k, \hat{t}_k) , (\hat{y}_k, \hat{s}_k) , and (\hat{z}_k, \hat{r}_k) replaced by (\bar{x}_k, \bar{t}_k) , (y_k, s_k) , and (z_k, r_k) , respectively.

We will show a contradiction for large k. Since $(\bar{x}_k, \bar{t}_k) \in U_{2\varepsilon}$, in view of (3.6) and (3.7), restricting our attention to large k, we may assume that $(y_k, s_k) \in U_{\varepsilon}$ and $(z_k, r_k) \in U_{\varepsilon}$.

Accordingly, we have

$$q_{k} \leq \chi^{+}(u, p_{k}, y_{k}, s_{k})g(p_{k}, X_{k}) - \delta,$$

$$q_{k} - b_{k} - \bar{b}_{k} \geq \chi^{-}(v, p_{k} + D\psi_{k}(\bar{x}_{k}) - a_{k} - \bar{a}_{k}, z_{k}, r_{k})$$

$$\cdot g(p_{k} + D\psi_{k}(\bar{x}_{k}) - a_{k} - \bar{a}_{k}, X_{k} + D^{2}\psi_{k}(\bar{x}_{k})).$$

If $\chi^+(u, p_k, y_k, s_k) = 0$, then

$$q_k + \delta \leq 0 \leq q_k - b_k - \bar{b}_k$$

Since $|b_k| + |\bar{b}_k| < 2\delta_k < \delta$, this is not the case. Therefore we have $\chi^+(u, p_k, y_k, s_k) = 1$, which yields

$$(3.15) u(x, s_k) \ge u(y_k, s_k) + p_k \cdot (x - y_k) \text{for } x \in \Omega_{s_k}.$$

From this we have

$$(3.16) \ \ u(x,s_k) - \frac{k}{2}(|y_k - \bar{x}_k|^2 + |s_k - \bar{t}_k|^2) \ge u_k(\bar{x}_k,\bar{t}_k) + p_k \cdot (x - y_k) \quad \text{ for } x \in \Omega_{s_k}.$$

We wish to show that for sufficiently large k,

(3.17)
$$\chi^{-}(v, p_k + D\psi_k(\bar{x}_k) - a_k - \bar{a}_k, z_k, r_k) = 1.$$

To this end, fix any $y \in \mathbf{R}^n$ so that $(y, r_k) \in U_{\varepsilon}$ and $y \neq z_k$. Since we are considering only large k, in view of (3.6) and (3.7), we may assume that

$$|(y+y_k-z_k,s_k)-(y,r_k)| \le |(y_k,s_k)-(\bar{x}_k,\bar{t}_k)| + |(\bar{x}_k,\bar{t}_k)-(z_k,r_k)| \le \varepsilon,$$

and hence

$$(y+y_k-z_k,s_k)\in\Omega.$$

We may assume as well that

$$|(y + \bar{x}_k - \bar{z}_k, \bar{t}_k) - (y, r_k)| = |(\bar{x}_k, \bar{t}_k) - (z_k, r_k)| \le \varepsilon,$$

and hence

$$(y + \bar{x}_k - z_k, \bar{t}_k) \in \Omega.$$

We insert $x = y + y_k - z_k$ into (3.16), to get

$$u_k(\bar{x}_k, \bar{t}_k) + p_k \cdot (y - z_k) \le u(y + y_k - z_k, s_k) - \frac{k}{2} (|\bar{x}_k - y_k|^2 + |\bar{t}_k - s_k|^2)$$

$$\le u_k(y + \bar{x}_k - z_k, \bar{t}_k).$$

Since (\bar{x}_k, \bar{t}_k) is a maximum point of

$$u_k(x,t) - v_k(x,t) + \psi_k(x) - (a_k + \bar{a}_k) \cdot x - (b_k + \bar{b}_k)t$$

over $\overline{\Omega}$ and $(y + \bar{x}_k - z_k, \bar{t}_k) \in \Omega$, we have

$$u_{k}(y + \bar{x}_{k} - z_{k}, \bar{t}_{k}) - v_{k}(y + \bar{x}_{k} - z_{k}, \bar{t}_{k}) + \psi_{k}(y + \bar{x}_{k} - z_{k})$$

$$- (a_{k} + \bar{a}_{k}) \cdot (y + \bar{x}_{k} - z_{k}) - (b_{k} + \bar{b}_{k})\bar{t}_{k}$$

$$\leq u_{k}(\bar{x}_{k}, \bar{t}_{k}) - v_{k}(\bar{x}_{k}, \bar{t}_{k}) + \psi_{k}(\bar{x}_{k}) - (a_{k} + \bar{a}_{k}) \cdot \bar{x}_{k} - (b_{k} + \bar{b}_{k})\bar{t}_{k}.$$

Furthermore, using the strict convexity of ψ_k , we have

$$\begin{split} v_k(y + \bar{x}_k - z_k, \bar{t}_k) &\geq v_k(\bar{x}_k, \bar{t}_k) + u_k(y + \bar{x}_k - z_k, \bar{t}_k) - u_k(\bar{x}_k, \bar{t}_k) \\ &\quad + \psi_k(y + \bar{x}_k - z_k) - \psi_k(\bar{x}_k) - (a_k + \bar{a}_k) \cdot (y - z_k) \\ &\geq v_k(\bar{x}_k, \bar{t}_k) + (p_k - a_k - \bar{a}_k) \cdot (y - z_k) + \psi_k(y + \bar{x}_k - z_k) - \psi_k(\bar{x}_k) \\ &\geq v_k(\bar{x}_k, \bar{t}_k) + (p_k - a_k - \bar{a}_k + D\psi_k(\bar{x}_k)) \cdot (y - z_k). \end{split}$$

Therefore, we get

$$\begin{aligned} v(y,r_k) &+ \frac{k}{2} \left(|z_k - \bar{x}_k|^2 + |r_k - \bar{t}_k|^2 \right) \ge v_k (y + \bar{x}_k - z_k, \bar{t}_k) \\ &> v_k (\bar{x}_k, \bar{t}_k) + (p_k - a_k - \bar{a}_k + D\psi_k(\bar{x}_k)) \cdot (y - z_k) \\ &= v(z_k, r_k) + \frac{k}{2} \left(|z_k - \bar{x}_k|^2 + |r_k - \bar{t}_k|^2 \right) \\ &+ (p_k - a_k - \bar{a}_k + D\psi_k(\bar{x}_k)) \cdot (y - z_k), \end{aligned}$$

and hence,

$$(3.18) v(y,r_k) > v(z_k,r_k) + (p_k - a_k - \bar{a}_k + D\psi_k(\bar{x}_k)) \cdot (y - z_k).$$

To complete the proof of (3.17), it remains to show that (3.18) holds for all $y \in \mathbf{R}^n$ with $(y, r_k) \notin U_{\varepsilon}$.

For this, we show first that $p_k \to 0$ as $k \to \infty$. To see this, we go back to (3.15) and observe that if $p_k \neq 0$, then, setting $x = y_k + \rho p_k/|p_k|$, with sufficiently small $\rho > 0$, we have

$$ho|p_k| \leq u\Big(y_k +
horac{p_k}{|p_k|}, s_k\Big) - u(y_k, s_k).$$

In view of (3.13), since $u \in \mathrm{USC}(\overline{\Omega})$, we get in the limit as $k \to \infty$,

$$\rho \limsup_{k \to \infty} |p_k| \le u(\hat{x} + \rho e, \hat{t}) - u(\hat{x}, \hat{t})$$

for some $(\hat{x}, \hat{t}) \in P$ and some $e \in S^{n-1} := \{x \in \mathbf{R}^n \mid |x| = 1\}$. Therefore, by assumption (2), we have

$$\rho \limsup_{k \to \infty} |p_k| \le o(\rho) \quad \text{as } \rho \searrow 0.$$

Hence, we get

$$\lim_{k\to\infty} p_k = 0.$$

Thus we may assume that $|p_k| \le 1$ and $|p_k - a_k - \bar{a}_k + D\psi_k(\bar{x}_k)| \le \min\{\gamma, 1\}$.

By (3.13), we know that $v(z_k, r_k) \to 0$ as $k \to \infty$. Hence, we may assume as well that $v(z_k, r_k) \le \gamma$. Now, (3.9) ensures that (3.18) holds for all $y \in \mathbf{R}^n$ with $(y, r_k) \notin U_{\varepsilon}$. This shows that (3.17) holds.

Now, we get

$$g(p_k + D\psi_k(\bar{x}_k) - a_k - \bar{a}_k, X_k + D^2\psi_k(\bar{x}_k)) \le q_k - b_k - \bar{b}_k \le g(p_k, X_k) - b_k - \bar{b}_k - \delta.$$

This and (3.14) give us a desired contradiction.

3.3. Distance between two evolving sets

In this subsection we study continuity properties of the distance between two evolving sets.

Let $u \in \mathrm{USC}(\mathbf{R}^N \times [0,\infty))$ and $v \in \mathrm{LSC}(\mathbf{R}^N \times [0,\infty))$ be respectively a viscosity subsolution and a viscosity supersolution of (L) in $\mathbf{R}^N \times (0,\infty)$.

Fix $\lambda \in \mathbf{R}$ and set

$$K = \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid v(z,t) \le \lambda\},$$

$$W = \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid u(z,t) < \lambda\},$$

$$K_t = \{z \mid (z,t) \in K\} \quad \text{and} \quad W_t = \{z \mid (z,t) \in W\} \quad \text{for } t \in [0,\infty).$$

For notational simplicity we consider the case when $\lambda = 0$, but the results in this subsection are valid for general λ .

Assume that

(3.19) K and W are bounded subsets of $\mathbf{R}^N \times [0, \infty)$ and that $K \neq \emptyset$.

We define the function $d:[0,\infty)\to[0,\infty]$ by

$$d(t) = \operatorname{dist}(K_t, W_t^c),$$

where $W_t^c = \mathbf{R}^N \setminus W_t$.

Theorem 3.4. (a) There is a constant $\tau > 0$ such that $d(t) < \infty$ for all $t \in [0, \tau]$ and $d(t) = \infty$ for all $t \in (\tau, \infty)$. (b) $d \in LSC([0, \tau])$. (c) d is left continuous in $(0, \tau]$.

Proof. Since v is lower semicontinuous, K is closed in $R^N \times [0, \infty)$ and hence K is compact. Since u is upper semicontinuous, W is a relatively open subset of $\mathbf{R}^N \times [0, \infty)$.

Accordingly, for any $t \in [0, \infty)$, K_t is a compact subset of \mathbf{R}^N and W_t^c is a non-empty closed subset of \mathbf{R}^N . Therefore, for any $t \in [0, \infty)$, as far as $K_t \neq \emptyset$, there is a point $(z_t^1, z_t^2) \in K_t \times W_t^c$ such that

$$dist(K_t, W_t^c) = |z_t^1 - z_t^2|.$$

We claim that for any $0 \le t < s < \infty$, we have

$$K_t \supset K_s$$
.

Indeed, noting that v satisfies

$$v_t \ge 0$$
 in $\mathbf{R}^N \times (0, \infty)$

in the viscosity sense, we see that for all $z \in \mathbf{R}^N$ and $0 \le t < s < \infty$,

$$v(z,t) \le v(z,s)$$
.

This immediately implies that $K_t \supset K_s$ for $0 \le t < s < \infty$.

Since K is compact, there is a $\tau \in (0, \infty)$ such that

$$K_{\tau} \neq \emptyset$$
 and $K_{t} = \emptyset$ for all $t \in (\tau, \infty)$.

From this, we get immediately

$$d(t) \begin{cases} < \infty & \text{for } t \in [0, \tau], \\ = \infty & \text{for } t \in (\tau, \infty). \end{cases}$$

Next, we prove assertion (b). Let $t \in [0, \tau]$ and $\{t_k\}_{k \in \mathbb{N}} \subset [0, \tau]$ be such that $t_k \to t$ as $k \to \infty$.

Set

$$\gamma = \liminf_{k \to \infty} d(t_k).$$

We may assume by selecting a subsequence of $\{t_k\}$ that

$$\gamma = \lim_{k \to \infty} d(t_k).$$

For each $k \in \mathbf{N}$ we choose a point $(z_k^1, z_k^2) \in K_{t_k} \times W_{t_k}^c$ so that

$$d(t_k) = |z_k^1 - z_k^2|.$$

Since $(z_k^1, t_k) \in K$, K is compact and W is bounded, we may assume by selecting a subsequence of $\{(z_k^1, z_k^2)\}$ that $\{(z_k^1, z_k^2)\}$ converges to a point $(z^1, z^2) \in \mathbf{R}^{2N}$. Since K and W^c is closed, we have

$$(z^1, t) \in K$$
 and $(z^2, t) \in W^c$.

That is,

$$z^1 \in K_t$$
 and $z^2 \in W_t^c$.

By the definition of d, we have

$$|d(t)| \le |z^1 - z^2| = \lim_{k \to \infty} |z_k^1 - z_k^2| = \lim_{k \to \infty} d(t_k) = \gamma,$$

which proves the required semicontinuity.

To prove (c), we show that there is an increasing continuous function $\omega : [0, \infty) \to [0, \infty)$ satisfying $\omega(0) = 0$ for which

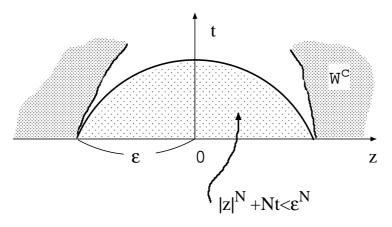
(3.20)
$$d(t+s) \ge d(t) - \omega(s) \quad \text{ for all } t, s \in [0, \infty).$$

The first step is to show that for each $\varepsilon > 0$ there exists $\delta \equiv \delta(\varepsilon) > 0$ such that (3.21) if $z \in \mathbf{R}^N$ and dist $(z, W_t^c) > \varepsilon$ then $z \in W_s$ for all $s \in [t, t + \delta)$.

Indeed, by translation, it is enough to show that for any $\varepsilon>0$, if dist $(0,W_0^c)>\varepsilon$, then

$$0 \in \bigcap_{s \in [0,\delta)} W_s$$
, with $\delta := \frac{\varepsilon^N}{N}$.

In the proof below we use the Gauss curvature flow of balls contained in W_0 at t=0.



Set

$$\varphi(z,t) = |z|^N + Nt - \varepsilon^N$$
 for all $(z,t) \in \mathbf{R}^N \times [0,\infty)$.

Note that

$$\varphi_t(z,t) = \lim_{\zeta \to z} G(D\varphi(\zeta,t), D^2\varphi(\zeta,t))$$
 for all $(z,t) \in \mathbf{R}^N \times (0,\infty)$.

Since $\sigma^+(u, p, z, t) \leq 1$ for all $(p, z, t) \in \mathbf{R}^{2N} \times (0, \infty)$, we see that u is a viscosity subsolution of

$$u_t(z,t) = G(Du(z,t), D^2u(z,t))$$
 in $\mathbf{R}^N \times (0,\infty)$.

We may assume that

$$u(z,t) = \begin{cases} 0 & \text{for } (z,t) \in W^c, \\ -\varepsilon^N & \text{for } (z,t) \in W. \end{cases}$$

Observe that

$$(u - \varphi)(z, 0) \le 0$$
 for all $z \in \mathbf{R}^N$,
 $(u - \varphi)(z, t) \le 0$ for all $(z, t) \in \mathbf{R}^N \times [\delta, \infty)$,
 $(u - \varphi)(z, t) \le 0$ for all $(z, t) \in B(0, \varepsilon)^c \times [0, \infty)$.

Suppose that

$$\sup(u-\varphi)>0.$$

Choose $\lambda > 0$ small enough so that the function $u(z,t) - \varphi(z,t) - \lambda t$ attains a positive maximum over $\mathbf{R}^N \times [0,\infty)$ at a point $(\hat{z},\hat{t}) \in B(0,\varepsilon) \times (0,\delta)$. Then we have

$$\varphi_t(\hat{z},\hat{t}) + \lambda \leq \lim_{z \to \hat{z}} G(D\varphi(z,\hat{t}), D^2\varphi(z,\hat{t})).$$

(The function φ is not admissible, but if $D\varphi(\hat{z},\hat{t})=0$, by testing φ from above by a collection of admissible test functions one may deduce the above inequality.) This is a contradiction, which proves that $u \leq \varphi$ in $\mathbf{R}^N \times [0,\infty)$. Since $\varphi(0,s) < 0$ for $s \in [0,\delta)$, we conclude that u(0,s) < 0 for $s \in [0,\delta)$, which guarantees that $0 \in W_s$ for $s \in [0,\delta)$. This proves (3.21).

Now, we show (3.20) for $\omega(s) = (Ns)^{\frac{1}{N}}$. Let $t, s \in [0, \infty)$. In view of assertion (a), if $d(t) = \infty$, then $d(t+s) = \infty$ and (3.20) holds. If $d(t) - \omega(s) \leq 0$, then it is clear that (3.20) holds. It remains to examine the case when $0 < d(t) - \omega(s) < \infty$. Choose any $0 < r < d(t) - \omega(s)$. Let $z \in \mathbf{R}^N$ satisfy dist $(z, K_t) \leq r$. Then we have dist $(z, W_t^c) \geq d(t) - r > \omega(s)$ and hence

$$z \in W_{t+\tau}$$
 for $\tau \in [0, \omega(s)^N/N] = [0, s]$.

Therefore, we have

$$\operatorname{dist}(K_t, W_{t+s}^c) \ge r,$$

which implies that $d(t + s) \ge d(t) - \omega(s)$.

Finally, assertion (c) follows from (b) and (3.20).

3.4. Completion of proof of Theorem 2.6

This subsection is totally devoted to the proof of Theorem 2.6.

Let $u \in \mathrm{USC}(\mathbf{R}^N \times [0,T))$ and $v \in \mathrm{LSC}(\mathbf{R}^N \times [0,T))$ be a viscosity subsolution and a viscosity supersolution of (L) in $\mathbf{R}^N \times (0,T)$, respectively, where $T \in (0,\infty]$.

Assume that v satisfies (A) and that $u(z,0) \leq v(z,0)$ for all $z \in \mathbf{R}^N$.

We want to show that $u \leq v$ on $\mathbf{R}^N \times (0,T)$. By adding positive constants to v, we need to show that if u(z,0) < v(z,0) for all $z \in \mathbf{R}^N$, then u < v on $\mathbf{R}^N \times (0,T)$.

We thus assume that u(z,0) < v(z,0) for all $z \in \mathbf{R}^N$, and will show that u < v on $\mathbf{R}^N \times (0,T)$.

Note that the inequality u < v on $\mathbf{R}^N \times (0,T)$ is equivalent to the inclusion

$$(3.22) \{(z,t) \in \mathbf{R}^N \times (0,T) \mid v(z,t) \le \lambda\} \subset \{(z,t) \in \mathbf{R}^N \times (0,T) \mid u(z,t) < \lambda\}$$

for all $\lambda \in \mathbf{R}$.

To show (3.22), we fix $\lambda \in \mathbf{R}$ and replace u and v by $u - \lambda$ and $v - \lambda$, respectively. We need to show that

$$(3.23) \{(z,t) \in \mathbf{R}^N \times (0,T) \mid v(z,t) \le 0\} \subset \{(z,t) \in \mathbf{R}^N \times (0,T) \mid u(z,t) < 0\}.$$

By relabeling u and v (see Proposition 2.5) if necessary, we may assume that u takes only two values -1, 0 and v takes only values 0, 1.

Fix any $0 < T_1 < T$ and define $u_1, v_1 : \mathbf{R}^N \times [0, \infty) \to \mathbf{R}$ by

$$u_1(z,t) = \begin{cases} u(z,t) & \text{for } 0 \le t \le T_1, \\ -1 & \text{for } T_1 < t < \infty, \end{cases}$$

and

$$v_1(z,t) = \begin{cases} v(z,t) & \text{for } 0 \le t \le T_1, \\ 1 & \text{for } T_1 < t < \infty. \end{cases}$$

It is easy to check that u_1 and v_1 are, respectively, a viscosity subsolution and a viscosity supersolution of (L) in $\mathbf{R}^N \times (0, \infty)$. Note as well that, since v satisfies (A), the set $\{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid v_1(x,t) \leq 0\}$ is compact.

We choose R > 0 so that

$${z \in \mathbf{R}^N \mid v(z,0) \le 0} \subset B(0,R).$$

Recall that the function

$$w(z,t) = |z|^N + Nt - (R+1)^N$$

is a viscosity subsolution of (L) in $\mathbf{R}^N \times (0, \infty)$. If we define $u_2 : \mathbf{R}^N \times [0, \infty) \to \mathbf{R}$ by

$$u_2(z,t) = \max\{u(z,t), w_1(z,t)\},$$

where

$$w_1(z,t) = \begin{cases} 0 & \text{if } |z|^N + Nt \ge (R+1)^N, \\ -1 & \text{if } |z|^N + Nt < (R+1)^N. \end{cases}$$

then $u_2 \in \mathrm{USC}(\mathbf{R}^N \times [0,\infty))$, u_2 is a viscosity subsolution of (L) in $\mathbf{R}^N \times (0,\infty)$ by Proposition 2.4, (a), $u_2(z,0) < v_1(z,0)$ for all $z \in \mathbf{R}^N$. Note that the set

$$\{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid u_2(z,t) < 0\}$$

is bounded.

To show (3.23) it is enough to prove that for any $T_1 \in (0,T)$,

$$\{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid v_1(z,t) \le 0\} \subset \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid u_2(z,t) < 0\}.$$

Hence, replacing u and v by u_2 and v_1 respectively if necessary, we may assume that $T = \infty$ and that

- (3.24) $\{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid v(z,t) \leq 0\}$ is a compact set,
- $(3.25) \quad \{z \in \mathbf{R}^N \times [0, \infty) \mid u(z, t) < 0\} \text{ is a bounded set.}$

We set

$$K = \{(z,t) \in \mathbf{R}^n \times [0,\infty) \mid v(z,t) \le 0\},\$$

$$W = \{(z,t) \in \mathbf{R}^N \times [0,\infty) \mid u(z,t) < 0\}.$$

Note that K is compact and W is relatively open in $\mathbf{R}^N \times [0, \infty)$.

We use the notation: $K_t = \{z \in \mathbf{R}^N \mid (z,t) \in K\}$ and $W_t = \{z \in \mathbf{R}^N \mid (z,t) \in W\}$ for $t \in [0,\infty)$. Define

$$d(t) = \operatorname{dist}(K_t, W_t^c) \quad \text{for } t \in [0, \infty).$$

Since $K_0 \subset W_0$, we have

$$d(0) > 0$$
,

and it is enough to show that, in order to prove (3.23),

$$d(t) \ge d(0)$$
 for all $t \in (0, \infty)$.

We prove this by arguing by contradiction. We thus suppose that for some $t_0 \in (0, \infty)$,

$$d(t_0) < d(0)$$
.

In view of Theorem 3.4, we may choose a $t_1 \in (0, t_0)$ so that

$$d(0) > d(t_1)$$
 and $\min_{t \in [0, t_1]} d(t) > 0$,

and choose $\psi \in C^2([0,t_1])$ so that

$$\psi'(t) > 0 \text{ for all } t \in [0, t_1],$$

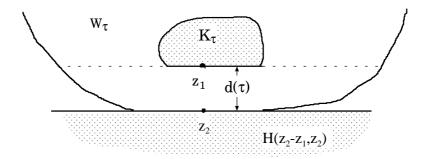
 $d + \psi$ attains a minimum over $[0, t_1]$ at some $\tau \in (0, t_1)$.

Next we show that for $(z^1, z^2) \in \mathbf{R}^{2N}$, if

(3.26)
$$z^1 \in K_{\tau}, \quad z^2 \in W_{\tau}^c, \quad \text{and} \quad |z^1 - z^2| = d(\tau),$$

then

(3.27)
$$H(z^2 - z^1, z^2) \subset W_{\tau}^c.$$



To do so, we fix $(z^1, z^2) \in \mathbf{R}^{2N}$ so that (3.26) is satisfied.

Define $\varphi \in C^2(\mathbf{R}^N \times [0, t_1])$ by

$$\varphi(z,t) = |z - z^1| + \psi(t).$$

Noting that $K_t \supset K_s$ for $0 \le t \le s$ and therefore $z^1 \in K_t$ for all $t \in [0, \tau]$, we see that for any $t \in [0, \tau]$ and $z \in W_t^c$,

$$\varphi(z,t) \ge d(t) + \psi(t).$$

We may assume by relabeling u again that

$$u(z,t) = \begin{cases} 0 & \text{if } (z,t) \in W^c, \\ -M & \text{if } (z,t) \in W, \end{cases}$$

where M > 0 is a constant to be chosen later.

Let $(z,t) \in \mathbf{R}^N \times [0,\tau]$. If $z \in W_t^c$, then

$$(u - \varphi)(z, t) \le -(d + \psi)(t) \le -(d + \psi)(\tau).$$

If $z \in W_t$, then

$$(u-arphi)(z,t) \leq -M - \psi(t).$$

We select M so that

$$M \ge \sup_{[0,t_1]} d.$$

Then we have

$$(u - \varphi)(z, t) \le -(d + \psi)(t) \le -(d + \psi)(\tau).$$

Observing that

$$(u-\varphi)(z^2,\tau) = -|z^2-z^1| - \psi(\tau) = -(d+\psi)(\tau).$$

we see that $u - \varphi$ attains a maximum over $\mathbf{R}^N \times [0, t_1]$ at (z^2, τ) .

Modifying the definition of φ outside a neighborhood of (z^2, τ) appropriately, we may assume that φ is an admissible function (see section 2 or [IS]). Thus we conclude that

$$0 < \psi'(\tau) \le \sigma^+(u, p, z^2, \tau)G(p, X),$$

where

$$p = rac{z^2 - z^1}{|z^2 - z^1|}$$
 and $X = rac{1}{|z^1 - z^2|}I - rac{(z^1 - z^2) \otimes (z^1 - z^2)}{|z^1 - z^2|^3}$.

This ensures that

$$\sigma^{+}(u, p, z^{2}, \tau) = 1,$$

which shows that $H(p, z^2) \subset W_{\tau}^c$, proving (3.27).

By translation and rotation in \mathbf{R}^N , we may assume that

$$z^1 = 0 \quad \text{and} \quad z^2 = -d(\tau)e_N,$$

where e_N denotes the unit vector of \mathbf{R}^N with unity as its N-th entry. (3.27) now reads

$$\mathbf{R}^n \times (-\infty, -d(\tau)] \subset W_{\tau}^c$$
 or equivalently $W_{\tau} \subset \mathbf{R}^n \times (-d(\tau), \infty)$.

From this, since $d(\tau) > 0$, by the definition of $d(\tau)$, we see that

$$(3.28) K_{\tau} \subset \mathbf{R}^{n} \times [0, \infty).$$

Define $Q \subset \mathbf{R}^n \times [0, \infty)$ by

$$Q=\{(x,t)\in\mathbf{R}^n\times[0,\infty)\mid (x,0,t)\in W\}.$$

and $u^-: Q \to \mathbf{R}$ by

$$u^{-}(x,t) = \inf\{a \in \mathbf{R} \mid \{x\} \times [a,0] \times \{t\} \subset W\}.$$

Define as well $u^+: \mathbf{R}^n \times [0, \infty) \to \mathbf{R} \cup \{\infty\}$ by

$$u^{+}(x,t) = \inf\{y \in \mathbf{R} \mid (x,y,t) \in K\}.$$

It is clear that $u^-(x,t) < 0$ for all $(x,t) \in Q$. Note as well that $(0,\tau) \in Q$, $u^+(0,\tau) = 0$, and $u^-(0,\tau) = -d(\tau)$. In particular, $u^+(0,\tau) - u^-(0,\tau) = d(\tau)$.

By Theorems 3.1 and 3.2, we know that $u^- \in \mathrm{USC}(Q)$ and $u^+ \in \mathrm{LSC}(\mathbf{R}^n \times [0,\infty), \mathbf{R} \cup \{\infty\})$. More importantly, u^- is a viscosity subsolution of (G) in $\mathrm{int}\,Q$ and u^+ is a viscosity supersolution of (G) in $\mathbf{R}^n \times (0,\infty)$.

We extend the definition of u^- to \overline{Q} by setting

$$u^{-}(x,t) = 0$$
 for $(x,t) \in \partial Q \setminus Q$.

Since $u^-(x,t) < 0$ for all $(x,t) \in Q$ and $u^- \in \mathrm{USC}(Q)$, we see immediately that $u^- \in \mathrm{USC}(\overline{Q})$.

Since $(x, u^-(x,t)) \in W_t^c$ for $(x,t) \in \overline{Q}$ and $(x, u^+(x,t)) \in K_t$, if $u^+(x,t) \in \mathbf{R}$, for $(x,t) \in \mathbf{R}^n \times [0,\infty)$, we have

$$|u^{+}(x,t) - u^{-}(x,t)| \ge d(t)$$
 if $u^{+}(x,t) \in \mathbf{R}$

for all $(x,t) \in \overline{Q}$.

We intend to apply Lemma 3.3 to functions u^+ and $u^- + d(\tau) + \varepsilon$, with sufficiently small $\varepsilon > 0$, to get a contradiction. To this end, we recall (3.28) and observe that $u^+(x,\tau) \geq 0$ for all $x \in \mathbf{R}^n$. Therefore, we have

$$u^{-}(x,\tau) \leq 0 \leq u^{+}(x,\tau)$$
 for all $x \in \mathbf{R}^{n}$ satisfying $(x,\tau) \in \overline{Q}$,

and furthermore,

$$(u^+ - u^-)(x, \tau) \ge d(\tau)$$
 for all $x \in \mathbf{R}^n$ satisfying $(x, \tau) \in \overline{Q}$.

Noting that the function $u^+ - u^-$ is lower semicontinuous on \overline{Q} , we find a constant $\sigma \in [0, \tau)$ such that

$$(u^+ - u^-)(x,t) > 0$$
 for all $(x,t) \in \overline{Q} \cap (\mathbf{R}^n \times [\sigma,\tau])$,

which guarantees that

$$(u^+ - u^-)(x,t) \ge d(t)$$
 for all $(x,t) \in \overline{Q} \cap (\mathbf{R}^n \times [\sigma,\tau])$.

Since K is compact and $u^+(x,\tau) \geq 0$ for all $x \in \mathbf{R}^n$, we may assume by reselecting σ if necessary that

(3.29)
$$u^{+}(x,t) > -d(\tau) \quad \text{for all } (x,t) \in \mathbf{R}^{n} \times [\sigma,\tau).$$

We modify the definition of $u^+(x,t)$ for $t > \tau$ by setting

$$u^+(x,t) = \infty$$
 for $t > \tau$.

The new function u^+ is again lower semicontinuous in $\mathbf{R}^n \times [0, \infty)$ and it is a viscosity supersolution of (G) in $\mathbf{R}^n \times (0, \infty)$. Also, we have for all $(x, t) \in \overline{Q}$,

$$(u^+ - u^-)(x,t) \ge \begin{cases} d(t) & \text{if } t \in [\sigma,\tau], \\ \infty & \text{if } t > \tau. \end{cases}$$

We set $\Omega = Q \cap (\mathbf{R}^n \times [\sigma, \infty))$. It is clear that Ω is a bounded, relatively open subset of $\mathbf{R}^n \times [\sigma, \infty)$. Note furthermore that $u^+ - u^-$ attains its minimum $d(\tau)$ over $\overline{\Omega}$ at the point $(0, \tau)$. We claim that

$$(3.30) (u^+ - u^-)(x,t) > d(\tau) \text{for all } (x,t) \in \partial \Omega.$$

To see this, let $(x,t) \in \partial \Omega$. We have

$$(u^+ - u^-)(x,t) \ge \begin{cases} d(t) > d(\tau) & \text{if } t < \tau, \\ \infty > d(\tau) & \text{if } t > \tau. \end{cases}$$

We may thus assume that $t = \tau$. Suppose that $(u^+ - u^-)(x, \tau) = d(\tau)$. Then, since $u^-(x, \tau) = 0$, setting $\zeta^1 = (x, d(\tau), \tau)$ and $\zeta^2 = (x, 0, \tau)$, we have

$$\zeta^1 \in K_{\tau}, \quad \zeta^2 \in W_{\tau}^c, \quad \text{ and } \quad |\zeta^1 - \zeta^2| = d(\tau).$$

Since (3.26) yields (3.27), we get

$$W_{\tau}^c \supset H(p,\zeta^2) \supset \mathbf{R}^n \times (-\infty,0)$$

for $p = -e_N$. In particular, we have

$$(0, -\frac{d(\tau)}{2}) \in W_{\tau}^c.$$

However, since $0 \in K_{\tau}$, we have

$$(0, -\frac{d(\tau)}{2}) \in W_{\tau}.$$

We thus get a contradiction, proving that $(u^+ - u^-)(x, \tau) > d(\tau)$. Thus we see that (3.30) holds.

Setting

$$E = \{(x,t) \in \mathbf{R}^n \times [\sigma,\infty) \mid (x,y,t) \in K \text{ for some } y \in \mathbf{R}\},\$$

which is a compact set, it is clear that

$$u^+(x,t) = \infty$$
 for $(x,t) \in (\mathbf{R}^n \times [\sigma,\infty)) \setminus E$.

Let $(x,t) \in E \setminus \Omega$. If $t > \tau$, then we have $u^+(x,t) = \infty$. If $t \leq \tau$ and if we suppose that $u^+(x,t) < d(t)$, then, since $|u^+(x,t)| < d(t)$ by (3.29) and $(x,u^+(x,t)) \in K_t$, we have $(x,0) \in W_t$ and hence $(x,t) \in \Omega$, which contradicts our choice of (x,t), This shows that if $t \leq \tau$, then $u^+(x,t) \geq d(t) \geq d(\tau) > 0$.

Now let $(x,t) \in \Omega$ be a minimum point of $u^+ - u^-$ over $\overline{\Omega}$. We know that $t = \tau$ and $(u^+ - u^-)(x,\tau) = d(\tau)$. Again, since (3.26) implies (3.27), we have

$$H(-e_N,z)\subset W^c_{\tau},$$

where $z = (x, u^-(x, \tau))$, which guarantees that $u^-(x, \tau) = -d(\tau)$ and $u^+(x, \tau) = 0$. Moreover, noting that $(x, u^+(x, \tau)) \in K_\tau$ and $(x, u^-(x, \tau)) \in W_\tau^c$, and that the open ball of \mathbf{R}^N with radius $d(\tau)$ centered at $(x, u^+(x, \tau))$ does not intersect with W_τ , we see that as $\mathbf{R}^n \ni \xi \to 0$,

$$u^{-}(x+\xi,\tau) \le u^{-}(x,\tau) + O(|\xi|^{2}).$$

The final step is just to apply Theorem 3.3 to the viscosity subsolution $u^- + d(\tau) + \varepsilon$ of (G) in Ω and the viscosity supersolution u^+ of (G) in $\mathbf{R}^n \times (\sigma, \infty)$, where $\varepsilon > 0$ is chosen so that $u^+ \geq u^- + d(\tau) + \varepsilon$ on $\partial \Omega$, to obtain a contradiction,

$$u^{-}(0,\tau) + d(\tau) + \varepsilon \le u^{+}(0,\tau) = u^{-}(0,\tau) + d(\tau).$$

4. Stability of viscosity solutions

In this section we prove Proposition 2.4 as well as Proposition 4.1 below.

Proposition 4.1. Let $\Omega = \mathbb{R}^N \times (0,T)$, with $T \in (0,\infty]$. (a) Let $\{u_k\}_{k \in \mathbb{N}} \subset \mathrm{USC}(\Omega)$ and $u \in \mathrm{USC}(\Omega)$. Assume that $u_k(\xi) \searrow u(\xi)$ for all $\xi \in \Omega$ as $k \to \infty$ and that u_k are viscosity subsolutions of (L) in Ω . Then u is a viscosity subsolution of (L) in Ω . (b) Let $\{u_k\}_{k \in \mathbb{N}} \subset \mathrm{LSC}(\Omega)$ and $u \in \mathrm{LSC}(\Omega)$. Assume that $u_k(\xi) \nearrow u(\xi)$ for all $\xi \in \Omega$ as $k \to \infty$, that u_k are viscosity supersolutions of (L) in Ω , and that all the u_k satisfy condition (A). Then u is a viscosity supersolution of (L) in Ω .

We need the following lemmas for the proof of Propositions 2.4, 2.5, and 4.1 and Theorem 2.7. Let $\Omega = \mathbb{R}^N \times (0,T)$, with $T \in (0,\infty)$.

Lemma 4.2. Let $\{v_k\}_{k\in\mathbb{N}}\subset \mathrm{USC}(\Omega),\ u\in\mathrm{USC}(\Omega),\ \xi\equiv(z,t)\in\Omega,\ \xi_k\equiv(z_k,t_k)\in\Omega,\ p_k\in\mathbf{R}^N,\ and\ p\in\mathbf{R}^N\setminus\{0\}.$ Assume that $v_k\leq u$ in Ω , and that $\xi_k\to\xi,\ v_k(\xi_k)\to u(\xi)$ and $p_k\to p$ as $k\to\infty$, and that $\sigma^+(u,p,\xi)=0$. Then $\sigma^+(v_k,p_k,\xi_k)=0$ for sufficiently large k.

Proof. Since $\sigma^+(u, p, \xi) = 0$, there is a point $y \in H(p, z)$ such that $u(y, t) < u(\xi)$. Since $u \in \mathrm{USC}(\Omega)$, there is a neighborhood U of (y, t) and a constant $\varepsilon > 0$ such that $u(\eta) \leq u(\xi) - 2\varepsilon$ for all $\eta \in U$. Since $v_k(\xi_k) \to u(\xi)$, we may assume that $u(\xi) \leq v_k(\xi_k) + \varepsilon$. Then $u(\eta) \leq v_k(\xi_k) - \varepsilon$ for all $\eta \in U$. Since $v_k \leq u$, we have $v_k(\eta) \leq v_k(\xi_k) - \varepsilon$ for all $\eta \in U$. By continuity, $U \cap H(p_k, \xi_k) \neq \emptyset$ and, accordingly, we have $\sigma^+(v_k, p_k, \xi_k) = 0$ for sufficiently large $k \in \mathbf{N}$.

Lemma 4.3. Let $\{u_k\}_{k\in\mathbb{N}}\subset \mathrm{USC}(\Omega),\ u\in\mathrm{USC}(\Omega),\ \xi\equiv(z,t)\in\Omega,\ \xi_k\equiv(z_k,t_k)\in\Omega,$ $p_k\in\mathbf{R}^N,\ and\ p\in\mathbf{R}^N\setminus\{0\}.$ Assume that $u_k(\eta)\searrow u(\eta)$ for all $\eta\in\Omega$ as $k\to\infty$ and that $\xi_k\to\xi,\ u_k(\xi_k)\to u(\xi),\ and\ p_k\to p$ as $k\to\infty,\ and\ that\ \sigma^+(u,p,\xi)=0.$ Then $\sigma^+(u_k,p_k,\xi_k)=0$ for sufficiently large k.

Proof. As in the previous proof, there is a compact neighborhood U of (y,t) and a constant $\varepsilon > 0$ such that $u(\eta) \leq v_k(\xi_k) - 2\varepsilon$ for all $\eta \in U$. Hence, $(v_k(\eta) - v_k(\xi_k) + 2\varepsilon)_+ \searrow 0$ for all $\eta \in U$ as $k \to \infty$ and therefore, in view of Dini's lemma, we see that $(v_k(\eta) - v_k(\xi_k) + 2\varepsilon)_+ \searrow 0$ uniformly for $\eta \in U$ as $k \to \infty$. Thus, we may assume further that $v_k(\eta) \leq v_k(\xi_k) - \varepsilon$ for all $\eta \in U$ and sufficiently large $k \in \mathbb{N}$. By continuity, $U \cap H(p_k, z_k) \neq \emptyset$ for large $k \in \mathbb{N}$. Thus, we have $\sigma^-(v_k, p_k, \xi_k) = 0$ for large $k \in \mathbb{N}$.

Lemma 4.4. Let $\varphi \in C^2(\mathbf{R}^{N+1})$. Assume that $\varphi(0) = 0$, $D\varphi(0) \neq 0$, where D denotes the gradient in the first N variables as usual, and $G(D\varphi(0), D^2\varphi(0)) > 0$. Then there are

a neighborhood U of 0 in \mathbf{R}^N and a constant $\varepsilon > 0$ such that for all $(z,t) \in U \times (-\varepsilon,\varepsilon)$, if $\varphi(z,t) \in (-\varepsilon,\varepsilon)$, then

$$\varphi(\zeta,t) > \varphi(z,t)$$
 for all $\zeta \in H(D\varphi(z,t),z) \cap U$.

Proof. By rotation, we may assume that $D\varphi(0) = (0, ..., 0, -|D\varphi(0)|)$. By the implicit function theorem, there are a neighborhood V of 0 in \mathbb{R}^n , a positive constant ε , and a function $g \in C^2(V \times (-\varepsilon, \varepsilon)^2)$ such that for any $x \in V$ and $(y, t, \lambda) \in (-\varepsilon, \varepsilon)^3$,

$$\varphi(x, y, t) = \lambda$$
 if and only if $y = g(x, t, \lambda)$.

Replacing V and ε by smaller ones if necessary, we may assume further that for any $x \in V$ and $(y, t, \lambda) \in (-\varepsilon, \varepsilon)^3$,

$$\varphi(x, y, t) < \lambda$$
 if and only if $y > g(x, t, \lambda)$.

We may assume as well that $D\varphi(z,t) \neq 0$ and $G(D\varphi(z,t), D^2\varphi(z,t)) > 0$ for all $(z,t) \in V \times (-\varepsilon,\varepsilon)^3$. This guarantees that for each $(t,\lambda) \in (-\varepsilon,\varepsilon)^2$, all the principal curvatures of the hypersurface $\{z \in V \mid \varphi(z,t) = \lambda\}$ at any point are positive and hence that $D^2g(x,t,\lambda) > 0$ for all $(x,t,\lambda) \in V \times (-\varepsilon,\varepsilon)^2$.

Fix
$$(\hat{x}, \hat{y}, t, \lambda) \in V \times (-\varepsilon, \varepsilon)^3$$
 so that $\hat{y} = g(\hat{x}, t, \lambda)$. Set $\hat{z} = (\hat{x}, \hat{y})$. Observe that

$$H(D\varphi(\hat{z},t),\hat{z})\cap (V\times (-\varepsilon,\varepsilon))$$

$$=\{(x,y)\in V\times (-\varepsilon,\varepsilon)\mid y\leq Dg(\hat{x},t,\lambda)\cdot (x-\hat{x})+\hat{y},\ (x,y)\neq \hat{z}\}.$$

We may assume that V is a convex subset of \mathbf{R}^n . Since $D^2g(x,t,\lambda) > 0$ in $V \times (-\varepsilon,\varepsilon)^2$, we see that

$$\{(x,y) \in V \times (-\varepsilon,\varepsilon) \mid y \le Dg(\hat{x},t,\lambda) \cdot (x-\hat{x}) + \hat{y}\}$$

$$\cap \{(x,y) \in V \times (-\varepsilon,\varepsilon) \mid y \ge g(x,t,\lambda)\} = \{\hat{z}\}.$$

Hence, we get

$$H(D\varphi(\hat{z},t),z)\cap (V\times(-\varepsilon,\varepsilon))\cap \{z\in\mathbf{R}^N\mid \varphi(z,t)\leq \lambda\}=\emptyset.$$

That is, if $z \in H(D\varphi(\hat{z},t),z) \cap (V \times (-\varepsilon,\varepsilon))$, then $\varphi(z,t) > \lambda$, which was to be proven.

Lemma 4.5. Let $\varphi \in C^2(\Omega)$, $\{v_k\}_{k \in \mathbb{N}} \subset LSC(\Omega)$, $u \in LSC(\Omega)$, $\xi \equiv (z,t) \in \Omega$, and r > 0. Assume that $B(\xi, r) \subset \Omega$. Let $\xi_k \equiv (z_k, t_k) \in B(\xi, r)$ for $k \in \mathbb{N}$, and assume that

 $u-\varphi$ and $v_k-\varphi$, with $k \in \mathbf{N}$, attain their minima over $B(\xi,r)$ at ξ and ξ_k , respectively. Assume as well that $v_k \geq u$ in Ω for all k, that $\xi_k \to \xi$ and $v_k(\xi_k) \to u(\xi)$ as $k \to \infty$, that $\sigma^-(u, D\varphi(\xi), \xi)G(D\varphi(\xi), D^2\varphi(\xi)) > 0$, and that u satisfies condition (A). Then $\sigma^-(v_k, D\varphi(\xi_k), \xi_k) = 1$ for sufficiently large k.

Proof. We set $p_k = D\varphi(\xi_k)$ for $k \in \mathbb{N}$ and $p = D\varphi(\xi)$. We may assume that $p_k \neq 0$ for all $k \in \mathbb{N}$.

Since $\sigma^-(u, p, \xi) = 1$, we have $u(y, t) > u(\xi)$ for all $y \in H(p, z)$. Choose a $\lambda > u(\xi)$ so that $\lambda < u(y, t)$ for some $y \in H(p, z)$. Thanks to condition (A), there is an R > 0 such that for all $(y, s) \in \Omega$,

$$u(y,s) \ge \lambda$$
 if $|y| \ge R$.

Accordingly, we have for all $(y, s) \in \Omega$ and $k \in \mathbb{N}$,

$$(4.1) v_k(y,s) \ge \lambda \text{if } |y| \ge R.$$

Since $v_k(\xi_k) \to u(\xi)$ and $u(\xi) < \lambda$, we may assume that

$$(4.2) v_k(\xi_k) < \lambda \text{for all } k \in \mathbf{N}.$$

Applying Lemma 4.4, we infer that there is a $\delta > 0$ such that if k is sufficiently large, then

(4.3)
$$\varphi(y, t_k) > \varphi(\xi_k) \quad \text{for all } y \in H(p_k, z_k) \cap B(z, \delta).$$

We may assume that $B(z, \delta) \times [t - \delta, t + \delta] \subset B(\xi, r)$ and $|t_k - t| \leq \delta$ for all $k \in \mathbb{N}$ and that inequality (4.3) holds for all $k \in \mathbb{N}$. From (4.3) and the assumption that ξ_k is a minimum point of $v_k - \varphi$ over $B(\xi, r)$, we get

$$(4.4) v_k(y,t_k) > v_k(\xi_k) \text{for all } y \in H(p_k,z_k) \cap B(z,\delta).$$

Now, we argue by contradiction. We thus suppose that

$$\liminf_{k \to \infty} \sigma^-(v_k, p_k, \xi_k) = 0,$$

and show a contradiction. By this assumption, we may assume by taking a subsequence that

$$\sigma^-(v_k, p_k, \xi_k) = 0$$
 for all $k \in \mathbf{N}$.

For each k, we may choose a point $y_k \in H(p_k, z_k)$ such that

$$(4.5) v_k(y_k, t_k) \le v_k(\xi_k).$$

Therefore, in view of (4.1) and (4.2), we see that $|y_k| \leq R$ for all $k \in \mathbb{N}$, and we may assume that $y_k \to y$ for some $y \in \mathbb{R}^N$ as $k \to \infty$. By continuity, we deduce that $y \in \overline{H}(p,z)$. From (4.4) and (4.5), we see that $y \notin B(z,\delta)$. Hence, $y \in H(p,z)$. However, from (4.5), we have

$$u(\xi) \ge \limsup_{k \to \infty} v_k(y_k, t_k) \ge \limsup_{k \to \infty} u(y_k, t_k) \ge u(y, t),$$

which shows that $\sigma^-(u, p, \xi) = 0$, a contradiction. Thus we conclude that $\sigma^-(v_k, p_k, \xi_k) = 1$ for sufficiently large $k \in \mathbb{N}$.

Lemma 4.6. Let $\varphi \in C^2(\Omega)$, $\{v_k\}_{k \in \mathbb{N}} \subset LSC(\Omega)$, $u \in LSC(\Omega)$, $\xi \equiv (z,t) \in \Omega$, and r > 0. Assume that $B(\xi,r) \subset \Omega$. Let $\xi_k \equiv (z_k,t_k) \in B(\xi,r)$ for $k \in \mathbb{N}$, and assume that $u - \varphi$ and $v_k - \varphi$, with $k \in \mathbb{N}$, attain their minima over $B(\xi,r)$ at ξ and ξ_k , respectively. Assume as well that $v_k(\zeta) \nearrow u(\zeta)$ for all $\zeta \in \Omega$ as $k \to \infty$, that $\xi_k \to \xi$ and $v_k(\xi_k) \to u(\xi)$ as $k \to \infty$, that $\sigma^-(u, D\varphi(\xi), \xi)G(D\varphi(\xi), D^2\varphi(\xi)) > 0$, and that all the v_k satisfy condition (A). Then $\sigma^-(v_k, D\varphi(\xi_k), \xi_k) = 1$ for sufficiently large k.

Proof. We set $p_k = D\varphi(\xi_k)$ for $k \in \mathbb{N}$ and $p = D\varphi(\xi)$. We may assume that $p_k \neq 0$ for all $k \in \mathbb{N}$.

Since $\sigma^-(u, p, \xi) = 1$, we have $u(y, t) > u(\xi)$ for all $\bar{y} \in H(p, z)$. By continuity, we may assume that there is a point $\bar{y} \in H(p, z)$ such that $\bar{y} \in H(p_k, z_k)$ for all $k \in \mathbb{N}$ and a constant $\varepsilon > 0$ such that $v_1(\bar{y}, t) > u(\xi) + \varepsilon$. By condition (A), there is an R > 0 such that for all $(y, s) \in \Omega$,

$$v_1(y,s) \ge u(\xi) + \varepsilon$$
 if $|y| \ge R$.

Accordingly, we have for all $(y, s) \in \Omega$ and $k \in \mathbb{N}$,

$$(4.6) v_k(y,s) \ge u(\xi) + \varepsilon \text{if } |y| \ge R.$$

Since $v_k(\xi_k) \to u(\xi)$, we may assume that

$$(4.7) v_k(\xi_k) < u(\xi) + \varepsilon \text{for all } k \in \mathbf{N}.$$

Using Lemma 4.4, we see that there is a $\delta > 0$ such that if k is sufficiently large, then

(4.8)
$$\varphi(y,t_k) > \varphi(\xi_k) \quad \text{for all } y \in H(p_k,z_k) \cap B(z,\delta).$$

We may assume that $B(z, \delta) \times [t - \delta, t + \delta] \subset B(\xi, r)$ and $|t_k - t| \leq \delta$ for all $k \in \mathbb{N}$ and that inequality (4.8) holds for all $k \in \mathbb{N}$. From (4.8), we get

$$(4.9) v_k(y, t_k) > v_k(\xi_k) \text{for all } y \in H(p_k, z_k) \cap B(z, \delta).$$

We argue by contradiction. We thus suppose that

$$\liminf_{k \to \infty} \sigma^{-}(v_k, p_k, \xi_k) = 0,$$

and show a contradiction. By this assumption, we may assume by taking a subsequence that

$$\sigma^-(v_k, p_k, \xi_k) = 0$$
 for all $k \in \mathbf{N}$.

For each k, we may choose a point $y_k \in H(p_k, z_k)$ such that

$$(4.10) v_k(y_k, t_k) \le v_k(\xi_k).$$

Therefore, in view of (4.6) and (4.7), we see that $|y_k| \leq R$ for all $k \in \mathbb{N}$, and we may assume that $y_k \to \hat{y}$ for some $\hat{y} \in \mathbb{R}^N$ as $k \to \infty$. By continuity, we deduce that $\hat{y} \in \overline{H}(p,z)$. From (4.9), we see that $\hat{y} \notin B(z,\delta/2)$. Hence, $\hat{y} \in H(p,z)$.

By assumption, if $m \leq k$, then $v_m(y_k, t_k) \leq v_k(y_k, t_k)$. Hence, from (4.10), we have

$$u(\xi) \ge \limsup_{k \to \infty} v_k(y_k, t_k) \ge \limsup_{m \to \infty} \liminf_{k \to \infty} v_m(y_k, t_k)$$
$$\ge \limsup_{m \to \infty} v_m(\hat{y}, t) = u(\hat{y}, t),$$

which is a contradiction. Thus we conclude that $\sigma^-(v_k, p_k, \xi_k) = 1$ for sufficiently large $k \in \mathbb{N}$, and completes the proof.

Proof of Proposition 4.1. Let $\{u_k\}$ and u be as in assertion (a) of Proposition 4.1. Let $\varphi \in \mathcal{A}_0(\Omega)$ and $\xi \in \Omega$. Assume that $u - \varphi$ has a strict maximum at ξ . Fix a compact neighborhood $V \subset \Omega$ of ξ . For each $k \in \mathbb{N}$ let ξ_k be a maximum point of $u_k - \varphi$ over V.

It is a standard observation that as $k \to \infty$,

$$\xi_k \to \xi$$
 and $u_k(\xi_k) \to u(\xi)$.

We may assume that $\xi_k \in \text{int } V$ for all $k \in \mathbb{N}$. Since u_k are viscosity subsolutions of (L) in Ω , we have

$$\varphi_t(\xi_k) \le \sigma^+(u_k, D\varphi(\xi_k), \xi_k)G(D\varphi(\xi_k), D^2\varphi(\xi_k))$$
 for $k \in \mathbf{N}$.

Using Lemma 4.3 and sending $k \to \infty$, we get

$$\varphi_t(\xi) \le \sigma^+(u, D\varphi(\xi), \xi)G(D\varphi(\xi), D^2\varphi(\xi)),$$

which proves assertion (a).

The proof of assertion (b) parallels the above with Lemma 4.6 in place of Lemma 4.3.

Proof of Proposition 2.4. We start with assertion (a). Let \mathcal{S} and u be as in Proposition 2.4, (a). Let $\varphi \in \mathcal{A}_0(\Omega)$ and $\xi \in \Omega$, and assume that $u^* - \varphi$ has a strict maximum at ξ . It is then standard to observe that there are sequences $\{v_k\}_{k \in \mathbb{N}} \subset \mathcal{S}$ and $\{\xi_k\}_{k \in \mathbb{N}} \subset \Omega$ such that as $k \to \infty$,

$$\xi_k \to \xi$$
 and $v_k(\xi_k) \to u^*(\xi)$,

and each of $v_k - \varphi$ attains a local maximum at ξ_k . Accordingly we have

$$\varphi_t(\xi_k) \le \sigma^+(v_k, D\varphi(\xi_k), \xi_k) G(D\varphi(\xi_k), D^2\varphi(\xi_k))$$
 for $k \in \mathbf{N}$.

Now, sending $k \to \infty$ and using Lemma 4.2, we see that

$$\varphi_t(\xi) \le \sigma^+(u^*, D\varphi(\xi), \xi)G(D\varphi(\xi), D^2\varphi(\xi)).$$

This proves assertion (a).

The proof of assertion (b) is similar to the above. Lemma 4.5 is now used instead of Lemma 4.2. We omit the details of the proof.

We turn to assertion (c). Let f_1 , f_2 , and u be as in assertion (c).

We argue by contradiction. We thus suppose that u_* is not a viscosity supersolution of (L). Then there would be a function $\varphi \in \mathcal{A}_0(\Omega)$ and a point $\hat{\xi} \in \Omega$ such that $u_* - \varphi$ has a strict minimum at $\hat{\xi}$ and

(4.11)
$$\varphi_t(\hat{\xi}) < \sigma^-(u_*, D\varphi(\hat{\xi}), \hat{\xi})G(D\varphi(\hat{\xi}), D^2\varphi(\hat{\xi})).$$

Here we may assume that $u_*(\hat{\xi}) = \varphi(\hat{\xi})$.

Since f_2 is continuous, we have

$$u^* \le f_2 \quad \text{in } \Omega,$$

and hence, since u^* is a viscosity subsolution of (L) in Ω by assertion (a), we see that $u = u^* \in USC(\Omega)$.

Fix a compact neighborhood $V \subset \Omega$ of $\hat{\xi}$. We claim that

(4.12) for sufficiently small $\varepsilon > 0$, the function $v_{\varepsilon} \in \mathrm{USC}(\Omega)$ defined by

$$v_{\varepsilon}(\xi) = \begin{cases} \max\{u(\xi), \varphi(\xi) + \varepsilon\} & \text{for } \xi \in V, \\ u(\xi) & \text{for } \xi \in \Omega \setminus V \end{cases}$$

is a viscosity subsolution of (L) in Ω .

We postpone the proof of this claim and, admitting (4.12) for the moment, we first complete the proof of assertion (c).

To this end, we observe that $u(\hat{\xi}) < f_2(\hat{\xi})$. Indeed, if this is not the case, then we have $u(\hat{\xi}) = f_2(\hat{\xi})$, which implies that $f_2 - \varphi$ attains its minimum at $\hat{\xi}$ and therefore yields

$$\varphi_t(\hat{\xi}) \ge \sigma^-(f_2, D\varphi(\hat{\xi}), \hat{\xi})G(D\varphi(\hat{\xi}), D^2\varphi(\hat{\xi})) \ge \sigma^-(u, D\varphi(\hat{\xi}), \hat{\xi})G(D\varphi(\hat{\xi}), D^2\varphi(\hat{\xi})).$$

This contradicts (4.11), which shows that $u(\hat{\xi}) < f_2(\hat{\xi})$. It is now clear that if $\varepsilon > 0$ is sufficiently small, then $v_{\varepsilon} \leq f_2$ in Ω . Since $v_{\varepsilon} \geq u \geq f_1$ in Ω , by the definition of u, we should have $v_{\varepsilon} \leq u$ in Ω . However, by the definition of v_{ε} , we have $v_{\varepsilon} \not\leq u$ in Ω , a contradiction.

It remains to prove (4.12). For this, we argue by contradiction. Suppose that there is a sequence $\varepsilon_k \searrow 0$ as $k \to \infty$ such that for each $k \in \mathbb{N}$, v_{ε_k} is not a viscosity subsolution of (L) in Ω . This ensures that there are sequences $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_0(\Omega)$ and $\{\xi_k\}_{k \in \mathbb{N}} \subset \Omega$ such that for each $k \in \mathbb{N}$, $v_{\varepsilon_k} - \psi_k$ has a maximum at ξ_k and

$$\psi_{k,t}(\xi_k) > \sigma^+(v_{\varepsilon_k}, D\psi_k(\xi_k), \xi_k)G(D\psi_k(\xi_k), D^2\psi_k(\xi_k))$$
 for all $k \in \mathbf{N}$.

If $v_{\varepsilon_k}(\xi_k) = u(\xi_k)$, then $u - \psi_k$ attains its maximum at ξ_k , which contradicts the above inequality. Thus, we must have $v_{\varepsilon_k}(\xi_k) = \varphi(\xi_k) + \varepsilon_k > u(\xi_k)$, which implies that $\varphi - \psi_k$ attains its maximum over V at ξ_k . Since $\xi_k \in V$ for all $k \in \mathbb{N}$, we may assume that $\xi_k \to \eta$ for some $\eta \in V$ as $k \to \infty$. We then have $\varphi(\eta) \geq u_*(\eta)$, which implies that $\eta = \xi$. Thus we may assume that $\xi_k \in \operatorname{int} V$ for all $k \in \mathbb{N}$. Therefore, we get

$$D\varphi(\xi_k) = D\psi_k(\xi_k)$$
 and $D^2\varphi(\xi_k) \le D^2\psi_k(\xi_k)$ for $k \in \mathbf{N}$,

and hence,

$$\varphi_t(\xi_k) > \sigma^+(v_{\varepsilon_k}, D\varphi(\xi_k), \xi_k)G(D\varphi(\xi_k), D^2\varphi(\xi_k))$$
 for all $k \in \mathbf{N}$.

Since
$$(v_{\varepsilon_k})_*(\xi_k) = v_{\varepsilon_k}(\xi_k)$$
 and $(v_{\varepsilon_k})_* \leq v_{\varepsilon_k}$ in Ω , we get
$$\varphi_t(\xi_k) > \sigma^-((v_{\varepsilon_k})_*, D\varphi(\xi_k), \xi_k)G(D\varphi(\xi_k), D^2\varphi(\xi_k)) \quad \text{for all } k \in \mathbf{N}.$$

Applying Lemma 4.5, we obtain in the limit as $k \to \infty$

$$\varphi_t(\hat{\xi}) \ge \sigma^-(u_*, D\varphi(\hat{\xi}), \hat{\xi})G(D\varphi(\hat{\xi}), D^2\varphi(\hat{\xi})).$$

This contradicts (4.11), which shows that (4.12) is valid.

We have the following proposition, the proof of which can be done in a way parallel to the proof of Proposition 2.4, (c), but with Lemma 4.2 in place of Lemma 4.5. We leave the reader to check the details of the proof.

Proposition 4.7. Let $\Omega = \mathbb{R}^N \times (0,T)$, with $T \in (0,\infty]$. Let $f_1, f_2 \in C(\Omega)$ be a viscosity subsolution and a viscosity supersolution of (L) in Ω , respectively. Assume that $f_1 \leq f_2$ in Ω . Set

$$u(\xi) = \inf\{v(\xi) \mid v \text{ is a viscosity supersolution of } (L) \text{ in } \Omega, f_1 \leq v \leq f_2 \text{ in } \Omega\}.$$

Then u^* is a viscosity subsolution of (L) in Ω .

5. Relabeling of level sets

In this section we prove Proposition 2.5, which allows us to relabel level sets by composing solutions of (L) with non-decreasing functions.

Proof of Proposition 2.5. Let $\Omega = \mathbf{R}^N \times (0,T)$, where $T \in (0,\infty]$, $\theta \in \mathrm{USC}(\mathbf{R})$ a non-decreasing function, and $u \in \mathrm{USC}(\Omega)$ a viscosity subsolution of (L) in Ω . Set $v = \theta \circ u$.

If θ is smooth and $\inf_{\mathbf{R}} \theta' > 0$, then, setting $\psi = \theta^{-1}$, we may compute formally

$$v_{t} = \theta'(u)u_{t} \leq \theta'(u)\sigma^{+}(u, Du, z, t)G(Du, D^{2}u)$$

$$= \theta'(u)\sigma^{+}(u, \psi'(v)Dv, z, t)G(\psi'(v)Dv, \psi'(v)D^{2}v + \psi''(v)Dv \otimes Dv)$$

$$= \theta'(u)\psi'(v)\sigma^{+}(v, Dv, z, t)G(Dv, D^{2}v) = \sigma^{+}(v, Dv, z, t)G(Dv, D^{2}v),$$

which can be justified by using test functions arguments.

The general case can be treated by approximations of θ . We choose a sequence $\{\theta_k\}_{k\in\mathbb{N}}\subset C^2(\mathbb{R})$ such that

$$\inf_{\mathbf{R}} \theta_k' > 0 \quad \text{ and, as } k \to \infty, \quad \theta_k(r) \searrow \theta(r) \quad \text{ for all } r \in \mathbf{R}.$$

As explained above, the functions $\theta_k \circ u$ are viscosity subsolutions of (L) in Ω and as $k \to \infty$,

$$\theta_k \circ u(z,t) \searrow \theta \circ u(z,t)$$
 for all $(z,t) \in \Omega$.

By Proposition 4.1, we may conclude that $\theta \circ u$ is a viscosity subsolution of (L) in Ω . The proof in the supersolutions case can be treated in the same way as above.

6. Existence of viscosity solutions

In this section we give a proof of Theorem 2.7.

Proof of Theorem 2.7. In view of Propositions 2.5 and 4.1 we may assume (see the outline of proof of Theorem 2.2 for the details) that h is bounded in \mathbf{R}^N and satisfies condition (A). As a result of this assumption, we see that $h \in \mathrm{BUC}(\mathbf{R}^N)$.

Recall that the function

$$w(z,t) := |z|^N + Nt$$
 for $(z,t) \in \mathbf{R}^N \times [0,\infty)$

is a viscosity supersolution of (L). For each $\varepsilon \in (0,1)$ there is a constant $B_{\varepsilon} > 0$ such that

$$|h(z) - h(\zeta)| \le \varepsilon + B_{\varepsilon}|z - \zeta|^N$$
 for all $z, \zeta \in \mathbf{R}^N$.

Define the function $f \in \mathrm{USC}(\mathbf{R}^N \times [0,\infty))$ by

$$f(z,t) = \inf\{h(\zeta) + \varepsilon + B_{\varepsilon}w(z-\zeta,t) \mid \zeta \in \mathbf{R}^N, \ \varepsilon \in (0,1)\}.$$

Observe that for each $\delta \in (0,1)$, the function f is bounded on $\mathbf{R}^n \times [0,\delta^{-1}]$ and hence for some B > 0 and for all $(z,t) \in \mathbf{R}^N \times [\delta,\delta^{-1}]$,

$$f(z,t) = \inf\{h(\zeta) + \varepsilon + B_{\varepsilon}w(z-\zeta,t) \mid \zeta \in \mathbf{R}^N, \ \varepsilon \in (0,1), \ B_{\varepsilon} \leq B\}.$$

Thus we see that $f \in \mathrm{BUC}(\mathbf{R}^N \times [\delta, \delta^{-1}])$ for all $\delta \in (0, 1)$. Furthermore, since $h(z) \leq f(z, t) \leq h(z) + \varepsilon + NB_{\varepsilon}t$ for all $(z, t) \in \mathbf{R}^N \times [0, \infty)$ and $\varepsilon \in (0, 1)$, and $h \in \mathrm{BUC}(\mathbf{R}^N)$, we see that $f \in \mathrm{BUC}(\mathbf{R}^N \times [0, T])$ for all $T \in (0, \infty)$.

Since h satisfies condition (A), so does the function f.

Note that v(z,t) := h(z) is a viscosity subsolution of (L).

Now, set

$$u(z,t) = \sup\{v(z,t) \mid v \text{ is a viscosity subsolution of (L)},$$

 $h(\zeta) \leq v(\zeta,s) \leq f(\zeta,s) \text{ for } (\zeta,s) \in \mathbf{R}^N \times [0,\infty)\}$
for $(z,t) \in \mathbf{R}^N \times [0,\infty).$

Then $u \in \mathrm{USC}(\mathbf{R}^N \times [0,\infty))$ and $h(z) \leq u_*(z,t) \leq u(z,t) \leq f(z,t)$ for $(z,t) \in \mathbf{R}^N \times [0,\infty)$. By Proposition 2.4, u is a viscosity subsolution of (L) and u_* is a viscosity supersolution of (L). By Theorem 2.6, we see that $u \leq u_*$ in $\mathbf{R}^N \times (0,\infty)$, from which follows that $u \in C(\mathbf{R}^N \times [0,\infty))$. It is now clear that u has all the required properties.

Appendix

We present a lemma which explains the relation between the function G in (L) and the Gauss curvature of level sets.

Lemma A. $\varphi \in C^2(\mathbf{R}^N)$, $\varphi(0) = 0$, $D\varphi(0) \neq 0$, $G(D\varphi(0), D^2\varphi(0)) > 0$. Then there is an $\varepsilon > 0$ such that the set $M = \{z \in \mathbf{R}^N \mid |z| < \varepsilon, \varphi(z) = 0\}$ is a C^2 hypersurface, and for each $z \in M$ all the principal curvatures $\kappa_1, ..., \kappa_n$ at z of the hypersurface with respect to the normal $-D\varphi(z)$ are positive and the Gauss curvature $\kappa_1 \cdots \kappa_n$ at z is given by

$$|D\varphi(z)|^{-1}G(D\varphi(z), D^2\varphi(z)).$$

Outline of proof. By the implicit function theorem, if $\varepsilon > 0$ is a small constant, then $M = \{z \in \mathbf{R}^N \mid |z| < \varepsilon, \varphi(z) = 0\}$ is a C^2 hypersurface.

Fix $\hat{z} \in M$. We choose an orthonormal basis $\{f_i\}_{i=1}^N$ such that $f_N = -D\varphi(\hat{z})/|D\varphi(\hat{z})|$. We may assume that M can be represented by the graph of a function $g: U \to \mathbf{R}$ in the new coordinate system, where U is a neighborhood of 0 in \mathbf{R}^n . That is, if $\xi = (\xi_1, ..., \xi_N) \in \mathbf{R}^N$ and $z = \xi_1 f_1 + \cdots + \xi_N f_N$, then

$$z \in M$$
 if and only if $(\xi_1, ..., \xi_n) \in U$, $\xi_N = g(\xi_1, ..., \xi_n)$.

By translation, we may assume that $\hat{z} = 0$. With this notation, the principal curvatures are the eigenvalues of $D^2g(0)$ by definition.

Differentiating

$$\varphi(\xi_1 f_1 + \dots + \xi_n f_n + g(\xi_1, \dots, \xi_n) f_N) = 0$$

twice, we deduce that

$$D^2g(0) = \frac{1}{|D\varphi(0)|}E^TD^2\varphi(0)E,$$

where $E = (f_1, ..., f_n)$ is understood as an $N \times n$ matrix and E^T denotes the transposed matrix of E.

Set $A = D^2 \varphi(0)$ and $F = (E, f_N)$. We calculate that

$$F\begin{pmatrix} E^T A E & 0 \\ 0 & 1 \end{pmatrix} F^T = (I - f_N f_N^T) A (I - f_N f_N^T) + f_N f_N^T.$$

From this we see that

$$A>0$$
 if and only if $(I-f_Nf_N^T)A(I-f_Nf_N^T)+f_Nf_N^T>0$,

and that

$$\det E^{T} A E = \det \{ (I - f_{N} f_{N}^{T}) A (I - f_{N} f_{N}^{T}) + f_{N} f_{N}^{T} \}.$$

Therefore, if $G(D\varphi(0), D^2\varphi(0)) > 0$, then A > 0, which means that the principal curvatures $\kappa_1, ..., \kappa_n$ are all positive and moreover

$$\prod_{i=1}^{n} \kappa_i = |D\varphi(0)| G(D\varphi(0), D^2\varphi(0)),$$

which completes the proof.

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References

- [A] B. Andrews, Gauss curvature flow: the fate of the rolling stones, Invent. math., 138 (1999), 151–161.
- [AEI] D. Adalsteinsson, L. C. Evans, and H. Ishii, The level set method for etching and deposition. Math. Models Methods Appl. Sci., 7 (1997), no. 8, 1153–1186.
- [C] B. Chow, Deforming convex hypersurfaces by the n th root of the Gaussian curvature, J. Diff. Geometry, **22** (1985), 117 138.
- [CEI] D. Chopp, L. C. Evans, and H. Ishii, Waiting time effects for Gauss curvature flows, preprint.
- [CGG] Y.-G. Chen, Y. Giga, and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Diff. Geometry, **33** (1991), 749 786.
- [CGG] M. G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1–67.

- [EL] I. Ekeland and G.Lebourg, Generic Frechet-differentiability and perturbed optimization problems in Banach spaces, Trans. Amer. Math. Soc. **224** (1976), no. 2, 193–216.
- [ES] L. C. Evans and J. Spruck, Motion of level sets by mean curvature I, J. Diff. Geometry, **33** (1991), 635 681.
- [F] W. Firey, Shapes of worn stones, Mathematika, 21 (1974), 1-11.
- [GG] M.-H. Giga and Y. Giga, Generalized motion by nonlocal curvature in the plane, Arch. Rational Mech. Anal., **159** (2001), 295 333.
- [H] R. Hamilton, Worn stones with flat sides. A tribute to Ilya Bakelman (College Station, TX, 1993), 69–78, Discourses Math. Appl., 3, Texas A & M Univ., College Station, TX, 1994.
- [I] H. Ishii, Gauss curvature flow and its approximation. Free boundary problems: theory and applications, II (Chiba, 1999), 198–206, GAKUTO Internat. Ser. Math. Sci. Appl., 14, Gakkōtosho, Tokyo, 2000.
- [IM1] H. Ishii and T. Mikami, A two dimensional random crystalline algorithm for Gauss curvature flow, Adv. in Appl. Probab., 34, 1-14, 2002.
- [IM2] H. Ishii and T. Mikami, A mathematical model of the wearing process of a nonconvex stone, SIAM J. Math.Anal., **33** (2001), no. 4, 860-876.
- [IM3] H. Ishii and T. Mikami, Motion of a graph by R-curvature, preprint.
- [IS] H. Ishii and P. E. Souganidis, Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor, Tôhoku Math. J., 47 (1995), 227 – 250.
- [OS] S. Osher and J. Sethian, Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations, J. Comput. Phys., **79** (1988), no. 1, 12–49.
- [T] K. Tso, Deforming a hypersurface by its Gauss-Kronecker curvature, Comm. Pure Appl. Math., **38** (1985), 867 882.