**A note on a paper by B-N**

Periodic problem:

$$\text{(PP)} \qquad u_t + H(x, Du) = f(t) \quad \text{for } (x,t) \in \mathbf{R}^n \times \mathbf{R},$$

where $f \in C(\mathbf{R})$ is periodic, i.e., $f(t+T) = f(t)$ for all $t \in \mathbf{R}$ and for some $T > 0$.

Stationary problem:

$$\text{(SP)} \qquad H(x, Dw) = \langle f \rangle \quad \text{for } x \in \mathbf{R}^n,$$

where $\langle f \rangle = (1/T) \int_0^T f(s)ds$.

Here we consider solutions of these equations in the class of bounded functions.

We introduce the function $F \in C^1(\mathbf{R})$ by setting

$$F(t) = \int_0^t (f(s) - \langle f \rangle)ds.$$

Note that $F$ is again $T$-periodic. For function $u$ on $\mathbf{R}^n \times \mathbf{R}$ we define $v$ by

$$v(x,t) = u(x,t) - F(t).$$

**Proposition 1.** *Under the above notation, $u$ is a solution of* (PP) *if and only if $v$ is a solution of*

$$\text{(PP}'\text{)} \qquad v_t + H(x, Dv) = \langle f \rangle \quad \textit{for } (x,t) \in \mathbf{R}^n \times \mathbf{R}.$$

**Proof.** We compute that if $u$ is a solution of (PP), then

$$v_t = u_t - F' = -H(x, Du) + f(t) - f(t) + \langle f \rangle = -H(x, Dv) + \langle f \rangle$$

and if $v$ is a solution of (PP$'$), then

$$u_t = v_t + F' = -H(x, Dv) + \langle f \rangle + f(t) - \langle f \rangle = -H(x, Du) + f(t). \qquad \square$$

**Proposition 2.** *If* (PP$'$) *has a bounded solution, then there exists a solution $w$ of* (SP).

**Proof.** Let $v$ be a bounded solution of (PP$'$). For $r \in \mathbf{R}$ define $v^r$ by $v^r(x,t) = v(x, t+r)$. Noting that $v^r$ is a solution of (PP$'$), we see that

$$v^+(x) := \sup_{t \in \mathbf{R}} v(x,t) \equiv \sup_{r \in \mathbf{R}} v^r(x,t)$$

is a subsolution of (PP′), which guarantees that $v^+$ is a subsolution of (SP). Simliarly, the function

$$v^-(x) = \inf_{t \in \mathbf{R}} v(x,t)$$

is a supersolution of (SP). Let $C > 0$ be a constant such that $\sup u \leq \inf u + C$. Then, we have $v^+ \leq v^- + C$ in $\mathbf{R}^n$. By the Perron method, we find a solution $w$ of (SP) such that $v^- + C \leq w \leq v^+$ in $\mathbf{R}^n \times \mathbf{R}$. □

**Remark.** If $w$ is a solution of (SP), then the function $u(x,t) := w(x) + F(t)$ is a $T$-periodic solution of (PP).