Let $\mathcal{P}$ ($\approx -\Delta$) denote one of Pucci operators. Consider the inequality

$$\mathcal{P}(D^2u) \le \mu|D| + f(x) \quad \text{in } \Omega, \qquad u \le 0 \quad \text{on } \partial\Omega,$$

where μ and f are nonnegative function belonging to $L^n(\Omega)$.

Proposition. *There are constants $A > 0$ and $B > 0$ such that*

$$\sup_{\Omega} u \le A\|f\|_n \exp\left(B\|\mu\|_n^n\right).$$

Let $\Gamma(u)$ be the upper contact set of u. Let $h := \sup_{\Omega} u^+$. Note that $B(0, ch) \subset Du(\Omega)$ for some constant $c > 0$ independent of u. We have

$$\int_{B(0,ch)} \phi(p)\,\mathrm{d}p \le \int_{\Gamma(\Omega)} \phi(p)\,\mathrm{d}p \le \int_{\Gamma(u)} \phi(Du(x))|\det D^2u(x)|\,\mathrm{d}x \tag{1}$$

for any nonnegative function $\phi \in C(\mathbf{R}^n)$. We choose

$$\phi(p) = \frac{1}{|p|^n + \|f\|_n^n},$$

to obtain

$$\int_{B(0,ch)} \frac{dp}{|p|^n + \|f\|_n^n} \le \int_{\Gamma(u)} \frac{|\det D^2u(x)|}{|Du(x)|^n + \|f\|_n^n}\,\mathrm{d}x. \tag{2}$$

Compute that

$$\int_{B(0,ch)} \frac{dp}{|p|^n + \|f\|_n^n} = C_n \int_0^{ch} \frac{r^{n-1}\,\mathrm{d}r}{r^n + \|f\|_n^n} = C_n \left[\frac{1}{n}\log(r^n + \|f\|_n^n)\right]_0^{ch}$$
$$= \frac{C_n}{n} \log \frac{(ch)^n + \|f\|_n^n}{\|f\|_n^n},$$

where C_n is a positive constant=measure of the unit sphere, depending only on n, and

$$\int_{\Gamma(u)} \frac{|\det D^2u(x)|}{|Du(x)|^n + \|f\|_n^n}\,\mathrm{d}x \le \int_{\Gamma(u)} \frac{C_0\mathcal{P}(D^2u)^n}{|Du(x)|^n + \|f\|_n^n}\,\mathrm{d}x$$
$$\le C_0 \int_{\Gamma(u)} \frac{(\mu|Du| + f)^n}{|Du(x)|^n + \|f\|_n^n}\,\mathrm{d}x$$
$$\le 2^{n-1}C_0 \left(\int_{\Gamma(u)} \frac{\mu^n|Du|^n}{|Du(x)|^n + \|f\|_n^n}\,\mathrm{d}x + \int_{\Gamma(u)} \frac{f^n}{|Du(x)|^n + \|f\|_n^n}\,\mathrm{d}x\right)$$
$$\le 2^{n-1}C_0 \left(\int_{\Gamma(u)} \mu^n\,\mathrm{d}x + \int_{\Gamma(u)} \frac{f^n}{\|f\|_n^n}\,\mathrm{d}x\right) \le 2^{n-1}C_0\left(\|\mu\|_n^n + 1\right),$$

where C_0 is a constant depending only on n and the ellipticity constants. Here we have used the fact that $D^2u \le 0$ on $\Gamma(u)$ and the inequality

$$|\det D^2 u|^{\frac{1}{n}} \le \frac{1}{n}(-\Delta u) \quad \text{on } \Gamma(u).$$

Now, from (2), we get

$$\frac{C_n}{n} \log \frac{(ch)^n + \|f\|_n^n}{\|f\|_n^n} \le 2^{n-1} C_0 \left(\|\mu\|_n^n + 1\right),$$

and hence

$$(ch)^n + \|f\|_n^n \le \exp\left(\frac{n 2^{n-1} C_0}{C_n} \left(\|\mu\|_n^n + 1\right)\right).$$

Thus we have

$$h \le c^{-1} \|f\|_n \exp\left(\frac{2^{n-1} C_0}{C_n} (\|\mu\|_n^n + 1)\right) = A \|f\|_n \exp\left(B \|\mu\|_n^n\right)$$

for some positive constants A, B.