ヤコビアンの計算に関するメモ

つぎの変換を考える．

$$\begin{cases} y_1 = \dfrac{2x_1}{x_1^2 + (x_2+1)^2} \\ y_2 = \dfrac{|x|^2 - 1}{x_1^2 + (x_2+1)^2} \end{cases}$$

$y_{ij} = \partial y_i / \partial x_j \ (i, j = 1, 2)$ とおく．

$$J := \det \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = y_{11} y_{22} - y_{12} y_{21}$$

を計算したい．以下の計算では，$A = x_1^2 + (x_2+1)^2$ とおく．

まず,

$$\begin{aligned} y_{11} &= \frac{2}{A} - \frac{2x_1 2x_1}{A^2} = \frac{2}{A^2}(A - 4x_1^2) = \frac{2}{A^2}\left(x_1^2 + (x_2+1)^2 - 4x_1^2\right) \\ &= \frac{2}{A^2}\left((x_2+1)^2 - x_1^2\right). \end{aligned}$$

つぎに,

$$\begin{aligned} y_{22} &= \frac{2x_2}{A} - \frac{2(|x|^2-1)(x_2+1)}{A^2} = \frac{2}{A^2}\left(x_2 A - (|x|^2-1)(x_2+1)\right) \\ &= \frac{2}{A^2}\left(x_2 x_1^2 + x_2(x_2+1)^2 - (|x|^2-1)(x_2+1)\right) \\ &= \frac{2}{A^2}\left(x_1^2 x_2 + x_2^3 + 2x_2^2 + x_2 - x_1^2 x_2 - x_2^3 + x_2 - x_1^2 - x_2^2 + 1\right) \\ &= \frac{2}{A^2}\left(x_2^2 + 2x_2 - x_1^2 + 1\right) = \frac{2}{A^2}\left((x_2+1)^2 - x_1^2\right). \end{aligned}$$

更に,

$$\begin{aligned} y_{12} &= -\frac{2x_1\, 2(x_2+1)}{A^2} = -\frac{4x_1(x_2+1)}{A^2}, \\ y_{21} &= \frac{2x_1}{A} - \frac{(|x|^2-1)2x_1}{A^2} = \frac{2}{A^2}\left(x_1 A - x_1(|x|^2-1)\right) \\ &= \frac{2}{A^2}\left(x_1^3 + x_1(x_2+1)^2 - x_1(|x|^2-1)\right) = \frac{2}{A^2}(2x_1x_2 + x_1 + x_1) \\ &= \frac{4x_1(x_2+1)}{A^2}. \end{aligned}$$

よって,

$$
\begin{aligned}
J &= \frac{4}{A^4}\left((x_2+1)^2 - x_1^2\right)^2 + \frac{16x_1^2(x_2+1)^2}{A^4} \\
&= \frac{4}{A^4}\left((x_2+1)^4 - 2(x_2+1)^2x_1^2 + x_1^4 + 4(x_2+1)^2x_1^2\right) \\
&= \frac{4}{A^2}\left((x_2+1)^4 + 2(x_2+1)^2x_1^2 + x_1^4\right) = \frac{4}{A^4}\left((x_2+1)^2 + x_1^2\right)^2 \\
&= \frac{4}{A^2}.
\end{aligned}
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すなわち,

$$
J = \left(\frac{2}{x_1^2 + (x_2+1)^2}\right)^2.
$$