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Relaxation of Hamilton-Jacobi equations

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Abstract

We study the relaxation of Hamilton-Jacobi equations. The relaxation in our terminology is the following phenomenon: the pointwise supremum over a certain collection of subsolutions, in the almost everywhere sense, of a Hamilton-Jacobi equation yields a viscosity solution of the "convexified" Hamilton-Jacobi equation. This phenomenon has recently been observed in [13] in eikonal equations. We show in this paper that this relaxation is a common phenomenon for a wide range of Hamilton-Jacobi equations.

1. Introduction

In this paper we study the Hamilton-Jacobi equation

$$H(x, u(x), Du(x)) = 0 \quad \text{ in } \Omega, \tag{1.1}$$

where Ω is an open subset of $\mathbf{R}^n$, H is a given real-valued function on $\Omega \times \mathbf{R} \times \mathbf{R}^n$, and u is a real-valued unknown function on Ω, and we are interested in an observation concerning (1.1) in [13] and its generalization, which we call the relaxation of Hamilton-Jacobi equations.

This observation in [13] is stated as follows. Let $H \in C(\mathbf{R}^n)$ be a function satisfying

$$\begin{cases} H(p) > 0 \quad \text{ for } p \in \mathbf{R}^n \setminus \{0\}, \\ H(\lambda p) = \lambda H(p) \quad \text{ for } (\lambda, p) \in [0, \infty) \times \mathbf{R}^n. \end{cases} \tag{1.2}$$

Let $\widehat{H}$ denote the convex envelope of the function H. Consider the eikonal equation

$$\widehat{H}(Du(x)) = 1 \quad \text{ in } \Omega \tag{1.3}$$

together with the Dirichlet boundary condition

$$u(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{1.4}$$

where Ω is assumed to be bounded. Let $u \in C(\overline{\Omega})$ be a (unique) viscosity solution of (1.3) and (1.4). Then we have:

$$u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ H(Dv(y)) \le 1 \ \text{ a.e. } y \in \Omega,\ v = 0 \ \text{ on } \partial\Omega\} \tag{1.5}$$

for all $x \in \Omega$, where for any subset W of $\mathbf{R}^n$, $\mathrm{Lip}(W)$ denotes the space of bounded and Lipschitz continuous functions on W.

It should be remarked that (1.5) is equivalent to the following formula

$$\begin{aligned} u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \le u \ \text{ in } \Omega,& \\ H(Dv(y)) \le 1 \ \text{ a.e. } \Omega\}& \quad \text{for } x \in \Omega. \end{aligned} \tag{1.6}$$

Indeed, it is clear that (1.5) implies that (1.6) holds. On the other hand, since $\widehat{H} \le H$ and $\widehat{H}$ is convex, if $v \in \mathrm{Lip}(\overline{\Omega})$, $H(Dv(y)) \le 1$ a.e. $y \in \Omega$, and $v = 0$ on $\partial\Omega$, then v is a viscosity subsolution of (1.3) and (1.4). By comparison, we see (e.g., [12]) that $v \le u$ in Ω, and hence that (1.6) implies (1.5).

It is well-known that if we replace, in the formula given by the right hand side of (1.5), the condition

$$H(Dv(y)) \le 1 \quad \text{a.e. } y \in \Omega$$

by the condition that v is a viscosity subsolution of

$$\begin{aligned} \widehat{H}(Dv(y)) = 1 \quad \text{in } \Omega \\ \big(\text{respectively,} \quad H(Dv(y)) = 1 \quad \text{in } \Omega\big), \end{aligned}$$

then the resulting formula gives a (unique) viscosity solution u of (1.3) and (1.4) (respectively, a (unique) viscosity solution of $H(Du(x)) = 1$ in Ω together with the boundary condition (1.4)).

We note here (see also the example presented just after the proof of Theorem 2.2 in [13]) that in general formula (1.6) does not give a subsolution of

$$H(Du(x)) = 1 \quad \text{a.e. } x \in \Omega. \tag{1.7}$$

To see this, we consider the case when $n = 2$, $H(p, q) = (|p|^{1/2} + |q|^{1/2})^2$, and $\Omega = \{(x, y) \in \mathbf{R}^2 \mid |x| + |y| < 1\}$. It is immediate to see that $\widehat{H}(p, q) = |p| + |q|$. It is not difficult to check that the function

$$u(x, y) = \frac{1 - |x| - |y|}{2}$$

is a viscosity solution of (1.3) in Ω and satisfies (1.4). For any $(x, y) \in \Omega$, if $x > 0$ and $y > 0$ for instance, then we have

$$Du(x, y) = (-\frac{1}{2}, -\frac{1}{2}),$$

and, in particular,

$$H(Du(x,y)) = 2.$$

Similarly we have

$$H(Du(x,y)) = 2$$

if $x \neq 0$ and $y \neq 0$. Hence we see that $H(Du(x,y)) = 2$ a.e. for $(x,y) \in \Omega$ and hence that u is not a subsolution of (1.7).

What we call the *relaxation* of Hamilton-Jacobi equations is the phenomenon that a viscosity solution of the eikonal equation with convexified Hamiltonian appears in the process of taking the pointwise supremum over a certain collection of subsolutions, in the almost everywhere sense, of the original eikonal equation, or a phenomenon similar to this for (1.1).

Since the viscosity solution $u \in C(\overline{\Omega})$ of (1.3)–(1.4) is Lipschitz continuous and $\widehat{H}$ is convex, u has the properties: (i) $u \in \mathrm{Lip}(\overline{\Omega})$; (ii) $\widehat{H}(Du(x)) \leq 1$ a.e. $x \in \Omega$; and (iii) $u = 0$ on $\partial\Omega$. Therefore the right hand side of (1.5), with H replaced by $\widehat{H}$, is attained by $u(x)$. Thus another way to state the relaxation of the eikonal equation, $H(Du) = 1$, is the following identity:

$$\begin{aligned} &\max\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ \widehat{H}(Dv(y)) \leq 1 \ \text{ a.e. } y \in \Omega,\ v = 0 \ \text{ on } \partial\Omega\} \\ &= \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ H(Dv(y)) \leq 1 \ \text{ a.e. } y \in \Omega,\ v = 0 \ \text{ on } \partial\Omega\}. \end{aligned} \tag{1.8}$$

That is, through the process of taking the pointwise supremum of all subsolutions of the eikonal equation $H(Du) = 1$ in the almost everywhere sense with the Dirichlet condition $u = 0$, one obtains the pointwise maximum of all subsolutions of the convexified eikonal equation $\widehat{H}(Du) = 1$ in the almost everywhere sense with the same Dirichlet condition. However, this view point is narrower than the previous one when more general situations are considered. In such a general situation, the supremum which corresponds to the left hand side of (1.8) may not give a Lipschitz continuous function on $\overline{\Omega}$ and hence the maximum may not be attained, but it may still give a continuous viscosity solution on $\overline{\Omega}$ of the associated Hamilton-Jacobi equation with convex Hamiltonian.

As is already noted, our purpose here is to show that the relaxation takes place in a wide class of Hamilton-Jacobi equations.

Some historical remarks here concerning the well-posedness for Hamilton-Jacobi equations are the developments of the theory of viscosity solutions of Hamilton-Jacobi equations ([8, 16, 6, 7]) and of general existence theories for solutions in the almost everywhere sense of Hamilton-Jacobi equations ([9, 17]). Also, an important remark may be the connection of the relaxation of eikonal equations to L^∞ optimization problems ([13, 3, 15, 11, 2, 4, 5]). Furthermore, it should be noted that relaxations in standard nonconvex variational problems has been studied for a long time ([10]). It is worth mentioning that in [18] a connection between a nonconvex eikonal equation and its convexified Hamilton-Jacobi equation is discussed in a different view point.

In Section 2 we formulate our results on the relaxation of Hamilton-Jacobi equations on bounded domains. A result in the case of the Cauchy problem for Hamilton-Jacobi equations is stated and proved in section 7 as a typical example of results for unbounded domains. We prepare some basic lemmas used in our proofs in section 3. Sections 4 and 5 are devoted to the proof of our main results. The formulation in section 2 is slightly abstract for sake of wider applicability, and the results in sections 2 are applied to some examples of Hamilton-Jacobi equations. Appendix collects a lemma on Lipschitz domains and an example of Hamilton-Jacobi equation relevant to our assumptions.

2. Main results

In this section we formulate our relaxation theorems.

We assume throughout this section that the set Ω is bounded.

In order to state our results, it is appropriate to use the set theoretic notation rather than the standard PDE notation.

First of all we explain how to transfer from the PDE notation to the set theoretic notation. For subsolutions of the Hamilton-Jacobi equation

$$H(x, u(x), Du(x)) = 0 \quad \text{in } \Omega, \tag{2.1}$$

where H is assumed to be continuous, the set

$$Z(x, r) = \{p \in \mathbf{R}^n \mid H(x, r, p) \leq 0\}, \tag{2.2}$$

where $(x, r) \in \overline{\Omega} \times \mathbf{R}$, is crucial, and the fact that u is a (classical) subsolution of (2.1) can be stated as

$$Du(x) \in Z(x, u(x)) \quad \text{for all } x \in \Omega.$$

Note as well that $u \in C(\Omega)$ is a viscosity subsolution of (2.1) if and only if

$$D^+u(x) \subset Z(x, u(x)) \quad \text{for } x \in \Omega.$$

The formulation of our first theorem is motivated by standard uniqueness theorems (see, e.g. [8, 6]) in viscosity solution theory.

Assume that $H \in C(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n)$ satisfies the following two conditions.

$$\left\{ \begin{array}{l} \text{For each } M > 0 \text{ there is a constant } \lambda_M > 0 \text{ such that for} \\ \text{each } (x, p) \in \overline{\Omega} \times B(0, M) \text{ the function} : r \mapsto H(x, r, p) - \lambda_M r \\ \text{is non-decreasing on } [-M, M]. \end{array} \right. \tag{2.3}$$

$$\left\{ \begin{array}{l} \text{For each } M > 0 \text{ there is a modulus } \omega_M \text{ for which} \\ \qquad |H(x, r, p) - H(y, r, q)| \leq \omega_M(|x - y|(|p| + 1) + |p - q|) \\ \text{for all } x, y \in \overline{\Omega},\ p, q \in \mathbf{R}^n,\ r \in [-M, M]. \end{array} \right. \tag{2.4}$$

It follows from (2.3) that for any $(x,p) \in \overline{\Omega} \times \mathbf{R}^n$, the function : $r \mapsto H(x,r,p)$ is non-decreasing on $\mathbf{R}$, which shows that for all $x \in \mathbf{R}^n$ and $r,s \in \mathbf{R}$,

$$Z(x,r) \subset Z(x,s) \quad \text{if } \ r \geq s. \tag{2.5}$$

Observe from (2.4) that for each $M > 0$, the collection of functions : $p \mapsto H(x,r,p)$, with $(x,r) \in \overline{\Omega} \times [-M,M]$, is equi-continuous on $\mathbf{R}^n$.

Let $M, R > 0$. For any $x, y \in \overline{\Omega}$, $r, s \in [-M-1, M+1]$, $\varepsilon, \gamma > 0$, $p \in B(0,R)$, and $q \in B(p,\varepsilon)$, if $|x-y| \leq \gamma$ and $r \geq s$, then we have

$$H(y,s,q) \leq H(x,r,p) + \omega_M((R+1)\gamma + \varepsilon) - \lambda_{M+1}(r-s). \tag{2.6}$$

Set

$$\sigma_M(t) = \lambda_{M+1}^{-1}\omega_M(t) \quad \text{ for all } t \geq 0,$$

and choose $\varepsilon_M > 0$ so that $\sigma_M(\varepsilon_M) \leq 1$. Then, from (2.6), for all $M, R > 0$, $\varepsilon, \gamma > 0$ satisfying $(R+1)\gamma + \varepsilon \leq \varepsilon_M$, $x, y \in \overline{\Omega}$, $(r,p) \in [-M,M] \times B(0,R)$, and $q \in B(p,\varepsilon)$, if $|x-y| \leq \gamma$, then we have

$$H(y, r - \sigma_M((R+1)\gamma + \varepsilon), p+q) \leq H(x,r,p),$$

which yields that for any $M, R > 0$, $\varepsilon, \gamma > 0$ satisfying $(R+1)\gamma + \varepsilon \leq \varepsilon_M$, $x, y \in \overline{\Omega}$, and $r \in [-M,M]$, if $|x-y| \leq \gamma$, then have

$$Z(x,r) \cap B(0,R) + B(0,\varepsilon) \subset Z(y, r - \sigma_M((R+1)\gamma + \varepsilon)). \tag{2.7}$$

Now, we formulate our first theorem on relaxation.

For each $(x,r) \in \overline{\Omega} \times \mathbf{R}$ let $Z(x,r)$ be a closed subset of $\mathbf{R}^n$, which may be an empty set.

The theorem concerns also the closed convex hull of the sets $Z(x,r)$. Thus we set

$$K(x,r) = \overline{\mathrm{co}}\, Z(x,r) \quad \text{ for } (x,r) \in \overline{\Omega} \times \mathbf{R}^n. \tag{2.8}$$

For $R > 0$ and $(x,r) \in \overline{\Omega} \times \mathbf{R}$ we define

$$Z_R(x,r) = Z(x,r) \cap B(0,R), \quad K_R(x,r) = K(x,r) \cap B(0,R).$$

Motivated by the above observation, we assume:

$$\begin{cases} \text{for any } r, s \in \mathbf{R} \text{ and } x \in \overline{\Omega}, \text{ if } r \geq s, \\ \qquad\qquad\qquad Z(x,r) \subset Z(x,s), \end{cases} \tag{2.9}$$

$$\begin{cases} \text{for each } M > 0 \text{ there are a modulus } \sigma_M \text{ and a constant} \\ \varepsilon_M > 0 \text{ such that for any } R > 0,\ \varepsilon, \gamma > 0,\ x, y \in \overline{\Omega}, \text{ and} \\ r \in [-M,M], \text{ if } (R+1)\gamma + \varepsilon \leq \varepsilon_M \text{ and } |x-y| \leq \gamma, \text{ then} \\ \text{have} \\ \qquad Z_R(x,r) + B(0,\varepsilon) \subset Z(y, r - \sigma_M((R+1)\gamma + \varepsilon)). \end{cases} \tag{2.10}$$

Moreover, we assume (2.11) and (2.12) below.

$$\begin{cases} \text{For each } M > 0 \text{ there are a modulus } \sigma_M \text{ and a constant } \\ \varepsilon_M > 0 \text{ such that for any } R > 0,\ \varepsilon, \gamma > 0,\ x, y \in \overline{\Omega}, \text{ and } \\ r \in [-M, M], \text{ if } (R+1)\gamma + \varepsilon \le \varepsilon_M \text{ and } |x-y| \le \gamma, \text{ then } \\ \text{have} \\ \qquad K_R(x,r) + B(0,\varepsilon) \subset K(y, r - \sigma_M((R+1)\gamma + \varepsilon)). \end{cases} \tag{2.11}$$

Unlike condition (2.10), this condition does not follow from (2.3) and (2.4), with the collection of $K(x,r) = \overline{\mathrm{co}}\,\{p \in \mathbf{R}^n \mid H(x,r,p) \le 0\}$. See for this Example 8.2 in the appendix.

$$\begin{cases} \text{For any } \varepsilon > 0 \text{ and } R > 0 \text{ there is a constant } \rho > 0 \text{ such that} \\ \text{for all } (x,r) \in \mathbf{R}^n \times [-R, R], \\ \qquad K_R(x,r) \subset \overline{\mathrm{co}}\, Z_\rho(x, r - \varepsilon). \end{cases} \tag{2.12}$$

We give examples of collections $\{Z(x,r)\}$ in Section 6, for which (2.11) and (2.12) are satisfied.

Next we need a regularity assumption on the boundary of Ω.

For $\gamma > 0$ we set

$$\Omega^\gamma = \{x \in \Omega \mid \operatorname{dist}(x, \Omega) < \gamma\},$$
$$\Omega_\gamma = \{x \in \Omega \mid \operatorname{dist}(x, \Omega^c) > \gamma\},$$

where $\Omega^c = \mathbf{R}^n \setminus \Omega$, and we assume that

$$\begin{cases} \text{there is a constant } \gamma_0 \in (0,1) \text{ and for each } \gamma \in (0, \gamma_0) \text{ a} \\ C^1 \text{ map } \psi_\gamma : \overline{\Omega^\gamma} \to \mathbf{R}^n \text{ such that } \psi_\gamma(\Omega^\gamma) \subset \Omega_\gamma \text{ and for all} \\ x \in \overline{\Omega^\gamma} \text{ and for some constant } C > 0 \text{ independent of } \gamma, \\ \qquad |\psi_\gamma(x) - x| \le C\gamma, \qquad \|D\psi_\gamma(x) - I\| \le C\gamma. \end{cases} \tag{2.13}$$

For instance, this condition is satisfied if Ω has Lipschitz boundary. See Proposition 8.1 in the appendix.

Theorem 2.1. *Let $\{Z(x,r) \mid (x,r) \in \overline{\Omega} \times \mathbf{R}\}$ be a collection of closed subsets of $\mathbf{R}^n$. Assume that Ω is bounded and that (2.9)–(2.13) hold. Let $u \in C(\overline{\Omega})$ satisfy*

$$D^+u(x) \subset K(x, u(x)) \quad \text{for all } x \in \Omega.$$

Then

$$u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \le u \ \text{ on } \overline{\Omega},\ Dv(y) \in Z(y, v(y)) \ \text{ a.e. } y \in \Omega\}$$

for all $x \in \overline{\Omega}$.

Remark 2.2. A natural way to define the notion of viscosity solution to the collection $\mathcal{Z} := \{Z(x,r) \mid (x,r) \in \overline{\Omega} \times \mathbf{R}\}$ is as follows: a function $u \in \mathrm{USC}(\Omega)$ (resp., $u \in \mathrm{LSC}(\Omega)$) is called a viscosity subsolution (resp., supersolution) of $\mathcal{Z}$ if

$$D^+u(x) \subset Z(x,u(x)) \quad \text{ for } x \in \Omega$$
$$(\text{resp., } D^-u(x) \subset \mathbf{R}^n \setminus \operatorname{int} Z(x,u(x)) \quad \text{ for } x \in \Omega).$$

A function $u \in C(\Omega)$ is called a viscosity solution of $\mathcal{Z}$ if it is both a viscosity subsolution and a viscosity supersolution of $\mathcal{Z}$. For instance, under assumption (2.10), using standard arguments, we can prove the comparison theorem: let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be a viscosity subsolution and a viscosity supersolution of $\mathcal{Z}$, respectively. Assume that $u \le v$ on $\partial\Omega$. Then $u \le v$ on $\overline{\Omega}$.

Next, we give another theorem similar to the above, which is motivated by the eikonal equation

$$H(Du(x)) = 1 \quad \text{ in } \Omega,$$

with Hamiltonian $H \in C(\mathbf{R}^n)$ satisfying (1.2).

In what follows, let $Z(x,r)$, with $(x,r) \in \overline{\Omega} \times \mathbf{R}$, be independent of r, and thus we may write $Z(x)$, $K(x)$, $Z_R(x)$, and $K_R(x)$ for $Z(x,r)$, $K(x,r)$, $Z_R(x,r)$, and $K_R(x,r)$, respectively.

For $\delta > 0$ define $\Delta(\delta)$ by

$$\Delta(\delta) = \{(x,y) \in \overline{\Omega} \times \overline{\Omega} \mid |x-y| \le \delta\}.$$

We assume:

$$\begin{cases} \text{there exists a } \gamma > 0 \text{ such that} \\ \qquad B(0,\gamma) \subset K(x) \quad \text{ for all } x \in \overline{\Omega}. \end{cases} \tag{2.14}$$

This condition can be replaced by the condition that there are a function $\psi \in C^1(\overline{\Omega}, \mathbf{R})$ and a constant $\gamma > 0$ such that $B(D\psi(x),\gamma) \subset K(x)$ for all $x \in \overline{\Omega}$. But, for simplicity of presentation, we have chosen (2.14) instead.

$$\begin{cases} \text{There exists a constant } \theta_0 > 1 \text{ and for each } \theta \in (1,\theta_0) \text{ constants } \delta_0 \equiv \delta_0(\theta) > 0 \text{ and } R_0 \equiv R_0(\theta) > 0 \text{ such that for all } R \ge R_0 \text{ and } (x,y) \in \Delta(\delta_0), \\ \qquad Z_R(x) \cap \partial\,(\overline{\mathrm{co}}\, Z_R(x)) \subset \theta Z(y). \end{cases} \tag{2.15}$$

We give a few of examples in Section 6, for which (2.14) and (2.15) are satisfied.

We define $L : \overline{\Omega} \times \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ by

$$L(x,\xi) = \sup\{\xi \cdot p \mid p \in Z(x)\}.$$

Let $u \in C(\overline{\Omega})$ be a given function and assume:

$$\begin{cases} \text{for each } \theta > 1 \text{ there is a } \delta_1 \equiv \delta_1(\theta) > 0 \text{ such that for all} \\ (x,y) \in (\partial\Omega)^2 \cap \Delta(\delta_1), \\ \qquad\qquad u(x) \le u(y) + \theta L(y, x-y). \end{cases} \tag{2.16}$$

This condition should be compared with the compatibility condition on the boundary data for the existence of a viscosity solution for the Dirichlet problem (see [13]).

Theorem 2.3. *Assume that Ω is bounded and that (2.14) and (2.15) hold. Let $u \in C(\overline{\Omega})$ satisfy*

$$D^+u(x) \subset K(x) \quad \textit{for all } x \in \Omega.$$

Assume that (2.16) holds with this u. Then

$$u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \le u \ \textit{ on } \overline{\Omega},\ Dv(y) \in Z(y) \textit{ a.e. } y \in \Omega\}$$

for all $x \in \overline{\Omega}$.

Remark 2.4. As explained in the Introduction, in [13] we stated the relaxation theorem in terms of the convex envelope $\hat{H}(p)$ of the given Hamitonian $H(p)$, where H is assumed to be positive for $p \ne 0$ and positively homogeneous of degree one. Under some additional assumptions one may state relaxation theorems in this paper in terms of the quasi-convex envelope of Hamiltonians. Let us recall the definition of the quasi-convex envelope of function $H \in C(\mathbf{R}^n)$. We call a function $G : \mathbf{R}^n \to \mathbf{R}$ *quasi-convex* if the set $\{p \in \mathbf{R}^n \mid G(p) \le a\}$ is convex for any $a \in \mathbf{R}$. Let $\mathcal{Q}$ denote the set of all lower semicontinuous quasi-convex functions on $\mathbf{R}^n$. Let $H : \mathbf{R}^n \to \mathbf{R}$. We assume that there is a function $G \in \mathcal{Q}$ such that $G \le H$ on $\mathbf{R}^n$. We define $\widetilde{H} : \mathbf{R}^n \to \mathbf{R}$ by

$$\widetilde{H}(p) = \sup\{G(p) \mid G \in \mathcal{Q},\ G \le H \text{ on } \mathbf{R}^n\}.$$

It follows that the function $\widetilde{H} \in \mathcal{Q}$ and satisfies $\widetilde{H} \le H$ on $\mathbf{R}^n$. We call $\widetilde{H}$ the quasi-convex envelope of the function H. We have the inclusion

$$\overline{\mathrm{co}}\,\{p \in \mathbf{R}^n \mid H(p) \le a\} \subset \{p \in \mathbf{R}^n \mid \widetilde{H}(p) \le a\}$$

for all $a \in \mathbf{R}$, but in general we do not have the identity

$$\overline{\mathrm{co}}\,\{p \in \mathbf{R}^n \mid H(p) \le a\} = \{p \in \mathbf{R}^n \mid \widetilde{H}(p) \le a\} \tag{2.17}$$

for $a \in \mathbf{R}$. For instance, consider the case when $n = 2$ and the function H given by

$$H(p,q) = \frac{q_+}{|p|+1} \quad \text{for } (p,q) \in \mathbf{R}^2,$$

where q_+ denotes the positive part, $\max\{q,0\}$, of $q \in \mathbf{R}$. It is easy to see that the quasi-convex envelope $\widetilde{H}$ of H is given by $\widetilde{H}(p,q) \equiv 0$, and that

$$\overline{\mathrm{co}}\,\{(p,q) \in \mathbf{R}^2 \mid H(p,q) \le 0\} = \mathbf{R} \times (-\infty,0] \ne \mathbf{R}^2 = \{p \in \mathbf{R}^n \mid \widetilde{H}(p) \le 0\}.$$

An assumption on H which guarantees that (2.17) holds for all $a \in \mathbf{R}$ is the coercivity of H. That is, if we assume that

$$\lim_{R\to\infty} \inf\{H(p) \mid p \in \mathbf{R}^n \setminus B(0,R)\} = \infty,$$

then we have (2.17) for all $a \in \mathbf{R}$. With these observations and notations, for instance, we may state a theorem similar to Theorem 2.3 as follows: Let $H \in C(\overline{\Omega} \times \mathbf{R}^n)$ and set

$$Z(x) = \{p \in \mathbf{R}^n \mid H(x,p) \le 0\}, \quad K(x) = \overline{\mathrm{co}}\, Z(x)$$

for $x \in \overline{\Omega}$. Let $\widetilde{H}(x,p)$ be the quasi-convex envelope of the function $H(x,p)$ with respect to p. Assume as in Theorem 2.3 that Ω is bounded and that (2.14) and (2.15) is satisfied. Moreover assume that (2.17) holds. Then, if $u \in C(\overline{\Omega})$ is a viscosity subsolution of

$$\widetilde{H}(x, Du(x)) = 0 \quad \text{in } \Omega$$

and (2.16) is satisifed, we have

$$u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \le u \text{ on } \overline{\Omega},\ H(y, Dv(y)) \le 0 \text{ a.e. for } y \in \Omega\}.$$

3. Preliminary lemmas

We prepare here for the proof of our main results by establishing a few basic lemmas.

Lemma 3.1. *Let K be a non-empty convex subset of $\mathbf{R}^n$ and set*

$$L(\xi) = \sup\{\xi \cdot p \mid p \in K\} \quad \text{for all } \xi \in \mathbf{R}^n.$$

Let U be an open subset of $\mathbf{R}^n$ and let $v \in C(\overline{U})$ satisfy

$$D^+v(x) \subset K \quad \text{for all } x \in U.$$

Let $x, y \in \overline{U}$, and assume that the open line segment $l_0(x,y) := \{tx+(1-t)y \mid t \in (0,1)\} \subset U$. Then

$$u(x) \le u(y) + L(x-y).$$

Proof. We may assume that $x \neq y$ and $L(x-y) < \infty$. By translation and rotation, we may assume that

$$y = 0, \qquad x = \rho e_n,$$

where $\rho > 0$ and e_n denotes the unit vector in $\mathbf{R}^n$ with unity as its n-th entry. We need to show that

$$u(\rho e_n) \leq u(0) + \rho L(e_n).$$

Define the function $v \in C([0,\rho])$ by

$$v(r) = u(re_n).$$

We show that

$$v'(r) \leq L(e_n) \quad \text{ for } r \in (0,\rho) \tag{3.1}$$

in the viscosity sense.

Let $\varphi \in C^1((0,\rho))$ and assume that $v - \varphi$ attains its strict maximum at $a \in (0,\rho)$. Choose a compact neighborhood $V \subset U$ of $ae_n \in U$. Let $\alpha > 0$ and consider the function

$$\Phi(x) := u(x) - \alpha(x_1^2 + \cdots + x_{n-1}^2) - \varphi(x_n)$$

on V. Let $x^\alpha \in V$ be a maximum point of Φ. It is standard to see that as $\alpha \to \infty$,

$$x^\alpha \to ae_n.$$

We are going to take the limit as $\alpha \to \infty$, and therefore we may assume that $x^\alpha \in \operatorname{int} V$ for all α under considerations. Hence, we have

$$p^\alpha := (2\alpha x_1^\alpha, ..., 2\alpha x_{n-1}^\alpha, \varphi'(x_n^\alpha)) \in D^+u(x^\alpha) \subset K.$$

Thus, by the definition of $L(e_n)$, we have

$$L(e_n) \geq p^\alpha \cdot e_n = \varphi'(x_n^\alpha),$$

and therefore, sending $\alpha \to \infty$, we get

$$\varphi'(a) \leq L(e_n),$$

which proves that (3.1) holds in the viscosity sense.

It is a standard fact that (3.1) yields

$$v(\rho e_n) \leq v(0) + \rho L(e_n) = v(0) + L(\rho e_n),$$

that is,

$$u(x) \leq u(y) + L(x-y),$$

which completes the proof. $\square$

Lemma 3.2. *Let $N \in \mathbf{N}$ and $f_1, ..., f_N \in \mathrm{Lip}(\Omega)$. Set*

$$f(x) = \max\{f_1(x), ..., f_N(x)\} \quad \text{for } x \in \Omega.$$

Then $f \in \mathrm{Lip}(\Omega)$ and f is almost everywhere differentiable. Moreover for almost every $x \in \Omega$,

$$Df(x) \in \{Df_1(x), ..., Df_N(x)\}.$$

Proof. It is easy to check that $f \in \mathrm{Lip}(\Omega)$. It is a well-known fact (Rademacher's theorem) that any Lipschitz continuous function (and hence, f) is almost everywhere differentiable.

Rademacher's theorem implies that almost everywhere, all of $f_1, ..., f_N$ and f are differentiable. Now let $y \in \Omega$ be a point where all of $f_1, ..., f_N$ and f are differentiable. By the definition of f, $f(y) = f_i(y)$ for some $i \in \{1, ..., N\}$. Then $f - f_i$ attains a local minimum at y, which yields

$$Df(y) = Df_i(y) \in \{Df_1(y), ..., Df_N(y)\},$$

completing the proof of the lemma. □

Lemma 3.3. *Let Z be a non-empty closed subset of $\mathbf{R}^n$. Define $L : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ by*

$$L(\xi) = \sup\{\xi \cdot p \mid p \in Z\}.$$

Let $\bar{\xi} \in \mathbf{R}^n$ be a point where L is differentiable. Then

$$DL(\bar{\xi}) \in Z \cap \partial(\overline{\mathrm{co}}\, Z)$$

This result is a key observation in [13, the proof of Theorem 2.2].

Proof. Let L be differentiable at $\bar{\xi} \in \mathbf{R}^n$ and let $\bar{p} = DL(\bar{\xi})$. For each $n \in \mathbf{N}$ select a $p_n \in Z$ so that

$$p_n \cdot \bar{\xi} + \frac{1}{n} > L(\bar{\xi}).$$

For each $\varepsilon > 0$ there is a $\delta > 0$ such that for all $h \in B(0, \delta)$,

$$L(\bar{\xi} + h) \le L(\bar{\xi}) + \bar{p} \cdot h + \varepsilon |h|.$$

Combining these two inequalities, we get

$$p_n \cdot (\bar{\xi} + h) < p_n \cdot \bar{\xi} + \frac{1}{n} + \bar{p} \cdot h + \varepsilon |h| \quad \text{for } h \in B(0, \delta),$$

and hence,

$$0 < (\bar{p} - p_n) \cdot h + \varepsilon |h| + \frac{1}{n} \quad \text{for } h \in B(0, \delta).$$

Hence, inserting $h = \delta(p_n - \bar{p})/|p_n - \bar{p}|$ if $p_n \neq \bar{p}$, we get

$$\delta |p_n - \bar{p}| \le \varepsilon \delta + \frac{1}{n},$$

which implies that $p_n \to \bar{p}$ as $n \to \infty$ and that $\bar{p} \in Z$.

Thus we conclude that the function : $p \mapsto \bar{\xi} \cdot p$ attains its maximum over Z at the unique point $\bar{p}$. Moreover we easily see that $\bar{p} \in \partial Z$.

Next, we note that

$$L(\xi) = \sup\{\xi \cdot p \mid p \in \overline{\mathrm{co}}\, Z\} \qquad \text{for } \xi \in \mathbf{R}^n.$$

Now the observation above shows that $\bar{p} \in \partial(\overline{\mathrm{co}}\, Z)$. Thus we see that $DL(\bar{\xi}) \in Z \cap \partial(\overline{\mathrm{co}}\, Z)$. $\square$

4. Proof of Theorem 2.1

We begin with the proof of a localized version, Theorem 4.1 below, of Theorem 2.1.

Let $\gamma > 0$ and let $\{Z(x,r) \mid (x,r) \in \overline{\Omega^\gamma} \times \mathbf{R}\}$ be a collection of closed subsets of $\mathbf{R}^n$, and we introduce the condition that

$$\begin{cases} \text{for each } R > 0 \text{ there are a modulus } \sigma_R \text{ and a constant } \varepsilon_R > 0 \\ \text{such that for any } \varepsilon \in (0, \varepsilon_R),\ x, y \in \overline{\Omega^\gamma}, \text{ and } r \in [-R, R], \text{ if} \\ |x-y| \le \varepsilon, \text{ then} \\ \qquad Z_R(x,r) + B(0,\varepsilon) \subset Z(y, r - \sigma_R(\varepsilon)). \end{cases} \tag{4.1}$$

Theorem 4.1. *Assume that Ω is bounded and that (2.9) and (2.12), with Ω^γ in place of Ω, and (4.1) hold. Let $u \in C(\overline{\Omega^\gamma})$ satisfy*

$$D^+u(x) \subset K(x, u(x)) \qquad \text{for all } x \in \Omega^\gamma.$$

Assume in addition that u is Lipschitz continuous on $\overline{\Omega^\gamma}$. Then

$$u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \le u \ \text{ on } \overline{\Omega},\ Dv(y) \in Z(y, v(y)) \ \text{ a.e. } y \in \Omega\}$$

for all $x \in \overline{\Omega}$.

Throughout this section we use the notation: for any $R > 0$ and $\varepsilon > 0$ $L_{R,\varepsilon}$ denotes the function: $\overline{\Omega^\gamma} \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ defined by

$$L_{R,\varepsilon}(x, r, \xi) = \sup\{\xi \cdot p \mid p \in Z_R(x,r) + B(0,\varepsilon)\},$$

where $\sup \emptyset = -\infty$.

Proof. We choose a constant $M > 0$ so that

$$|u(x)| \le M, \qquad |u(x) - u(y)| \le M|x-y| \qquad \text{for all } x, y \in \Omega^\gamma. \tag{4.2}$$

Fix $\varepsilon \in (0, 1 \wedge \gamma)$. Note that $B(y, \varepsilon) \subset \Omega^\gamma$ for all $y \in \overline{\Omega}$. Fix $\alpha > 0$ so that

$$(M+1)\alpha \le \frac{\varepsilon}{4}.$$

In view of (2.12) we may choose $R \geq M+1$ so that for all $(x,r) \in \overline{\Omega^\gamma} \times [-M-1, M+1]$,

$$K_M(x,r) \subset \overline{\mathrm{co}}\, Z_R(x, r-\alpha).$$

Let ε_R and σ_R be the positive constant and the function from (4.1), respectively, and fix $\beta \in (0, \alpha \wedge \varepsilon_R)$ so that

$$\sigma_R(\beta) \leq \frac{\varepsilon}{4}, \qquad (2R+1)\beta \leq \varepsilon.$$

Observe by virtue of (4.1) that for all $x, y \in \overline{\Omega^\gamma}$ and $r \in \mathbf{R}$, if $|x-y| \leq \beta$ and $|r| \leq M+1$, then

$$Z_R(x,r) + B(0,\beta) \subset Z(y, r - \sigma_R(\beta)). \tag{4.3}$$

Noting that $u(x) \geq u(y) - M\beta$ for all $y \in \overline{\Omega}$ and $x \in B(y,\beta)$, we deduce by using (2.9) and (4.3) that for all $y \in \overline{\Omega}$ and $x \in B(y,\beta)$,

$$\begin{aligned} D^+u(x) \subset K_M(x,u(x)) &\subset \overline{\mathrm{co}}\, Z_R(x, u(x) - \alpha) \\ &\subset \overline{\mathrm{co}}\, Z_R(x, u(y) - M\beta - \alpha) \\ &\subset \overline{\mathrm{co}}\, Z_R(y, u(y) - (M+1)\alpha - \sigma_R(\beta)). \end{aligned} \tag{4.4}$$

Here we used the observation that $M \geq u(y) - M\beta - \alpha \geq -M - \alpha(M+1) > -M - \frac{1}{4}$. We write

$$\delta = (M+1)\alpha + \sigma_R(\beta),$$

and note that $0 < \delta \leq \frac{\varepsilon}{2} < \frac{1}{2}$.

From (4.3) it follows that for all $x \in \overline{\Omega}$ and $z \in B(x,\beta)$,

$$Z_R(z, u(z) - \delta) + B(0,\beta) \subset Z(x, u(x) - 2\delta). \tag{4.5}$$

By virtue of Lemma 3.1, we see from (4.4) that for all $y \in \overline{\Omega}$,

$$u(x) \leq u(y) + L_{R,0}(y, u(y) - \delta) \quad \text{ for all } x \in B(y,\beta). \tag{4.6}$$

Let $\mu \in (0,\varepsilon)$ be a constant to be fixed later on, and choose a finite subset Y_μ of $\overline{\Omega}$ such that

$$\overline{\Omega} \subset \bigcup_{y \in Y_\mu} B(y,\mu).$$

Define $f_\mu : \overline{\Omega} \to \mathbf{R}$ by

$$f_\mu(x) = \min\{u(y) + L_{R,\beta}(y, u(y) - \delta, x - y) \mid y \in Y_\mu,\ |y - x| \leq \beta\}.$$

For each $x \in \overline{\Omega}$, there is a $y \in Y_\mu \cap B(x,\mu)$ such that

$$\begin{aligned} f_\mu(x) &= u(y) + L_{R,\beta}(y, u(y) - \delta, x - y) \\ &\leq u(x) + L_{R,0}(y, u(y) - \delta, 0) + M\mu + (R+1)\mu \\ &\leq u(x) + (2R+1)\mu. \end{aligned} \tag{4.7}$$

For any $x \in \overline{\Omega}$, using (4.6), we get

$$\begin{aligned}\min\{u(y) + L_{R,\beta}(y, u(y) - \delta, x - y) \mid y \in \overline{\Omega},\ \frac{\beta}{2} \le |y - x| \le \beta\} \\ \ge \min\{u(x) + \beta|x - y| \mid y \in \overline{\Omega},\ \frac{\beta}{2} \le |y - x| \le \beta\} \\ = u(x) + \frac{\beta^2}{2}. \qquad (4.8)\end{aligned}$$

We fix $\mu > 0$ so that $(2R + 1)\mu < \frac{\beta^2}{2}$. By virtue of (4.7) and (4.8), for all $x \in \overline{\Omega}$ we have

$$f_\mu(x) = \min\Big\{u(y) + L_{R,\beta}(y, u(y) - \delta, x - y) \ \Big|\ y \in Y_\mu \cap B\Big(x, \frac{\beta}{2}\Big)\Big\}. \quad (4.9)$$

Let $a \in \overline{\Omega}$ and $x \in \overline{\Omega} \cap B\left(a, \frac{\beta}{4}\right)$. Since

$$B\Big(x, \frac{\beta}{2}\Big) \subset B\Big(a, \frac{3\beta}{4}\Big) \subset B(x, \beta),$$

from (4.9) we have

$$f_\mu(x) = \min\Big\{u(z) + L_{R,\beta}(z, u(z) - \delta, x - z) \ \Big|\ z \in Y_\mu \cap B\Big(a, \frac{3\beta}{4}\Big)\Big\}.$$

Therefore, we have for any $x, y \in B\left(a, \frac{\beta}{4}\right) \cap \overline{\Omega}$,

$$|f_\mu(x) - f_\mu(y)| \le (R + 1)|x - y|, \qquad (4.10)$$

which implies that $f_\mu \in \mathrm{Lip}(\overline{\Omega})$ since $\overline{\Omega}$ is bounded. Also, using (4.5), in view of Lemmas 3.2 and 3.3 we have

$$\begin{aligned}Df_\mu(x) &\in \bigcup\Big\{D_\xi L_{R,0}(z, u(z) - \delta, x - z) \ \Big|\ z \in Y_\mu \cap B\Big(a, \frac{3\beta}{4}\Big)\Big\} + B(0, \beta) \\ &\subset \bigcup\{Z_R(z, u(z) - \delta) \mid z \in B(x, \beta)\} + B(0, \beta) \\ &\subset Z(x, u(x) - 2\delta) \quad \text{ a.e. } x \in B\Big(a, \frac{\beta}{4}\Big) \cap \Omega.\end{aligned}$$

Thus we have

$$Df_\mu(x) \in Z(x, u(x) - 2\delta) \subset Z(x, u(x) - \varepsilon) \quad \text{ a.e. } x \in \Omega.$$

Let $x, y \in \overline{\Omega}$ satisfy $|x - y| \le \beta$. Then we have

$$u(y) + L_{R,\beta}(y, u(y) - \delta, x - y) \ge u(x) - M\beta - (R + 1)\beta \ge u(x) - (2R + 1)\beta.$$

Hence, we have

$$f_\mu(x) \ge u(x) - (2R + 1)\beta \quad \text{ for all } x \in \overline{\Omega}.$$

This together with (4.7) and our choice of β yields

$$|f_\mu(x) - u(x)| \le (2R+1)\beta \le \varepsilon \quad \text{for all } x \in \overline{\Omega}.$$

We now define $g \in \mathrm{Lip}(\overline{\Omega})$ by

$$g(x) = f_\mu(x) - 2\varepsilon.$$

Then, we have

$$g(x) \le u(x) - \varepsilon \quad \text{for all } x \in \overline{\Omega},$$

and hence,

$$Dg(x) \in Z(x, u(x) - 2\delta) \subset Z(x, g(x)) \quad \text{a.e. } x \in \Omega.$$

Finally, noting that

$$u(x) - 3\varepsilon \le g(x) \le u(x) - \varepsilon \quad \text{for all } x \in \overline{\Omega},$$

we conclude the proof. $\square$

Proof of Theorem 2.1. Let $\gamma_0 \in (0,1)$ be the constant from (2.13). Let $0 < \gamma < \gamma_0$. We consider the sup-convolution of u. That is, we define $u^\gamma : \mathbf{R}^n \to \mathbf{R}$ by

$$u^\gamma(x) = \sup\Big\{ u(y) - \frac{1}{2\gamma^2}|x-y|^2 \;\Big|\; y \in \overline{\Omega} \Big\}.$$

It is well-known that there is a modulus ν such that for all $x \in \Omega_{\gamma\nu(\gamma)}$,

$$D^+u^\gamma(x) \subset \bigcup \{ D^+u(y) \mid y \in B(x, \gamma\nu(\gamma)) \}, \tag{4.11}$$

$$D^+u^\gamma(x) \subset B\Big(0, \frac{\nu(\gamma)}{\gamma}\Big), \tag{4.12}$$

$$u^\gamma(x) \in [u(x), u(x) + \nu(\gamma)]. \tag{4.13}$$

We may assume that $\nu(t) < 1$ for all $t \in (0, \gamma_0)$.

Let ψ_γ be the function from condition (2.13). Define the function $U^\gamma : \overline{\Omega^\gamma} \to \mathbf{R}$ by $U^\gamma = u^\gamma \circ \psi_\gamma$.

Noting that $D^+U^\gamma(x) = D\psi_\gamma(x)^* D^+u^\gamma \circ \psi_\gamma(x)$, where for any matrix A, A^* denotes the transposed matrix of A, and using (2.13), we see from (4.11)–(4.13) that for all $x \in \Omega^\gamma$,

$$D^+U^\gamma(x) \subset \bigcup \{ D^+u(y) \mid y \in B(\psi_\gamma(x), \gamma\nu(\gamma)) \} + B(0, C\nu(\gamma)), \tag{4.14}$$

$$D^+U^\gamma(x) \subset B\Big(0, (1+C\gamma)\frac{\nu(\gamma)}{\gamma}\Big), \tag{4.15}$$

$$U^\gamma(x) \in [u \circ \psi_\gamma(x), u \circ \psi_\gamma(x) + \nu(\gamma)]. \tag{4.16}$$

We set

$$R = (1+C\gamma)\frac{\nu(\gamma)}{\gamma} \quad \text{and} \quad M = \max_{\overline{\Omega}} |u|.$$

By (4.14) and (4.15) we get for all $x \in \Omega^\gamma$,

$$D^+U^\gamma(x) \subset \bigcup\{K_R(y,u(y)) \mid y \in B(\psi_\gamma(x),\gamma\nu(\gamma))\} + B(0,C\nu(\gamma)).$$

Hence, noting that for all $x \in \Omega^\gamma$ and $y \in B(\psi_\gamma(x),\gamma\nu(\gamma))$,

$$u(y) \ge u(\psi_\gamma(x)) - m(\gamma\nu(\gamma)) \ge U^\gamma(x) - \nu(\gamma) - m(\gamma\nu(\gamma)),$$

where m denotes the modulus of continuity of u and defining the modulus μ_1 by setting $\mu_1(t) = \nu(t) + m(t\nu(t))$ for $t \ge 0$, we get for all $x \in \Omega^\gamma$,

$$D^+U^\gamma(x) \subset \bigcup\{K_R(y,U^\gamma(x)-\mu_1(\gamma)) \mid y \in B(\psi_\gamma(x),\gamma\nu(\gamma))\}+B(0,C\nu(\gamma)).$$

Furthermore, noting that for all $x \in \Omega^\gamma$ and $y \in B(\psi_\gamma(x),\gamma\nu(\gamma))$,

$$|x-y| \le |x-\psi_\gamma(x)| + |\psi_\gamma(x)-y| \le C\gamma + \gamma\nu(\gamma),$$

defining the modulus μ_2 by

$$\mu_2(t) = \mu_1(t) + \sigma_{M+1}\big([(1+Ct)(\nu(t)/t)+1][Ct+t\nu(t)] + C\nu(t)\big),$$

where σ_M is the modulus from (2.11), and using (2.11), we get for all $x \in \Omega^\gamma$,

$$D^+U^\gamma(x) \subset K_{R+1}(x, U^\gamma(x) - \mu_2(\gamma)).$$

Here and henceforth we assume that γ_0 is small enough so that $\mu_2(\gamma_0) < 1 \wedge \varepsilon_M$ and $C\nu(\gamma_0) < 1 \wedge \varepsilon_M$, where ε_M is the constant from (2.11).

Now, using (2.12), we find a constant $\rho > 0$ depending on R, M, and $\mu_2(\gamma)$ such that for all $x \in \Omega^\gamma$,

$$D^+U^\gamma(x) \subset \overline{\mathrm{co}}\, Z_\rho(x, U^\gamma(x) - 2\mu_2(\gamma))$$

Next, define $V^\gamma : \overline{\Omega^\gamma} \to \mathbf{R}$ by

$$V^\gamma(x) = U^\gamma(x) - 2\mu_2(\gamma).$$

Observe from (4.16) that for all $x \in \overline{\Omega}$,

$$u(x) - m(C\gamma) - 2\mu_2(\gamma) \le V^\gamma(x) \le u(x) + \nu(\gamma) + m(C\gamma) + 2\mu_2(\gamma).$$

We may assume by replacing μ_2 if necessary that $2\mu_2(t) \ge \nu(t) + m(Ct)$ for $t \ge 0$, so that $V^\gamma(x) \le u(x)$ for all $x \in \overline{\Omega}$.

Noting that V^γ is Lipschitz continuous on $\overline{\Omega^{\gamma/2}}$ and applying Theorem 4.1, with V^γ and $\Omega^{\gamma/2}$ in place of u and Ω^γ, respectively, we conclude that for all $x \in \overline{\Omega}$,

$$\begin{aligned} V^\gamma(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \le V^\gamma \ \text{ on } \overline{\Omega}, \\ Dv(y) \in Z(y, v(y)) \ \text{ a.e. } y \in \Omega\}. \end{aligned}$$

Finally, noting that $u(x) = \sup\{V^\gamma(x) \mid \gamma \in (0,\gamma_0)\}$ for all $x \in \overline{\Omega}$, we finish the proof. □

5. Proof of Theorem 2.3

First of all we remark that under assumption (2.14), we have

$$L(x,\xi) \geq \gamma|\xi| \quad \text{ for all } (x,\xi) \in \overline{\Omega} \times \mathbf{R}^n.$$

For $R > 0$ we define the function $L_R : \overline{\Omega} \times \mathbf{R}^n \to \mathbf{R}$ by

$$L_R(x,\xi) = \sup\{\xi \cdot p \mid p \in Z_R(x)\}.$$

In order to prove Theorem 2.3, we need the following two lemmas.

Lemma 5.1. *Let θ_0, k_0, and δ_0 be from (2.15). Let $\theta \in (1, \theta_0)$, $(x,y) \in \Delta(\delta_0(\theta))$, and $k \in \mathbf{N}$. If $k \geq k_0$, then*

$$\overline{\mathrm{co}}\, Z_k(x) \subset \theta\, \overline{\mathrm{co}}\, Z_k(y), \tag{5.1}$$

$$\overline{\mathrm{co}}\, Z(x) \subset \theta\, \overline{\mathrm{co}}\, Z(y). \tag{5.2}$$

Proof. Let θ, x, y, and k be as above. From (2.15) we have

$$Z_k(x) \cap \partial(\overline{\mathrm{co}}\, Z_k(x)) \subset \theta Z_k(y).$$

Since

$$\overline{\mathrm{co}}\,\big(Z_k(x) \cap \partial(\overline{\mathrm{co}}\, Z_k(x))\big) = \overline{\mathrm{co}}\, Z_k(x),$$

we get

$$\overline{\mathrm{co}}\, Z_k(x) \subset \theta\, \overline{\mathrm{co}}\, Z_k(y),$$

and therefore,

$$\overline{\mathrm{co}}\, Z(x) \subset \theta\, \overline{\mathrm{co}}\, Z(y). \qquad \square$$

Lemma 5.2. *Let $\theta_1 = \theta_0^2$, where θ_0 is from (2.15). Let u be the function from Theorem 2.3. For each $\theta \in (1, \theta_1)$ there is a $\delta_2 \equiv \delta_2(\theta) > 0$ such that for all $(x,y) \in \Delta(\delta_2)$,*

$$u(x) \leq u(y) + \theta L(y, x-y).$$

Proof. Fix $\theta \in (1, \theta_1)$, and set $\sigma = \theta^{1/2}$. Note that $\sigma \in (1, \theta_0)$. By (2.16), we have for $(x,y) \in (\partial\Omega)^2 \cap \Delta(\delta_1(\sigma))$,

$$u(x) \leq u(y) + \sigma L(y, x-y). \tag{5.3}$$

By (5.2), we have for $(x,y) \in \Delta(\delta_0(\sigma))$,

$$K(x) \subset \sigma K(y). \tag{5.4}$$

By Lemma 3.1 and (5.4), we see that for $(x,y) \in \Delta(\delta_0(\sigma))$, if $l_0(x,y) \in \Omega$, then

$$u(x) \leq u(y) + \sigma L(y, x-y). \tag{5.5}$$

Set $\delta_2 = \min\{\delta_0(\sigma), \delta_1(\sigma)\}$. Fix $(x,y) \in \Delta(\delta_2)$. If $l_0(x,y) \subset \Omega$, then we have from (5.5)

$$u(x) \leq u(y) + \sigma L(y, x-y). \tag{5.6}$$

Otherwise we have

$$l_0(x,y) \cap \partial\Omega \neq \emptyset.$$

In this case choose $0 \le t_0 \le t_1 \le 1$ so that

$$\begin{gathered}(1-t)x + ty \in \Omega \quad \text{ for all } t \in (0,t_0) \cup (t_1,1),\\ (1-t_0)x + t_0 y,\ (1-t_1)x + t_1 y \in \partial\Omega.\end{gathered}$$

Set $\bar{x} = (1-t_0)x + t_0 y$ and $\bar{y} = (1-t_1)x + t_1 y$. Note that $\bar{x}, \bar{y} \in \partial\Omega$ and $l_0(x,\bar{x}) \cup l_0(\bar{y},y) \subset \Omega$. Note as well that since

$$x - \bar{x} = t_0(x-y), \quad \bar{x} - \bar{y} = (t_1 - t_0)(x-y), \quad \bar{y} - y = (1-t_1)(x-y),$$

we have

$$\begin{cases} L(y, x-\bar{x}) = t_0 L(y, x-y), \\ L(y, \bar{x}-\bar{y}) = (t_1-t_0) L(y, x-y), \\ L(y, \bar{y}-y) = (1-t_1) L(y, x-y), \end{cases} \tag{5.7}$$

and

$$|x-\bar{x}| = t_0|x-y|, \quad |\bar{x}-\bar{y}| = (t_1-t_0)|x-y|, \quad |\bar{y}-y| = (1-t_1)|x-y|$$

From (5.5), we have

$$\begin{aligned} u(x) &\le u(\bar{x}) + \sigma L(\bar{x}, x-\bar{x}), \\ u(\bar{y}) &\le u(y) + \sigma L(y, \bar{y}-y). \end{aligned}$$

From (5.3), we have

$$u(\bar{x}) \le u(\bar{y}) + \sigma L(\bar{y}, \bar{x}-\bar{y}).$$

Using (5.4), we get

$$\begin{aligned} u(x) &\le u(\bar{x}) + \sigma^2 L(y, x-\bar{x}), \\ u(\bar{y}) &\le u(y) + \sigma^2 L(y, \bar{y}-y), \\ u(\bar{x}) &\le u(\bar{y}) + \sigma^2 L(y, \bar{x}-\bar{y}). \end{aligned}$$

Furthermore, using (5.7), we get

$$\begin{aligned} u(x) &\le u(\bar{x}) + \theta t_0 L(y, x-y), \\ u(\bar{y}) &\le u(y) + \theta(1-t_1) L(y, x-y), \\ u(\bar{x}) &\le u(\bar{y}) + \theta(t_1-t_0) L(y, x-y). \end{aligned}$$

Hence,

$$u(x) \le u(y) + \theta L(y, x-y).$$

This together with (5.6) completes the proof. □

Proof of Theorem 2.3. By adding a constant we may assume that $u \geq 0$ on $\overline{\Omega}$.

We claim that there is a $k_1 \in \mathbf{N}$ such that if $k \geq k_1$, then $L_k \in C(\overline{\Omega} \times \mathbf{R}^n)$.

To see this fix any $\theta_2 \in (1, \theta_0)$ and set $k_1 \equiv k_0(\theta_2)$, where θ_0 and k_0 are from (2.15). Let $k \in \mathbf{N}$ satisfy $k \geq k_1$. Fix $\theta \in (1, \theta_0)$ and let $\delta = \delta_0(\theta_2)$, where δ_0 is from (2.15).

Using (5.1), we compute that for $(x, y) \in \Delta(\delta)$ and $\xi \in \mathbf{R}^n$,

$$\begin{aligned} L_k(x, \xi) &= \theta \max\{\xi \cdot q \mid q \in \theta^{-1} Z_k(x)\} \\ &= \theta \max\{\xi \cdot q \mid q \in \theta^{-1} \overline{\mathrm{co}}\, Z_k(x)\} \\ &\leq \theta \max\{\xi \cdot q \mid q \in \overline{\mathrm{co}}\, Z_k(y)\} = \theta L_k(y, \xi) \\ &= L_k(y, \xi) + (\theta - 1) L_k(y, \xi) \leq L_k(y, \xi) + (\theta - 1) k |\xi|, \end{aligned}$$

which yields by symmetry

$$|L_k(x, \xi) - L_k(y, \xi)| \leq (\theta - 1) k |\xi|.$$

Noting that

$$|L_k(x, \xi) - L_k(x, \eta)| \leq \max\{|(\xi - \eta) \cdot p| \mid p \in Z_k(x)\} \leq k |\xi - \eta|$$

for all $x \in \overline{\Omega}$ and $\xi, \eta \in \mathbf{R}^n$, we conclude that L_k is continuous on $\overline{\Omega} \times \mathbf{R}^n$.

Henceforth fix $\theta > 1$ and $\delta > 0$ so that $\theta < \theta_0^2$ and $\delta \leq \min\{\delta_0(\theta), \delta_2(\theta)\}$, where θ_0 and δ_0 are from (2.15) and δ_2 is from Lemma 5.2, respectively.

For $k \in \mathbf{N}$, with $k \geq k_0$, define $f_k, g_k : \overline{\Omega} \to \mathbf{R}$ by

$$\begin{aligned} f_k(x) &= \min\{u(y) + \theta^2 L_k(y, x - y) \mid y \in \overline{\Omega},\ |y - x| \leq \delta\}, \\ g_k(x) &= \min\{u(y) + \theta^2 L_k(y, x - y) \mid y \in \overline{\Omega},\ \frac{\delta}{2} \leq |y - x| \leq \delta\}. \end{aligned}$$

Fix $\varepsilon > 0$. We show that there is an $\alpha \in \mathbf{N}$ such that

$$g_\alpha(x) \geq u(x) + \frac{\theta(\theta - 1)\gamma\delta}{4} \quad \text{for all } x \in \overline{\Omega}, \tag{5.8}$$

$$f_\alpha(x) \geq u(x) - \varepsilon \quad \text{for all } x \in \overline{\Omega}. \tag{5.9}$$

We henceforth write

$$\nu = \theta(\theta - 1)\gamma\delta.$$

Note by Lemma 5.2 that for all $(x, y) \in \Delta(\delta)$,

$$u(x) - u(y) \leq \theta L(y, x - y).$$

Also we have

$$\gamma |x - y| \leq L(y, x - y) \quad \text{for all } x, y \in \overline{\Omega}.$$

We claim that there is an $\alpha \in \mathbf{N}$ such that for $(x, y) \in \Delta(\delta)$,

$$\max\{\theta^{-1}(u(x) - u(y)), \gamma |x - y|\} \leq \min\{\frac{\varepsilon}{\theta}, \frac{\nu}{4\theta^2}\} + L_\alpha(y, x - y). \tag{5.10}$$

To see this, set

$$h(x,y)=\max\{\theta^{-1}(u(x)-u(y)),\gamma|x-y|\}\quad\text{ for } (x,y)\in\varDelta(\delta).$$

We know that

$$h(x,y)\le L(y,x-y)\quad\text{ for all } (x,y)\in\varDelta(\delta),$$

and it is clear that as $k\to\infty$,

$$L_k(x,\xi)\nearrow L(x,\xi)\quad\text{ for all } (x,\xi)\in\overline{\varOmega}\times\mathbf{R}^n.$$

Hence, as $k\to\infty$,

$$(h(x,y)-L_k(y,x-y))_+\searrow 0\quad (x,y)\in\varDelta(\delta).$$

Note that if $k\ge k_1$, then $L_k\in C(\overline{\varOmega}\times\mathbf{R}^n)$. Thus, thanks to Dini's lemma, we see that as $k\to\infty$,

$$(h(x,y)-L_k(y,x-y))_+\searrow 0\quad\text{ uniformly for } (x,y)\in\varDelta(\delta),$$

which proves that there is an $\alpha\in\mathbf{N}$ such that (5.10) holds.

Fix such an $\alpha\in\mathbf{N}$. We show that (5.8) and (5.9) hold with this choice of α.

Compute that for $(x,y)\in\varDelta(\delta)$, if $|x-y|\ge\frac{\delta}{2}$, then

$$\begin{aligned}u(x)+\frac{\nu}{2}&\le u(x)+\theta(\theta-1)\gamma|x-y|\\&\le u(x)+\theta(\theta-1)\big(\frac{\nu}{4\theta^2}+L_\alpha(y,x-y)\big)\\&\le u(y)+\frac{\nu}{4\theta}+\theta L_\alpha(y,x-y)+\theta(\theta-1)\big(\frac{\nu}{4\theta^2}+L_\alpha(y,x-y)\big)\\&=u(y)+\frac{\nu}{4}+\theta^2L_\alpha(y,x-y),\end{aligned}$$

that is,

$$u(x)+\frac{\nu}{4}\le u(y)+\theta^2L_\alpha(y,x-y).$$

Consequently, we have

$$g_\alpha(x)\ge u(x)+\nu/4\quad\text{ for } x\in\overline{\varOmega},$$

which shows that (4.8) holds. Also, by (5.10) we have

$$u(x)-u(y)\le\varepsilon+\theta L_\alpha(y,x-y)\le\varepsilon+\theta^2L_\alpha(y,x-y)$$

for all $(x,y)\in\varDelta(\delta)$, and hence,

$$u(x)-\varepsilon\le f_\alpha(x)\quad\text{ for all } x\in\overline{\varOmega},$$

which shows that (5.9) holds.

For each $\beta \in (0, \delta)$ choose a finite subset Y_β of $\overline{\Omega}$ such that

$$\overline{\Omega} \subset \bigcup_{y \in Y_\beta} B(y, \beta).$$

For $\beta > 0$ define $f_{\alpha\beta} : \overline{\Omega} \to \mathbf{R}$ by

$$f_{\alpha\beta}(x) = \min\{u(y) + \theta^2 L_k(y, x-y) \mid y \in Y_\beta,\ |y-x| \le \delta\}.$$

Since u is uniformly continuous on $\overline{\Omega}$, there is a $\beta \in (0, \delta)$ such that for all $(x, y) \in \Delta(\beta)$,

$$|u(x) - u(y)| + \theta^2 \alpha \beta \le \varepsilon.$$

Henceforth we fix such a $\beta \in (0, \delta)$. Observe that for each $x \in \overline{\Omega}$ there is a $y \in Y_\beta \cap B(x, \delta)$ and that for all $x \in \overline{\Omega}$ and $y \in Y_\beta \cap B(x, \delta)$,

$$\begin{aligned} f_{\alpha\beta}(x) &\le u(y) + \theta^2 L_\alpha(y, x-y) \\ &\le u(x) + u(y) - u(x) + \theta^2 \alpha |x-y| \le u(x) + \varepsilon. \end{aligned} \tag{5.11}$$

Since $f_\alpha(x) \le f_{\alpha\beta}(x)$ for all $x \in \overline{\Omega}$, it follows immediately from (5.9) and (5.11) that

$$|u(x) - f_{\alpha\beta}(x)| \le \varepsilon \quad \text{for all } x \in \overline{\Omega}. \tag{5.12}$$

We assume that $\varepsilon < \nu/4$ and show that

$$f_{\alpha\beta} \in \mathrm{Lip}(\overline{\Omega}). \tag{5.13}$$

Note by (5.8) and (5.11) that

$$g_\alpha(x) \ge u(x) + \nu/4 > f_{\alpha\beta}(x) \quad \text{for all } x \in \overline{\Omega}.$$

Hence, we have for all $x \in \overline{\Omega}$,

$$\begin{aligned} f_{\alpha\beta}(x) &= \min\{u(y) + \theta^2 L_\alpha(y, x-y) \mid y \in Y_\beta,\ |x-y| \le \delta\} \\ &< \min\{u(y) + \theta^2 L_\alpha(y, x-y) \mid y \in \overline{\Omega},\ \frac{\delta}{2} \le |y-x| \le \delta\}, \end{aligned}$$

and therefore,

$$f_{\alpha\beta}(x) = \min\{u(y) + \theta^2 L_\alpha(y, x-y) \mid y \in Y_\beta,\ |x-y| \le \frac{\delta}{2}\}.$$

Accordingly, we have for all $(x, z) \in \Delta(\delta/4)$,

$$f_{\alpha\beta}(x) = \min\{u(y) + \theta^2 L_\alpha(y, x-y) \mid y \in Y_\beta,\ |y-z| \le 3\delta/4\}, \tag{5.14}$$

since $B(x, \delta/2) \subset B(z, 3\delta/4) \subset B(x, \delta)$, and moreover,

$$\begin{aligned} f_{\alpha\beta}(x) - f_{\alpha\beta}(z) &= \min\{u(y) + \theta^2 L_\alpha(y, x-y) \mid y \in Y_\beta,\ |y-z| \le 3\delta/4\} \\ &\quad - \min\{u(y) + \theta^2 L_\alpha(y, z-y) \mid y \in Y_\beta,\ |y-z| \le 3\delta/4\} \\ &\le \max\{\theta^2 (L_\alpha(y, x-y) - L_\alpha(y, z-y)) \mid y \in Y_\beta,\ |y-z| \le 3\delta/4\} \\ &\le \theta^2 \alpha |x-z|, \end{aligned}$$

which shows that $f_{\alpha\beta}$ is Lipschitz continuous on $\overline{\Omega}$.

Next we want to show that

$$Df_{\alpha\beta}(x) \in \theta^3 Z_\alpha(x) \quad \text{a.e. } x \in \overline{\Omega}.$$

Using (5.14), the finiteness of the set Y_β, and (2.15), in view of Lemmas 3.2 and 3.3 we get

$$\begin{aligned} Df_{\alpha\beta}(x) &\in \theta^2 \{D_\xi L_\alpha(y, x-y) \mid y \in Y_\beta,\ |y-x| \le \delta\} \\ &\subset \bigcup \big\{\theta^2 \left[Z_\alpha(y) \cap \partial\left(\overline{\mathrm{co}}\, Z_\alpha(y)\right)\right] \mid y \in Y_\beta \cap B(x,\delta)\big\} \\ &\subset \theta^3 Z_\alpha(x) \quad \text{a.e. } x \in \Omega. \end{aligned}$$

Finally we set

$$g(x) = \theta^{-3}(f_{\alpha\beta}(x) - \varepsilon) \quad \text{for all } x \in \overline{\Omega}.$$

Then

$$g(x) \le \theta^{-3} u(x) \le u(x) \quad \text{for all } x \in \overline{\Omega}.$$

On the other hand, we have

$$\begin{aligned} g(x) &\ge \theta^{-3}(f(x) + \varepsilon) - 2\varepsilon\theta^{-3} \\ &\ge u(x) - 2\varepsilon\theta^{-3} \quad \text{for all } x \in \overline{\Omega}. \end{aligned}$$

Thus, we see that for each $\varepsilon > 0$ there is a function $v \in \mathrm{Lip}(\overline{\Omega})$ such that

$$\begin{gathered} v(x) \le u(x) \quad \text{for all } x \in \overline{\Omega}, \\ v(x) + \varepsilon > u(x) \quad \text{for all } x \in \overline{\Omega}, \\ Dv(x) \in Z(x) \quad \text{a.e. } x \in \Omega. \end{gathered}$$

As an easy consequence, we get

$$u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \le u \ \text{ on } \overline{\Omega},\ Dv(y) \in Z(y) \ \text{ a.e. } y \in \Omega\}$$

for all $x \in \overline{\Omega}$, which concludes the proof. □

6. Examples

We examine a few cases of Hamilton-Jacobi equations

$$H(x, u(x), Du(x)) = 0 \quad \text{in } \Omega,$$

for which either of Theorems 2.1 or 2.3 is applied.

6.1. Examples in view of Theorem 2.1

First we consider the case when $H \in C(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n)$ satisfies:

$$\begin{cases} \text{for any } r \in \mathbf{R}, \\ \lim_{R\to\infty} \inf\{H(x,r,p) \mid x \in \mathbf{R}^n,\ p \in \mathbf{R}^n \setminus B(0,R)\} = \infty. \end{cases} \tag{6.1}$$

As in section 2 we define

$$Z(x,r) = \{p \in \mathbf{R}^n \mid H(x,r,p) \le 0\} \quad \text{for } (x,r) \in \overline{\Omega} \times \mathbf{R}.$$

Propostion 6.1. *Assume that (2.3) and (6.1) hold and that Ω is bounded. Then the collection $\{Z(x,r) \mid (x,r) \in \overline{\Omega} \times \mathbf{R}\}$ satisfies (2.9)–(2.12).*

Proof. As noted in section 2, since for each $(x,p) \in \overline{\Omega} \times \mathbf{R}^n$ the function : $r \mapsto H(x,r,p)$ is non-decreasing on $\mathbf{R}$, we deduce that (2.9) is satisfied.

Next we show that H satisfies condition (2.10). To this end we prove the following assertion (6.2) which is stronger than (2.10).

$$\begin{cases} \text{for each } M > 0 \text{ there are a modulus } \sigma_M \text{ and a constant} \\ \varepsilon_M > 0 \text{ such that for any } \varepsilon \in (0,\varepsilon_M],\ (x,y) \in \Delta(\varepsilon), \text{ and} \\ r \in [-M,M], \\ \qquad Z(x,r) + B(0,\varepsilon) \subset Z(y, r - \sigma_M(\varepsilon)). \end{cases} \tag{6.2}$$

Fix $M > 0$ and, in view of (6.1), choose $\rho > 0$ so that

$$H(x,-M-1,p) > 0 \quad \text{for all } (x,p) \in \overline{\Omega} \times (\mathbf{R}^n \setminus B(0,\rho)).$$

Then, in view of the monotonicity of $H(x,r,p)$ in r, we have

$$H(x,r,p) > 0 \quad \text{for all } (x,r,p) \in \overline{\Omega} \times [-M-1,\infty) \times (\mathbf{R}^n \setminus B(0,\rho)).$$

Moreover, for $(x,r) \in \overline{\Omega} \times [-M-1,\infty)$, we have

$$Z(x,r) = Z_\rho(x,r).$$

Now, since H is uniformly continuous on $\overline{\Omega} \times [-M-1,M] \times B(0,\rho+1)$, there is a modulus ω such that for all $x,y \in \overline{\Omega}$, $r \in [-M-1,M]$, and $p,q \in B(0,\rho+1)$,

$$|H(x,r,p) - H(y,r,q)| \le \omega(|x-y| + |p-q|). \tag{6.3}$$

Define the modulus σ_M by $\sigma_M(t) = \lambda_{M+1}^{-1}\omega(2t)$, and choose a constant $\varepsilon_M \in (0,1]$ so that $\sigma_M(\varepsilon_M) \le 1$.

Let $\varepsilon \in (0,\varepsilon_M]$, $(x,r) \in \overline{\Omega} \times [-M,M]$, $y \in \overline{\Omega} \cap B(x,\varepsilon)$, $p \in Z(x,r)$, and $q \in B(0,\varepsilon)$. Using (2.3) and (6.3), we compute that

$$\begin{aligned} H(y, r - \sigma_M(\varepsilon), p+q) &\le -\lambda_{M+1}\sigma_M(\varepsilon) + H(y,r,p+q) \\ &\le -\omega(2\varepsilon) + H(x,r,p) + \omega(|x-y| + |q|) \\ &\le -\omega(2\varepsilon) + \omega(2\varepsilon) = 0, \end{aligned}$$

and conclude that $p+q \in Z(y, r-\sigma_M(\varepsilon))$, which shows that (6.2) holds.

Let σ_M be the moduli and $\varepsilon_M > 0$ be the constants from (6.2). From (6.2), we see that for any $M > 0$, $\varepsilon \in (0, \varepsilon_M]$, $(x,y) \in \varDelta(\varepsilon)$, and $r \in [-M, M]$, we have

$$\begin{aligned} K(x,r) + B(0,\varepsilon) &= \overline{\mathrm{co}}\,[Z(x,r) + B(0,\varepsilon)] \\ &\subset \overline{\mathrm{co}}\,Z(y, r-\sigma_M(\varepsilon)) = K(y, r-\sigma_M(\varepsilon)). \end{aligned}$$

This shows that (2.11) is satisfied with our $Z(x,r)$.

Let $M > 0$, and $\rho > 0$ be a constant fixed as above. For $(x,r) \in \overline{\varOmega} \times [-M, M]$, since

$$Z(x,r) \subset B(0,\rho),$$

we have

$$K(x,r) = \overline{\mathrm{co}}\,Z(x,r) = \overline{\mathrm{co}}\,Z_\rho(x,r),$$

which shows together with (2.9) that (2.12) holds. □

Next we consider the case when $H \in C(\overline{\varOmega} \times \mathbf{R} \times \mathbf{R}^n)$ satisfies the condition:

$$\begin{cases} \text{for each } M > 0 \text{ there is a modulus } \omega_M \text{ such that for any} \\ x, y \in \overline{\varOmega},\ r \in [-M, M], \text{ and } p, q \in \mathbf{R}^n, \\ \qquad |H(x,r,p) - H(y,r,q)| \le \omega_M(|x-y| + |p-q|). \end{cases} \tag{6.4}$$

Observe that if $G \in C(\mathbf{R} \times \mathbf{R}^n)$, $f \in C(\overline{\varOmega})$, and for each $M > 0$ the function G is uniformly continuous on $[-M, M] \times \mathbf{R}^n$, then the function

$$H(x,r,p) = G(r,p) - f(x)$$

satisfies (6.4).

Propostion 6.2. *Assume that $H \in C(\overline{\varOmega} \times \mathbf{R} \times \mathbf{R}^n)$ satisfies (2.3) and (6.4) and that $\varOmega$ is bounded. Then the collection $\{Z(x,r) \mid (x,r) \in \overline{\varOmega} \times \mathbf{R}^n\}$ satisfies (2.9)–(2.12).*

Proof. Arguments parallel to the proof of Proposition 6.1 guarantee that (2.9), (2.10), and (6.2) hold with our current $Z(x,r)$. Therefore, (2.9)–(2.11) hold.

We intend to show that (2.12) holds. Fix $\varepsilon > 0$ and $R > 0$. We are going to prove that there is a constant $\rho > 0$ such that for all $(x,r) \in \mathbf{R}^n \times [-R, R]$,

$$K_R(x,r) \subset \overline{\mathrm{co}}\,Z_\rho(x, r-4\varepsilon).$$

Here we may assume that $4\varepsilon < 1$.

By virtue of (6.2), we can choose a constant $\delta \in (0,1)$ so that for all $(x,y) \in \varDelta(\delta)$ and $r \in [-R-1, R+1]$,

$$Z(x,r) + B(0,\delta) \subset Z(y, r-\varepsilon).$$

By a compactness argument, we can choose a finite sequence $X := \{x_1, ..., x_N\} \subset \overline{\Omega}$ such that for each $x \in \overline{\Omega}$ there is an $x_i \in X$ such that for any $r \in [-R-1, R]$,

$$Z(x, r) \subset Z(x_i, r - \varepsilon), \qquad Z(x_i, r) + B(0, \delta) \subset Z(x, r - \varepsilon). \tag{6.5}$$

We select a finite sequence $T := \{t_1, ..., t_M\} \subset [-R-1, R+1]$ so that $-R-1 = t_1 \le t_2 \le \cdots \le t_M = R$ and $t_{i+1} - t_i \le \varepsilon$ for all $i = 1, 2, ..., M$. We may assume by relabeling (for instance, by counting some of elements multiply) either X or T that $M = N$.

Fix $(x, t) \in \overline{\Omega} \times [-R, R]$. Choose $x_i \in X$ and $t_j \in T$ so that (6.5) holds for all $r \in [-R-1, R+1]$ and $t \ge t_j \ge t - 2\varepsilon$. Observe that

$$K_R(x, t) \subset K_R(x_i, t - \varepsilon) \subset K_R(x_i, t_j - \varepsilon). \tag{6.6}$$

Fix $(i, j) \in \{1, ..., N\}^2$. Since $K_R(x_i, t_j)$ is compact, we can select a finite sequence $\{p_1, ..., p_L\} \subset K_R(x_i, t_j - \varepsilon)$ so that

$$K_R(x_i, t_j - \varepsilon) \subset \bigcup_{k=1}^{L} B(p_k, \delta/2). \tag{6.7}$$

For each $k \in \{1, ..., L\}$ we choose finite sequences $\{p_{k1}, ..., p_{km}\} \subset Z(x_i, t_j - \varepsilon)$ and $\{\lambda_{k1}, ..., \lambda_{km}\} \subset [0, 1]$ so that

$$\Big| p_k - \sum_{\alpha=1}^{m} \lambda_{k\alpha} p_{k\alpha} \Big| \le \frac{\delta}{2},$$
$$\sum_{\alpha=1}^{m} \lambda_{k\alpha} = 1.$$

Set $\rho_k = \max\{|p_{k1}|, ..., |p_{km}|\}$. Then we have

$$\sum_{\alpha=1}^{m} \lambda_{k\alpha} p_{k\alpha} \in \overline{\mathrm{co}}\, Z_{\rho_k}(x_i, t_j - \varepsilon),$$

and hence,

$$p_k \in \overline{\mathrm{co}}\, Z_{\rho_k}(x_i, t_j - \varepsilon) + B(0, \delta/2).$$

Recalling that the ρ_i depend on i, j, we set $\rho_{ij} = \max\{\rho_1, ..., \rho_L\}$. Then, using (6.7) and (6.5), we get

$$K_R(x_i, t_j - \varepsilon) \subset \overline{\mathrm{co}}\, Z_{\rho_{ij}}(x_i, t_j - \varepsilon) + B(0, \delta) \subset \overline{\mathrm{co}}\, Z_{\rho_{ij}+1}(x, t_j - 2\varepsilon).$$

Furthermore, setting $\rho = \max\{\rho_{ij} \mid i, j = 1, ..., N\} + 1$ and using (6.6), we have

$$K_R(x, t) \subset \overline{\mathrm{co}}\, Z_\rho(x, t_j - 2\varepsilon) \subset \overline{\mathrm{co}}\, Z_\rho(x, t - 4\varepsilon),$$

which completes the proof. □

6.2. Examples in view of Theorem 2.3

Let $H \in C(\mathbf{R}^n)$ and consider the Hamilton-Jacobi equation

$$H(Du(x)) = 0 \quad \text{in } \Omega.$$

As usual we define

$$Z = \{p \in \mathbf{R}^n \mid H(p) \le 0\}.$$

Assume that

$$\text{int}\, \overline{\text{co}}\, Z \ne \emptyset. \tag{6.8}$$

We fix a point $\bar{p} \in \mathbf{R}^n$ and a constant $\gamma > 0$ so that $B(\bar{p}, \gamma) \subset \overline{\text{co}}\, Z$.
We now assume that

$$\begin{cases} \text{there are a constant } \lambda_0 \in (0,1) \text{ and an integer } k_0 \in \mathbf{N} \text{ such} \\ \text{that for each integer } k \ge k_0 \text{ and each point } p \in Z \cap \partial[\overline{\text{co}}\, Z \cap \\ B(\bar{p}, k)] \text{ the function : } t \mapsto H(tp + (1-t)\bar{p}) \text{ is non-decreasing} \\ \text{on } (\lambda_0, 1]. \end{cases} \tag{6.9}$$

We define

$$\begin{aligned} \bar{Z} &= Z - \bar{p}, \\ \bar{K} &= K - \bar{p} \ (= \overline{\text{co}}\, \bar{Z}), \\ \bar{L}(\xi) &= L(\xi) - \bar{p} \cdot \xi \end{aligned}$$

for $x \in \overline{\Omega}$ and $\xi \in \mathbf{R}^n$.

Propostion 6.3. *Assume that (6.8) and (6.9) hold. Then (2.14) and (2.15) hold with $\bar{Z}$ and $\bar{K}$ in place of $Z(x)$ and $K(x)$, respectively.*

Proof. It is obvious that (2.14) holds with $Z(x) = \bar{Z}$.
Now let $k_0 \in \mathbf{N}$ and $\lambda_0 \in (0,1)$ be from (6.9). For any integer $k \ge k_0 \in \mathbf{N}$, any point $p \in \bar{Z} \cap \partial[\overline{\text{co}}\, \bar{Z} \cap B(0,k)]$, and any $t \in (\lambda_0, 1]$, we have

$$\bar{p} + p \in Z \cap \partial[Z \cap B(\bar{p}, k)],$$

and we know from (6.9) that the function: $t \mapsto H(tp + \bar{p})$ is non-decreasing on $(\lambda_0, 1]$ and therefore,

$$tp + \bar{p} \in Z.$$

That is, for any $k \ge k_0$, $p \in \bar{Z} \cap \partial[\overline{\text{co}}\, \bar{Z} \cap B(0,k)]$, and $t \in (\lambda_0, 1]$, we have $p \in t^{-1}\bar{Z}$, which shows that (2.15) holds with $Z(x) = \bar{Z}$. □

Let $u \in C(\overline{\Omega})$ satisfy

$$D^+u(x) \subset K \quad \text{for all } x \in \Omega.$$

Set

$$\bar{u}(x) = u(x) - \bar{p} \cdot x \quad \text{for all } x \in \overline{\Omega}.$$

Then we have

$$D^+\bar{u}(x) = D^+u(x) - \bar{p} \subset \bar{K} \quad \text{ for all } x \in \Omega.$$

On the other hand, if $\theta > 1$, $\delta > 0$, $(x, y) \in \Delta(\delta)$, and

$$u(x) - u(y) \leq \theta L(y, x - y),$$

then we have

$$\begin{aligned}\bar{u}(x) - \bar{u}(y) &\leq \theta \bar{L}(y, x - y) + (\theta - 1)\bar{p} \cdot (x - y) \\ &\leq \theta \bar{L}(y, x - y) + (\theta - 1)|\bar{p}||x - y| \\ &\leq \big(\theta + \gamma^{-1}(\theta - 1)|\bar{p}|\big)\bar{L}(y, x - y).\end{aligned}$$

These observations together with Proposition 6.3 and Theorem 2.3 yield the following proposition.

Propostion 6.4. *Assume that Ω is bounded and that (6.8) and (6.9) hold. Let $u \in C(\overline{\Omega})$ satisfy*

$$D^+u(x) \subset K \quad \textit{ for all } x \in \Omega.$$

Assume also that (2.16) holds with $L(x) = L$. Then

$$u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \leq u \ \text{ on } \overline{\Omega},\ Dv(y) \in Z \ \text{ a.e. } y \in \Omega\}$$

for all $x \in \overline{\Omega}$.

Now let $H \in C(\overline{\Omega} \times \mathbf{R}^n)$ and consider the Hamilton-Jacobi equation

$$H(x, Du(x)) = 0 \quad \text{ in } \Omega.$$

Define

$$Z(x) = \{p \in \mathbf{R}^n \mid H(x, p) \leq 0\}, \quad K(x) = \overline{\mathrm{co}}\, Z(x) \quad \text{ for } x \in \overline{\Omega}.$$

Assume that Ω is bounded and that

$$\begin{cases} \text{there is a function } \psi \in C^1(\overline{\Omega}) \text{ such that} \\ \qquad H(x, D\psi(x)) < 0 \quad \text{ for all } x \in \overline{\Omega}. \end{cases} \tag{6.10}$$

$$\lim_{R \to \infty} \inf\{H(x, p) \mid (x, p) \in \overline{\Omega} \times (\mathbf{R}^n \setminus B(0, R))\} > 0. \tag{6.11}$$

From these assumptions, we see that there are constants $0 < \gamma < R < \infty$ such that

$$B(D\psi(x), \gamma) \subset Z(x) \subset B(D\psi(x), R) \quad \text{ for all } x \in \overline{\Omega}. \tag{6.12}$$

We now assume that

$$\begin{cases} \text{there is a constant } \theta_0 > 1 \text{ and for each } \theta \in (1, \theta_0) \text{ a constant} \\ \eta \equiv \eta(\theta) > 0 \text{ such that for all points } x \in \overline{\Omega} \text{ and } p \in Z(x) \cap \\ \partial K(x), \\ \qquad H(x, \theta^{-1}p + (1-\theta^{-1})D\psi(x)) \leq -\eta. \end{cases} \tag{6.13}$$

We set

$$\bar{Z}(x) = Z(x) - D\psi(x), \quad \bar{K}(x) = K(x) - D\psi(x)$$

for $x \in \overline{\Omega}$.

Propostion 6.5. *Assume that Ω is bounded and that (6.10), (6.11), and (6.13) hold. Then (2.14) and (2.15) hold with $\bar{Z}(x)$ and $\bar{K}(x)$ in place of $Z(x)$ and $K(x)$, respectively.*

Proof. We have already seen in (6.12) that (2.14) holds with $\bar{K}(x)$ in place of $K(x)$.

To see (2.15), with $\bar{Z}(x)$ in place of $Z(x)$, fix any $\theta \in (1, \theta_0)$, and choose a $\delta > 0$ so that for all $(x, y) \in \Delta(\delta),\ p \in B(0, R)$,

$$|H(x, D\psi(x) + p) - H(y, D\psi(y) + p)| \leq \eta(\theta),$$

where R and $\eta(\theta)$ are constants from (6.12) and (6.13), respectively. Fix any points $x \in \overline{\Omega}$ and $p \in \bar{Z}(x) \cap \partial\bar{K}(x)$, and observe that

$$D\psi(x) + p \in Z(x) \cap \partial K(x),$$

and hence, according to (6.13), $H(x, \theta^{-1}p + D\psi(x)) \leq -\eta(\theta)$. Now, let $y \in \overline{\Omega}$ satisfy $|x - y| \leq \delta$. Then we have

$$H(y, D\psi(y) + \theta^{-1}p) \leq H(x, D\psi(x) + \theta^{-1}p) + \eta(\theta) \leq 0,$$

and therefore,

$$\theta^{-1}p + D\psi(y) \in Z(y).$$

That is, for any $(x, y) \in \Delta(\delta)$, $p \in \bar{Z}(x) \cap \partial[\overline{\mathrm{co}}\,\bar{Z}(x)]$, and $\theta \in (1, \theta_0)$, we have $p \in \theta\bar{Z}(y)$, which shows that (2.15) holds with $Z(x)$ replaced by $\bar{Z}(x)$. □

Now, the following proposition is an easy consequence of Theorem 2.3 and Proposition 6.5, as so was Proposition 6.4 from Theorem 2.3 and Proposition 6.3.

Propostion 6.6. *Assume that Ω is bounded and that (6.10), (6.11), and (6.13) hold. Let $u \in C(\overline{\Omega})$ satisfy*

$$D^+u(x) \subset K(x) \quad \text{for all } x \in \Omega.$$

Assume that (2.16) holds. Then

$$u(x) = \sup\{v(x) \mid v \in \mathrm{Lip}(\overline{\Omega}),\ v \leq u \ \text{ on } \overline{\Omega},\ Dv(y) \in Z(y) \ \text{ a.e. } y \in \Omega\}$$

for all $x \in \overline{\Omega}$.

7. Cauchy problem

In this section, as a typical case of unbounded domains we discuss the Cauchy problem

$$u_t(x,t) + H(x, D_x u(x,t)) = 0 \quad \text{for } (x,t) \in \mathbf{R}^n \times (0,T) \tag{7.1}$$

with initial condition

$$u(x,0) = g(x) \quad \text{for } x \in \mathbf{R}^n, \tag{7.2}$$

where $T \in (0,\infty)$ is a given constant, g is a given continuous function on $\mathbf{R}^n$, and $D_x u$ denotes the spatial gradient, $(\partial u/\partial x_1, ..., \partial u/\partial x_n)$, of u.

We use in this section the notation: for $\delta > 0$,

$$\varDelta(\delta) = \{(x,y) \in \mathbf{R}^{2n} \mid |x-y| \le \delta\}.$$

We assume that

$$H \in C(\mathbf{R}^{2n}), \tag{7.3}$$

$$\lim_{R\to\infty} \inf \Big\{ \frac{H(x,p)}{|p|} \mid (x,p) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus B(0,R)) \Big\} = \infty, \tag{7.4}$$

and

$$H \in \mathrm{BUC}(\mathbf{R}^n \times B(0,R)) \quad \text{for all } R > 0. \tag{7.5}$$

We remark that conditions (7.4) and (7.5) imply that there are convex functions $H_1,\ H_2 \in C(\mathbf{R}^n)$ such that

$$H_1(p) \le H(x,p) \le H_2(p) \quad \text{for all } (x,p) \in \mathbf{R}^{2n}, \tag{7.6}$$

$$\lim_{|p|\to\infty} \frac{H_i(p)}{|p|} = \infty \quad \text{for } i = 1,2. \tag{7.7}$$

We introduce the sets $Z(x,r),\ K(x,r) \subset \mathbf{R}^{n+1}$, with $(x,r) \in \mathbf{R}^n \times \mathbf{R}$, by

$$Z(x,r) = \{(p,q) \in \mathbf{R}^n \times \mathbf{R} \mid q + H(x,p) \le r\} \quad \text{and} \quad K(x,r) = \overline{\mathrm{co}}\, Z(x,r).$$

Note that this notation differs from that of previous sections.

A natural inclusion associated with (7.1), i.e. the relaxed problem, is

$$D^+ u(x,t) \subset K(x,0) \quad \text{for } (x,t) \in \mathbf{R}^n \times (0,T), \tag{7.8}$$

where the superdifferential $D^+ u$ of u is taken with respect to (x,t).

Let $\widehat{H}$ denote the function on $\mathbf{R}^{2n}$ such that for each $x \in \mathbf{R}^n$ the function $\widehat{H}(x,\cdot)$ is the convex envelope of the function $H(x,\cdot)$. That is,

$$\widehat{H}(x,p) = \sup\{a \cdot p + b \mid (a,b) \in \mathbf{R}^n \times \mathbf{R},\ a \cdot q + b \le H(q) \ \text{ for all } q \in \mathbf{R}^n\}.$$

Observe that

$$H_1(p) \le \widehat{H}(x,p) \le H_2(p) \quad \text{for } (x,p) \in \mathbf{R}^{2n}, \tag{7.9}$$

and that

$$\widehat{H} \in \mathrm{BUC}(\mathbf{R}^n \times B(0,R)) \quad \text{for all } R > 0. \tag{7.10}$$

Noting that for all $(x,r) \in \mathbf{R}^{n+1}$,

$$K(x,r) = \{(p,q) \in \mathbf{R}^{n+1} \mid q + \widehat{H}(x,p) \le r\},$$

we see that inclusion (7.8) for $u \in C(\mathbf{R}^n \times (0,T))$ is equivalent to saying that u is a viscosity subsolution of

$$u_t(x,t) + \widehat{H}(x, D_x u(x,t)) = 0 \quad \text{for } (x,t) \in \mathbf{R}^n \times (0,T). \tag{7.11}$$

The main result in this section is the following:

Theorem 7.1. *Assume that (7.3)–(7.5) hold. Let $u \in \mathrm{BUC}(\mathbf{R}^n \times (0,T))$. Assume that u is a viscosity subsolution of (7.11). Then, for all $(x,t) \in \mathbf{R}^n \times (0,T)$,*

$$\begin{aligned} u(x,t) = \sup\{v(x,t) \mid v \in \mathrm{Lip}(\mathbf{R}^n \times (0,T)),\ v \le u \textit{ in } \mathbf{R}^n \times (0,T),\\ v_s(y,s) + H(y, D_y v(y,s)) \le 0 \textit{ a.e. } \mathbf{R}^n \times (0,T)\}. \end{aligned} \tag{7.12}$$

Recalling (see, for instance, [1]) that the Cauchy problem (7.11) and (7.2) has a unique viscosity solution in $\mathrm{BUC}(\mathbf{R}^n \times [0,T))$ provided $g \in \mathrm{BUC}(\mathbf{R}^n)$, we state:

Corollary 7.2. *Assume that $g \in \mathrm{BUC}(\mathbf{R}^n)$ and that (7.3)–(7.5) hold. Let $u \in \mathrm{BUC}(\mathbf{R}^n \times [0,T))$ be the viscosity solution of (7.11) satisfying (7.2). Then for all $(x,t) \in \mathbf{R}^n \times (0,T)$,*

$$\begin{aligned} u(x,t) = \sup\{v(x,t) \mid v \in \mathrm{Lip}(\mathbf{R}^n \times [0,T)),\ v(\cdot,0) \le g \textit{ on } \mathbf{R}^n,\\ v_s(y,s) + H(y, D_y v(y,s)) \le 0 \textit{ a.e. } \mathbf{R}^n \times (0,T)\}. \end{aligned} \tag{7.13}$$

Proof. We write $w(x,t)$ for the right hand side of (7.13). Theorem 7.1 yields that $u \le w$ on $\mathbf{R}^n \times (0,T)$. Let $v \in \mathrm{Lip}(\mathbf{R}^n \times [0,T))$ be a function which satisfies

$$v_t(x,t) + H(x, D_x v(x,t)) \le 0 \quad \text{a.e. } \mathbf{R}^n \times (0,T)$$

and $v(\cdot,0) \le g$ on $\mathbf{R}^n$. Then, since $\widehat{H} \le H$, it is clear that

$$v_t(x,t) + \widehat{H}(x, D_x v(x,t)) \le 0 \quad \text{a.e. } \mathbf{R}^n \times (0,T).$$

Since $\widehat{H}(x,\cdot)$ is convex, as is well-known, v is a viscosity subsolution of (7.11). By comparison, we have $v \le u$ on $\mathbf{R}^n \times (0,T)$, from which we see that $w \le u$ on $\mathbf{R}^n \times (0,T)$. Thus we have $u = w$ on $\mathbf{R}^n \times (0,T)$. □

We need the following two lemmas.

Lemma 7.3. *Assume that (7.3)–(7.5) hold. For any $R > 0$ and $\varepsilon > 0$ there exist constants $\delta > 0$ depending only on ε and the modulus of continuity of H on $\mathbf{R}^n \times B(0, R+1)$ such that for any $(x, y) \in \Delta(\delta)$,*

$$Z_R(x,0) + B(0,\delta) \subset Z_{R+1}(y,\varepsilon), \tag{7.14}$$

where $Z_L(x,r) = Z(x,r) \cap B(0,L)$.

Proof. Fix $\varepsilon > 0$ and $R > 0$. Let ω denote the modulus of continuity of H on $\mathbf{R}^n \times B(0, R+1)$.

Fix a constant $\delta > 0$ so that $\delta + \omega(2\delta) \le \min\{\varepsilon, 1\}$. Fix $(\xi, \eta) \in B(0,\delta)$, $(x,y) \in \mathbf{R}^{2n}$ so that $|x-y| \le \delta$, and $(p,q) \in Z_R(x,0)$.

Noting that $|p+\xi| \le R+1$, we compute that

$$q + \eta + H(y, p+\xi) \le q + H(x,p) + \eta + \omega(|x-y| + |\xi|) \le \delta + \omega(2\delta) \le \varepsilon,$$

and hence that

$$(p+\xi, q+\eta) \in Z_{R+1}(y,\varepsilon),$$

which finishes the proof. □

Lemma 7.4. *Assume that (7.3)–(7.5) hold. For any $R > 0$ and $\varepsilon > 0$ there exist constants $\delta > 0$ and $\rho > 0$ depending only on R, ε, the modulus of continuity of $\widehat{H}$ on $\mathbf{R}^n \times B(0,R+1)$, and H_i, with $i = 1,2$, where H_i are from (7.6), such that for any $(x,y) \in \Delta(\delta)$,*

$$K_R(x,0) \subset \operatorname{co} Z_\rho(y,\varepsilon), \tag{7.15}$$

where $K_R(x,r) = K(x,r) \cap B(0,R)$.

Proof. Fix $R > 0$. We first prove that there is a constant $\rho > 0$ depending only on R and H_i, with $i = 1,2$, such that for all $x \in \mathbf{R}^n$,

$$K_R(x,0) \subset \operatorname{co} Z_\rho(x,0). \tag{7.16}$$

To see this, we fix $x \in \mathbf{R}^n$ and $(\bar{p}, \bar{q}) \in K_R(x,0)$. Choose $(a,b) \in \mathbf{R}^{n+1}$ so that

$$\begin{aligned} a \cdot \bar{p} + b &= \widehat{H}(x,\bar{p}), \\ a \cdot p + b &\le \widehat{H}(x,p) \quad \text{for all } p \in \mathbf{R}^n. \end{aligned}$$

Setting

$$B = \{p \in \mathbf{R}^n \mid H(x,p) = a \cdot p + b\},$$

we claim that

$$\bar{p} \in \operatorname{co} B. \tag{7.17}$$

If this is not the case, from the separation theorem, we see that there is a $(\alpha, \beta) \in \mathbf{R}^{n+1}$ such that

$$\alpha \cdot p + \beta < 0 < \alpha \cdot \bar{p} + \beta \quad \text{for all } p \in \operatorname{co} B.$$

Choosing $\varepsilon > 0$ small enough, in view of (7.4), we have

$$\varepsilon(\alpha \cdot p + \beta) \leq H(x,p) - a \cdot p - b \quad \text{for all } p \in \mathbf{R}^n.$$

Hence we have

$$\widehat{H}(x,p) \geq (a + \varepsilon\alpha) \cdot p + b + \varepsilon\beta \quad \text{for all } p \in \mathbf{R}^n,$$

and consequently,

$$\widehat{H}(x,\hat{p}) \geq (a + \varepsilon\alpha) \cdot \bar{p} + \beta + \varepsilon\beta = \widehat{H}(x,\hat{p}) + \varepsilon(\alpha \cdot \bar{p} + \beta),$$

which yields a contradiction, $\alpha \cdot \bar{p} + \beta \leq 0$, proving (7.17).

Next, we wish to show that there is a constant $l_R > 0$ depending only on $R > 0$ and H_i, with $i = 1, 2$, such that

$$|a| + |b| \leq l_R. \tag{7.18}$$

Indeed, if $a \neq 0$, then, since

$$\widehat{H}(x, \bar{p} + \frac{a}{|a|}) \geq a \cdot (\bar{p} + \frac{a}{|a|}) + b = \widehat{H}(x,\bar{p}) + |a|,$$

we have

$$|a| \leq \max_{B(0,R+1)} H_2 - \min_{B(0,R)} H_1.$$

Therefore, noting that

$$|b| \leq |\widehat{H}(x,\bar{p})| + |a \cdot \bar{p}|,$$

we conclude that (7.18) holds for some constant l_R depending only on $R > 0$ and H_i, with $i = 1, 2$.

Now, we show that there is a constant $m_R > 0$ depending only on $R > 0$ and H_i, with $i = 1, 2$, such that

$$B \subset B(0, m_R).$$

To see this, fix any $p \in B$ and observe that

$$H_1(p) \leq H(x,p) = a \cdot p + b,$$

and

$$\frac{H_1(p)}{|p| + 1} \leq |a| + |b| \leq l_R.$$

In view of (7.7), we find a constant $m_R > 0$ depending only on R and H_i, with $i = 1, 2$, such that $B \subset B(0, m_R)$.

Now that $\bar{p} \in \operatorname{co} B$, there are points $p_1, ..., p_m \in B \subset B(0, m_R)$ and positive numbers $\lambda_1, ..., \lambda_m$, with $m \in \mathbf{N}$, such that

$$\bar{p} = \sum_{i=1}^{m} \lambda_i p_i \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = 1.$$

Note that

$$(p_i, H(x,p_i)) \in Z(x,0) \quad \text{for all } i = 1, ..., m,$$

and that

$$\widehat{H}(x,\bar{p}) = a \cdot \bar{p} + b = \sum_{i=1}^{m} \lambda_i (a \cdot p_i + b) = \sum_{i=1}^{m} \lambda_i H(x,p_i).$$

This last equality shows that

$$\begin{aligned}(\bar{p}, \widehat{H}(x,\bar{p})) &= \sum_{i=1} \lambda_i (p_i, H(x,p_i)) \\ &\in \ \mathrm{co}\,\{(p,q) \in Z(x,0) \mid p \in B(0,m_R),\ |q| \le M_R\},\end{aligned}$$

where $M_R := \max_{B(0,R)}(|H_1| + |H_2|)$. Since $\bar{q} \le \widehat{H}(x,\bar{p})$, in view of the property that for any $r > 0$, $Z(x,0) + (0,-r) = Z(x,-r) \subset Z(x,0)$, then $(p, q - r) \in Z(x,0)$, we see that

$$\begin{aligned}(\bar{p}, \bar{q}) &= (\bar{p}, \widehat{H}(x,\bar{p})) + (0, \bar{q} - \widehat{H}(x,\bar{p})) \\ &\in \ \mathrm{co}\,\{(p,q) \in Z(x,0) \mid p \in B(0,m_R),\ |q| \le 2M_R + R\}.\end{aligned}$$

Here we have used the estimate

$$|\bar{q} - \widehat{H}(x,\bar{p})| \le |\bar{q}| + |\widehat{H}(x,\bar{p})| \le R + M_R.$$

Thus we have shown that (7.16) holds with $\rho = m_R + 2M_R + R$.

Now, we observe that Lemma 7.3, with $\widehat{H}$ in place of H, yields that there is a constant $\delta > 0$ depending only on ε and the modulus of continuity of H on $\mathbf{R}^n \times B(0, R+1)$ such that for all $(x,y) \in \Delta(\delta)$,

$$K_R(x,0) \subset K_{R+1}(y,\varepsilon),$$

which obviously implies that

$$K_R(x,0) \subset K_R(y,\varepsilon).$$

This combined with (7.16) yields that for all $(x,y) \in \Delta(\delta)$,

$$K_R(x,0) \subset K_R(y,\varepsilon) = K_R(y,0) + (0,\varepsilon) \subset \ \mathrm{co}\,Z_\rho(y,0) + (0,\varepsilon) = \ \mathrm{co}\,Z_\rho(y,\varepsilon),$$

completing the proof. $\square$

Proof of Theorem 7.1. We first show that we may assume that u is defined and bounded Lipschitz continuous on $\mathbf{R}^n \times (-\delta, T+\delta)$ for some constant $\delta > 0$ and that

$$u_t(x,t) + \widehat{H}(x, D_x u(x,t)) \le 0 \quad \text{in } \mathbf{R}^n \times (-\delta, T+\delta) \tag{7.19}$$

in the viscosity sense. Indeed, we have

$$\begin{aligned}u(x,t) = \sup\{&v(x,t) \mid v \in \mathrm{Lip}(\mathbf{R}^n \times (-\delta, T+\delta)) \text{ for some } \delta > 0, \\ &v \text{ is a viscosity solution of (7.19)},\ v \le u \text{ on } \mathbf{R}^n \times (0,T)\}.\end{aligned} \tag{7.20}$$

To see this, we solve the Cauchy problem

$$w_t(x,t) + \widehat{H}(x, D_x w(x,t)) \le 0 \quad \text{in } \mathbf{R}^n \times (T, T+1)$$

with the initial condition

$$w(x,T) = \lim_{t \nearrow T} u(x,t) \quad \text{for } x \in \mathbf{R}^n. \tag{7.21}$$

Due to [1], there is a unique viscosity solution $w \in \mathrm{BUC}(\mathbf{R}^n \times [T, T+1))$ for which (7.21) holds. We extend the domain of definition of w to $\mathbf{R}^n \times (0, T+1)$ by setting

$$w(x,t) = u(x,t) \quad \text{for } (x,t) \in \mathbf{R}^n \times (0,T).$$

It is easy to see that $w \in \mathrm{BUC}(\mathbf{R}^n \times (0, T+1))$, and moreover it is rather standard to see that w is a viscosity subsolution of

$$w_t(x,t) + \widehat{H}(x, D_x w(x,t)) = 0 \quad \text{for } (x,t) \in \mathbf{R}^n \times (0, T+1).$$

Fix any $\varepsilon > 0$. Since $w \in \mathrm{BUC}(\mathbf{R}^n \times (0, T+1))$, there is a constant $\delta \in (0, 1/2)$ such that

$$\begin{aligned} u(x,t) - 2\varepsilon = w(x,t) - 2\varepsilon &\le w(x, t-\delta) - \varepsilon \\ &\le w(x,t) = u(x,t) \quad \text{for all } (x,t) \in \mathbf{R}^n \times (0,T). \end{aligned} \tag{7.22}$$

It is clear that the function $z(x,t) := w(x, t-\delta) - 2\varepsilon$ is defined and bounded continuous on $\mathbf{R}^n \times (-\delta, T+\delta)$ and a viscosity solution of (7.19).

The next step is to take the sup-convolution of z in the t variable. That is, for $\gamma > 0$, we consider the function

$$z^\gamma(x,t) = \sup\{ z(x,s) - \frac{1}{2\gamma}(t-s)^2 \mid s \in (-\delta, T+\delta) \} \quad \text{for } (x,t) \in \mathbf{R}^{n+1}.$$

If $\gamma > 0$ is small enough, then z^γ is a viscosity solution of (7.19) in $\mathbf{R}^n \times (-\delta/2, T+\delta/2)$ and

$$z(x,t) \le z^\gamma(x,t) \le z(x,t) + \varepsilon \quad \text{for all } (x,t) \in \mathbf{R}^n \times (-\delta, T+\delta). \tag{7.23}$$

Also, for each $\gamma > 0$, the collection of functions $z^\gamma(x, \cdot)$, with $x \in \mathbf{R}^n$, is equi-Lipschitz continuous on $(-\delta/2, T+\delta/2)$. Since

$$H_1(D_x z^\gamma(x,t)) \le L_\gamma \quad \text{for } (x,t) \in \mathbf{R}^n \times (-\delta/2, T+\delta/2)$$

in the viscosity sense, where $L_\gamma > 0$ is a uniform Lipschitz bound of the functions $z^\gamma(x,\cdot)$ on $(-\delta/2, T+\delta/2)$, we see that the functions $z^\gamma(\cdot, t)$ are Lipschitz continuous on $\mathbf{R}^n$ with a Lipschitz bound independent of $t \in (-\delta/2, T+\delta/2)$.

Now, writing $U(x,t)$ for the right hand side of (7.20), using (7.22) and (7.23), we see that for sufficiently small $\gamma > 0$ and for all $(x,t) \in \mathbf{R}^n \times (0,T)$, we have

$$u(x,t) \ge z(x,t) + \varepsilon \ge z^\gamma(x,t),$$

and hence,

$$U(x,t) \geq z^\gamma(x,t) \geq z(x,t) \geq u(x,t) - 3\varepsilon,$$

proving (7.20).

In what follows we assume that $u \in \mathrm{Lip}(\mathbf{R}^n \times (-\delta, T+\delta))$ and it satisfies (7.19) in the viscosity sense.

We now follow the line of arguments in the proof of Theorem 4.1. It is worth mentioning that conditions (7.14) and (7.15) in the proof below are somehow similar to (4.1) and (2.12) in the proof of Theorem 4.1.

Let $R > 0$ be a Lipschitz bound of the function u. Fix any $\varepsilon > 0$. Thanks to Lemma 7.4, there are constants $\rho > 0$ and $\gamma > 0$ such that for any $(x,y) \in \Delta(\gamma)$,

$$K_R(x,0) \subset \mathrm{co}\, Z_\rho(y,\varepsilon),$$

which implies that for all $r \in \mathbf{R}$,

$$K_R(x,r) \subset \mathrm{co}\, Z_\rho(y, r+\varepsilon).$$

We reselect $\gamma > 0$ so that for any $(x,y) \in \Delta(3(\rho+1)\gamma)$,

$$K_R(x,r) \subset \mathrm{co}\, Z_\rho(y, r+\varepsilon). \tag{7.24}$$

Lemma 7.3 guarantees that we may assume that for all $r \in \mathbf{R}$ and $(x,y) \in \Delta(3(\rho+1)\gamma)$,

$$Z_\rho(x,r) + B(0,\gamma) \subset Z_{\rho+1}(y, r+\varepsilon). \tag{7.25}$$

Let $\mu \in (0,\gamma)$ be a constant to be fixed later. Choose a set $Y_\mu \subset \mathbf{R}^n \times (-\delta, T+\delta)$ so that

$$\#(Y_\mu \cap B(0,L)) < \infty \quad \text{ for all } L > 0,$$

and

$$\bigcup_{(y,s)\in Y_\mu} B(y,s;\mu) \supset \mathbf{R}^n \times (-\delta, T+\delta).$$

We write $Q = \mathbf{R}^n \times (0,T)$ and $Q_\delta = \mathbf{R}^n \times (-\delta, T+\delta)$. We set

$$L(\xi,\eta;y) = \sup\{\xi \cdot p + \eta q \mid (p,q) \in Z_\rho(y,\varepsilon) + B(0,\gamma)\}$$

for $\xi, y \in \mathbf{R}^n$ and $\eta \in \mathbf{R}$ and

$$v(x,t;y,s) = u(y,s) + L(x-y, t-s; y)$$

for $(x,t) \in Q_\delta$ and $(y,s) \in Y_\mu$.

We set $\beta = (\rho+1)\gamma$. By Lemma 3.1, we have for all $(x,t) \in Q_\delta$ and $(y,s) \in Y_\mu \cap B(x,t;3\beta)$,

$$u(x,t) + \gamma(|x-y|^2 + (t-s)^2)^{1/2} \leq v(x,t;y,s), \tag{7.26}$$

since

$$D^+u(x,t) \subset K_R(x,0) \subset \mathrm{co}\, Z_\rho(y,\varepsilon)$$

by (7.24).

Note that the functions $v(\cdot;y,s)$ are Lipschitz continuous with $\rho+1$ as a Lipschitz bound.

Also, we observe that for all $(x,t)\in Q_\delta$ and $(y,s)\in Y_\mu\cap B(x,t;3\beta)$, if the function $v(\cdot;y,s)$ is differentiable at (x,t), then

$$Dv(x,t;y,s)\in Z_\rho(y,\varepsilon)+B(0,\gamma)\subset Z_{\rho+1}(x,2\varepsilon),$$

by (7.25), i.e.,

$$v_t(x,t;y,s)+H(x,D_xv(x,t;y,s))\le 2\varepsilon. \tag{7.27}$$

Consequently, for any $(y,s)\in Y_\mu$, we have

$$v_t(x,t;y,s)+H(x,D_xv(x,t;y,s))\le 2\varepsilon \quad \text{a.e. } (x,t)\in Q_\delta\cap B(y,s;3\beta).$$

We define $v:Q\to\mathbf{R}$ by

$$v(x,t)=\min\{v(x,t;y,s)\mid (y,s)\in Y_\mu\cap B(x,t;3\beta)\}.$$

By reselecting $\gamma>0$ if necessary, we may assume that $3\beta\le\delta$. We fix $\mu\in(0,\gamma)$ so that $(2\rho+1)\mu<\min\{\gamma\beta,\varepsilon\}$.

Fix any $(\bar{x},\bar{t})\in Q$. We show that for all $(x,t)\in B(\bar{x},\bar{t};\beta)\cap Q$,

$$v(x,t)=\min\{v(x,t;y,s)\mid (y,s)\in Y_\mu\cap B(\bar{x},\bar{t};2\beta)\}, \tag{7.28}$$

which guarantees that v is Lipschitz continuous on $B(\bar{x},\bar{t};\beta)\cap Q$, with $\rho+1$ as a Lipschitz bound, and that

$$v_t(x,t)+H(x,D_xv(x,t))\le 2\varepsilon \quad \text{a.e. } (x,t)\in B(\bar{x},\bar{t};\beta)\cap Q.$$

These show immediately that v is a Lipschitz continuous function on Q and satisfies

$$v_t(x,t)+H(x,D_xv(x,t))\le 2\varepsilon \quad \text{a.e. } (x,t)\in Q.$$

To see (7.28), fix any $(x,t)\in B(\bar{x},\bar{t};\beta)\cap Q$ and write $w(x,t)$ for the right hand side of (7.28).

Since $B(\bar{x},\bar{t};2\beta)\subset B(x,t;3\beta)$, we see that $v(x,t)\le w(x,t)$.

We note that $Y_\mu\cap B(x,t;\beta)\ne\emptyset$ and recall that u and $L(\cdot;y,s)$ are Lipschitz continuous with R and $\rho+1$ as their Lipschitz bounds, which yields that for any $(y,s)\in Y_\mu\cap B(x,t;\beta)$,

$$\begin{aligned} v(x,t)&\le u(y,s)+L(x-y,t-s;y)\le u(x,t)+R\mu+(\rho+1)\mu \\ &\le u(x,t)+(2\rho+1)\mu. \end{aligned} \tag{7.29}$$

On the other hand, from (7.26) we see that for all $(y,s)\in Y_\mu\cap(B(x,t;3\beta)\setminus B(x,t;\beta))$,

$$v(x,t;y,s)\ge u(x,t)+\gamma\beta.$$

Since $(2\rho+1)\mu<\gamma\beta$, from what we just observed, we have

$$v(x,t)=\min\{v(x,t;y,s)\mid (y,s)\in Y_\mu\cap B(x,t;\beta)\}.$$

Noting that $B(x, r; \beta) \subset B(\bar{x}, \bar{t}; 2\beta)$, we see that

$$v(x,t) \geq \min\{v(x,t;y,s) \mid (y,s) \in Y_\mu \cap B(\bar{x},\bar{t};2\beta)\} = w(x,t).$$

Thus we obtain $v(x,t) = w(x,t)$.

From (7.26) and (7.29), we have

$$u(x,t) \leq v(x,t) < u(x,t) + \varepsilon \quad \text{for } (x,t) \in Q.$$

If we set $v^\varepsilon(x,t) = v(x,t) - \varepsilon(2t+1)$, then the function v^ε has the properties:

$$\begin{gathered} u(x,t) - \varepsilon(2T+1) \leq v^\varepsilon(x,t) \leq u(x,t) \quad \text{for all } (x,t) \in Q, \\ v^\varepsilon \in \mathrm{Lip}(Q), \\ v^\varepsilon_t(x,t) + H(x, D_x v^\varepsilon(x,t)) \leq 0 \quad \text{a.e. } (x,t) \in Q. \end{gathered}$$

These show that

$$\begin{aligned} u(x,t) = \sup\{v(x,t) \mid v \in \mathrm{Lip}(Q),\ v_t(x,t) + H(x, D_x v(x,t)) \leq 0 \ \text{ a.e. } Q, \\ v \leq u \text{ on } Q\}, \end{aligned}$$

which completes the proof. $\square$

8. Appendix

Here we prove the following assertion.

Propostion 8.1. *Let Ω be a bounded open subset of $\mathbf{R}^N$ with Lipschitz boundary. Then condition (2.13) holds.*

Proof. Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with Lipschitz boundary.

It is then immediate to see that for all $z \in \overline{\Omega}$, there exist $r > 0$ and $v \in B(0,1)$ such that

$$B(x + tv, rt) \subset \Omega \text{ for all } t \in (0,r) \text{ and } x \in \overline{\Omega} \cap B(z,r).$$

Next, by the compactness of $\overline{\Omega}$, there exist finite sequences $\{z_j\}_{j=1}^N \subset \Omega$, $\{r_j\}_{j=1}^N \subset (0,\infty)$, and $\{v_j\}_{j=1}^N \subset B(0,1)$ such that

$$\overline{\Omega} \subset \bigcup_{j=1}^N B(z_j, \frac{r_j}{4}),$$

and

$$B(x + tv_j, r_j t) \subset \Omega \quad \text{for all } t \in (0, r_j],\ x \in \overline{\Omega} \cap B(z_j, r_j),\ \text{and } j \in \{1, ..., N\}.$$

By an argument based on a partition of unity, we see that there exists a sequence $\{\phi_j\}_{j=1}^N \subset C_0^\infty(\mathbf{R}^n)$ such that

$$\phi_j(x) \geq 0 \quad \text{for all } x \in \mathbf{R}^n \text{ and } j \in \{1, ..., N\},$$
$$\operatorname{spt}\phi_j \subset B(z_j, \frac{r_j}{2}) \quad \text{for all } j \in \{1, ..., N\},$$
$$\sum_{j=1}^N \phi_j(x) = 1 \qquad \text{for all } x \in \overline{\Omega}.$$

Set

$$\delta = \min_j r_j \quad \text{and} \quad \phi(x) = \sum_{j=1}^N \phi_j(x) v_j \quad \text{for } x \in \mathbf{R}^n.$$

We now intend to show that

$$B(x + t\phi(x), \frac{\delta t}{N}) \subset \Omega \quad \text{for all } x \in \overline{\Omega} \text{ and } t \in (0, \frac{\delta}{2}]$$

Fix $x \in \overline{\Omega}$ and $0 < t \leq \frac{\delta}{2}$. By relabeling if needed, we may assume that

$$\phi_1(x) > 0, \ldots, \phi_m(x) > 0, \qquad \phi_{m+1}(x) = \cdots = \phi_N(x) = 0$$

for some $m \in \{1, \ldots, N\}$.

Set

$$x_k = x + t \sum_{j<k} \phi_j(x) v_j,$$

for $k = 1, 2, \ldots, m$. It follows that

$$|x_k - x| \leq t \sum_{j<k} \phi_j(x) \leq t \leq \frac{\delta}{2}.$$

We claim that

$$x_k \in \overline{\Omega} \qquad \text{for all } k = 1, \ldots, m.$$

To see this, we first note that $x_1 = x \in \overline{\Omega}$. Next, assume that $x_k \in \overline{\Omega}$ for some $k < m$. Noting that

$$x \in \operatorname{spt}\phi_k \subset B(z_k, \frac{r_k}{2}),$$

we see that

$$|x_k - z_k| \leq |x - z_k| + |x_k - x| \leq \frac{r_k}{2} + \frac{\delta}{2} \leq r_k,$$

and therefore,

$$B(x_k + t\phi_k(x)v_k, r_k t\phi_k(x)) \subset \Omega,$$

which shows that

$$x_{k+1} = x_k + t\phi_k(x)v_k \in \Omega.$$

The induction argument now ensures that $x_k \in \overline{\Omega}$ for all $k \leq m$.

Next, note that

$$x + t\phi(x) = x_m + t\phi_m(x)v_m.$$

Since

$$x \in \operatorname{spt}\phi_m \subset B(z_m, \frac{r_m}{2}),$$

we get

$$|x_m - z_m| \leq |x - z_m| + |x_m - x| \leq \frac{\delta}{2} + \frac{r_m}{2} \leq r_m.$$

Noting that

$$x_m \in \overline{\Omega} \cap B(z_m, r_m),$$

we see that

$$B(x_m + t\phi_m(x)v_m, r_m t\phi_m(x)) \subset \Omega.$$

We may assume

$$\phi_1(x) \leq \cdots \leq \phi_m(x).$$

Then, since

$$\sum_{j=1}^{N} \phi_j(x) = 1,$$

we have

$$\phi_m(x) \geq \frac{1}{N}.$$

Hence,

$$B(x + t\phi(x), \frac{\delta t}{N}) \subset \Omega.$$

Thus we have shown that

$$B(x + t\phi(x), \frac{\delta t}{N}) \subset \Omega \quad \text{ for all } x \in \overline{\Omega} \text{ and } t \in (0, \frac{\delta}{2}].$$

To complete the proof, we let $\gamma > 0$, set

$$\psi_\gamma(x) = x + \frac{4\gamma N}{\delta}\phi(x),$$

and will show that (2.13) holds with an appropriate $\gamma_0 > 0$.

Since

$$\phi \in C_0^\infty(\mathbf{R}^n, \mathbf{R}),$$

there is a constant $M > 0$ such that $\|D\phi\|_\infty \leq M$.

Fix $x \in \overline{\Omega^\gamma}$. There are $y \in \overline{\Omega}$ and $\zeta \in B(0,1)$ such that $x = y + \gamma\zeta$. Let $\eta \in B(0,1)$. Then we have

$$x + \psi_\gamma(x) + \gamma\eta = y + \psi_\gamma(y) + \gamma z,$$

where

$$z = \zeta + \eta + \frac{4N}{\delta}(\phi(x) - \phi(y)).$$

Note that

$$|z| \le 2 + \frac{4MN}{\delta}|x-y| \le 2 + \frac{4\gamma MN}{\delta},$$

and observe that if $\gamma \le \frac{\delta}{2MN}$, then $|z| \le 2+2 = 4$, and that if $\gamma \le \min\{\frac{\delta}{2MN}, \frac{\delta^2}{8N}\}$, then

$$x + \psi_\gamma(x) + \gamma\eta = y + \frac{4\gamma N}{\delta}\phi(y) + \gamma z \in B\Big(y + \frac{4\gamma N}{\delta}\phi(y), \frac{\delta}{N}\cdot\frac{4\gamma N}{\delta}\Big) \subset \Omega.$$

That is, if

$$0 < \gamma \le \min\{\frac{\delta}{2MN}, \frac{\delta^2}{8N}\} \quad \text{and} \quad x \in \overline{\Omega}^\gamma,$$

then we have

$$x + \psi_\gamma(x) + B(0,\gamma) \subset \Omega,$$

which guarantees that

$$x + \psi_\gamma(x) \subset \Omega_\gamma.$$

Setting

$$\gamma_0 = \min\{\frac{\delta}{2MN}, \frac{\delta^2}{8N}\},$$

we conclude that

$$\psi_\gamma(\overline{\Omega}^\gamma) \subset \Omega_\gamma \qquad \text{for all } \gamma \in (0,\gamma_0],$$
$$|\psi_\gamma(x) - x| = \frac{4\gamma N}{\delta}|\phi(x)| \le \|\phi\|_\infty \frac{4\gamma N}{\delta} = C\gamma,$$

and

$$\|D\psi_\gamma(x) - I\| = \frac{4\gamma N}{\delta}\|D\phi(x)\| \le C\gamma.$$

for some constant $C > 0$, independent of γ, which completes the proof. □

The following example gives a Hamiltonian H for which (2.3) and (2.4) hold, but (2.11) does not hold.

Example 8.2. Consider the case where $n = 2$, $\Omega = (-1,1)\times(-1,1)$, and

$$H(x,r,p) = r + \min\{p_2, x_1p_1 + p_2\}.$$

We observe that $p \in Z(x,r) \equiv \{p \in \mathbf{R}^2 \mid H(x,r,p) \le 0\}$ if and only if $p_2 \le -r$ or $p_2 \le -r - x_1p_1$. Hence

$$Z(x,r) = \begin{cases} Z_+(x_1,r) & \text{if } x_1 > 0, \\ \mathbf{R}\times(-\infty,-r] & \text{if } x_1 = 0, \\ Z_-(x_1,r) & \text{if } x_1 < 0, \end{cases}$$

where

$$Z_+(x_1,r) = \{p \in (-\infty,0]\times\mathbf{R} \mid p_2 \le -r - x_1p_1\} \cup ((0,\infty)\times(-\infty,-r]),$$
$$Z_-(x_1,r) = ((-\infty,0]\times(-\infty,-r]) \cup \{p \in (0,\infty)\times\mathbf{R} \mid p_2 \le -r - x_1p_1\}.$$

Also, note that

$$K(x,r) \equiv \overline{\text{co}}\, Z(x,r) = \begin{cases} \mathbf{R}^2 & \text{if } x_1 \neq 0, \\ \mathbf{R} \times (-\infty, -r] & \text{if } x_1 = 0. \end{cases}$$

Let $R > 0$. We have

$$K_R(x,r) \equiv Z(x,r) \cap B(0,R) = \begin{cases} B(0,R) & \text{if } x_1 \neq 0, \\ \{p \in B(0,R) \mid p_2 \leq -r\} & \text{if } x_1 = 0. \end{cases}$$

Let $\varepsilon \in (0,1)$, $\delta > 0$, $\rho > 0$, and $x_\varepsilon = (\varepsilon, 0)$, and observe that

$$K_R(x_\varepsilon, 0) + B(0,\varepsilon) = B(0, R+\varepsilon),$$

and moreover that

$$K_R(x_\varepsilon, 0) + B(0,\varepsilon) \subset K(0,-\delta)$$

if and only if $\delta \geq R + \varepsilon$. This shows that the collection $\{K(x,r)\}$ does not satisfy condition (2.11). On the other hand, the Hamiltonian H satisfies conditions (2.3) and (2.4).

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References

1. G. Barles, Remarques sur des résultats d'existence pour les équations de Hamilton-Jacobi du premier ordre, *Ann. Inst. Henri Poincaré* **2** (1985), no. 1, 21–32.
2. E. N. Barron, Viscosity solutions and analysis in L^∞. *Nonlinear analysis, differential equations and control* (Montreal, QC, 1998), NATO Sci. Ser. C Math. Phys. Sci., 528, Kluwer Acad. Publ., Dordrecht, 1999, pp. 1–60
3. T. Bhattacharya, E. DiBenedetto, and J. Manfredi, Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems, *Rend. Sem. Mat. Univ. Pol. Torino*, (1989), 15–68.
4. E. N. Barron, R. R. Jensen, and C. Y. Wang, The Euler equation and absolute minimizers of L^∞ functionals, *Arch. Ration. Mech. Anal.* **157** (2001), no. 4, 255–283.
5. M. G. Crandall, L. C. Evans, and R. F. Gariepy, Optimal Lipschitz extensions and the infinity laplacian, *Calc. Var. Partial Differential Equations* **13** (2001), no. 2, 123–139.
6. M. G. Crandall, H. Ishii, and P.-L. Lions, Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited, *J. Math. Soc. Japan 39* (1987), no. 4, 581–596.
7. M. G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67.

8. M. G. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **277** (1983), no. 1, 1–42.
9. B. Dacorogna and P. Marcellini, *Implicit partial differential equations.* Progress in Nonlinear Differential Equations and their Applications, 37. Birkhäuser Boston, Inc., Boston, MA, 1999
10. I. Ekeland and R. Temam, *Convex analysis and variational problems.* Translated from the French. Studies in Mathematics and its Applications, Vol. 1. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1976
11. T. Ishibashi and S. Koike, On fully nonlinear PDEs derived from variational problems of L^p norms, *SIAM J. Math. Anal.* **33** (2001), no.3, 545-569.
12. H. Ishii, A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type, *Proc. Amer. Math. Soc.* **100** (1987), no. 2, 247–251.
13. H. Ishii and P. Loreti, Relaxation in an L^∞-optimization problem, to appear in *Proc. Roy. Soc. Edinburgh, Sect. A.*
14. R. Jensen, Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient, *Arch. Rational Mech. Anal.* **123** (1993), no. 1, 51–74.
15. B. Kawohl, On a family of torsion creep problems, *J. reine angrew. Math.* **410** (1990), 1–22.
16. P.-L. Lions, *Generalized solutions of Hamilton-Jacobi equations* Research Notes in Math. 69, Pitman, Boston 1982
17. S. Müller and M. A. Sychev, Optimal existence theorems for nonhomogeneous differential inclusions, *J. Funct. Anal.* **181** (2001), no. 2, 447–475.
18. A. Siconolfi, Metric character of Hamilton-Jacobi equations, preprint.

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