Representation formulas for solutions of Hamilton-Jacobi equations with convex Hamiltonians

Hitoshi ISHII * and Hiroyoshi MITAKE †

Abstract. We establish general representation formulas for solutions of Hamilton-Jacobi equations with convex Hamiltonians. In order to treat representation formulas on general domains, we introduce a notion of ideal boundary similar to the Martin boundary [Ma] in potential theory. We apply such representation formulas to investigate maximal solutions, in certain classes of functions, of Hamilton-Jacobi equations. Part of the results in this paper has been announced in [Mi].

AMS Subject Classification (2000): Primary 35F30; Secondary 35C99, 35F20

Keywords: representation formula, Hamilton-Jacobi equations, Aubry sets, weak KAM theory, state constraint problem

1. Introduction and preliminaries

Let Ω be an open connected subset of $\mathbf{R}^n$ and $H : \Omega \times \mathbf{R}^n \to \mathbf{R}$ be a given function. We consider the Hamilton-Jacobi equation

$$H(x, Du) = 0 \quad \text{in } \Omega. \tag{1.1}$$

In this paper we establish a couple of general representation formulas for solutions of (1.1), introduce an ideal boundary associated with (1.1) which is analogous to Martin boundaries in potential theory (see [Ma]), and, as applicatons of our representation formulas, study maximal solutions of (1.1) having data prescribed on the Aubry set.

Our starting point was to study the formula for solutions of eikonal equations given in P.-L. Lions [Li], which is stated as follows: let $H(x,p) = |p| - f(x)$, where $f \in C(\overline{\Omega})$, $f \geq 0$ in Ω, and f vanishes at a finite number of distinct points $x_1, ..., x_N$. Assume that Ω is bounded and regular enough. Using the notation d_H which will be

* Department of Mathematics, Faculty of Education and Integrated Arts and Sciences, Waseda University, Nishi-waseda 1-6-1, Shinjuku, Tokyo 169-8050, Japan. Suppoted in part by the Grant-in-Aids for Scientific Research, JSPS, No. 18204009.

† Department of Mathematical Sciences, Graduate School of Science and Engineering, Waseda University, Ohkubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan.

defined later in this section, for any $\alpha_1, ..., \alpha_N \in \mathbf{R}$, if $|\alpha_i - \alpha_j| \leq d_H(x_i, x_j)$ and $|\alpha_i| \leq d_H(x, x_i)$ for all $i, j = 1, 2, ..., N$ and $x \in \partial\Omega$, then the function u defined by $u(x) = \min\{\alpha_i + d_H(x, x_i), d_H(x, y) \mid i = 1, ..., N,\ y \in \partial\Omega\}$ is continuous on $\overline{\Omega}$ and is a (unique) viscosity solution of (1.1) which satisfies $u = 0$ on $\partial\Omega$ and $u(x_i) = \alpha_i$ for all i.

While investigating the formula, we learned that weak KAM theory (see for instance [F1, F2, FSi]), developed in the last decade, is partly concerned with representation formulas for viscosity solutions of Hamilton-Jacobi equations. A typical representation formula obtained there is stated as follows: if Ω is (replaced by) the torus $\mathbf{R}^n/\mathbf{Z}^n$ or n-dimensional smooth compact manifold and if H is a convex and coercive Hamiltonian, then any viscosity solution u of (1.1) can be represented as $u(x) = \min\{u(y) + d_H(x, y) \mid y \in \mathcal{A}_\Omega\}$ for all $x \in \Omega$, where $\mathcal{A}_\Omega$ is the Aubry set associated with (1.1), which will be explained later in this section. We remark that the zeros of f of eikonal equations in the result mentioned above comprise the Aubry set.

Our representation formulas are more general than those mentioned above in the sense that the domain Ω may not be either smooth or bounded. The notion of ideal boundary is useful in this generality, which we develop here. Generally speaking, an important feature in representation formulas for solutions is to give us an intuition on the structure of solutions. We employ our representation results, to study maximal viscosity solutions of (1.1) which take the prescribed values on the Aubry set and the state constraint problem for (1.1). We are aware of the work [AGW] in which optimal control problems or Hamilton-Jacobi equations are treated in the viewpoint of ideal boundaries and max-plus algebra, but our approach is different from [AGW] and it is not obvious if the results in [AGW] cover some of our main results.

The study of representation of solutions of Hamilton-Jacobi equations has indeed a long history. As is well-known now, value functions in optimal control and differetial games are (viscosity) solutions of the corresponding dynamic programming (or Bellman-Isaacs) equations, which means that solutions of Hamilton-Jacobi equations of the Bellman-Isaacs type can be represented as the value functions of the corresponding optimal control or differetial games. We refer the reader to [BC, FSo, Li] for general overviews on optimal control, differential games and Hamilton-Jacobi equations, and also to [EI, I2] and the references therein for representation formulas for solutions of Hamilton-Jacobi equations. For the evolution equation of the form $\partial u/\partial t + H(D_x u) = 0$ under certain convexity conditions, such representation formulas in simple forms for solutions $u = u(x, t)$ have been known since [H, K, La, O] and are called as Hopf-Lax formulas or Hopf-Lax-Oleinik formulas. For recent developments in this direction, we refer the reader to [ABI] and the references therein.

Some of results in this paper can be extended to Hamilton-Jacobi equations of the form $H(x, u, Du) = 0$, where $H(x, u, p)$ is assumed to be nondecreasing in the variable

u. We shall return to this point in a future work.

Before we proceed our discussions, we give the plan of this paper. In the rest of this section we give our main assumptions on H, the definitions of d_H and the Aubry set $\mathcal{A}_\Omega$, and a collection of standard propositions on viscosity solutions which are needed in this paper. In Section 2 we establish our basic representation formula for viscosity solutions of (1.1). In Section 3, under a regularity assumption on Ω we give a simple formula for solutions of the Dirichlet problem for (1.1). Section 4 is devoted to defining an ideal boundary which is appropriate to (1.1). In Section 5 we give a definition of "boundary value" of the Dirichlet type on the ideal boundary for solutions of (1.1). We study maximal solutions of (1.1) having prescribed data on the Aubry set in Section 6 and the state constraint problem for (1.1) in Section 7. In the appendix we prove the uniform continuity of the functions u on Ω with bounded gradients Du under a mild regularity condition on Ω.

Part of the results in this paper has been announced in [Mi].

Now, let $\mathcal{S}_H$ (resp., $\mathcal{S}_H^+$ or $\mathcal{S}_H^-$) denote the space of continuous viscosity solutions (resp., viscosity supersolutions or viscosity subsolutions) of (1.1). If necessary, we write $\mathcal{S}_H(\Omega)$ and $\mathcal{S}_H^\pm(\Omega)$ for $\mathcal{S}_H$ and $\mathcal{S}_H^\pm$, respectively, in order to refer the domain Ω under consideration. We note that $u + c \in \mathcal{S}_H$ (resp., $u + c \in \mathcal{S}_H^\pm$) for all $(u, c) \in \mathcal{S}_H \times \mathbf{R}$ (resp., $(u, c) \in \mathcal{S}_H^\pm \times \mathbf{R}$).

We will make the following assumptions.

(A1) $H \in C(\Omega \times \mathbf{R}^n)$.

(A2) For each $x \in \Omega$ the function: $p \mapsto H(x, p)$ is convex on $\mathbf{R}^n$.

(A3) For each compact subset $K \subset \Omega$, there exists a constant $R_K > 0$ such that $H(x, p) > 0$ for all $(x, p) \in K \times (\mathbf{R}^n \setminus B(0, R_K))$.

(A4) $\mathcal{S}_H^-(\Omega) \neq \emptyset$.

When the Hamiltonian H satisfies (A3) (resp., (A2)), we say that H is coercive (resp., convex). The coercivity assumption (A3) is adapted in this paper to guarantee that any family of viscosity subsolutions of (1.1) is equi-Lipschitz continuous on compact subsets of Ω. Indeed, by strengthening the continuity of $H(x, p)$ in x (for instance, assuming that $|H(x, p) - H(y, p)| \leq C|x - y|(|p| + 1)$ for all $x, y \in \Omega$, $p \in \mathbf{R}^n$ and some constant $C > 0$), we may replace (A3) by a weaker condition on H which guarantees the local equi-continuity of any family of viscosity subsolutions of (1.1).

We recall some of standard propositions on viscosity solutions. For general references on these results and their proofs, we refer the reader to [B, BCD, CIL].

Proposition 1.1. *Assume that* (A1) *and* (A3) *hold. For each compact $K \subset \Omega$ there is a constant $C_K > 0$ depending only on K and H such that $|v(x) - v(y)| \leq C_K|x - y|$ for all $v \in \mathcal{S}_H^-$ and $x, y \in K$.*

In the above assertion, if K is convex, then we may choose R_K from (A3) as a Lipschitz constant C_K for $v \in \mathcal{S}_H^-$.

Proposition 1.2. *Assume that* (A1) *holds. Let* $S \subset \mathcal{S}_H^-$ *(resp.,* $S \subset \mathcal{S}_H^+$*) and set* $u(x) = \sup\{v(x) \mid v \in S\}$ *(resp.,* $u(x) = \inf\{v(x) \mid v \in S\}$*) for* $x \in \Omega$*. Suppose that* $u \in C(\Omega)$*. Then* $u \in \mathcal{S}_H^-$ *(resp.,* $u \in \mathcal{S}_H^+$*).*

Proposition 1.3. *Assume that* (A1)–(A3) *hold. Let* $S \subset \mathcal{S}_H^-$ *and set* $u(x) = \inf\{v(x) \mid v \in S\}$ *for* $x \in \Omega$*. Assume that* $u(x) \in \mathbf{R}$ *for some* $x \in \Omega$*. Then* $u \in \mathcal{S}_H^-$*. If, in addition,* $S \subset \mathcal{S}_H$*, then* $u \in \mathcal{S}_H$*.*

The following comment may be useful to see that Proposition 1.3 is a consequence of Proposition 1.2: by the theory of semicontinuous viscosity solutions due to Barron-Jensen (see [BJ, B, I3, BCD]), we have $v \in \mathcal{S}_H^-$ if and only if $H(x,p) \leq 0$ for any $x \in \Omega$ and $p \in D^-v(x)$, and therefore, $v \in \mathcal{S}_H^-$ if and only if $v \in \mathcal{S}_{-H}^+$.

For any $u \in C(\Omega)$ and $y \in \Omega$, $D^+u(y)$ (resp., $D^-u(y)$) denotes the superdifferential (resp., subdifferential) of u at y. Let $B(a,r)$ stand for the closed ball of radius r and with center at $a \in \mathbf{R}^n$.

In what follows we always *assume* that (A1)–(A4) hold. Following [FSi] with small variations, we will introduce the Aubry set for the Hamiltonian H. We define the function $d_H : \Omega \times \Omega \to \mathbf{R}$ by

$$d_H(x,y) = \sup\{v(x) - v(y) \mid v \in \mathcal{S}_H^-\}.$$

To see that d_H is well-defined, we set $S = \{v - v(y) \mid v \in \mathcal{S}_H^-\}$ and note that, due to (A4) and Proposition 1.1, S is a nonempty, locally equi-Lipschitz continuous family of functions on Ω. Note also that $\phi(y) = 0$ for all $\phi \in S$ and Ω is connected. Therefore, thanks to the Ascoli-Arzela theorem, S is precompact in $C(\Omega)$. Thus, the function d_H is a continuous function on $\Omega \times \Omega$ and satisfies $d_H(x,x) = 0$ for all $x \in \Omega$. Furthermore, by definition, we have $u(x) - u(y) \leq d_H(x,y)$ for all $u \in \mathcal{S}_H^-$ and $x, y \in \Omega$.

Now, we fix any $y \in \Omega$ and set $u(x) = d_H(x,y)$ for $x \in \Omega$. We see by Propositions 1.1 and 1.2 that u is locally Lipschitz continuous on Ω and $u \in \mathcal{S}_H^-$. We argue as in the proof of Perron's method for viscosity solutions (see [B, BCD, CIL]), to find that $u \in \mathcal{S}_H(\Omega \setminus \{y\})$. Next, we note by the definition of d_H that $u(x) - u(y) = u(x) - u(z) + u(z) - u(y) \leq d_H(x,z) + d_H(x,z)$ for all $u \in \mathcal{S}_H^-$ and $x, y, z \in \Omega$, to conclude that $d_H(x,y) \leq d_H(x,z) + d_H(z,y)$ for all $x, y, z \in \Omega$. In particular, we see that d_H is locally Lipschitz continuous on $\Omega \times \Omega$. The following theorem summarizes these observations.

Theorem 1.4. (a) $d_H(x,x) = 0$ *for all* $x \in \Omega$ *and* d_H *is locally Lipschitz continuous on* $\Omega \times \Omega$*.* (b) $u(x) - u(y) \leq d_H(x,y)$ *for all* $u \in \mathcal{S}_H^-$ *and* $x, y \in \Omega$*.* (c) $d_H(\cdot, y) \in \mathcal{S}_H^-(\Omega)$ *for* *all* $y \in \Omega$. (d) $d_H(\cdot, y) \in \mathcal{S}_H(\Omega \setminus \{y\})$ *for all* $y \in \Omega$. (e) $d_H(x, y) \leq d_H(x, z) + d_H(z, y)$ *for all* $x, y, z \in \Omega$.

Now, we define the (projected) *Aubry* set $\mathcal{A}_\Omega$ for the Hamiltonian H by

$$\mathcal{A}_\Omega = \{y \in \Omega \mid d_H(\cdot, y) \in \mathcal{S}_H(\Omega)\}.$$

We note that $\mathcal{A}_\Omega$ is a closed subset of Ω. Indeed, let $\{y_j\}_{j\in\mathbf{N}} \subset \mathcal{A}_\Omega$ be a sequence converging to $y \in \Omega$. By the continuity of d_H (Theorem 1.4, (a)) and the stability of the viscosity property under uniform convergence, we see that $d_H(\cdot, y) \in \mathcal{S}_H(\Omega)$ and conclude that $y \in \mathcal{A}_\Omega$ and that $\mathcal{A}_\Omega$ is a closed subset of Ω. We write $\Omega_0 = \Omega \setminus \mathcal{A}_\Omega$ in what follows. Notice that it may happen that $\Omega_0 = \emptyset$.

Theorem 1.5. *Let* $K \subset \Omega_0$ *be a compact set and* $u \in C(K)$*. Set* $U = \operatorname{int} K$*. Assume that* $u \leq v$ *on* ∂K *and that* $u \in \mathcal{S}_H^-(U)$ *and* $v \in \mathcal{S}_H^+(U)$*. Then* $u \leq v$ *in* K*.*

Outline of proof. Since $K \cap \mathcal{A}_\Omega = \emptyset$, for each $z \in K$ we may choose a constant $r_z > 0$ and a function $\varphi_z \in C^1(\Omega)$ such that $B(z, r_z) \subset \Omega_0$, $H(x, D\varphi_z(x)) < 0$ for all $x \in B(z, r_z)$, $\varphi_z(z) > 0 = d_H(z, z)$, and $\varphi_z(x) < d_H(x, z)$ for all $x \in \Omega \setminus B(z, r_z)$. For each $z \in K$, we set $\psi_z(x) = \max\{d_H(x, z), \varphi_z(x)\}$ for $x \in \Omega$ and observe that $\psi_z \in \mathcal{S}_H^-(\Omega)$ and that $H(x, D\psi_z(x)) < 0$ in a neighborhood V_z of z in the classical sense. By the compactness of K, there is a finite sequence $\{z_j\}_{j=1}^J$ such that $K \subset \bigcup_{j=1}^J V_{z_j}$. We define the function $\psi \in C(\Omega)$ by $\psi(x) = \frac{1}{J}\sum_{j=1}^J \psi_{z_j}(x)$ and observe by convexity (A2) that $\psi \in \mathcal{S}_{H+\delta}^-(V)$ for some neighborhood V of K and some constant $\delta > 0$. Regularizing ψ by mollification, if necessary, we may assume that $\psi \in C^1(V)$. In view of Proposition 1.1, we may apply the classical comparison result (see e.g. [I1]), to conclude that $u \leq v$ in K. □

The convexity and covering arguments in the above proof are already classical in weak KAM theory, which we refer to the proofs of Theorem 3.3 or Proposition 6.1 in [FSi].

We give a variational formula for the function d_H. We write L for the the Lagrangian of H, i.e., the function on $\Omega \times \mathbf{R}^n$ defined by

$$L(x, \xi) = \sup\{\xi \cdot p - H(x, p) \mid p \in \mathbf{R}^n\},$$

where $\xi \cdot p$ stands for the Euclidean inner product of $\xi, p \in \mathbf{R}^n$. Let $\mathcal{C}(x, t; y, 0)$ denote the set of all absolutely continuous functions $\gamma : [0, t] \to \Omega$ such that $\gamma(t) = x$ and $\gamma(0) = y$.

Proposition 1.6. *Let* $x, y \in \Omega$*. Then*

$$d_H(x, y) = \inf\left\{\int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s \;\middle|\; t > 0,\ \gamma \in \mathcal{C}(x, t; y, 0)\right\}.$$

For a proof we refer the reader to [I4, L, BCD]. Finally, we refer the reader to [FSi, F1, F2, I4] and the references therein for general properties of Aubry sets, weak KAM theory, and related topics.

2. A representation formula

We begin with

Lemma 2.1. *Let* $\{(y_j, c_j)\}_{j\in\mathbf{N}} \subset \Omega_0 \times \mathbf{R}$ *and* $\phi \in C(\Omega)$*. Assume that* $d_H(\cdot, y_j) + c_j \to \phi$ *in* $C(\Omega)$ *as* $j \to \infty$*. Then the following three conditions are equivalent:*

(a) $\phi \notin \mathcal{S}_H$.

(b) $\phi = d_H(\cdot, y) + c$ *for some* $(y, c) \in \Omega_0 \times \mathbf{R}$.

(c) $\{y_j\}_{j\in\mathbf{N}}$ *has a subsequence converging to a point in* Ω_0.

Proof. We first prove that (c) implies (b). For this, we assume that there is a subsequence $\{y_{j_k}\}_{k\in\mathbf{N}}$ of $\{y_j\}$ converging to a point $y_0 \in \Omega_0$. By Theorem 1.4, (a), the function d_H is continuous on $\Omega\times\Omega$. Therefore, we have $\lim_{k\to\infty} c_{j_k} = \lim_{k\to\infty}(d_H(y_0, y_{j_k}) + c_{j_k}) = \phi(y_0)$ and moreover $\phi(x) = \lim_{k\to\infty}(d_H(x, y_{j_k}) + c_{j_k}) = d_H(x, y_0) + \phi(y_0)$ for all $x \in \Omega$. Thus we see that (c) implies (b). Next, we see immediately from the definition of Ω_0 and $\mathcal{A}_\Omega$ that (b) implies (a).

Lastly, we prove that (a) implies (c). We suppose that (c) does not hold, and show that (a) does not hold, i.e., $\phi \in \mathcal{S}_H$. There are three possible cases. The first case is when $|y_j| \to \infty$ as $j \to \infty$. Fix any open bounded subset U of Ω such that $\overline{U} \subset \Omega$. If $j \in \mathbf{N}$ is large enough, then $y_j \in \Omega_0 \setminus U$ and hence $d_H(\cdot, y_j) + c_j \in \mathcal{S}_H(U)$. By the stability of the viscosity property under uniform convergence, we see that $\phi \in \mathcal{S}_H(U)$. Because of the arbitrariness of the choice of U, we find that $\phi \in \mathcal{S}_H(\Omega)$. In other cases, there is a subsequence of $\{y_j\}$ converging a point $y_0 \in \partial\Omega_0$. Note that $\partial\Omega_0 \subset \mathcal{A}_\Omega \cup \partial\Omega$. The second case is when $y_0 \in \partial\Omega$. The argument for the first case applies to this case, to conclude that $\phi \in \mathcal{S}_H$. The third case is when $y \in \mathcal{A}_\Omega$. By the continuity of d_H, we find that $\phi = d_H(\cdot, y_0) + c$ for some $c \in \mathbf{R}$ and hence $\phi \in \mathcal{S}_H$. Thus we always have $\phi \in \mathcal{S}_H$. □

We set $\mathcal{D}_0 = \{d_H(\cdot, y) + c \mid (y, c) \in \Omega_0 \times \mathbf{R}\}$ and $\mathcal{B}_0 = \overline{\mathcal{D}}_0 \cap \mathcal{S}_H$, where $\overline{\mathcal{D}}_0$ denotes the closure of $\mathcal{D}_0$ in $C(\Omega)$, i.e., in the topology of locally uniform convergence on Ω.

Proposition 2.2. *We have* $\mathcal{B}_0 = \overline{\mathcal{D}}_0 \setminus \mathcal{D}_0$.

Proof. Since $\mathcal{S}_H \cap \mathcal{D}_0 = \emptyset$, it is clear that $\mathcal{B}_0 \subset \overline{\mathcal{D}}_0 \setminus \mathcal{D}_0$. On the other hand, if $\phi \in \overline{\mathcal{D}}_0 \setminus \mathcal{D}_0$, then, by Lemma 2.1, we have $\phi \in \mathcal{S}_H$. Therefore, $\overline{\mathcal{D}}_0 \setminus \mathcal{D}_0 \subset \mathcal{B}_0$. □

We set $\mathcal{B} = \mathcal{B}_0 \cup \{d_H(\cdot, y) \mid y \in \mathcal{A}_\Omega\}$. Note that $\mathcal{B} \subset \mathcal{S}_H$. We are now in position to state one of our main results.

Theorem 2.3. *Let* $u \in \mathcal{S}_H$*. Then*

$$u(x) = \inf\{\phi(x) + \sup_{\Omega}(u - \phi) \mid \phi \in \mathcal{B}\} \quad \text{for all } x \in \Omega. \tag{2.1}$$

We remark here that it may happen that $\sup_\Omega(u-\phi) = \infty$ for some $\phi \in \mathcal{B}$ in formula (2.1).

Proof. Let v denote the function given by the right hand side of (2.1), and we first observe that $u(x) \le v(x)$ for all $x \in \Omega$.

Next, we show that the reversed inequality $u \ge v$ holds. We choose a sequence $\{\Omega_j\}_{j\in\mathbf{N}}$ of bounded open subsets of Ω_0 such that $\overline{\Omega}_j \subset \Omega_{j+1}$ for all $j \in \mathbf{N}$, and $\bigcup_{j\in\mathbf{N}} \Omega_j = \Omega_0$. We define the functions $v_j \in C(\Omega)$, with $j \in \mathbf{N}$, by

$$v_j(x) = \inf\{d_H(x,y) + u(y) \mid y \in \partial\Omega_j\}.$$

In view of Theorem 1.4, (d) and Proposition 1.3, we see that $v_j \in \mathcal{S}_H(\Omega \setminus \partial\Omega_j) \cap \mathcal{S}_H^-(\Omega)$ for all $j \in \mathbf{N}$. By the definition of d_H, we have $u(x) \le d_H(x,y) + u(y)$ for all $x, y \in \Omega$ and hence $u(x) \le v_j(x)$ for all $x \in \Omega$ and $j \in \mathbf{N}$. It is clear by the definition of v_j that $v_j(x) \le u(x)$ for all $x \in \partial\Omega_j$ and $j \in \mathbf{N}$. Consequently, we have $u = v_j$ on $\partial\Omega_j$ for all $j \in \mathbf{N}$. Using Theorem 1.5, we see that $v_j = u$ in Ω_j for all $j \in \mathbf{N}$.

Now, we fix any $y \in \Omega$. Consider first the case when $y \in \Omega_0$. We may assume by reselecting $\{\Omega_j\}$ if necessary that $y \in \Omega_1$. By the above observation, we may choose a sequence $\{y_j\}_{j\in\mathbf{N}} \subset \Omega_0$ such that $y_j \in \partial\Omega_j$ and $u(y) = u(y_j) + d_H(y, y_j)$ for all $j \in \mathbf{N}$. Proposition 1.1 assures that the sequence $\{d_H(\cdot, y_j) + u(y_j)\}$ is locally equi-Lipschitz continuous on compact subsets of Ω. Thus, noting that all the members of this sequence have the value $u(y)$ at y and that Ω is connected, we see that the sequence is precompact in $C(\Omega)$. Therefore we may assume by replacing the sequence by its subsequence if necessary that it converges in $C(\Omega)$ to a function $\phi \in C(\Omega)$. By Lemma 2.1, we have $\phi \in \mathcal{S}_H$ and moreover $\phi \in \mathcal{B}_0$. It is clear that $\phi(y) = u(y)$. Since $u(x) \le u(y_j) + d_H(x, y_j)$ for all $x \in \Omega$ and $j \in \mathbf{N}$, we have $u(x) \le \phi(x)$ for all $x \in \Omega$. Accordingly, we have $u(y) = \phi(y) = \phi(y) + \sup_\Omega(u - \phi)$ and conclude that $u(y) \ge v(y)$.

We next consider the case when $y \in \mathcal{A}_\Omega$. We have $d_H(\cdot, y) \in \mathcal{B}$ and $d_H(y,y) + \sup_\Omega(u - d_H(\cdot, y)) = u(y)$. From these we get $u(y) \ge v(y)$, and we finish the proof. □

3. The Dirichlet problem

In this section we are concerned with the Dirichlet problem for (1.1).

We assume throughout this section that Ω is bounded and the following condition is satisfied.

(A5) The function d_H is uniformly continuous on $\Omega \times \Omega$.

Condition (A5) guarantees that d_H can be extended uniquely to a function on the closure $\overline{\Omega} \times \overline{\Omega}$ by continuity. We may thus assume that $d_H \in C(\overline{\Omega} \times \overline{\Omega})$.

We pause for a while to discuss condition (A5). To this end, we introduce a coercivity condition on H.

(A3′) There exists an $R > 0$ such that $H(x,p) > 0$ for all $(x,p) \in \Omega \times (\mathbf{R}^n \setminus B(0,R))$.

This ensures that for any $u \in \mathcal{S}_H^-$, we have $|Du| \le R$ in Ω in the viscosity sense. This inequality in the viscosity sense is equivalent to the fact that u is locally Lipschitz continuous in Ω and $|Du| \le R$ a.e. in Ω. A remark here is that a sufficient condition for (A5) to hold is the condition that there is a modulus ω such that for any function $u \in C(\Omega)$ such that $|Du| \le R$ in Ω in the viscosity sense, we have $|u(x) - u(y)| \le \omega(|x-y|)$ for all $x, y \in \Omega$. Indeed, under this hypothesis, we have

$$|d_H(x,y)| \le \omega(|x-y|) \quad \text{for all } x, y \in \Omega,$$

and moreover, by the triangle inequality applied twice,

$$\begin{aligned} d_H(x,y) - d_H(x',y') &\le d_H(x,x') + d_H(x',y') + d_H(y',y) - d_H(x',y') \\ &\le \omega(|x-x'|) + \omega(|y-y'|) \quad \text{for all } x, y, x', y' \in \Omega, \end{aligned}$$

which shows the uniform continuity of d_H on $\Omega \times \Omega$. Here and henceforth we call a function $\omega \in C([0,\infty))$ a modulus if it is nondecreasing and vanishes at the origin 0.

A typical case where (A5) holds is given as follows. In addition to the boundedness of Ω and the coercivity (A3′) on H, let us assume that $\partial\Omega$ is locally represented as the graph of a continuous function, i.e,

(A6) for each $z \in \partial\Omega$ there are neighborhoods U and V of z, a C^1-diffeomorphism $\Phi : U \to V$, and a function $b \in C(\mathbf{R}^{n-1})$ such that

$$\Phi(\Omega \cap U) = \{(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid x_n > b(x')\} \cap V.$$

Indeed, it is not hard to check (see Proposition A.1 in the appendix for the details) that the functions $d_H(\cdot, y)$, with $y \in \Omega$, are uniformly continuous on Ω with a common modulus of continuity ω, and consequently, condition (A5) holds.

We now return to our main theme.

Theorem 3.1. *Let $u \in \mathcal{S}_H(\Omega)$. Then u can be extended uniquely to a function on $\overline{\Omega}$ by continuity and satisfies as a continuous function on $\overline{\Omega}$*

$$u(x) = \min\{d_H(x,y) + u(y) \mid y \in \mathcal{A}_\Omega \cup \partial\Omega_0\} \quad \textit{for all } x \in \Omega. \tag{3.1}$$

Proof. Let ω be a modulus of continuity of d_H. Noting that $u(x) - u(y) \le d_H(x,y) = d_H(x,y) - d_H(y,y) \le \omega(|x-y|)$ for all $x, y \in \Omega$, we find that u is uniformly continuous on Ω and hence it can be extended uniquely to a function on $\overline{\Omega}$ by continuity.

It is easy to see that $\overline{\mathcal{D}}_0 = \{d_H(\cdot, y) + c \mid (y,c) \in \overline{\Omega}_0 \times \mathbf{R}\}$, and therefore, by Proposition 2.2, that $\mathcal{B}_0 = \{d_H(\cdot,y) + c \mid (y,c) \in \partial\Omega_0 \times \mathbf{R}\}$. Since $u(x) - u(y) \le d_H(x,y)$ for all $x, y \in \overline{\Omega}$, we see that $\sup_\Omega (u - d_H(\cdot, y)) = u(y)$ for all $y \in \overline{\Omega}$. Theorem 2.3 now ensures that (3.1) holds. □

Next, we assume that

$$\mathcal{A}_\Omega = \emptyset \tag{3.2}$$

and let $g \in C(\partial\Omega)$. We consider the Dirichlet problem

$$H(x, Du) = 0 \quad \text{in } \Omega, \tag{3.3}$$

$$u = g \quad \text{on } \partial\Omega. \tag{3.4}$$

If $u \in C(\overline{\Omega})$ is a solution of (3.3) and (3.4), i.e., $u \in \mathcal{S}_H$ and it satisfies (3.4) pointwise, then we have

$$g(x) - g(y) = u(x) - u(y) \leq d_H(x, y) \quad \text{for all } x, y \in \partial\Omega.$$

This is a necessary condition for the solvability of (3.3) and (3.4). Indeed, we have

Theorem 3.2. *Assume that* (3.2) *holds and that* $g \in C(\partial\Omega)$ *satisfies*

$$g(x) - g(y) \leq d_H(x, y) \quad \textit{for all } x, y \in \partial\Omega. \tag{3.5}$$

Then the function u *on* $\overline{\Omega}$ *defined by*

$$u(x) = \min\{g(y) + d_H(x, y) \mid y \in \partial\Omega\} \tag{3.6}$$

is continuous and satisfies (3.3) *in the viscosity sense and* (3.4) *pointwise.*

We note in view of Theorems 3.1 and 3.2 and (3.2) that the function u given by (3.6) is a unique solution of (3.3) and (3.4). We refer the reader to Theorem 5.3, vi) of [Li] for a classical result similar to the above.

Proof. The family $\{g(y)+d_H(\cdot, y) \mid y \in \partial\Omega\}$ is uniformly bounded and equi-continuous on $\overline{\Omega}$. Therefore, u is a continuous function on $\overline{\Omega}$. It follows from Proposition 1.3 that $u \in \mathcal{S}_H$. It is clear by the definition of u that $u(x) \leq g(x)$ for all $x \in \partial\Omega$. On the other hand, by (3.5) we have $g(x) \leq u(x)$ for all $x \in \partial\Omega$. Thus we see that u satisfies (3.4) pointwise. □

Without assumption (3.2), in view of Theorem 3.1, a natural boundary-interior value problem is as follows. Let $g \in C(\mathcal{A}_\Omega \cup \partial\Omega_0)$ and consider the problem

$$H(x, Du) = 0 \quad \text{in } \Omega, \tag{3.7}$$

$$u = g \quad \text{on } \mathcal{A}_\Omega \cup \partial\Omega_0. \tag{3.8}$$

Theorem 3.3. *Let* g *satisfy*

$$g(x) - g(y) \leq d_H(x, y) \quad \textit{for all } x, y \in \mathcal{A}_\Omega \cup \partial\Omega_0. \tag{3.9}$$

Then the function u *on* $\overline{\Omega}$ *given by*

$$u(x) = \min\{g(y) + d_H(x, y) \mid y \in \mathcal{A}_\Omega \cup \partial\Omega_0\}$$

is continuous and is a unique solution of (3.7) *and* (3.8)*.*

In the above theorem we mean by a solution of (3.7) and (3.8) a function $u \in \mathcal{S}_H$ which satisfies (3.8) pointwise. The proof of the above theorem is similar to that of the previous theorem, and we omit giving it here. The above theorem generalizes the representation formula mentioned in the introduction (Proposition 5.4 of [Li]). See also Theorem 6.7 of [FSi] for a previous result on the torus.

4. Ideal boundary

In this section we give a couple of observations analogous to those in the theory of Martin boundaries (see [Ma]) in potential theory. We will be concerned with an ideal boundary of Ω_0 rather than those of Ω. The quotient set $\mathcal{B}_0/\mathbf{R}$ will be interpreted as our ideal boundary of Ω_0 as we will see in what follows.

In view of the fact that our equation $H(x, Du(x)) = 0$ depends on u only through the gradient Du, it is natural to introduce the quotient space $C(\Omega)/\mathbf{R}$. Let $\pi : C(\Omega) \to C(\Omega)/\mathbf{R}$ denote the projection defined by $\phi \mapsto \{\phi + c \mid c \in \mathbf{R}\}$. Let ρ be a standard distance on $C(\Omega)$ and ρ^π the distance on $C(\Omega)/\mathbf{R}$ induced by ρ, i.e., $\rho^\pi(\xi, \eta) := \inf\{\rho(\phi, \psi) \mid \phi \in \xi,\ \psi \in \eta\}$. For instance, we may choose ρ to be the one defined by

$$\rho(\phi, \psi) = \sum_{k\in\mathbf{N}} \frac{1}{2^k} \frac{\sup_{U_k} |\phi - \psi|}{1 + \sup_{U_k} |\phi - \psi|},$$

where the sequence $\{U_k\}_{k\in\mathbf{N}} \subset \Omega$ is chosen so that each U_k is a bounded open subset of Ω, $\overline{U}_k \subset U_{k+1}$ for all $k \in \mathbf{N}$, and $\bigcup_{k\in\mathbf{N}} U_k = \Omega$. We define the mapping $d^\pi : \Omega \to C(\Omega)/\mathbf{R}$ by setting $d^\pi(y) = \pi(d_H(\cdot, y))$. It follows from Theorem 1.4, (a) that $d^\pi : (\Omega, \rho_E) \to (C(\Omega)/\mathbf{R}, \rho^\pi)$ is continuous, where ρ_E denotes the Euclidean distance on $\mathbf{R}^n$, i.e., $\rho_E(x, y) = |x - y|$ for $x, y \in \mathbf{R}^n$.

Lemma 4.1. *Let* $y, z \in \Omega$*. Then the following three conditions are equivalent:* (a) $d^\pi(y) = d^\pi(z)$*.* (b) $d_H(y, z) + d_H(z, y) = 0$*.* (c) $d_H(\cdot, y) = d_H(\cdot, z) + d_H(z, y)$*.*

Proof. Assume that $d^\pi(y) = d^\pi(z)$. We may choose a constant $c \in \mathbf{R}$ such that $d_H(x, y) - d_H(x, z) = c$ for all $x \in \Omega$. This implies that $d_H(z, y) = c$ and $d_H(y, z) = -c$, from which $d_H(y, z) + d_H(z, y) = 0$. Next, we assume that $d_H(y, z) + d_H(z, y) = 0$. Then, for any $x \in \Omega$, applying the triangle inequality twice, we get

$$d_H(x, y) \le d_H(x, z) + d_H(z, y) \le d_H(x, y) + d_H(y, z) + d_H(z, y) = d_H(x, y),$$

which shows that for any $x \in \Omega$, $d_H(x, y) = d_H(x, z) + d_H(z, y)$. Finally, if $d_H(x, y) = d_H(x, z) + d_H(z, y)$ for all $x \in \Omega$, then we get immediately $d^\pi(y) = d^\pi(z)$. □

Lemma 4.2. *The mapping $d^\pi : \Omega_0 \to C(\Omega)/\mathbf{R}$ is injective.*

Proof. Let $y, z \in \Omega_0$ be such that $y \neq z$. We argue by contradiction and therefore suppose that $d^\pi(y) = d^\pi(z)$. By Lemma 4.1, we have $d_H(\cdot, y) = d_H(\cdot, z) + d_H(z, y)$. In view of Theorem 1.4, (d), we see that $d_H(\cdot, y) \in \mathcal{S}_H(\Omega \setminus \{y\})$ and $d_H(\cdot, z) \in \mathcal{S}_H(\Omega \setminus \{z\})$. Therefore, we have $d_H(\cdot, y) = d_H(\cdot, z) + d_H(z, y) \in \mathcal{S}_H(\Omega \setminus \{y\}) \cap \mathcal{S}_H(\Omega \setminus \{z\}) = \mathcal{S}_H(\Omega)$, which implies that $y, z \in \mathcal{A}_\Omega$. This contradiction proves that $d^\pi(y) \neq d^\pi(z)$. □

Let us introduce the distance ρ_0 on Ω_0 by setting $\rho_0(x, y) = \rho^\pi(d^\pi(x), d^\pi(y))$.

Proposition 4.3. *The bijection: $x \mapsto x$ is a homeomorphism from (Ω_0, ρ_E) to (Ω_0, ρ_0).*

Proof. Let $\{x_j\}_{j \in \mathbf{N}} \subset \Omega_0$ and $x_0 \in \Omega_0$. Observe first that if $\rho_E(x_j, x_0) \to 0$ as $j \to \infty$, then, by Theorem 1.4, (a), we have

$$\rho_0(x_j, x_0) \leq \rho(d_H(\cdot, x_j), d_H(\cdot, x_0)) \to 0 \quad \text{as } j \to \infty.$$

Next, assume that $\rho_0(x_j, x_0) \to 0$ as $j \to \infty$, from which we see that $\rho(d_H(\cdot, x_j) + c_j, d_H(\cdot, x_0)) \to 0$ as $j \to \infty$ for some sequence $\{c_j\} \subset \mathbf{R}$. We suppose that $\rho_E(x_j, x_0) \not\to 0$ as $j \to \infty$. By passing to a subsequence, we may assume that $\rho_E(x_j, x_0) \geq \delta$ for all $j \in \mathbf{N}$ and some constant $\delta > 0$, which assures that $d_H(\cdot, x_j) \in \mathcal{S}_H(\Omega \cap \operatorname{int} B(x_0, \delta))$ for all $j \in \mathbf{N}$. By the stability of the viscosity property, we see that $d_H(\cdot, x_0) \in \mathcal{S}_H(\Omega \cap \operatorname{int} B(x_0, \delta))$ and therefore $x_0 \in \mathcal{A}_\Omega$, which is a contradiction. Thus we must have $\rho_E(x_j, x_0) \to 0$ as $j \to \infty$, and we finish the proof. □

Let $(\widehat{\Omega}_0, \rho_0)$ denote the completion of (Ω_0, ρ_0). Let $\overline{\mathcal{D}_0/\mathbf{R}}$ denote the closure of $\mathcal{D}_0/\mathbf{R}$ in the complete metric space $(C(\Omega)/\mathbf{R}, \rho^\pi)$. It follows from Lemma 4.2 and the definition of ρ_0 that $d^\pi : (\Omega_0, \rho_0) \to (\mathcal{D}_0/\mathbf{R}, \rho^\pi)$ is isometric. Therefore, d^π can be extended uniquely to an isometric homeomorphism $d^\pi : (\widehat{\Omega}_0, \rho_0) \to (\overline{\mathcal{D}_0/\mathbf{R}}, \rho^\pi)$.

Proposition 4.4. *The set $(\widehat{\Omega}_0, \rho_0)$ is compact.*

Proof. Since $d^\pi : (\widehat{\Omega}_0, \rho_0) \to (\overline{\mathcal{D}_0/\mathbf{R}}, \rho^\pi)$ is isometric, it is enough to show that $\overline{\mathcal{D}_0/\mathbf{R}}$ is compact. We pick a point $z \in \Omega$ and observe by Proposition 1.1 and the connectedness of Ω that the family $\mathcal{D}_z := \{d_H(\cdot, y) - d_H(z, y) \mid y \in \Omega_0\}$ is precompact in $C(\Omega)$, which guarantees that $\overline{\mathcal{D}_0/\mathbf{R}}$ is compact. □

Proposition 4.5. *Ω_0 is an open subset of $\widehat{\Omega}_0$.*

Proof. Since $d^\pi : (\widehat{\Omega}_0, \rho_0) \to (\overline{\mathcal{D}_0/\mathbf{R}}, \rho^\pi)$ is isometric, it is enough to show that $\mathcal{D}_0/\mathbf{R}$ is an open subset of $(\overline{\mathcal{D}_0/\mathbf{R}}, \rho^\pi)$. Recalling that $\mathcal{B}_0 = \overline{\mathcal{D}}_0 \setminus \mathcal{D}_0$ by Proposition 2.2, we need only to show that for each $x \in \Omega_0$,

$$\inf\{\rho^\pi(d^\pi(x), \pi(\phi)) \mid \phi \in \mathcal{B}_0\} > 0.$$

But, if this is not the case, we find a point $x_0 \in \Omega_0$ and sequences $\{\phi_j\}_{j\in\mathbf{N}} \subset \mathcal{B}_0$ and $\{c_j\}_{j\in\mathbf{N}} \subset \mathbf{R}$ such that $\rho(d_H(\cdot, x_0), \phi_j + c_j) \to 0$ as $j \to \infty$, and, by the stability of the viscosity property, we deduce that $d_H(\cdot, x_0) \in \mathcal{S}_H$, which is a contradiction. The proof is complete. □

We set $\Delta_0 = \widehat{\Omega}_0 \setminus \Omega_0$. This set Δ_0 is the ideal boundary of Ω_0 which we are concerned with in this paper. According to Propositions 4.4 and 4.5, $\widehat{\Omega}_0$ is a compactification of Ω_0 and Δ_0 is the boundary of the open subset Ω_0 of $\widehat{\Omega}_0$. By the definition of Δ_0 and the fact that $\mathcal{B}_0 = \overline{\mathcal{D}}_0 \setminus \mathcal{D}_0$, for each $y \in \Delta_0$ there corresponds a $\phi \in \mathcal{B}_0$, unique up to an additive constant, and conversely for each $\phi \in \mathcal{B}_0$ so does a unique $y \in \Delta_0$ such that for any $\{y_j\}_{j\in\mathbf{N}} \subset \Omega_0$, $y_j \to y$ in $\widehat{\Omega}_0$ if and only if $d_H(\cdot, y_j) + c_j \to \phi$ in $C(\Omega)$ for some $\{c_j\} \subset \mathbf{R}$ as $j \to \infty$.

A few remarks are in order: the inclusion "$\partial\Omega_0 \subset \Delta_0$" holds in the sense that for each $z \in \partial\Omega_0$ there exist $y \in \Delta_0$ and a sequence $\{y_j\} \subset \Omega_0$ such that $y_j \to y$ in $\widehat{\Omega}_0$ and $y_j \to z$ in $\mathbf{R}^n$ as $j \to \infty$. Next, if $\mathcal{A}_\Omega \neq \emptyset$ and $\Omega_0 \neq \emptyset$, then $\mathcal{A}_\Omega$ and Δ_0 have a nonempty intersection in the sense that $d^\pi(\mathcal{A}_\Omega) \cap d^\pi(\Delta_0) \neq \emptyset$. Indeed, since Ω is connected, we easily deduce that there is a sequence $\{y_j\} \subset \Omega_0$ such that $y_j \to y$ in Ω as $j \to \infty$ for some $y \in \mathcal{A}_\Omega$. It is then clear that $\pi(d_H(\cdot, y)) \in d^\pi(\mathcal{A}_\Omega) \cap d^\pi(\Delta_0)$. In what follows we use the notation $\Delta_\mathcal{A} := \{y \in \Delta_0 \mid d^\pi(y) \in d^\pi(\mathcal{A}_\Omega)\}$. The following example shows that the inclusion "$\mathcal{A}_\Omega \subset \Delta_0$" does not hold in general.

Example 4.1. Let $n = 1$ and $\Omega = \mathbf{R}$. Consider the Hamiltonian $H \in C(\mathbf{R}^2)$ defined by $H(x, p) = (|p| - 1)_+ - x_+$, where r_+ denotes the positive part, $\max\{r, 0\}$, of $r \in \mathbf{R}$. It is easy to see that the function d_H is given by

$$d_H(x, y) = \left| \int_y^x (1 + t_+)\,\mathrm{d}t \right|.$$

By checking the viscosity property of d_H, it is not hard to see that $\mathcal{A}_\Omega = (-\infty, 0]$, that $\pi(d_H(\cdot, y)) \neq \pi(d_H(\cdot, z))$ for any $y, z \in \mathbf{R}$, with $y \neq z$, and that $d^\pi(\Delta_0)$ has only two elements $\pi(d_H(\cdot, 0))$ and $\pi(\phi)$, where ϕ is the function on $\mathbf{R}$ given by

$$\phi(x) := -\int_0^x (1 + t_+)\,\mathrm{d}t = \lim_{y\to\infty} (d_H(x, y) - d_H(0, y)).$$

Thus we conclude that $\pi(d_H(\cdot, y)) \notin d^\pi(\Delta_0)$ for all $y \in (-\infty, 0) \equiv \mathcal{A}_\Omega \setminus \{0\}$.

5. The boundary data on the ideal boundary

In this section we intend to define the Dirichlet boundary value in a broad sense on the ideal boundary for each $u \in \mathcal{S}_H$. Let $y \in \Delta_0$. We call it a *proper* boundary point of Ω_0 if there exists a sequence $\{y_j\}_{j\in\mathbf{N}} \subset \Omega_0$ such that for any $\phi \in d^\pi(y)$,

$$\phi(y_j) + d_H(\cdot, y_j) \to \phi \quad \text{in } C(\Omega) \text{ as } j \to \infty. \tag{5.1}$$

We see easily that for such a $\{y_j\}$, we have $y_j \to y$ in $(\widehat{\Omega}_0, \rho_0)$ as $j \to \infty$. We denote by Δ_0^* the set of the proper boundary points of Ω_0. We set $\mathcal{B}_0^* = \{\phi \in \mathcal{B}_0 \mid \pi(\phi) \in d^\pi(\Delta_0^*)\}$ and $\mathcal{B}^* = \mathcal{B}_0^* \cup \{d_H(\cdot, y) \mid y \in \mathcal{A}_\Omega\}$.

Remark. The authors do not know if $\Delta_0^* = \Delta_0$ in general.

The following theorem is a (possibly refined) version of Theorem 2.3.

Theorem 5.1. *Let $u \in \mathcal{S}_H$. Then*

$$u(x) = \min\{\phi(x) + \sup_\Omega (u - \phi) \mid \phi \in \mathcal{B}^*\} \quad \text{for all } x \in \Omega. \tag{5.2}$$

We formulate the main part of the proof of the above theorem as a lemma.

Lemma 5.2. *Let $(u, y_0) \in \mathcal{S}_H \times \Omega_0$. Then there exist a function $\phi \in \mathcal{B}_0$ and a sequence $\{y_j\}_{j\in\mathbf{N}} \subset \Omega_0$ such that*

$$u(y_j) = d_H(y_j, y_k) + u(y_k) \quad \text{for all } j, k \in \{0\} \cup \mathbf{N} \text{ satisfying } j \le k, \tag{5.3}$$

$$d_H(\cdot, y_j) + u(y_j) \to \phi \quad \text{in } C(\Omega) \quad \text{as } j \to \infty. \tag{5.4}$$

The following proof is similar to that of Theorem 2.3.

Proof. We choose a sequence $\{\Omega_j\}_{j\in\mathbf{N}}$ of bounded open subsets of Ω_0 such that $y_0 \in \Omega_1$, $\overline{\Omega}_j \subset \Omega_{j+1}$ for all $j \in \mathbf{N}$, and $\bigcup_{j\in\mathbf{N}} \Omega_j = \Omega_0$, and define the functions $v_j \in C(\Omega)$, with $j \in \mathbf{N}$, by

$$v_j(x) = \inf\{d_H(x, y) + u(y) \mid y \in \partial\Omega_j\}.$$

We know by the proof of Theorem 2.3 that $v_j = u$ in $\overline{\Omega}_j$ for all $j \in \mathbf{N}$.

We choose inductively a point $y_j \in \partial\Omega_j$ for each $j \in \mathbf{N}$ so that $v_j(y_{j-1}) = d_H(y_{j-1}, y_j) + u(y_j)$. Since $y_{j-1} \in \Omega_j$, we have

$$u(y_{j-1}) = v_j(y_{j-1}) = d_H(y_{j-1}, y_j) + u(y_j) \text{ for all } j \in \mathbf{N}. \tag{5.5}$$

To see that (5.3) holds, let $j, k \in \{0\} \cup \mathbf{N}$ satisfy $j \le k$. If $j = k$, then (5.3) is obviously satisfied. Assume instead that $j < k$. Using (5.5), we get

$$u(y_j) = \sum_{m=j+1}^{k} d_H(y_{m-1}, y_m) + u(y_k) \ge d_H(y_j, y_k) + u(y_k).$$

On the other hand, we have $u(y_j) \le d_H(y_j, y_k) + u(y_k)$ for all j, k, and we conclude that (5.3) holds.

Now set $\phi_j = d_H(\cdot, y_j) + u(y_j)$ for $j \in \mathbf{N}$ and observe that the family $\{\phi_j\}_{j\in\mathbf{N}}$ is precompact in $C(\Omega)$. Thus, passing to a subsequence of $\{y_j\}$ if necessary, we may assume that $\{\phi_j\}$ converges in $C(\Omega)$ to a function $\phi \in C(\Omega)$ as $j \to \infty$.

Noting that $\{y_j\}$ does not have a subsequence converging to a point $y \in \Omega_0$, we see by Lemma 2.1 that $\phi \in \mathcal{S}_H$. It is clear that $\phi \in \overline{\mathcal{D}}_0$. Thus we conclude that $\phi \in \mathcal{B}_0$. □

Remark. The sequence $\{y_j\}$ in the above proof can be regarded as a discrete version of etremal curve for the solution u (see for instance [I4]).

Outline of proof of Theorem 5.1. Let v denote the function on Ω defined by the right hand side of (5.2). Clearly we have $u(x) \le v(x)$ for all $x \in \Omega$. Fix any $y_0 \in \Omega$ and show that $u(y_0) \ge v(y_0)$. As in the proof of Theorem 2.3, if $y_0 \in \mathcal{A}_\Omega$, then we have $u(y_0) = d_H(y_0, y_0) + \sup_\Omega(u - d_H(\cdot, y_0)) \ge v(y_0)$. Next assume that $y_0 \in \Omega_0$ and let $\phi \in \mathcal{B}_0$ and $\{y_j\}$ be as in Lemma 5.2. From (5.3) and (5.4) we see that $u(y_j) = \phi(y_j)$ for all $j \in \mathbf{N} \cup \{0\}$. Therefore, by (5.4), we have $\phi \in \mathcal{B}_0^*$. Since $u \le d_H(\cdot, y_j) + u(y_j)$ on Ω for all $j \in \mathbf{N}$, we get $u \le \phi$ on Ω. Hence we obtain $u(y_0) = \phi(y_0) = \phi(y_0) + \sup_\Omega(u - \phi)$ and consequently $u(y_0) \ge v(y_0)$. The proof is now complete. □

Lemma 5.3. *Let $u \in \mathcal{S}_H^-(\Omega)$, $y \in \Delta_0^*$, and $\phi \in d^\pi(y)$. Then*

$$\sup_\Omega(u - \phi) = \lim_{r \to +0} \sup\{(u - \phi)(\xi) \mid \xi \in \Omega_0,\ \rho_0(\xi, y) < r\}. \tag{5.6}$$

Proof. We choose a sequence $\{y_j\}_{j \in \mathbf{N}} \subset \Omega_0$ so that $\phi(y_j) + d_H(\cdot, y_j) \to \phi$ in $C(\Omega)$ as $j \to \infty$. Observe that for any $x \in \Omega$ and $j \in \mathbf{N}$,

$$(u - \phi)(x) \le u(y_j) + d_H(x, y_j) - \phi(x) = (u - \phi)(y_j) + \phi(y_j) + d_H(x, y_j) - \phi(x),$$

and hence

$$(u - \phi)(x) \le \liminf_{j \to \infty} (u - \phi)(y_j) \le \lim_{r \to +0} \sup\{(u - \phi)(\xi) \mid \xi \in \Omega_0,\ \rho_0(\xi, y) < r\}.$$

Consequently we get

$$\sup_\Omega(u - \phi) \le \lim_{r \to +0} \sup\{(u - \phi)(\xi) \mid \xi \in \Omega_0,\ \rho_0(\xi, y) < r\}.$$

On the other hand, it is clear that

$$\sup_\Omega(u - \phi) \ge \lim_{r \to +0} \sup\{(u - \phi)(\xi) \mid \xi \in \Omega_0,\ \rho_0(\xi, y) < r\}.$$

Thus we see that (5.6) is valid. □

Let $u \in \mathcal{S}_H(\Omega)$ and $y \in \Delta_0^*$. The value given by

$$\lim_{r \to +0} \sup\{(u - \phi)(\xi) \mid \xi \in \Omega_0,\ \rho_0(\xi, y) < r\}$$

for $\phi \in d^\pi(y)$ represents somehow the behavior of u near the proper boundary point y of Ω_0. This value depends on the choice of $\phi \in d^\pi(y)$. We introduce the function $g(u, y)$ on Ω given by

$$g(u, y)(x) = \phi(x) + \lim_{r \to +0} \sup\{(u - \phi)(\xi) \mid \xi \in \Omega_0,\ \rho_0(\xi, y) < r\},$$

which does not depend on the choice of $\phi \in d^\pi(y)$. We remark that the value $g(u, y)(x)$ can be infinity for all $x \in \Omega$ and for some $(u, y) \in \mathcal{S}_H \times \Delta_0^*$. We regard this function $g(u, y)$ as the boundary data of u at the proper boundary point y. We set $g(u, y) := d_H(\cdot, y) + u(y) \equiv d_H(\cdot, y) + \lim_{\Omega \ni x \to y}(u(x) - d_H(x, y))$ for $(u, y) \in \mathcal{S}_H \times \mathcal{A}_\Omega$. Then we have the following theorem as a corollary of Theorem 5.1 and Lemma 5.3.

Theorem 5.4. *Let $u \in \mathcal{S}_H$. Then*

$$u(x) = \min\{g(u,y)(x) \mid y \in \Delta_0^* \cup \mathcal{A}_\Omega\} \quad \textit{for } x \in \Omega.$$

In the definition of proper boundary points, condition (5.1) is assumed to hold for a sequence $\{y_j\}$, but not for all the sequences $\{y_j\}$ such that $y_j \to y$ as $j \to \infty$. Example 5.1 below is concerned with this point.

Example 5.1. Let $\Omega = \mathbf{R}^2$ and let U denote the open disc with radius $1/4$ and center at the origin. Let $H(x,p) = |p| - f(x)$, where f is the function given by

$$f(x) = \max_{k \in \mathbf{Z}} f_0(x - ke_1), \qquad \text{with } f_0(x) = \operatorname{dist}(x, \mathbf{R}^2 \setminus U),$$

and $e_1 := (1,0) \in \mathbf{R}^2$. Define the function ϕ on Ω by $\phi = \frac{1}{2}f^2$. It is not hard to see that ϕ is a viscosity solution of $H(x, D\phi) = 0$ in Ω and that $d_H(\cdot, y) = \phi$ for all $y \in \Omega \setminus \bigcup_{k \in \mathbf{Z}}(ke_1 + U)$ and $\mathcal{A}_\Omega = \Omega \setminus \bigcup_{k \in \mathbf{Z}}(ke_1 + U)$. Also, it is easily seen that for any $k \in \mathbf{Z}$,

$$d_H(x, ke_1) = \begin{cases} \phi(0) - \phi(x) & \text{if} \quad x \in ke_1 + U, \\ \phi(0) + \phi(x) & \text{otherwise.} \end{cases}$$

For $j \in \mathbf{N}$ we put $y_j = (\frac{1}{4} - \frac{1}{4j})e_1 \in U$ and $z_j = je_1$. Then $y_j, z_j \in \Omega_0 := \Omega \setminus \mathcal{A}_\Omega$ for all $j \in \mathbf{N}$. Observe that $d_H(\cdot, y_j) \to d_H(\cdot, \frac{1}{4}e_1) = \phi$ and $d_H(\cdot, z_j) \to \phi(0) + \phi$ in $C(\Omega)$ as $j \to \infty$ and consequently that $y_j \to y$ and $z_j \to y$ for the point $y \in \Delta_0$ which is characterized by $\phi \in d^\pi(y)$ as $j \to \infty$. Noting that $\phi(y_j) + d_H(\cdot, y_j) \to \phi(\frac{1}{4}e_1) + d_H(\cdot, \frac{1}{4}e_1) = \phi$ in $C(\Omega)$ as $j \to \infty$, we see that $y \in \Delta_0^*$. However, we have $\phi(z_j) + d_H(\cdot, z_j) = \phi(0) + d_H(\cdot, z_j) \to 2\phi(0) + \phi$ in $C(\Omega)$ as $j \to \infty$ and $2\phi(0) + \phi \neq \phi$.

The next example describes a very typical situation in which the classical Dirichlet data do not make sense.

Example 5.2. Let $\Omega = \{x \in \mathbf{R}^2 \mid x_1^2 + x_2^2 < 1\} \setminus ([0,1) \times \{0\})$ and $H(x,p) = |p| - 1$. First of all we note that $\mathcal{A}_\Omega = \emptyset$. It is easy to check that the function d_H can be extended uniquely to a continuous function on $\Omega \times \overline{\Omega} \setminus ((0,1] \times \{0\})$. On the other hand, for any $z \in (0,1] \times \{0\}$, we have exactly two different "limits" $d_H^\pm(\cdot, z)$ at z of the functions $d_H(\cdot, y)$. Here the functions $d_H^\pm(\cdot, z) \in C(\Omega)$ are defined respectively by

$$d_H^+(x,z) = \lim_{\mathbf{R} \times (0,\infty) \ni y \to z} d_H(x,y),$$
$$d_H^-(x,z) = \lim_{\mathbf{R} \times (-\infty,0) \ni y \to z} d_H(x,y).$$

For $x = (x_1, x_2) \in \Omega$ and $y = (y_1, 0) \in (0,1] \times \{0\}$, we have $d_H^+(x,y) = |x - y|$ and $d_H^-(x,y) = y_1 + |x|$ if $x_2 \geq 0$ and $d_H^+(x,y) = y_1 + |x|$ and $d_H^-(x,y) = |x-y|$ if $x_2 \leq 0$. For instance, if we set $u = d_H^+(\cdot, y)$ or $u = d_H^-(\cdot, y)$, with $y = (y_1, 0) \in (0,1] \times \{0\}$, the function $u(x)$ does not have the limit as $x \to y$. Heuristically, each point $y \in (0,1] \times \{0\}$ corresponds to two points in the ideal boundary Δ_0.

We now return to Theorem 3.1, which was obtained as an easy consequence of Theorem 2.3. Here we intend to prove Theorem 3.1 via Theorem 5.4.

We thus assume that Ω is bounded and (A5) holds. Recall that d_H is a continuous function on $\overline{\Omega} \times \overline{\Omega}$.

Let $y \in \Delta_0$ and $\phi \in d^\pi(y)$. We may choose sequences $\{y_j\}_{j\in\mathbf{N}} \subset \Omega_0$, converging to y in $\widehat{\Omega}_0$, and $\{c_j\}_{j\in\mathbf{N}} \subset \mathbf{R}$ so that

$$d_H(\cdot, y_j) + c_j \to \phi \quad \text{in } C(\Omega) \text{ as } j \to \infty. \tag{5.7}$$

Since Ω is bounded, we may assume by passing to a subsequence if necessary that $y_j \to \xi$ for some $\xi \in \overline{\Omega}_0$ as $j \to \infty$. Then, from (5.7) we find that $c_j \to c$ for some $c \in \mathbf{R}$ as $j \to \infty$ and $\phi = d_H(\cdot, \xi) + c$, from which we deduce that $\xi \in \partial\Omega_0$ and, as $j \to \infty$,

$$\phi(y_j) + d_H(\cdot, y_j) \to \phi \quad \text{in } C(\Omega).$$

Therefore we have $d^\pi(y) = \pi(d_H(\cdot, \xi))$ for some $\xi \in \partial\Omega_0$ and $y \in \Delta_0^*$. Similarly, we infer that for any $\xi \in \partial\Omega_0$, there exists a $y \in \Delta_0$ such that $d^\pi(y) = \pi(d_H(\cdot, \xi))$. That is, we have $d^\pi(\Delta_0) = d^\pi(\Delta_0^*) = \{\pi(d_H(\cdot, \xi)) \mid \xi \in \partial\Omega_0\}$.

Next let $u \in \mathcal{S}_H(\Omega)$. We may assume by assumption (A5) that $u \in C(\overline{\Omega})$. We want to compute $g(u, y)$ for $y \in \Delta_0^*$. Fix any $y \in \Delta_0$ and $\xi \in \partial\Omega_0$ so that $d^\pi(y) = \pi(d_H(\cdot, \xi))$. We calculate

$$\begin{aligned} g(u, y)(x) =& d_H(x, \xi) + \lim_{r\to+0} \sup\{u(z) - d_H(z, \xi) \mid z \in \Omega_0,\ \rho_0(z, y) < r\} \\ =& d_H(x, \xi) + \max\{u(z) - d_H(z, \xi) \mid z \in \partial\Omega_0,\ d^\pi(y) = \pi(d_H(\cdot, z))\}. \end{aligned}$$

We note that the function $u(z) - d_H(z, \xi)$ of z attains a maximum at $z = \xi$. Therefore we get $g(u, y) = d_H(\cdot, \xi) + u(\xi) - d_H(\xi, \xi) = u(\xi) + d_H(\cdot, \xi)$. On the other hand, we have $g(u, y) = u(y) + d_H(\cdot, y)$ for $y \in \mathcal{A}_\Omega$ by definition.

We may now apply Theorem 5.4, to conclude that

$$u(x) = \inf\{u(y) + d_H(x, y) \mid y \in \partial\Omega_0 \cup \mathcal{A}_\Omega\} \quad \text{for } x \in \Omega,$$

which is the conclusion of Theorem 3.1.

6. Maximal solutions

In this section, as easy applications of Theorems 2.3 and 5.4, we study maximal solutions of (1.1) with prescribed data on $\mathcal{A}_\Omega$. We always assume here that $\mathcal{A}_\Omega \neq \emptyset$.

Let $g : \mathcal{A}_\Omega \to \mathbf{R}$ be a continuous function. Theorem 2.3 suggests that the function u on Ω given by

$$u(x) = \inf\{g(y) + d_H(x, y) \mid y \in \mathcal{A}_\Omega\} \tag{6.1}$$

should be a (unique) maximal solution of (1.1) among those which satisfy $u \le g$ on $\mathcal{A}_\Omega$. Here, by definition, $u \in C(\Omega)$ is a maximal solution of (1.1) among those which satisfy $u \le g$ on $\mathcal{A}_\Omega$ if $u \in \mathcal{S}_H$, $u \le g$ on $\mathcal{A}_\Omega$, and $u \ge v$ on Ω for all $v \in \mathcal{S}_H(\Omega)$ which satisfy $v \le g$ on $\mathcal{A}_\Omega$. To make (6.1) meaningful, we have to assume that

$$\inf\{g(y) + d_H(z, y) \mid y \in \mathcal{A}_\Omega\} > -\infty \quad \text{for some } z \in \Omega, \tag{6.2}$$

so that $u(z) > -\infty$ and moreover, thanks to Proposition 1.1, $u \in C(\Omega)$.

Proposition 6.1. *Assume that* (6.2) *holds. Then the function* u *defined by* (6.1) *is a* (*unique*) *maximal solution of* (1.1) *among those which satisfy* $u \le g$ *on* $\mathcal{A}_\Omega$. *Moreover, if the inequality*

$$g(x) \le g(y) + d_H(x, y) \quad \textit{for all } x, y \in \mathcal{A}_\Omega, \tag{6.3}$$

holds, then $u = g$ *on* $\mathcal{A}_\Omega$.

Of course, uniqueness of such a maximal solution is a direct consequence of the definition of maximal solutions.

Proof. By Propositions 1.3, we have $u \in \mathcal{S}_H(\Omega)$. If $v \in \mathcal{S}_H(\Omega)$ satisfies $v \le g$ on $\mathcal{A}_\Omega$, then, by Theorem 2.3, we get

$$\begin{aligned} v(x) &\le \inf\{v(y) + d_H(x, y) \mid y \in \mathcal{A}_\Omega\} \\ &\le \inf\{g(y) + d_H(x, y) \mid y \in \mathcal{A}_\Omega\} = u(x) \quad \text{for all } x \in \Omega. \end{aligned}$$

We thus conclude that u is a maximal solution of (1.1) among those which satisfy $u \le g$ on $\mathcal{A}_\Omega$. Finally we observe that if (6.3) holds, then

$$u(x) = \inf\{g(y) + d_H(x, y) \mid y \in \mathcal{A}_\Omega\} \ge g(x) \quad \text{for all } x \in \mathcal{A}_\Omega,$$

and therefore $u(x) = g(x)$ for all $x \in \mathcal{A}_\Omega$. □

We remark that under assumption (6.2), the function u defined by (6.1) is the maximal subsolution of (1.1) among those which satisfy $u \le g$ on $\mathcal{A}_\Omega$. To see this, we need only to recall that $v(x) \le v(y) + d_H(x, y)$ for all $v \in \mathcal{S}_H^-$ and $x, y \in \Omega$.

We now treat the case where $\Omega = \mathbf{R}^n$. We are concerned with a growth condition on functions at infinity which selects the maximal solution of (1.1) among those which satisfy $u = g$ on $\mathcal{A}_\Omega$. We assume hereafter that g satisfies (6.3).

Theorem 6.2. *Let* $\Omega = \mathbf{R}^n$. *Let* $u \in \mathcal{S}_H(\Omega)$ *satisfy* $u = g$ *on* $\mathcal{A}_\Omega$. *Assume that*

$$\lim_{r \to \infty} \inf\{u(y) + d_H(z, y) \mid y \in \Omega_0,\ |y| \ge r\} = \infty$$

for some $z \in \Omega$. *Then* u *is the maximal solution of* (1.1) *among those which satisfy* $u = g$ *on* $\mathcal{A}_\Omega$.

Proof. In view of Theorem 5.4 and Proposition 6.1, we need only to show that $g(u, y)(x) = \infty$ for any $x \in \Omega$ and $y \in \Delta_0^* \setminus \Delta_{\mathcal{A}}$. Fix any $x \in \Omega$ and $y \in \Delta_0^* \setminus \Delta_{\mathcal{A}}$.

We may choose a sequence $\{y_j\}_{j\in\mathbf{N}} \subset \Omega_0$ so that $\phi(y_j) + d_H(x, y_j) \to \phi(x)$ as $j \to \infty$. Since $d^\pi(y) \notin d^\pi(\mathcal{A}_\Omega)$, we infer that $|y_j| \to \infty$ as $j \to \infty$. Thus we have

$$\begin{aligned} g(u, y)(x) &= \phi(x) + \lim_{r\to+0} \sup\{(u - \phi)(\xi) \mid \xi \in \Omega_0,\ \rho_0(\xi, y) < r\} \\ &\geq \phi(x) + \limsup_{j\to\infty} [u(y_j) + d_H(x, y_j) - \phi(y_j) - d_H(x, y_j)] \\ &= \phi(x) - \phi(x) + \limsup_{j\to\infty} [u(y_j) + d_H(x, y_j)] = \infty, \end{aligned}$$

from which we conclude that u is the maximal solution of (1.1) among those which satisfy $u = g$ on $\mathcal{A}_\Omega$. □

Next, we treat the case where Ω is bounded and (A5) is satisfied. We may assume that $d_H \in C(\overline{\Omega} \times \overline{\Omega})$ and that if $u \in \mathcal{S}_H^-(\Omega)$, then $u \in C(\overline{\Omega})$.

We assume furthermore that $H \in C(\overline{\Omega}\times\mathbf{R}^n)$. Let $Q \subset \overline{\Omega}$ be a locally compact subset of the compact set $\overline{\Omega}$. We say by definition that $u \in C(Q)$ is a (viscosity) supersolution of $H(x, Du) = 0$ in Q or, in other words, a (viscosity) solution of $H(x, Du) \geq 0$ in Q if whenever $(\varphi, y) \in C^1(Q) \times Q$ and $u - \varphi$ attains a minimum over Q at y, then $H(y, D\varphi(y)) \geq 0$. We write $u \in \mathcal{S}_H^+(Q)$ if u is a supersolution of $H(x, Du) = 0$ in Q.

Lemma 6.3. *We have $d_H(\cdot, y) \in \mathcal{S}_H^+(\overline{\Omega} \setminus \{y\})$ for all $y \in \overline{\Omega}$.*

We remark that the set $\overline{\Omega}\setminus\{y\}$ is locally compact. The above Lemma is a consequence of Theorem II.2 of [CL], and we refer to [CL] for the proof.

Proposition 6.4. *Let $g \in C(\mathcal{A}_\Omega)$ and let $u \in C(\overline{\Omega})$ be the function defined by* (6.1)*. Then $u \in \mathcal{S}_H^+(\overline{\Omega})$.*

Proof. Since $d_H(\cdot, y) \in \mathcal{S}_H(\Omega) \cap \mathcal{S}_H^+(\overline{\Omega} \setminus \{y\})$ for $y \in \mathcal{A}_\Omega$, we deduce that $d_H(\cdot, y) \in \mathcal{S}_H^+(\overline{\Omega})$. We invoke Proposition 1.2 with Ω replaced by the closed set $\overline{\Omega}$, which is still valid, to conclude that $u \in \mathcal{S}_H^+(\overline{\Omega})$. □

The problem of finding a function $u \in C(\overline{\Omega})$ which satisfies $u \in \mathcal{S}_H^-(\Omega) \cap \mathcal{S}_H^+(\overline{\Omega})$ is called the state constraint problem in connection with state constraint problems (see [S]) in optimal control. Proposition 6.4 tells us that the function u given by (6.1) is a solution of the state constraint problem. The following example shows that there are solutions of the state constraint problem other than the function u given by (6.1).

Example 6.1. Let $n = 1$ and $\Omega = (-\pi, \pi)$. Let $H \in C([-\pi, \pi] \times \mathbf{R})$ be the function given by $H(x, p) = |p| - |\sin x|$. Note that the function d_H is given by

$$d_H(x, y) = \left| \int_y^x |\sin t| \, \mathrm{d}t \right|.$$

From this we see that $\mathcal{A}_\Omega = \{0\}$. Let $g : \{0\} \to \mathbf{R}$ be given by $g(0) = 0$. Then the function u defined by (6.1) is explicitly written as $u(x) = 1 - \cos x$ for $x \in [-\pi, \pi]$. It is easy to check that $-u$ is another solution of the state constraint problem which satisfies $u = g$ at the origin.

In the above example the behavior of the function $H(x,p)$ near $x = \pm\pi$ is similar to that at $x = 0$. In order to get uniqueness of solutions of the state constraint problem, we have to take care of the boundary points of Ω when we define the Aubry set. This will be the subject of the next section.

7. The state constraint problem

In this section we continue to discuss the state constraint problem (see [S]). The state constraint problem is to seek for a solution u of the inclusion $u \in \mathcal{S}_H^-(\Omega) \cap \mathcal{S}_H^+(\overline{\Omega})$. In other words, the problem is to seek for a function $u \in C(\overline{\Omega})$ which satisfies in the viscosity sense both

$$H(x, Du) \le 0 \ \text{ in } \Omega \quad \text{and} \quad H(x, Du) \ge 0 \ \text{ on } \overline{\Omega}. \tag{7.1}$$

Throughout this section, we assume as before that Ω is bounded, (A5) holds, and $H \in C(\overline{\Omega} \times \mathbf{R}^n)$. We may thus assume that $d_H \in C(\overline{\Omega} \times \overline{\Omega})$ and we may regard any $u \in \mathcal{S}_H^-(\Omega)$ as a continuous function on $\overline{\Omega}$.

We modify the definition of the Aubry set by introducing

$$\mathcal{A}_{\overline{\Omega}} = \{y \in \overline{\Omega} \mid d_H(\cdot, y) \in \mathcal{S}_H^+(\overline{\Omega})\}.$$

By Lemma 6.3, we see that for any $y \in \Omega$, $y \in \mathcal{A}_\Omega$ if and only if $y \in \mathcal{A}_{\overline{\Omega}}$. We infer as in the case of $\mathcal{A}_\Omega$ that $\mathcal{A}_{\overline{\Omega}}$ is a closed subset of $\overline{\Omega}$.

Let $g \in C(\mathcal{A}_{\overline{\Omega}})$ and set

$$u(x) = \inf\{g(y) + d_H(x,y) \mid y \in \mathcal{A}_{\overline{\Omega}}\} \quad \text{for } x \in \overline{\Omega}. \tag{7.2}$$

If

$$g(x) - g(y) \le d_H(x,y) \quad \text{for all } x, y \in \mathcal{A}_{\overline{\Omega}}, \tag{7.3}$$

then we have $u(x) = g(x)$ for all $x \in \mathcal{A}_{\overline{\Omega}}$. By the same reasoning as in the proof of Proposition 6.4, we get the following.

Proposition 7.1. *Let u be the function given by* (7.2)*. Then $u \in C(\overline{\Omega})$ and $u \in \mathcal{S}_H(\Omega) \cap \mathcal{S}_H^+(\overline{\Omega})$. That is, u is a solution of the state constraint problem* (7.1)*.*

The main result in this section is now stated as follows. We assume in the rest that H is coercive on $\overline{\Omega}$, that is, H satisfies condition (A3$'$), and that Ω satisfies (A6).

Theorem 7.2. *Let $g \in C(\mathcal{A}_{\overline{\Omega}})$ satisfy* (7.3)*. Then there exists at most one solution u of* (7.1) *which satisfies $u = g$ on $\mathcal{A}_{\overline{\Omega}}$.*

It is obvious that Theorem 7.2 is a consequence of the following comparison theorem.

Theorem 7.3. *Let $u \in \mathcal{S}_H^-(\Omega)$ and $v \in \mathcal{S}_H^+(\overline{\Omega})$. Assume that $u \le v$ on $\mathcal{A}_{\overline{\Omega}}$. Then $u \le v$ in Ω.*

For the proof of the above theorem, we need the following lemma.

Lemma 7.4. *For each compact $K \subset \overline{\Omega} \setminus \mathcal{A}_{\overline{\Omega}}$, there exist a function $\psi \in C(K)$ and a constant $\delta > 0$ such that $\psi \in \mathcal{S}_{H+\delta}^-(\operatorname{int} K)$.*

Proof. Reviewing the proof of Theorem 1.5, we realize that we only need to find functions $w \in C(\overline{\Omega})$ and $f \in C(\overline{\Omega})$ for each $y \in K$ such that $f \ge 0$ in $\overline{\Omega}$, $f(y) > 0$, and $w \in \mathcal{S}_{H+f}^-(\Omega)$.

Let $y \in K$. As we have seen in the proof of Theorem 1.5, we already know that if $y \in \Omega$, then there is such a pair of functions w and f. Assume instead that $y \in \partial\Omega$. Set $v = d_H(\cdot, y)$. Since $y \notin \mathcal{A}_{\overline{\Omega}}$, we may choose a function $\varphi \in C^1(\overline{\Omega})$ such that $v - \varphi$ attains a strict minimum at y and $H(y, D\varphi(y)) < 0$. We may assume that $v(y) < \varphi(y)$ and, for some constant $r > 0$, we have $H(x, D\varphi(x)) < 0$ for all $x \in \overline{\Omega} \cap B(0, r)$ and $v(x) > \varphi(x)$ for all $x \in \overline{\Omega} \setminus B(y, r)$. We define the function $w \in C(\overline{\Omega})$ by setting $w(x) = \max\{v(x), \varphi(x)\}$ for $x \in \overline{\Omega}$. We observe that $w \in \mathcal{S}_H^-(\Omega \cap U)$ for some neighborhood U of $B(y, r)$, by Proposition 1.2, and that $w = v$ in $\Omega \setminus B(y, r)$ and hence $w \in \mathcal{S}_H^-(\Omega \setminus B(y, r))$, which guarantees that $w \in \mathcal{S}_H^-(\Omega)$. Now, since $w = \varphi$ in a neighborhood V, relative to $\overline{\Omega}$, of y, we easily find a function $f \in C(\overline{\Omega})$ such that $f \ge 0$ in $\overline{\Omega}$, $f(y) > 0$, and $w \in \mathcal{S}_{H+f}^-(\Omega)$, which completes the proof. □

Proof of Theorem 7.3. Fix any $\varepsilon > 0$. We may choose an open neighborhood V of $\mathcal{A}_{\overline{\Omega}}$ so that $u \le v + \varepsilon$ on $\overline{\Omega} \cap V$. It is enough to show that $u \le v + \varepsilon$ on $\overline{\Omega} \setminus V$.

We follow the line of proof of Theorem 1.5. We set $v_\varepsilon = v + \varepsilon$ and $K = \overline{\Omega} \setminus V$. By Lemma 7.4, there are a function $\psi \in C(K)$ and a constant $\delta > 0$ such that $\psi \in \mathcal{S}_{H+\delta}^-(\operatorname{int} K)$. Fix any $\lambda \in (0, 1)$ and set $u_\lambda = (1 - \lambda)u + \lambda\psi$. We have $u_\lambda \in \mathcal{S}_{H+\lambda\delta}^-(\operatorname{int} K)$. It is then enough to prove that $u_\lambda \le v_\varepsilon + \lambda \max_K |\psi|$ on K.

By abuse of notation, we write u and v for u_λ and $v_\varepsilon + \lambda \max_K |\psi|$, respectively. We have $u \in \mathcal{S}_{H+\lambda\delta}^-(\operatorname{int} K)$, $v \in \mathcal{S}_H^+(\overline{\Omega})$, and $u - v \le 0$ on $\overline{\Omega} \cap V$, and we wish to show that $\max_K(u - v) \le 0$.

To prove that $\max_K(u - v) \le 0$, we argue by contradiction, and thus assume that $\max_K(u - v) > 0$. If $\max_K(u - v) > \max_{\partial K}(u - v)$, then Theorem 1.5 yields a contradiction. Therefore we may assume that $\max_{\partial K}(u - v) = \max_K(u - v)$. Let $z \in \partial K$ be a maximum point of $u - v$. Since $u \le v$ on $\overline{\Omega} \cap \overline{V}$, we have $z \notin \overline{V}$ and hence, $z \in \partial\Omega \setminus \overline{V}$. We may thus choose a constant $r > 0$ so that $B(z, r) \cap K = B(z, r) \cap \overline{\Omega}$. By adding a smooth function to u, we may assume that $u - v$ attains a strict maximum at z over $B(z, r) \cap \overline{\Omega}$ and $u \in \mathcal{S}_{H+c}^-(\Omega \cap \operatorname{int} B(z, r))$ for some constant $c > 0$. By (A6), after a C^1 change of variables if necessary, we may assume that $B(z,r)\cap\Omega = B(z,r)\cap\{(x',x_n)\in\mathbf{R}^{n-1}\times\mathbf{R}\mid x_n > b(x_n)\}$ for some continuous function b on $\mathbf{R}^{n-1}$.

Let $\delta > 0$ and set $u_\delta(x) = u(x+\delta e_n)$ for $x \in -\delta e_n+\Omega$, where $e_n := (0,...,0,1) \in \mathbf{R}^n$. Replacing r by a smaller positive number if necessary, we may assume that there is a $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then we have $B(z,r)\cap\overline{\Omega} \subset -\delta e_n + \Omega$. Henceforth we assume that $0 < \delta < \delta_0$. We may choose a bounded open neighborhood W of $B(z,r)\cap\overline{\Omega}$ such that $\overline{W} \subset -\delta e_n + \Omega$. It is easy to see that u_δ satisfies $H(x + \delta e_n, Du_\delta(x)) \le -c$ in W in the viscosity sense. Noting that H is uniformly continuous on $\Omega \times B(0,R)$ for every $R > 0$ and that there is an $R > 0$ such that $H(x,p) > 0$ if $|p| > R$, we may assume by replacing δ_0 by a smaller positive number that $H(x,Du_\delta(x)) \le -c/2$ in W in the viscosity sense. We may also assume that all the maximum points of $u_\delta - v$ over $B(z,r)\cap\overline{\Omega}$ lie in $B(z,r/2)\cap\overline{\Omega}$.

We now just need to follow the standard proof of comparison theorems on viscosity solutions. Let $\alpha > 0$ and consider the function $\Phi(x,y) := u_\delta(x) - v(y) - \alpha|x-y|^2$ on $\overline{W}\times(B(z,r)\cap\overline{\Omega})$. Let (x_α, y_α) be a maximum point of the function Φ. We may choose a sequence $\alpha_j \to \infty$ so that $x_{\alpha_j} \to x$ for some $x \in B(z,r)\cap\overline{\Omega}$, and observe that $y_{\alpha_j} \to x$ as $j \to \infty$ and that x is a maximum point of $u_\delta - v$ over $B(z,r)\cap\overline{\Omega}$, which guarantees that $x \in B(z,r/2)$. Choosing j large enough, we may assume that $y_{\alpha_j} \in \operatorname{int} B(z,r)$ and $x_{\alpha_j} \in W$. Now, by the viscosity property, we get $H(x_{\alpha_j}, 2\alpha_j(x_{\alpha_j} - y_{\alpha_j})) \le -c/2$ and $H(y_{\alpha_j}, 2\alpha_j(x_{\alpha_j} - y_{\alpha_j})) \ge 0$. The former of these inequalities assures that the sequence $\{\alpha_j(x_{\alpha_j} - y_{\alpha_j})\}$ is bounded in $\mathbf{R}^n$ and therefore, by sending $j \to \infty$ along a subsequence, we get $H(x,p) \le -c/2$ and $H(x,p) \ge 0$ for some $p \in \mathbf{R}^n$, which is a contradiction. This completes the proof. □

The above idea of enlarging the domain of definition of subsolutions appears already in the proof of Theorem III.5 of [CL].

Acknowledgments. The authors are grateful to Prof. Hidehiro Kaise for discussions which hinted at the use of ideal boundary and to Prof. Antonio Siconolfi for critical comments on an early version of this paper. They thank the referee for many useful comments on the previous version of this paper.

Appendix

Let $R > 0$ and $\mathcal{U}$ denote the set of all functions $u \in C(\Omega)$ such that $|Du| \le R$ in Ω in the viscosity sense.

Proposition A.1. *Assume that Ω is bounded and that* (A6) *holds. Then the set $\mathcal{U}$ is uniformly equi-continuous on Ω, i.e., there exists a modulus ω such that $|u(x)-u(y)| \le \omega(|x-y|)$ for all $u \in \mathcal{U}$ and $x,y \in \Omega$.*

Proof. We fix any $z \in \partial\Omega$, and choose $r > 0$ and $b \in C(\mathbf{R}^{n-1})$ so that, after a C^1 change of variables, we have

$$\Omega \cap B(z,r) = \{(x', x_n) \in \mathbf{R}^n \mid x_n > b(x')\} \cap B(z,r).$$

Here and henceforth, we write $x = (x', x_n)$ for $x \in \mathbf{R}^n$, where $x' \in \mathbf{R}^{n-1}$. Let ω_z be the modulus of continuity of b on the ball $B(z', r) \subset \mathbf{R}^{n-1}$. We choose a $\delta \in (0, r/2)$ so that $\omega_z(\delta) < r/4$.

Let $x, y \in \Omega \cap B(z, \delta/2)$, and set $\varepsilon := |x - y|$, $\xi := (x', x_n + \omega_z(\varepsilon))$, and $\eta := (y', y_n + \omega_z(\varepsilon))$. Consider the line segments $[x, \xi] := \{tx + (1-t)\xi \mid 0 \le t \le 1\}$, $[y, \eta] := \{ty + (1-t)\eta \mid 0 \le t \le\}$, and $[\xi, \eta] := \{t\xi + (1-t)\eta \mid 0 \le t \le 1\}$. Noting that $\xi, \eta \in B(z, r/2)$, we see easily that $[x, \xi] \cup [y, \eta] \subset \Omega$ and that $[\xi, \eta] \subset B(z, r/2)$. Observe that for any $t \in [0,1]$,

$$\begin{aligned} b(tx' + (1-t)y') &\le \min\{b(x') + \omega_z(\varepsilon), b(y') + \omega_z(\varepsilon)\} \\ &< \min\{x_n, y_n\} + \omega_z(\varepsilon) \le t(x_n + \omega_z(\varepsilon)) + (1-t)(y_n + \omega_z(\varepsilon)), \end{aligned}$$

which reads $b(t\xi' + (1-t)\eta') < t\xi_n + (1-t)\eta_n$. We thus conclude that

$$[x, \xi] \cup [\xi, \eta] \cup [y, \eta] \subset \Omega,$$

and furthermore that

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u(\xi)| + |u(\xi) - u(\eta)| + |u(\eta) - u(y)| \\ &\le R(\varepsilon + 2\omega_z(\varepsilon)) = R(|x-y| + 2\omega_z(|x-y|)). \end{aligned}$$

By the standard compactness argument, we find an open neighborhood V of $\partial\Omega$, relative to Ω, and a modulus ω_0 such that $|u(x) - u(y)| \le \omega_0(|x-y|)$ for all $u \in \mathcal{U}$ and $x, y \in V$. We choose a compact neighborhood $W \subset \Omega$ of $\Omega \setminus V$ and observe that the collection $\mathcal{U}$ is equi-Lipschitz continuous on W. It is now easy to conclude that $\mathcal{U}$ is uniformly equi-continuous on Ω. □

References

[AGW] M. Akian, S. Gaubert, and C. Walsh, The max-plus Martin boundary, arXiv:math.MG/0412408 v2.

[ABI] O. Alvarez, E. N. Barron and H. Ishii, Hopf-Lax formulas for semicontinuous data. Indiana Univ. Math. J. **48** (1999), no. 3, 993–1035.

[B] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi.* Mathématiques & Applications (Berlin), **17**. Springer-Verlag, Paris, 1994.

[BCD] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia.* Systems & Control: Foundations & Applications. Birkhauser Boston, Inc., Boston, MA, 1997.

[BJ] E. N. Barron and R. Jensen, Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians. Comm. Partial Differential Equations **15** (1990), no. 12, 1713–1742.

[CIL] M. G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67.

[CL] I. Capuzzo-Dolcetta and P.-L. Lions, Hamilton-Jacobi equations with state constraints. Trans. Amer. Math. Soc. **318** (1990), no. 2, 643–683.

[EI] L. C. Evans and H. Ishii, Differential games and nonlinear first order PDE on bounded domains. Manuscripta Math. 49 (1984), no. 2, 109–139.

[F1] A. Fathi, Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. C. R. Acad. Sci. Paris Ser. I Math. **324** (1997), no. 9, 1043–1046.

[F2] A. Fathi, *Weak KAM theorem in Lagrangian dynamics*, version 2003.

[FSi] A. Fathi and A. Siconolfi, PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians. Calc. Var. Partial Differential Equations **22** (2005), no. 2, 185–228.

[FSo] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions.* Second edition. Stochastic Modelling and Applied Probability, 25. Springer, New York, 2006.

[H] E. Hopf, Generalized solutions of non-linear equations of first order. J. Math. Mech. **14** (1965) 951–973.

[I1] H. Ishii, A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type. Proc. Amer. Math. Soc. **100** (1987), no. 2, 247–251.

[I2] H. Ishii, Representation of solutions of Hamilton-Jacobi equations. Nonlinear Anal. **12** (1988), no. 2, 121–146.

[I3] H. Ishii, A generalization of a theorem of Barron and Jensen and a comparison theorem for lower semicontinuous viscosity solutions. Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), no. 1, 137–154.

[I4] H. Ishii, Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space, Ann. Inst. H. Poincaré Anal. Non Linéaire (2007), doi: 10.1016/j.anihpc.2006.09.002.

[K] S. N. Kružkov, Generalized solutions of nonlinear first order equations with several independent variables. II. Mat. Sb. (N.S.) **72 (114)** (1967) 108–134 (Math. USSR–Sbornik, **1** (1967), 93–116).

[La] P. D. Lax, Hyperbolic systems of conservation laws. II. Comm. Pure Appl. Math. **10** (1957) 537–566.

[Li] P.-L. Lions, *Generalized solutions of Hamilton-Jacobi equations.* Research Notes in Mathematics, 69. Pitman, Boston, Mass.-London, 1982.

[Ma] R. S. Martin, Minimal positive harmonic functions. Trans. Amer. Math. Soc. **49** (1941), 137–172.

[Mi] H. Mitake, A representation formula for solutions of Hamilton-Jacobi equation. *Viscosity solution theory of differential equations and its developments.* Sūrikaisekikenkyūsho Kōkyūroku No. 1481 (2006), 32–42.

[O] O. A. Oleĭnik, Discontinuous solutions of non-linear differential equations. Uspehi Mat. Nauk **12** (1957), no. 3 (75), 3–73 (Amer. Math. Soc. Transl. (2) **26** (1963), 95–172).

[S] H. M. Soner, Optimal control with state-space constraint. I. SIAM J. Control Optim. **24** (1986), no. 3, 552–561.