1. $\langle x \rangle = (|x|^2+1)^{\frac{1}{2}}$ とおく.

$$\frac{\partial}{\partial x_i}\langle x \rangle = \langle x \rangle^{-1} x_i ,$$

$$\frac{\partial}{\partial x_i}\langle x \rangle^{2-n} = (2-n)\langle x \rangle^{-n} x_i$$

$$\frac{\partial^2}{\partial x_i^2}\langle x \rangle^{2-n} = (2-n)\left(-n\langle x \rangle^{-n-2} x_i^2 + \langle x \rangle^{-n}\right)$$

$$\Delta \langle x \rangle^{2-n} = (2-n)\left(-n\langle x \rangle^{-n-2}|x|^2 + n\langle x \rangle^{-n}\right)$$

$$= (2-n)n\left(-|x|^2+|x|^2+1\right)\langle x \rangle^{-n-2}$$

$$= -n(n-2)\langle x \rangle^{-n-2} < 0$$

2. $$\int \langle x \rangle^{2-n} m_R \, dx = \int \langle x \rangle^{2-n} \Delta f_R \, dx$$

$$= \int \Delta \langle x \rangle^{2-n} f_R \, dx = n(n-2)\int \langle x \rangle^{-n-2}(-f_R)\, dx$$

$$= n(n-2)\int \langle x \rangle^{-n-2}|f_R|\, dx$$

$0 \geq f_R \geq f$ より, $|f_R| \leq |f|$.

$$\therefore \int \langle x \rangle^{2-n} m_R \, dx \leq n(n-2)\int \langle x \rangle^{-n-2}|f|\, dx$$

3. f は subharmonic なので，

$$f(0) \le [f]_{0,r} \quad (r>0)$$

ただし，$[f]_{0,r} = |S^{n-1}|^{-1} \int_{S^{n-1}} f(r\omega)\,d\omega$.

$f \le 0$ より，

$$|f(0)| \ge [|f|]_{0,r} \quad (\forall r>0).$$

4. 極座標変換で，

$$\int \langle x \rangle^{-n-2} |f|\,dx = \cancel{\int_0^\infty r^{n-1}\,dr \int_{S^{n-1}} \underbrace{|f(r\omega)|\,d\omega}_{\left(\frac{1}{(1+r^2)^{\frac{1}{2}}}\right)^{n+2}}}$$

$$\cancel{= \left(\int_0^1 + \int_1^\infty\right) dr \int \cdots}$$

$$\cancel{\le \int_0^1 r^{n-1}}$$

$(\langle x \rangle \ge 1$ かつ $\langle x \rangle \ge |x|$ を用いて$)$

$$= \int_{B_{0,1}} \langle x \rangle^{-n-2} |f|\,dx + \int_{\mathbb{R}^n \setminus B_{0,1}} \langle x \rangle^{-n-2} |f|\,dx$$

$$\le \int_{B_{0,1}} |f|\,dx + \int_{\mathbb{R}^n \setminus B_{0,1}} |x|^{-n-2} |f|\,dx$$

$$= \int_{B_{0,1}} |f|\,dx + \int_1^\infty r^{-n-2} \cdot r^{n-1}\,dr \int_{S^{n-1}} |f(r\omega)|\,d\omega$$

$$= \int_{B_{0,1}} |f|\,dx + \int_1^\infty r^{-3} [f]_{0,r} |S^{n-1}|\,dr$$

$$\leq \int_{B_{0,1}} |f|\,dx + |S^{n-1}|\,|f(0)| \int_1^\infty r^{-3}\,dr$$

$$= \int_{B_{0,1}} |f|\,dx + \frac{|S^{n-1}|\,|f(0)|}{2}$$

$$\therefore \int \langle x \rangle^{2-n} m_R\,dx \leq n(n-2)\left(\int_{B_{0,1}} |f|\,dx + \frac{1}{2}|S^{n-1}|\,|f(0)| \right)$$

$$\cancel{(1+|x|^2)^{\frac{1}{2}}}$$

$$\cancel{1+|x| \leq \sqrt{2}\,(1+|x|^2)^{\frac{1}{2}}}\ \cancel{\text{より}}$$

$$\cancel{\langle x \rangle^{2-n} \leq \left(\frac{1+|x|}{\sqrt{2}}\right)^{2-n} = 2^{\frac{n-2}{2}}\ (1+|x|)}$$

$$(1+|x|^2)^{\frac{1}{2}} \leq 1+|x| \quad \text{より}$$

$$(1+|x|)^{2-n} \leq \langle x \rangle^{2-n}.$$

R に依らない.

よって

$$\int (1+|x|)^{2-n} m_R\,dx \leq n(n-2)\left(\int_{B_{0,1}} |f|\,dx + \frac{1}{2}|S^{n-1}|\,|f(0)| \right)$$

$$\therefore \int (1+|x|)^{2-n} m\,dx \leq \boxed{\text{上と同じ}}$$