A PDE approach to state-constraint problems in Hilbert spaces*

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Abstract. We study state-constraint problems with bounded continuous dynamics and the corresponding Hamilton-Jacobi-Bellman equations in real Hilbert spaces. We prove that under appropriate assumptions, the value functions of the state-constraint problems are uniform continuous and characterized as the unique viscosity solutions of the associated Hamilton-Jacobi-Bellman equations with the state-constraint boundary condition.

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§1 Introduction

In this paper we investigate state-constraint problems in optimal control and the corresponding Hamilton-Jacobi-Bellman equations in real Hilbert spaces.

State-constraint problems are natural and important subjects in optimal control and they have been studied extensively in finite dimensional spaces where the states are governed by ordinary differential equations (see, e.g., [4, 19, 5, 14]) and for problems with specific partial differential equations as their state equations.

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However, there are only a few references concerning the general theory of state-constraint problems and the corresponding Hamilton-Jacobi-Bellman equations in infinite dimensional spaces. See, e.g., [16].

Our purpose here is to develop this theory a little further, so that the resulting theory covers state-constraint problems in infinite dimensional Hilbert spaces with bounded continuous dynamics. Because of the boundedness of the dynamics, the class of control problems does not cover optimal control of systems governed by partial differential equations and the level of art in this paper is thus like in the papers [7, 8]. The reader who is interested in optimal control of systems governed by partial differential equations should consult [9, 10, 11, 21].

Our approach is based on the theory of viscosity solutions of Hamilton-Jacobi-Bellman equations and the line of arguments adapted here lies between those of [14] and [19, 5].

The paper is organized as follows. In Section 2 we establish a general comparison theorem for viscosity solutions of Hamilton-Jacobi equations with a kind of Neumann type boundary conditions. For the proof, we use the method developed in [14], which relies on the existence of a certain kind of test functions. The construction of such test functions in infinite dimensions differs from that in finite dimensions because the standard mollification techniques are not available in infinite dimensions. We use instead the sup-inf or inf-sup convolutions as in [17]. In Section 3 we study state-constraint problems with infinite horizon, and establish the uniform continuity of the corresponding value functions and characterize the value functions as unique viscosity solutions of the state-constraint problem for the associated Hamilton-Jacobi-Bellman equations. In Section 4 we study finite horizon problems and establish the uniform continuity of the value functions and their characterization as state-constraint problems of the corresponding Hamilton-Jacobi-Bellman equations.

§2 A comparison theorem

In this section we will establish a general comparison theorem for solutions of Hamilton-Jacobi equations. Our arguments which will follow depends largely on this comparison theorem.

Let Ω be an open subset of $\mathcal{H}$ and Γ a relatively open subset of $\partial\Omega$. By $|x|$ and $\langle x, y\rangle$ we denote the norm of $x \in \mathcal{H}$ and the inner product of $x, y \in \mathcal{H}$, respectively. Set $I = \partial\Omega \setminus \Gamma$. Let H be a real continuous function on $\overline{\Omega} \times \mathbf{R} \times \mathcal{H}$ and ξ a continuous map of $\overline{\Omega}$ into $\mathcal{H}$.

We shall consider viscosity solutions of

$$\text{(2.1)} \qquad \begin{cases} H(x, u(x), Du(x)) \leq 0 & \text{in } \Omega \\ -\langle \xi(x), Du(x)\rangle \leq K & \text{on } \Gamma, \end{cases}$$

and those of

$$\text{(2.2)} \qquad H(x, u(x), Du(x)) \geq \theta \quad \text{in } \Omega \cup \Gamma,$$

where K and θ are given positive constants. Hereafter $Du(x)$ denotes, when it is understood in the classical sense, the Fréchet derivative of u at x, which is regarded as an element of $\mathcal{H}$ via the Riesz theorem.

The boundary condition in (2.1) is understood here in the strong sense of the terminology of [6]. To be precise, let us give the definition of viscosity solution of (2.1): a function $u : \Omega \cup \Gamma \to \mathbf{R}$ is called a viscosity solution of (2.1) if it is locally bounded and satisfies $H(x, u^*(x), p) \leq 0$ for all $x \in \Omega$ and $p \in D^+u^*(x)$ and $-\langle \xi(x), p\rangle \leq K$ for all $x \in \Gamma$ and $p \in D^+u^*(x)$. Here the upper semicontinuous envelope u^* of u is defined by

$$u^*(x) = \lim_{r \searrow 0} \sup\{u(y) \mid y \in \Omega \cup \Gamma,\ |y - x| < r\} \quad (x \in \Omega \cup \Gamma).$$

We refer the reader to [4, 6] for the definition of viscosity solution of (2.2) together with a general scope of the theory of viscosity solutions.

We assume:

(2.3) $$|H(x, r, p) - H(y, r, p)| \leq \omega(|x - y|(|p| + 1)) \quad (x, y, p \in \mathcal{H},\ r \in \mathbf{R});$$

(2.4) $H(x, r, p)$ is nondecreasing in $r \in \mathbf{R}$ for each $(x, p) \in \mathcal{H} \times \mathcal{H}$;

(2.5) $$|H(x, r, p) - H(x, r, q)| \leq L|p - q| \quad (x, p, q \in \mathcal{H},\ r \in \mathbf{R}).$$

Here ω is a continuous real-valued function on $[0, \infty)$, with $\omega(0) = 0$, and L is a positive constant.

For each $\varepsilon > 0$ we write

$$\Gamma_\varepsilon = \{x \in \Gamma \mid \operatorname{dist}(x, I) > \varepsilon\},$$

and assume:

(2.6) for each $\varepsilon > 0$ there is a constant $\delta > 0$ such that

$$B(x + t\xi(z), \delta t) \subset \overline{\Omega} \quad (\, z \in \Gamma_\varepsilon,\ x \in B(z, \delta) \cap \overline{\Omega},\ 0 \leq t \leq \delta\,);$$

(2.7) ξ is bounded Lipschitz continuous on $\overline{\Omega}$.

Theorem 2.1. *Under assumptions* (2.3)–(2.7), *if* $u : \overline{\Omega} \to \mathbf{R}$ *and* $v : \overline{\Omega} \to \mathbf{R}$ *are a bounded viscosity solution of* (2.1) *and a bounded viscosity solution of* (2.2), *respectively, and if*

(2.8) $$\lim_{r \searrow 0} \sup\{u(x) - v(y) \mid x, y \in \overline{\Omega},\ |x - y| < r,\ \operatorname{dist}(x, I) < r\} \leq 0,$$

then

$$\lim_{r\searrow 0}\sup\{u(x)-v(y)\mid x,y\in\overline{\Omega},\ |x-y|<r\}\le 0.$$

Lemma 2.2. *Assume that* (2.6) *and* (2.7) *hold. Then for each* $\varepsilon>0$ *there is a bounded function* $\psi\in \mathrm{C}^1(\overline{\Omega})$ *such that* $D\psi$ *is bounded and Lipschitz continuous on* $\overline{\Omega}$ *and such that*

$$\langle \xi(x), D\psi(x)\rangle \ge 1 \quad (x\in\Gamma_\varepsilon).$$

Lemma 2.3. *Under assumptions* (2.6) *and* (2.7), *for each* $\varepsilon>0$ *there are a function* $w\in \mathrm{C}^{1,1}(\overline{\Omega}\times\overline{\Omega})$ *and constants* $r\in(0,1)$ *and* $C>0$ *such that*

$$\langle \xi(x), D_1w(x,y)\rangle \le C|x-y|^2 \quad (x\in\Gamma_\varepsilon,\ y\in\overline{\Omega}\cap B(x,r))$$

and for all $x,y\in\overline{\Omega}$,

$$\begin{gathered}|x-y|^2\le w(x,y)\le C|x-y|^2;\\ |Dw(x,y)|\le C|x-y|;\\ |D_1w(x,y)+D_2w(x,y)|\le C|x-y|^2.\end{gathered}$$

In the above and henceforth D_1w and D_2w denote the first and second components of the Fréchet derivative Dw, respectively. More precisely, $D_1w=P_1Dw$ and $D_2w=P_2Dw$, with P_1 and P_2 denoting the projections $\mathcal{H}\times\mathcal{H}\ni(p,q)\mapsto p$ and $\mathcal{H}\times\mathcal{H}\ni(p,q)\mapsto q$, respectively.

Lemma 2.4. *Let* $\mathcal{O}$ *be an open subset of* $\mathcal{H}$. *Let* $\eta:\mathcal{O}\to\mathcal{H}$ *and* $g:\mathcal{O}\to\mathbf{R}$ *be uniformly continuous. Let* $v:\mathcal{O}\to\mathbf{R}$ *be a Lipschitz continuous function. Then the following two inequalities are equivalent:*

$$\langle \eta(x),p\rangle\le g(x)\quad (x\in\mathcal{O},\ p\in D^+v(x));$$

$$\langle \eta(x),p\rangle\le g(x)\quad (x\in\mathcal{O},\ p\in D^-v(x)).$$

We refer for the proof of this lemma to [13].

Proof of Lemma 2.2. In view of Kirszbraun's theorem (see, e.g., [18]), we may assume that ξ is defined on $\mathcal{H}$ and is a bounded Lipschitz continuous function on $\mathcal{H}$. Fix a constant $M>0$ so that

$$|\xi(x)|\le M,\quad |\xi(x)-\xi(y)|\le M|x-y|\quad (x,y\in\mathcal{H}).$$

Fix $\varepsilon > 0$ and select $\delta > 0$ so that

$$B(x + t\xi(z), \delta t) \subset \overline{\Omega} \quad (\, z \in \Gamma_\varepsilon,\ x \in B(z,\delta) \cap \overline{\Omega},\ 0 \le t \le \delta\,).$$

For $\gamma > 0$ set

$$N_\gamma = \{x \in \mathcal{H} \mid \operatorname{dist}(x, \Gamma_\varepsilon) < \gamma\}.$$

For $x \in \mathcal{H}$ let $X(t) \equiv X(t;x)$ denote the solution of

$$\dot{X}(t) = \xi(X(t)) \;\; (t > 0), \qquad X(0) = x,$$

where $\dot{X}(t)$ denotes the derivative dX/dt evaluated at t. Observe that

$$|X(t) - x| \le Mt \quad (t > 0),$$

and

$$|X(t) - [x + t\xi(x)]| \le M \int_0^t |X(s) - x| ds \le M^2 t^2 \quad (t > 0).$$

We may assume that $\delta \le 1$ and $M \ge 1$. Fix any $x \in N_\gamma$ and choose $z \in \Gamma_\varepsilon$ so that $|x - z| < \gamma$. Let $t \ge 0$. Observe that if

$$M(Mt + \gamma) \le \frac{\delta}{4} \quad \text{and} \quad \gamma \le \frac{\delta t}{4},$$

then

$$|X(t;x) - [z + t\xi(z)]| \le |X(t;x) - [x + t\xi(x)]| + \gamma + M\gamma t \le M(Mt+\gamma)t + \gamma \le \frac{\delta t}{2},$$

i.e., $X(t;x) \in B(z + t\xi(z), \delta t/2)$.

Now fix $\gamma > 0$ and $\tau > 0$ so that

$$M(M\tau + \gamma) \le \frac{\delta}{4} \quad \text{and} \quad \gamma \le \frac{\delta\tau}{4}.$$

Note that $\tau < \delta$ and $\gamma < 1/4$. Then we have

$$X(\tau;x) \in B\left(z + \tau\xi(z), \frac{\delta\tau}{2}\right).$$

Since $B(z + \tau\xi(z), \delta\tau) \subset \overline{\Omega}$, we see that $B(X(\tau;x), \delta\tau/2) \subset \overline{\Omega}$ and hence $X(\tau;x) \notin N_\gamma$.

Define the Lipschitz continuous function ζ on $\mathcal{H}$ by

$$\zeta(x) = \left(1 - \frac{2}{\gamma}\,\mathrm{dist}\,(x, \Gamma_\varepsilon)\right)_+ .$$

Define the function $v : \mathcal{H} \to \mathbf{R}$ by

$$v(x) = -\int_0^\tau \zeta(X(t;x))dt.$$

It is easily seen that v is bounded and Lipschitz continuous on $\mathcal{H}$.

Noting that $X(\tau;x) \notin N_\gamma$ for $x \in N_\gamma$, we see that there is a constant $\sigma \in (0,\tau)$ such that

$$\zeta(X(t;x)) = 0 \quad (x \in N_\gamma,\ \sigma \le t \le \tau).$$

Using the dynamic programming principle, wee see that v is a viscosity solution of

$$-\langle \xi(x), Dv(x)\rangle + \zeta(x) = 0 \quad (x \in N_\gamma).$$

Next we regularize the function v by using the inf-sup convolutions (see [17]). Let $0 < \mu < \nu$. Define $w : \mathcal{H} \to \mathbf{R}$ by

$$\begin{aligned} w(x) = (v^\nu)_\mu(x) &= \inf_{y\in\mathcal{H}} \left(v_\nu(y) + \frac{1}{2\mu}|x-y|^2 \right) \\ &= \inf_{y\in\mathcal{H}} \sup_{z\in\mathcal{H}} \left(v(z) - \frac{1}{2\nu}|y-z|^2 + \frac{1}{2\mu}|x-y|^2 \right). \end{aligned}$$

According to [17], w is a $\mathrm{C}^{1,1}$ function on $\mathcal{H}$.

Let $x \in \mathcal{H}$ and $p \in D^+ v^\nu(x)$. As a well-known property of inf-convolutions, we have

$$p \in D^- v(x + \nu p).$$

Noting that $|p| \le \mathrm{Lip}\,(v)$, we choose ν so that $\mathrm{Lip}\,(v)\nu < \gamma/4$. Here and henceforth $\mathrm{Lip}\,(v)$ denotes the Lipschitz seminorm of v, i.e.,

$$\mathrm{Lip}\,(v) = \sup\left\{ \frac{|v(x) - v(y)|}{|x-y|} \;\middle|\; x \ne y \right\}.$$

Since v is a viscosity solution of

$$-\langle \xi(x), Dv(x)\rangle + \zeta(x) = 0 \quad (x \in N_\gamma),$$

we see that

$$-\langle \xi(x+\nu p), p\rangle + \zeta(x+\nu p) \le 0 \quad (x \in N_{3\gamma/4},\ p \in D^+ v^\nu(x)).$$

Hence,

$$-\langle \xi(x), p\rangle + \zeta(x) \le L\nu \quad (x \in N_{3\gamma/4},\ p \in D^+ v^\nu(x)),$$

where $L := \mathrm{Lip}\,(\xi)\,\mathrm{Lip}\,(v)^2 + \mathrm{Lip}\,(\zeta)\,\mathrm{Lip}\,(v)$. By virtue of Lemma 2.4, we have

$$-\langle \xi(x), p\rangle + \zeta(x) \le L\nu \quad (x \in N_{3\gamma/4},\ p \in D^- v^\nu(x)).$$

Now let $x \in N_{\gamma/2}$ and $p = Dw(x)$. Again we have

$$p \in D^- v^\nu(x - \mu p),$$

as a well-known property of sup-convolutions. Since $|p| \le \mathrm{Lip}\,(v)$ for all $p \in D^- v^\nu(x)$ with $x \in \mathcal{H}$, we see that

$$-\langle \xi(x), Dw(x)\rangle + \zeta(x) \le 2L\nu \quad (x \in N_{\gamma/2}).$$

Choosing ν small enough so that $2L\nu \le 1/2$, we see that the function $2w(x)$ has all the required properties. □

Proof of Lemma 2.3. As in the previous proof, we assume that ξ is a bounded Lipschitz continuous map of $\mathcal{H}$.

Fix $\varepsilon > 0$ and let $\delta > 0$ be a constant such that

$$B(x + t\xi(z), \delta t) \subset \overline{\Omega} \quad (\, z \in \Gamma_\varepsilon,\ x \in B(z,\delta) \cap \overline{\Omega},\ 0 \le t \le \delta\,).$$

Set

$$\mathcal{O} = \{x \in \mathcal{H} \mid |\xi(x)| > \delta/2\}.$$

Since $B(x + \delta\xi(x), \delta^2) \subset \overline{\Omega}$ for $x \in \Gamma_\varepsilon$ and hence $x \notin \mathrm{int}\, B(x + \delta\xi(x), \delta^2)$ for $x \in \Gamma_\varepsilon$, we see that $|\xi(x)| \ge \delta$ for all $x \in \Gamma_\varepsilon$ and therefore that $\Gamma_\varepsilon \subset \mathcal{O}$. Dividing $\xi(x)$ by $|\xi(x)|$ and multiplying a cut-off function, we can choose a bounded Lipschitz continuous map $\eta : \mathcal{H} \to \mathcal{H}$ such that

$$\eta(x) = \frac{\xi(x)}{|\xi(x)|} \quad (x \in \mathcal{O}).$$

Replacing δ by a smaller constant if necessary, we may assume that

$$B(x + t\eta(x), \delta t) \subset \overline{\Omega} \quad (\, z \in \Gamma_\varepsilon,\ x \in B(z,\delta) \cap \overline{\Omega},\ 0 \le t \le \delta\,). \tag{2.9}$$

In what follows let ρ and σ be two constants satisfying $0 < \sigma < \rho < 1$ which will be fixed later on. We write

$$K := \underset{t\geq 0}{\cup} B(te_1, \sigma t) \equiv \{(x_1, x_2) \in \mathbf{R}^2 \mid |x_2| \leq \frac{\sigma}{\sqrt{1-\sigma^2}} x_1\}$$

and

$$L := \underset{t\geq 0}{\cup} B(te_1, \rho t) \equiv \{(x_1, x_2) \in \mathbf{R}^2 \mid |x_2| \leq \frac{\rho}{\sqrt{1-\rho^2}} x_1\},$$

with e_1 denoting the unit vector $(1, 0) \in \mathbf{R}^2$.

According to [14], there is a convex function $v \in \mathrm{C}(\mathbf{R}^2) \cap \mathrm{C}^{1,1}(\mathbf{R}^2 \setminus \{0\})$ such that

$$v(tx) = tv(x) \quad \text{for } x \in \mathbf{R}^2 \text{ and } t \geq 0;$$

$$v(x) > 0 \quad \text{if } x \neq 0;$$

$$v(x_1, -x_2) = v(x_1, x_2) \quad ((x_1, x_2) \in \mathbf{R}^2);$$

$$\langle q, Dv(x)\rangle \leq 0 \quad \text{for all } q \in K \text{ and } x \in L^c.$$

We define $w \in \mathrm{C}(\mathcal{H} \times \mathcal{H})$ by

$$w(x, y) = v\big(\langle (x-y), \eta(y)\rangle, |x - y - \langle (x-y), \eta(y)\rangle|\big)^2.$$

For each $y \in \mathcal{H}$ define the continuous linear map $Q(y) : \mathcal{H} \to \mathcal{H}$ by

$$Q(y) = I - \eta(y) \otimes \eta(y).$$

With this notation, we have

$$w(x, y) = v\big(\langle (x-y), \eta(y)\rangle, |Q(y)(x-y)|\big)^2.$$

Observe that

$$\big(\langle (x-y), \eta(y)\rangle\big)^2 + |Q(y)(x-y)|^2 = |x-y|^2 + (|\eta(y)|^2 - 1)\big(\langle (x-y), \eta(y)\rangle\big)^2 \geq \frac{3|x-y|^2}{4},$$

to conclude that for some constant $\varepsilon_1 > 0$,

$$w(x, y) \geq \varepsilon_1 |x-y|^2.$$

Similarly, we see that for some constant $C_1 > 0$,

$$w(x, y) \leq C_1 |x-y|^2.$$

Now fix any $x, y \in \mathcal{H}$ and choose $p, q \in \mathcal{H}$ such that $(p, q) \in D^+ w(x, y)$. We want to show that for some constant $C_2 > 0$,

$$|p + q| \leq C_2 |x - y|^2;$$

$$|p| + |q| \leq C_2 |x - y|. \tag{2.10}$$

We write

$$z_1 = \langle (x - y), \eta(y) \rangle, \quad z_2 = |Q(y)(x - y)| \quad \text{and} \quad z = (z_1, z_2).$$

By calculus, we have

$$p = 2v(z) \left\{ D_1 v(z) \eta(y) + D_2 v(z) \frac{Q(y)(x - y)}{|Q(y)(x - y)|} \right\},$$

where $D_1 v(z) := \partial v(z)/\partial z_1$ and $D_2 v(z) := \partial v(z)/\partial z_2$. Here we understand that the second term in the braces above vanishes if $z_2 = 0$. In this regard, note that $D_2 v(x_1, 0) = 0$ for all $x_1 \in \mathbf{R}$. Thus, choosing a constant $C_3 > 0$ so that $C_3 \geq 2\|Dv\|_\infty$, we get

$$|p| \leq C_3 v(x, y).$$

In the above and hereafter, given a function h on a set X (or a map h of X to $\mathcal{H}$), we write $\|h\|_\infty$ for $\sup_{x \in X} |h(x)|$.

Next, we observe that as $h \to 0$,

$$|w(x, y + h) - w(x - h, y)| \leq 2v(z)|Dv(z)||x - y|(M + M^2)|h| + O(|h|^2),$$

where $M := \max\{\|\eta\|_\infty, \mathrm{Lip}\,(\eta)\}$. From this it follows that

$$|p + q| \leq 2v(z)|Dv(z)|(M + M^2)|x - y|.$$

Now we choose $r > 0$ so that $r^2 < 4\delta$ and

$$\{x \in \mathcal{H} \mid \mathrm{dist}\,(x, \Gamma_\varepsilon) < 2r^2\} \subset \mathcal{O}.$$

Then define the open subset Λ of $\mathcal{H}^2$ by

$$\Lambda = \{(x, y) \in \mathcal{H} \times \mathcal{H} \mid \mathrm{dist}\,(x, \Gamma_\varepsilon) < r^2,\ |x - y| < r^2,\ \min_{0 \leq t \leq r} (|x - [y + t\eta(y)]| - rt) > 0\}.$$

Note that if $(x, y) \in \Lambda$ then $x, y \in \mathcal{O}$.

Next we fix

$$\rho = \frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \sigma = \frac{\rho}{2}.$$

We set $s = \text{Lip}\,(\eta) r^2$. By choosing $r > 0$ small enough, we may assume that

$$\frac{s}{1-s} \leq \frac{\sigma}{\sqrt{1-\sigma^2}}.$$

Assume that $(x, y) \in \Lambda$. We are about to prove that

$$\langle p, \eta(x) \rangle \leq 0.$$

Set

$$q_1 = \langle \eta(x), \eta(y) \rangle, \quad q_2 = \left\langle \eta(x), \frac{Q(y)(x-y)}{|Q(y)(x-y)|} \right\rangle, \quad \text{and} \quad q = (q_1, q_2),$$

and note that

$$\langle \eta(x), p \rangle = 2v(z)(q_1 D_1 v(z) + q_2 D_2 v(z)) = 2v(z) \langle q, Dv(z) \rangle.$$

Since

$$1 \geq q_1 \geq 1 - \text{Lip}\,(\eta)|x-y| \geq 1-s$$

and

$$\langle \eta(y), Q(y)(x-y) \rangle = 0,$$

we have

$$\frac{|q_2|}{q_1} \leq \frac{1}{1-s} \left| \left\langle (\eta(x) - \eta(y)), \frac{Q(y)(x-y)}{|Q(y)(x-y)|} \right\rangle \right| \leq \frac{s}{1-s} \leq \frac{\sigma}{\sqrt{1-\sigma^2}}.$$

Hence, $q \in K$.

We want to show that $z \in L^c$. If $z_1 \leq 0$, then it is immediate that $z \in L^c$. Assume that $z_1 > 0$. Noting that $z_1 \leq r^2 < r$, we find that if $z_2/z_1 \leq r$, then

$$|x - [y + z_1 \eta(y)]| = |Q(y)(x-y)| = z_2 \leq r z_1,$$

which is impossible since $(x, y) \in \Lambda$. That is, we have

$$z_2/z_1 > r = \rho/\sqrt{1-\rho^2}$$

and $z \in L^c$. By our choice of v, we conclude that $\langle p, \eta(x)\rangle \le 0$.

The final step is to regularize w by inf-sup convolutions. Let $0 < \mu < \nu$ be two small constants to be fixed later on. Define $\psi : \mathcal{H}\times\mathcal{H} \to \mathbf{R}$ by $\psi = (w^\nu)_\mu$. If $\nu < (4C_1)^{-1}$, then ψ is well-defined and $\psi \in \mathrm{C}^{1,1}(\mathcal{H}\times\mathcal{H})$. We assume that $\nu < (4C_1)^{-1}$.

Noting that

$$w^\nu(x,y) \ge w(x,y) \ge \varepsilon_1|x-y|^2,$$

we have

$$\psi(x,y) \ge \inf_{x',y'\in\mathcal{H}} \left(\varepsilon_1|x'-y'|^2 + \frac{1}{2\mu}\left(|x-x'|^2+|y-y'|^2\right)\right),$$

and, calculating the right hand side of the inequality above, we get

$$\psi(x,y) \ge \frac{\varepsilon_1}{1+4\varepsilon_1\mu}|x-y|^2.$$

Since

$$w(x,y) \le C_1|x-y|^2,$$

we have

$$w^\nu(x,y) \le \sup_{x',y'\in\mathcal{H}} \left(C_1|x'-y'|^2 - \frac{1}{2\nu}\left(|x-x'|^2+|y-y'|^2\right)\right).$$

Calculating the maximum value of the right hand side of the above, we get

$$w^\nu(x,y) \le \frac{C_1}{1-4\nu C_1}|x-y|^2 \quad (x,y\in\mathcal{H}),$$

and consequently,

$$\psi(x,y) \le \frac{C_1}{1-4\nu C_1}|x-y|^2 \quad (x,y\in\mathcal{H}).$$

For $i = 1,2$ we define

$$\Lambda_i = \{(x,y)\in\mathcal{H}\times\mathcal{H} \mid \mathrm{dist}\,(x,\Gamma_\varepsilon) < 2^{-i}r^2, |x-y| < 2^{-i}r^2, \\ \min_{0\le t\le r}\left(|x-[y+t\eta(y)]| - 2^i rt\right) > 0\}$$

Note that $\Lambda \supset \Lambda_1 \supset \Lambda_2$ and that Λ_i $(i=1,2)$ are open subsets of $\mathcal{H}^2$.

Fix any $(x,y)\in\Lambda_1$ and $(p,q)\in D^+w^\nu(x,y)$. We have

$$(p,q) \in D^+w(\hat{x},\hat{y}),$$

where $\hat{x} = x + \nu p$ and $\hat{y} = y + \nu q$. Using (2.10), we compute that

$$|p| + |q| \le C_2|\hat{x} - \hat{y}| \le C_2(|x - y| + \nu(|p| + |q|))$$

and

$$|\hat{x} - \hat{y}| \le |x - y| + \nu(|p| + |q|).$$

If $2\nu C_2 \le 1$, then we have

$$|p| + |q| \le 2C_2|x - y|$$

and

$$|\hat{x} - \hat{y}| \le (1 + 2\nu C_2)|x - y|,$$

and furthermore that for $0 \le t \le r$,

$$\begin{aligned}
|\hat{x} - [\hat{y} + t\eta(\hat{y})]| &\ge |x - [y + t\eta(y)]| - \nu|p - q| - t\,\mathrm{Lip}\,(\eta)\nu|q| \\
&\ge |x - [y + t\eta(y)]| - \nu(|p| + |q|)(1 + r\,\mathrm{Lip}\,(\eta)) \\
&\ge |x - [y + t\eta(y)]| - \nu C_4|x - y| \\
&\ge |x - [y + t\eta(y)]| - \nu C_4(|x - y - t\eta(y)| + t) \\
&\ge (1 - \nu C_4)|x - [y + t\eta(y)]| - \nu C_4 t.
\end{aligned}$$

where $C_4 := 2C_2(1 + r\,\mathrm{Lip}\,(\eta))$. We now fix ν so that

$$\nu C_1 < \frac{1}{4} \quad \text{and} \quad \nu C_4 \le \frac{r}{2r + 1}.$$

Then, since $2\nu C_2 < 1$, we have

$$|p| + |q| \le 2C_2|x - y|;$$

$$|\hat{x} - \hat{y}| \le 2|x - y| < r^2;$$

$$|p + q| \le C_2|\hat{x} - \hat{y}|^2 \le 4C_2|x - y|^2;$$

$$\mathrm{dist}\,(\hat{x}, \Gamma_\varepsilon) \le \mathrm{dist}\,(x, \Gamma_\varepsilon) + \nu|p| < 2^{-1}r^2 + 2\nu C_2|x - y| \le r^2;$$

$$|\hat{x} - [\hat{y} + t\eta(\hat{y})]| > rt \quad (0 \le t \le r).$$

Thus we have $(\hat{x}, \hat{y}) \in \Lambda$ and hence,

$$\begin{aligned}
0 \ge \langle p, \eta(\hat{x})\rangle &\ge \langle p, \eta(x)\rangle - \mathrm{Lip}\,(\eta)\nu|p|^2 \\
&\ge \langle p, \eta(x)\rangle - 4C_2^2\,\mathrm{Lip}\,(\eta)\nu|x - y|^2.
\end{aligned}$$

This together with Lemma 2.4 yields that

$$\langle p, \eta(x)\rangle \leq 4C_2^2 \operatorname{Lip}(\eta)\nu|x-y|^2 \quad ((x,y) \in \Lambda_1,\ (p,q) \in D^- w^\nu(x,y)).$$

Now let $(x,y) \in \Lambda_2$ and $(p,q) = D\psi(x,y)$. We have

$$(p,q) \in D^- w^\nu(\bar{x}, \bar{y}),$$

where $\bar{x} = x - \mu p$ and $\bar{y} = y - \mu q$. To proceed, we fix

$$\mu = \frac{\nu}{2},$$

which will be convenient for our computations below. As before we compute

$$|\bar{x} - \bar{y}| \leq |x-y| + \mu(|p| + |q|) \leq |x-y| + 2\mu C_2 |\bar{x} - \bar{y}|,$$

and get

$$|\bar{x} - \bar{y}| \leq 2|x-y| \leq 2^{-1} r^2,$$

since $2\mu C_2 \leq 1/2$. Using this, we also have

$$|p| + |q| \leq 2C_2 |\bar{x} - \bar{y}| \leq 4C_2|x-y|;$$

$$|p+q| \leq 4C_2 |\bar{x} - \bar{y}|^2 \leq 16 C_2 |x-y|^2.$$

Moreover we have

$$\operatorname{dist}(\bar{x}, \Gamma_\varepsilon) \leq \operatorname{dist}(x, \Gamma_\varepsilon) + \mu|p| < 4^{-1} r^2 + 4^{-1} r^2 = 2^{-1} r^2;$$

$$|\bar{x} - [\bar{y} + t\eta(\bar{y})]| \geq (1 - 2\mu C_4)|x - [y + t\eta(y)]| - 2\mu C_4 t > 2rt.$$

Thus we have $(\bar{x}, \bar{y}) \in \Lambda_2$ and hence,

$$\begin{aligned} 0 &\geq \langle p, \eta(\bar{x})\rangle - \nu C_2^2 \operatorname{Lip}(\eta)|\bar{x} - \bar{y}|^2 \\ &\geq \langle p, \eta(x)\rangle - 16\mu C_2^2 \operatorname{Lip}(\eta)|x-y|^2 - 4\nu C_2^2 \operatorname{Lip}(\eta)|x-y|^2 \\ &\geq \langle p, \eta(x)\rangle - 12\nu C_2^2 \operatorname{Lip}(\eta)|x-y|^2. \end{aligned}$$

To conclude the proof, we claim that the function $\varepsilon_1^{-1}(1 + 4\varepsilon_1\mu)\psi$ has all the required properties. To check this, we only need to show that if $x \in \Gamma_\varepsilon$ and $y \in \operatorname{int} B(x, 4^{-1}r^2) \cap \overline{\Omega}$, then $(x,y) \in \overline{\Lambda}_2$. Indeed, choosing a sequence $\{x_n\} \subset \Omega^c$ converging to x as $n \to \infty$ and noting that

$$B(y + t\eta(y), \delta t) \subset \overline{\Omega} \quad (0 \le t \le \delta),$$

we have

$$|x_n - [y + t\eta(y)]| > 4rt \quad (0 \le t \le r).$$

Since $|x_n - y| < r^2/4$ and $|x_n - x| < r^2/4$ for sufficiently large n, we see that $(x,y) \in \overline{\Lambda}_2$. □

Proof of Theorem 2.1. We need to show that for each $\eta > 0$,

$$\lim_{r \searrow 0} \sup\{u(x) - v(y) \mid x, y \in \overline{\Omega},\ |x-y| < r\} \le \eta.$$

We fix any $\eta > 0$, suppose that

$$\lim_{r \searrow 0} \sup\{u(x) - v(y) \mid x, y \in \overline{\Omega},\ |x-y| < r\} > 2\eta,$$

and will show a contradiction.

Selecting $\varepsilon > 0$ small enough, by (2.8) we have

$$u(x) - v(y) < \eta \quad (\, x, y \in \overline{\Omega},\ \operatorname{dist}(x, I) < 2\varepsilon,\ \operatorname{dist}(y, I) < 2\varepsilon, |x-y| < \varepsilon\,). \tag{2.11}$$

Let ψ be the function from Lemma 2.2 with ε given above. Let $\mu > 0$ be a constant to be fixed later. Define functions $\tilde{u}$ and $\tilde{v}$ by

$$\tilde{u}(x) = u^*(x) + \mu\psi(x) \quad \text{and} \quad \tilde{v}(x) = v_*(x) + \mu\psi(x),$$

where u^* and v_* denote the upper and lower semicontinuous envelopes of u and v, respectively. Set

$$\widetilde{H}(x, r, p) = H(x, r - \mu\psi(x), p - \mu D\psi(x)),$$

and observe that $\tilde{u}$ and $\tilde{v}$, respectively, satisfy

$$\begin{cases} \widetilde{H}(x, \tilde{u}, D\tilde{u}) \le 0 & \text{in } \Omega, \\ -\langle \xi(x), D\tilde{u} \rangle + \mu\langle \xi(x), D\psi(x) \rangle \le K & \text{on } \Gamma, \end{cases}$$

and

$$\widetilde{H}(x, \tilde{v}, D\tilde{v}) \ge \theta \quad \text{on } \Omega \cup \Gamma$$

in the viscosity sense. In view of Lemma 2.2, we can fix $\mu > 0$ so that $\tilde{u}$ satisfies

$$-\langle \xi(x), D\tilde{u} \rangle \le -1 \quad \text{on } \Gamma_\varepsilon$$

in the viscosity sense.

By virtue of (2.11), there is $\gamma \in (0, \varepsilon)$ such that

$$\tilde{u}(x) - \tilde{v}(y) < \eta \quad (\, x, y \in \overline{\Omega}, \ \operatorname{dist}(x, I) < 2\varepsilon, \ \operatorname{dist}(y, I) < 2\varepsilon, \ |x - y| < \gamma\,). \tag{2.12}$$

Let w, $r > 0$, and C be the function and constants from Lemma 2.3. We may assume that $r < \gamma$. Let $\delta > 0$ and consider the function

$$\Phi(x, y) = \tilde{u}(x) - \tilde{v}(y) - \frac{1}{\delta} w(x, y)$$

on $\overline{\Omega} \times \overline{\Omega}$. Setting

$$m(\delta) = \sup_{\overline{\Omega} \times \overline{\Omega}} \Phi,$$

we observe that $m(\delta) \searrow m$ as $\delta \searrow 0$ for some m and moreover that

$$m = \lim_{t \searrow 0} \sup\{\tilde{u}(x) - \tilde{v}(y) \mid x, y \in \overline{\Omega}, \ |x - y| < t\} > 2\eta. \tag{2.13}$$

For each $\delta > 0$ we select $\nu \in (0, \delta)$ so that

$$\sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} [\Phi(x, y) - \nu(|x|^2 + |y|^2)] > m(\delta) - \delta.$$

By the standard optimization technique due to Stegall [20] and Ekeland-Lebourg [12], there are $a, b \in \mathcal{H}$ satisfying $a, b \in B(0, \nu)$ and depending on δ such that the function

$$\Phi(x, y) - \nu(|x|^2 + |y|^2) - \langle a, x \rangle - \langle b, y \rangle$$

attains a maximum at some point $(x_\delta, y_\delta) \in \overline{\Omega} \times \overline{\Omega}$. Set

$$\tilde{m}(\delta) = \Phi(x_\delta, y_\delta) - \nu(|x_\delta|^2 + |y_\delta|^2) - \langle a, x_\delta \rangle - \langle b, y_\delta \rangle,$$

and we may assume by replacing ν by a smaller positive constant that $|m(\delta) - \tilde{m}(\delta)| \le 2\delta$.

Since

$$\begin{aligned} \tilde{m}(\delta) &\le \tilde{u}(x_\delta) - \tilde{v}(y_\delta) - \frac{1}{\delta} w(x_\delta, y_\delta) + \nu(|x_\delta| - |x_\delta|^2 + |y_\delta| - |y_\delta|^2) \\ &\le \tilde{u}(x_\delta) - \tilde{v}(y_\delta) - \frac{1}{\delta} w(x_\delta, y_\delta) - \frac{\nu}{2}(|x_\delta|^2 + |y_\delta|^2) + \nu, \end{aligned}$$

and

$$\tilde{m}(2\delta) \ge m(2\delta) - 4\delta \ge \tilde{u}(x_\delta) - \tilde{v}(y_\delta) - \frac{1}{2\delta} w(x_\delta, y_\delta) - 4\delta,$$

we have

$$\frac{1}{2\delta}w(x_\delta, y_\delta) + \frac{\nu}{2}(|x_\delta|^2 + |y_\delta|^2) \le \tilde{m}(2\delta) - \tilde{m}(\delta) + 5\delta.$$

From this we see that as $\delta \to 0$,

$$\frac{1}{\delta}|x_\delta - y_\delta|^2 + \nu(|x_\delta|^2 + |y_\delta|^2) \to 0. \tag{2.14}$$

We are interested in the limit as $\delta \to 0$. Hence, in view of (2.13) and (2.14), we may assume that $|x_\delta - y_\delta| < r$ and that $\tilde{u}(x_\delta) - \tilde{v}(y_\delta) > 2\eta$. These together with (2.12) imply that $\operatorname{dist}(x_\delta, I) \ge 2\varepsilon$ or $\operatorname{dist}(y_\delta, I) \ge 2\varepsilon$. Thus we see that if $x_\delta \in \partial\Omega$, then $x_\delta \in \Gamma_\varepsilon$. If this is the case, we must have

$$\langle \xi(x_\delta), -\delta^{-1}D_1 w(x_\delta, y_\delta) + a + 2\nu x_\delta \rangle \le -1,$$

but this is impossible for δ sufficiently small since

$$\langle \xi(x_\delta), -\delta^{-1}D_1 w(x_\delta, y_\delta) + a + 2\nu x_\delta \rangle \ge -\frac{C}{\delta}|x_\delta - y_\delta|^2 - \|\xi\|_\infty(\nu + 2\nu|x_\delta|) \to 0$$

as $\delta \to 0$. Thus we find that $x_\delta \in \Omega$ for δ sufficiently small. We may assume henceforth that $x_\delta \in \Omega$. We see as well that if $y_\delta \in \partial\Omega$, then $y_\delta \in \Gamma_\varepsilon$.

We have

$$\widetilde{H}(x_\delta, \tilde{u}(x_\delta), \delta^{-1}D_1 w(x_\delta, y_\delta) + 2\nu x_\delta + a) \le 0;$$

$$\widetilde{H}(y_\delta, \tilde{v}(y_\delta), -\delta^{-1}D_2 w(x_\delta, y_\delta) - 2\nu y_\delta - b) \ge \theta.$$

Here we may assume that

$$\tilde{u}(x_\delta) - \mu\psi(x_\delta) > t := \tilde{v}(y_\delta) - \mu\psi(y_\delta).$$

Hence, setting $p = D_1 w(x_\delta, y_\delta)$ and $q = D_2 w(x_\delta, y_\delta)$, we get

$$\begin{aligned}
\theta \le\, & H(y_\delta, t, \delta^{-1}p - \mu D\psi(y_\delta) + 2\nu x_\delta + a) - H(x_\delta, t, -\delta^{-1}q - \mu D\psi(x_\delta) - 2\nu y_\delta - b) \\
\le\, & \omega(|x_\delta - y_\delta|(\delta^{-1}|p| + \mu\|D\psi\|_\infty + 1)) \\
& + L\left(\delta^{-1}|p + q| + \mu|D\psi(x_\delta) - D\psi(y_\delta)| + 2\nu(|x_\delta| + |y_\delta|) + |a| + |b|\right).
\end{aligned}$$

Sending $\delta \to 0$, we get a contradiction, which completes the proof. □

§3 Infinite horizon problems

In this section we discuss state-constraint problems with infinite horizon.

Let $f : \mathcal{H} \times A \to \mathbf{R}$, $g : \mathcal{H} \times A \to \mathcal{H}$, and $c : \mathcal{H} \times A \to \mathbf{R}$ be given continuous functions, where A is a metric space.

In order to get a certain generality of results, we introduce a sequence $\{U_n\}_{n\in\mathbf{N}}$ of open subsets of $\mathcal{H}$ with the properties:

$$U_n + B(0,1) \subset U_{n+1} \quad \text{and} \quad B(0,n) \subset U_n \qquad (n \in \mathbf{N}).$$

Typical examples of $\{U_n\}$ are given as the cases where $U_n = \operatorname{int} B(0,n)$ for all $n \in \mathbf{N}$ and where $U_n = \mathcal{H}$ for all $n \in \mathbf{N}$.

We make the assumptions on f, g, and c:

(3.1) f, g, and c are bounded on $\mathcal{H}$;

(3.2) for each $n \in \mathbf{N}$ there are a function $\omega_n \in \mathrm{C}([0,\infty))$, with $\omega_n(0) = 0$, and a constant $L_n > 0$ such that for all $x, y \in U_n$ and $a \in A$,

$$\begin{cases} |f(x,a) - f(y,a)| \le \omega_n(|x-y|), \\ |c(x,a) - c(y,a)| \le \omega_n(|x-y|), \\ |g(x,a) - g(y,a)| \le L_n|x-y|, \end{cases}$$

(3.3) there is a constant $\lambda_0 > 0$ such that $c(x,a) \ge \lambda_0$ for all $(x,a) \in \mathcal{H} \times A$.

In order to describe our optimal control problem, we give the notation first. Let $\mathcal{A}$ denote the set of all measurable functions $\alpha : [0,\infty) \to A$. For $x \in \mathcal{H}$ and $\alpha \in \mathcal{A}$ we consider the initial value problem

$$\dot{X}(t) = g(X(t), \alpha(t)) \quad (t > 0), \qquad X(0) = x.$$

The solution $X(t)$ will be denoted by $X(t; x, \alpha)$.

Let Ω be an open subset of $\mathcal{H}$. For $x \in \overline{\Omega}$ we define

$$\mathcal{A}(x) = \{\alpha \in \mathcal{A} \mid X(t; x, \alpha) \in \overline{\Omega} \text{ for all } t \ge 0\},$$

and

$$V(x) = \inf_{\alpha \in \mathcal{A}(x)} \int_0^\infty e(t; x, \alpha) f(X(t), \alpha(t))\,dt,$$

where $X(t) := X(t; x, \alpha)$ and

$$e(t; x, \alpha) = \exp\Big\{-\int_0^t c(X(s), \alpha(s))\,ds\Big\}.$$

The main theme of optimal control is to seek, in our setting, for a control $\hat{\alpha} \in \mathcal{A}(x)$ for which we have

$$V(x) = \int_0^\infty e(t; x, \hat{\alpha}) f(X(t; x, \hat{\alpha}), \hat{\alpha}(t)) dt.$$

Such an control $\hat{\alpha}$ is called an optimal control and may not exists in general. Our concern here is restricted to characterizing the function V as a viscosity solution of the corresponding dynamic programming equations. The function V is called the value function associated with our optimal control problem.

It may happen in general that $\mathcal{A}(x) = \emptyset$ for some $x \in \overline{\Omega}$, and this is not the case on which we are going to discuss in this paper. In this respect we introduce the condition:

(3.4) For each $n \in \mathbf{N}$ there are a map $\xi_n : U_n \to \mathcal{H}$ and a constant $\delta_n > 0$ having the properties:

a) ξ_n is bounded and Lipschitz continuous on U_n;
b) $\xi_n(z) \in \operatorname{co}\{g(z, a) \mid a \in A\}$ for $z \in \partial\Omega \cap U_n$;
c) $B(x + t\xi_n(z), \delta_n t) \subset \overline{\Omega}$ for $z \in \partial\Omega \cap U_n$, $x \in \overline{\Omega} \cap U_n$, and $0 \le t \le \delta_n$.

Here and in what follows $\operatorname{co} K$ denotes the convex hull of the subset K of a real vector space.

Proposition 3.1. *Under assumptions* (3.1), (3.2), *and* (3.4), *the set* $\mathcal{A}(x)$ *is non-empty for all* $x \in \overline{\Omega}$.

We remark as a consequence of this proposition that if (3.1)–(3.4) hold, then

$$|V(x)| \le \int_0^\infty e^{-\lambda_0 t} \|f\|_\infty dt \le 1/\lambda_0 \quad (x \in \overline{\Omega}).$$

Theorem 3.2. *Assume that* (3.1)–(3.4) *hold. Then the function* $u(x) := V(x)$ *is a viscosity solution of*

$$\begin{cases} H(x, u(x), Du(x)) \ge 0 & (x \in \overline{\Omega}), \\ H(x, u(x), Du(x)) \le 0 & (x \in \Omega). \end{cases} \tag{3.5}$$

In addition, let $n \in \mathbf{N}$ *and* ξ_n *be from* (3.4), *and set* $K = \|f\|_\infty + \|c\|_\infty \|V\|_\infty$. *Then* u *satisfies the inequality*

$$-\langle \xi_n(x), Du(x,t) \rangle \le K \qquad \text{on } \partial\Omega \cap U_n \tag{3.6}$$

in the viscosity sense.

We need the following lemma to prove Proposition 3.1 and Theorem 3.2.

Lemma 3.3. *Assume that* (3.1), (3.2), *and* (3.4) *hold. Let* $n \in \mathbf{N}$ *and* ξ_n *be from* (3.4). *Then for each* $\varepsilon > 0$ *there is a constant* $\sigma > 0$ *such that for each* $z \in \partial\Omega \cap U_n$ *there is* $\beta \in \mathcal{A}$ *such that*

$$X(s; x, \beta) \in B(x + s\xi_n(z), \varepsilon s) \quad (\, x \in B(z, \sigma),\ s \in (0, \sigma)\,).$$

Assuming that Lemma 3.3 has already been proved, we first give the proof of Proposition 3.1 and Theorem 3.2.

Proof of Proposition 3.1. First of all define the function $\tau : [0, \infty) \to (0, 1)$ as follows. Let $R \geq 0$ and let n be the smallest natural number such that $R + 1 \leq n$. Let δ_n be from (3.4) and σ from Lemma 3.3 with $\varepsilon = \delta_n$. We may assume that $\sigma \leq \delta_n$. Observe from Lemma 3.3 that if $x \in \overline{\Omega}$ and $\operatorname{dist}(x, \partial\Omega \cap B(0, R + 1)) < \sigma$, then there is $\beta \in \mathcal{A}$ such that

$$X(t; x, \beta) \subset \overline{\Omega} \quad (0 \leq t \leq \sigma).$$

We select $\delta > 0$ so that $\delta \|g\|_\infty \leq 1$. Now, for all $x \in \overline{\Omega} \cap B(0, R)$ and $\alpha \in \mathcal{A}$, if $\operatorname{dist}(x, \partial\Omega \cap B(0, R + 1)) \geq \sigma$ then we have

$$X(t; , x, \alpha) \in B(x, t\|g\|_\infty) \subset \overline{\Omega} \quad (0 \leq t \leq \delta).$$

Hence, for $x \in \overline{\Omega} \cap B(0, R)$, then there is $\beta \in \mathcal{A}$ such that

$$X(t; x, \beta) \in \overline{\Omega} \quad (0 \leq t \leq \min\{\delta, \sigma\}).$$

We define τ by setting $\tau(R) = \min\{\sigma, \delta\}$. We may assume that the function τ is nonincreasing.

Fix $x \in \overline{\Omega}$. We will build a control $\alpha \in \mathcal{A}(x)$. By our choice of τ, for each $z \in \overline{\Omega}$ there is $\beta_z \in \mathcal{A}$ such that

$$X(t; z, \beta_z) \in \overline{\Omega} \quad (0 \leq t \leq \tau(|z|)). \tag{3.7}$$

Set $t_0 = 0$ and $x_0 = x$. Define $t_k > 0$ and $x_k \in \mathcal{H}$ $(k \in \mathbf{N})$ inductively by

$$t_k = t_{k-1} + \tau(|x_{k-1}|) \quad \text{and} \quad x_k = X(\tau(|x_{k-1}|); x_{k-1}, \beta_{x_{k-1}}).$$

To see that this definition is indeed well-defined, we need to show that $x_k \in \overline{\Omega}$ for all $k \in \mathbf{N}$. Fix $k \in \mathbf{N}$ and suppose that $x_{k-1} \in \overline{\Omega}$, which is true for $k = 1$. Now (3.7) yields immediately that $x_k \in \overline{\Omega}$ and moreover that

$$X(t; x_{k-1}, \beta_{x_{k-1}}) \in \overline{\Omega} \quad (\, 0 \leq t \leq \tau(|x_{k-1}|)\,). \tag{3.8}$$

Next define the map $\alpha : [0, t_\infty) \to A$, where $t_\infty := \lim_{n\to\infty} t_n$, by

$$\alpha(t) = \begin{cases} \beta_x(t) & (0 \le t < t_1), \\ \beta_{x_1}(t - t_1) & (t_1 \le t < t_2), \\ \beta_{x_2}(t - t_2) & (t_2 \le t < t_3), \\ \vdots & \qquad \vdots \end{cases}$$

We want to prove that $\alpha \in \mathcal{A}(x)$. Let $k \in \mathbf{N}$ and $t_{k-1} \le t < t_k$. Observe that

$$\begin{aligned} X(t; x, \alpha) &= X(t - t_{k-1}; X(t_{k-1}, x, \alpha), \alpha(\cdot + t_{k-1})) \\ &= X(t - t_{k-1}; X(t_{k-1}, x, \alpha), \beta_{x_{k-1}}), \end{aligned}$$

and further by induction that

$$X(t; x, \alpha) = X(t - t_{k-1}; x_{k-1}, \beta_{x_{k-1}}).$$

From this and (3.8) we see that $X(t; x, \alpha) \in \overline{\Omega}$. This show that $X(t; x, \alpha) \in \overline{\Omega}$ for all $0 \le t < t_\infty$.

It remains to shows that $t_\infty = \infty$. Suppose for the moment that $t_\infty < \infty$, which yields that $\lim_{k\to\infty} \tau(|x_k|) = 0$. On the other hand, since $X(t; x, \alpha) = x + \int_0^t g(X(s; x, \alpha), \alpha(s))ds$, we have

$$|X(t; x, \alpha) - x| \le t\|g\|_\infty \le t_\infty \|g\|_\infty \quad (0 \le t < t_\infty),$$

which implies that $\inf_{k\in\mathbf{N}} \tau(|x_k|) > 0$. This is a contradiction, which proves that $t_\infty = \infty$. □

Remark 3.4. It is now clear that in Lemma 3.3 the control β can be chosen so that $\beta \in \mathcal{A}(z)$.

Proof of Theorem 3.2. Since it can be proved in a more or less standard way that u is a viscosity solution of (3.5), we omit giving the proof here.

We show that u satisfies (3.6) in the viscosity sense. Let $\varphi \in \mathrm{C}^1(\overline{\Omega})$ and $\hat{x} \in \partial\Omega \cap U_n$ and assume that $u^* - \varphi$ has a maximum over $\overline{\Omega} \cap U_n$ at $\hat{x}$. We may assume that $(u^* - \varphi)(\hat{x}) = 0$ and that $D\varphi$ is bounded on $\overline{\Omega}$.

We choose a sequence $x_k \in \overline{\Omega}$ so that $x_k \to \hat{x}$ as $k \to \infty$. Fix any $\varepsilon > 0$. According to Lemma 3.3, there is $\sigma \in (0, \delta_n)$ such that if $x_k \in B(\hat{x}, \sigma)$, then there exists $\beta_k \in \mathcal{A}(x_k)$ for which we have

$$X(t; x_k, \beta_k) \in B(x_k + t\xi_n(\hat{x}), \varepsilon t) \quad (0 \le t < \sigma).$$

We may assume that $x_k \in B(\hat{x}, \sigma)$ for all $k \in \mathbf{N}$.

Let $t \in (0, \sigma)$ and $k \in \mathbf{N}$. By the dynamic programming principle, we have

$$u(x_k) = \inf_{\alpha \in \mathcal{A}(x_k)} \left(\int_0^t e(s; x_k, \alpha) f(X(s), \alpha(s)) ds + e(t; x_k, \alpha) u(X(t)) \right),$$

where $X(s) := X(s; x_k, \alpha)$. We may assume by choosing $\sigma > 0$ small enough that $X(s) \in U_n$ for all $\alpha \in \mathcal{A}(x_k)$ and $s \in [0, t]$. Hence, we have

$$u(x_k) \leq \int_0^t e(s; x_k, \beta_k) f(X_k(s), \beta_k(s)) ds + e(t; x_k, \beta_k) \varphi(X_k(t)),$$

where $X_k(s) := X(s; x_k, \beta_k)$. Noting that if we set $q := X_k(t) - (x_k + t\xi_n(\hat{x}))$, then $q \in B(0, \varepsilon t)$, we find that

$$u(x_k) \leq \int_0^t e(s; x_k, \beta_k) f(X_k(s), \beta_k(s)) ds + e(t; x_k, \beta_k) \varphi(x_k + t\xi_n(\hat{x})) + \varepsilon t \|D\varphi\|_\infty.$$

Moreover, setting $d_k := u(x_k) - \varphi(x_k)$ and $Y_k(s) := x_k + s\xi_n(\hat{x}))$, we have

$$\begin{aligned} d_k \leq& \int_0^t e(s; x_k, \beta_k) \Big(-c(X_k(s), \beta_k(s)) \varphi(Y_k(s)) + \langle D\varphi(Y_k(s)), \xi_n(\hat{x}) \rangle + f(X_k(s), \beta_k(s)) \Big) ds \\ &+ \varepsilon t \|D\varphi\|_\infty \\ \leq& \int_0^t e^{-\lambda_0 s} \Big(\|c\|_\infty |\varphi(Y_k(s))| + \langle D\varphi(Y_k(s)), \xi_n(\hat{x}) \rangle + \|f\|_\infty \Big) ds + \varepsilon t \|D\varphi\|_\infty. \end{aligned}$$

Sending first $k \to \infty$, and then dividing this by t and sending $t \to 0$, we get

$$0 \leq \|c\|_\infty |\varphi(\hat{x})| + \langle \xi_n(\hat{x}), D\varphi(\hat{x}) \rangle + \|f\|_\infty + \varepsilon \|D\varphi\|_\infty.$$

Noting that $\varepsilon > 0$ is arbitrary in the above, we conclude the proof. □

Proof of Lemma 3.3. Select sequences $\gamma_i > 0$ and $a_i \in A$ $(i = 1, ..., m)$ so that

$$\xi_n(z) = \sum_{i=1}^m \gamma_i g(z, a_i).$$

Set $I = \{1, ..., m\}$ and $h(i) = g(z, a_i)$ for $i = 1, ..., m$. The first step of the proof is to prove that there is a measurable function $\beta : [0, \infty) \to I$ such that

$$\int_0^t h(\beta(s)) ds \in B(t\bar{h}, \varepsilon t) \quad (t \geq 0), \tag{3.9}$$

where $\bar{h} = \sum_{i=1}^{m} \gamma_i h(i)$.

Set $\gamma_0 = 0$, and define $\zeta : [0, 1) \to I$ by

$$\zeta(t) = i \quad \text{if} \;\; i \in \mathbf{N} \text{ and } \sum_{k=0}^{i-1} \gamma_k \le t < \sum_{k=0}^{i} \gamma_k.$$

Extend the domain of definition of ζ to $[0, \infty)$ by periodicity, i.e.,

$$\zeta(t) = \zeta(t-k) \quad \text{if} \;\; k \in \mathbf{N} \text{ and } k \le t < k+1.$$

For $k \in \mathbf{N}$ define $\zeta_k : [0, \infty) \to I$ by setting

$$\zeta_k(t) = \zeta(kt).$$

Observe that for $k \in \mathbf{N}$,

$$\int_0^{jk^{-1}} h(\zeta_k(s))ds = k^{-1} \int_0^{j} h(\zeta(s))ds = jk^{-1} \sum_{i=1}^{m} \gamma_i h(i) = jk^{-1}\bar{h},$$

and furthermore that for $k, j \in \mathbf{N}$ and $(j-1)k^{-1} \le t < jk^{-1}$,

$$\begin{aligned} \int_0^{t} h(\zeta_k(s))ds = & \int_0^{(j-1)k^{-1}} h(\zeta_k(s))ds + \int_{(j-1)k^{-1}}^{t} h(\zeta_k(s))ds \\ = & (j-1)k^{-1}\bar{h} + \int_{(j-1)k^{-1}}^{t} h(\zeta_k(s))ds. \end{aligned}$$

Hence, we have

$$\left| \int_0^{t} h(\zeta_k(s))ds - t\bar{h} \right| \le 2k^{-1} \max |h(i)|.$$

Fix $\varepsilon > 0$. Choose $k \in \mathbf{N}$ so that

$$2k^{-1} \max |h(i)| < \varepsilon.$$

Define $\beta : (0, \infty) \to I$ by

$$\beta(t) = \zeta_k(2^{-j}t) \quad \text{if} \;\; j \in \mathbf{Z} \text{ and } 2^j \le t < 2^{j+1}.$$

Let $p \in \mathbf{Z}$ and $2^p \le t < 2^{p+1}$. Compute that

$$\begin{aligned}\int_0^t h(\beta(s))ds &= \sum_{q<p} \int_{2^q}^{2^{q+1}} h(\beta(s))ds + \int_{2^p}^t h(\beta(s))ds \\ &= \sum_{q<p} 2^q \int_1^2 h(\zeta_k(s))ds + 2^p \int_1^{2^{-p}t} h(\zeta_k(s))ds \\ &= \sum_{q<p} 2^q \int_0^1 h(\zeta_k(s))ds + 2^p \int_0^{2^{-p}t-1} h(\zeta_k(s))ds \\ &= \sum_{q<p} 2^q \bar{h} + 2^p \int_0^{2^{-p}t-1} h(\zeta_k(s))ds = 2^p \bar{h} + 2^p \int_0^{2^{-p}t-1} h(\zeta_k(s))ds.\end{aligned}$$

Therefore, we have

$$\left| \int_0^t h(\beta(s))ds - t\bar{h} \right| \le 2^{p+1}k^{-1} \max |h(i)| \le 2k^{-1} \max |h(i)| t \le \varepsilon t.$$

We have proved (3.9).

Choose $L > 0$ so that

$$|g(x,a) - g(y,a)| \le L|x-y| \quad (x, y \in U_{n+1},\ a \in A).$$

Let $\delta \in (0,1)$. Fix any $x \in B(z,\delta)$, so that $x \in U_{n+1}$. Define $\alpha \in \mathcal{A}$ by $\alpha(t) = a_{\beta(t)}$ and set $X(t) := X(t; x, \alpha)$. We have

$$|X(s) - x| \le \|g\|_\infty s \quad (s \ge 0),$$

and hence,

$$|X(s) - z| \le (\|g\|_\infty + 1)\delta \quad (0 \le s \le \delta).$$

We may assume that $(\|g\|_\infty + 1)\delta \le 1$, so that $X(t) \in U_{n+1}$ for all $\le t \le \delta$.

By (3.9) we have

$$\int_0^t g(z, \alpha(s))ds \in B(t\bar{\xi}_R(z), \varepsilon t) \quad (t \ge 0).$$

This yields that for $0 \le t \le \delta$,

$$\int_0^t g(X(s), \alpha(s))ds \in B(t\xi_n(z), \varepsilon t + L(\|g\|_\infty + 1)\delta t).$$

We fix $\delta > 0$ so that

$$(\|g\|_\infty + 1)\delta \leq 1 \quad \text{and} \quad L(\|g\|_\infty + 1)\delta \leq \varepsilon,$$

and conclude that for all $0 \leq t \leq \delta$,

$$X(t) = x + \int_0^t g(X(s), \alpha(s))ds \in B(x + t\xi_n(z), 2\varepsilon t). \qquad \square$$

Theorem 3.5. *Assume that* (3.1)–(3.4) *hold. Then the value function* $u(x) := V(x)$ *is uniformly continuous on* $\overline{\Omega} \cap U_n$ *for every* $n \in \mathbf{N}$.

Proof. We fix $n \in \mathbf{N}$ and need to show that

$$\lim_{r \searrow 0} \sup\{u(x) - u(y) \mid x, y \in \overline{\Omega} \cap U_n,\ |x - y| < r\} \leq 0. \tag{3.10}$$

To this end, we set

$$\zeta(x) = \operatorname{dist}(x, U_n) \quad (x \in \mathcal{H}),$$

fix $\gamma > 0$, and define

$$v(x) = u(x) + \gamma\big(\lambda_0 \zeta(x) + (1 + \|g\|_\infty)\big) \quad (x \in \overline{\Omega}).$$

Using Theorem 3.2 and the fact that $D^- v(x) \subset D^- u(x) + B(0, \gamma\lambda_0)$, we easily see that v is a viscosity solution of

$$H(x, v(x), Dv(x)) \geq \gamma \quad (x \in \overline{\Omega}).$$

Next we fix $k \in \mathbf{N}$ so large that $\gamma\lambda_0\zeta(x) > 2\|u\|_\infty$ for all $x \in \partial U_k$. Define

$$G = \Omega \cap U_k \quad \text{and} \quad I = \overline{\Omega} \cap \partial U_k,$$

and observe that

$$\lim_{r \searrow 0} \sup\{u(x) - v(y) \mid x, y \in \overline{G},\ \operatorname{dist}(x, I) < r,\ |x - y| < r\} \leq 0.$$

An application of Theorem 2.1 yields that

$$\lim_{r \searrow 0} \sup\{u(x) - v(y) \mid x, y \in \overline{G},\ |x - y| < r\} \leq 0,$$

and hence that

$$\lim_{r\searrow 0} \sup\{u(x)-u(y) \mid x,y\in \overline{\Omega}\cap U_n,\ |x-y|<r\} \le 0,$$

completing the proof. □

Theorem 3.6. *Assume that* (3.1)–(3.4) *hold. Let* $u:\overline{\Omega}\to \mathbf{R}$ *and* $v:\overline{\Omega}\to\mathbf{R}$ *be, respectively, a bounded viscosity solution of*

$$H(x,u(x),Du(x)) \le 0 \quad \text{in } \Omega, \tag{3.11}$$

and a bounded viscosity solution of

$$H(x,v(x),Dv(x)) \ge 0 \quad \text{in } \overline{\Omega}. \tag{3.12}$$

Assume that u *is uniformly continuous on* $\overline{\Omega}\cap U_n$ *for all* $n\in\mathbf{N}$. *Then* $u\le v$ *on* $\overline{\Omega}$.

This theorem can be proved by the method of the proof of Theorem 3.5 with obvious modifications provided with the following lemma at hand. It is thus left to the reader to give the details of the proof.

Lemma 3.7. *Under the hypotheses of Theorem 3.6, let* u *be the function from Theorem 3.6. Let* $n\in\mathbf{N}$ *and* ξ_n *be from* (3.4). *Then*

$$-\langle \xi_n(x), Du(x)\rangle \le K \quad \text{on } \partial\Omega\cap U_n$$

in the viscosity sense, where $K:=\|c\|_\infty\|u\|_\infty+\|f\|_\infty$.

Proof. We borrow an argument from [15] for the proof of this theorem.

We argue by contradiction and thus suppose that there were a point $\hat{x}\in\partial\Omega\cap U_n$ and a function $\varphi\in C^1(B(\hat{x},r)\cap\overline{\Omega})$, with $r>0$, such that

$$-\langle \xi_n(\hat{x}), D\varphi(\hat{x})\rangle > K \tag{3.13}$$

and such that

$$u(\hat{x})=\varphi(\hat{x}) \quad \text{and} \quad (u-\varphi)(x)\le -|x-\hat{x}|^2 \quad (x\in B(\hat{x},r)\cap\overline{\Omega}).$$

We will obtain a contradiction.

We may assume that $B(\hat{x},r)\subset U_n$ and that $D\varphi$ is bounded on $B(\hat{x},r)\cap\overline{\Omega}$. We set

$$\psi(x)=\varphi(x)-r^2,$$

and observe that $u \le \psi$ on $\partial B(\hat{z}, r) \cap \overline{\Omega}$ and $u(\hat{x}) > \psi(\hat{x})$.

Set

$$\eta = \xi_n(\hat{x}).$$

Choosing $r > 0$ small enough, we have

$$B(x + s\eta, rs) \subset \overline{\Omega} \quad (x \in B(\hat{x}, r) \cap \overline{\Omega},\ 0 \le s \le r). \tag{3.14}$$

In view of (3.4), we see that there are finite sequences $a_i \in A$ and $\lambda_i > 0$, with $i = 1, ..., m$, such that

$$\xi_n(\hat{x}) = \sum_{i=1}^{m} \lambda_i g(\hat{x}, a_i) \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = 1.$$

We set

$$\xi(x) = \sum_{i=1}^{m} \lambda_i g(x, a_i) \quad (x \in B(\hat{x}, r) \cap \overline{\Omega}).$$

In view of (3.13) and (3.14), we may assume that

$$-\langle \xi(x), D\varphi(x) \rangle \ge K + r \quad (\, x \in B(\hat{x}, r) \cap \overline{\Omega}\,),$$

and that

$$B(x + s\xi(x), rs) \subset \overline{\Omega} \quad (x \in B(\hat{x}, r) \cap \overline{\Omega},\ 0 \le s \le r).$$

These together imply that ψ is a viscosity solution of

$$-\langle \xi(x), D\varphi(x) \rangle \ge K + r \quad (\, x \in \operatorname{int} B(\hat{x}, r) \cap \overline{\Omega}\,).$$

On the other hand, since u is a viscosity solution of

$$H(x, u, Du) \le 0 \quad \text{ in } \Omega,$$

it is also a viscosity solution of

$$-\langle \xi(x), Du(x) \rangle \le K \quad \text{ in } \Omega.$$

We now claim that $u \le \psi$ on $B(\hat{x}, r) \cap \overline{\Omega}$, which yields $u(\hat{x}) \le \psi(\hat{x})$, a contradiction.

To prove this claim, we follow the standard argument introduced by [19] for proving comparison theorems for state-constraint problems.

Thus we consider the function

$$\Phi_\alpha(x,y) = u(x) - \psi(y) - |\alpha(x-y) - \eta|^2 - \langle p, x\rangle - \langle q, y\rangle$$

on $B(\hat{x}, r) \cap \overline{\Omega}$ with $p, q \in \mathcal{H}$ satisfying

$$\alpha^2(|p| + |q|) \le 1.$$

We may choose p, q so that Φ_α attains a maximum at some point (x_α, y_α).

It is clear that as $\alpha \to \infty$,

$$|\alpha(x_\alpha - y_\alpha) - \eta|^2$$

stays bounded. In particular, we have

$$x_\alpha - y_\alpha \to 0 \quad \text{as } \alpha \to \infty.$$

We are only concerned with large α and hence we may assume that α is as large as we desire. If

$$\limsup_{\alpha\to\infty} \max\{|x_\alpha - \hat{x}|, |y_\alpha - \hat{x}|\} = r,$$

then we get

$$\limsup_{n\to\infty} \sup \Phi_{\alpha_n} \le 0,$$

since $u \le \psi$ on $\partial B(\hat{x}, r) \cap \overline{\Omega}$ and u, ψ are uniformly continuous on $B(\hat{x}, r) \cap \overline{\Omega}$, from which it follows that

$$\sup_{B(\hat{x},r)\cap\overline{\Omega}} (u - \psi) \le 0.$$

If this is not the case, we may assume that for all α under considerations, we have

$$x_\alpha \in B(\hat{x}, \rho) \quad \text{and} \quad y_\alpha \in B(\hat{x}, \rho)$$

for some $\rho \in (0, r)$.

Selecting $\delta \in (0, r)$ small enough, by (3.14) we have

$$B(x + t\eta, \delta t) \subset B(\hat{x}, r) \cap \overline{\Omega} \quad (\, x \in B(\hat{x}, \rho) \cap \overline{\Omega},\ 0 \le t \le \delta\,).$$

We may assume that $1/\alpha \le \delta$, so that $y_\alpha + \alpha^{-1}\eta \in B(\hat{x}, r) \cap \overline{\Omega}$. Hence, we have

$$\Phi_\alpha(x_\alpha, y_\alpha) \ge \Phi_\alpha(y_\alpha + \alpha^{-1}\eta, y_\alpha),$$

and therefore,

$$|\alpha(x_\alpha - y_\alpha) - \eta|^2 \le \omega(|x_\alpha - y_\alpha - \alpha^{-1}\eta|) + M/\alpha,$$

for some constant $M > 0$, where ω is the modulus of continuity of u. This implies that as $\alpha \to \infty$,

$$|\alpha(x_\alpha - y_\alpha) - \eta|^2 \to 0.$$

Set $\zeta_\alpha = \alpha(x_\alpha - y_\alpha) - \eta$. Then

$$x_\alpha = y_\alpha + \alpha^{-1}(\eta + \zeta_\alpha),$$

and we may assume that $x_\alpha \in \operatorname{int} B(y_\alpha + \alpha^{-1}\eta, \alpha^{-1}\delta) \subset \Omega$. Since $x_\alpha, y_\alpha \in \operatorname{int} B(\hat{x}, r)$, we have

$$-2\alpha\langle \xi(x_\alpha), (\alpha(x_\alpha - y_\alpha) - \eta) + p\rangle \le K;$$
$$-2\alpha\langle \xi(y_\alpha), \alpha(x_\alpha - y_\alpha) - \eta - q\rangle \ge K + r.$$

From these we get

$$\begin{aligned} r &\le 2\,\mathrm{Lip}\,(\xi)|\alpha(x_\alpha - y_\alpha)||\alpha(x_\alpha - y_\alpha) - \eta| + 2\alpha\|\xi\|_\infty(|p| + |q|) \\ &\le 2\,\mathrm{Lip}\,(\xi)(|\alpha(x_\alpha - y_\alpha) - \eta| + |\eta|)|\alpha(x_\alpha - y_\alpha) - \eta| + 2\alpha\|\xi\|_\infty(|p| + |q|). \end{aligned}$$

Sending $\alpha \to \infty$, we get a contradiction, which proves that $\sup_{B(\hat{x},r)\cap\overline{\Omega}}(u - \psi) \le 0$. □

§4 Finite horizon problems

We treat here state-constraint problems with finite horizon on the same themes as in the previous section. Many of arguments parallel those for infinite horizon problems which will be often omitted presenting here.

Set $\mathcal{H}_1 = \mathcal{H} \times \mathbf{R}$ and let $f : \mathcal{H}_1 \times A \to \mathbf{R}$, $g : \mathcal{H}_1 \times A \to \mathcal{H}$, $c : \mathcal{H}_1 \to \mathbf{R}$, and $h : \mathcal{H} \to \mathbf{R}$ be given continuous functions. As in the previous section let $\{U_n\}$ be a sequence of open subsets of $\mathcal{H}$ satisfying

$$U_n + B(0,1) \subset U_{n+1} \quad \text{and} \quad B(0,n) \subset U_n \qquad (n \in \mathbf{N}).$$

Our assumptions on f, g, c, and h are as follows:

(4.1) f, g, c, and h are bounded on $\mathcal{H}_1$;

(4.2) for each $n \in \mathbf{N}$ there are a function $\omega_n \in \mathrm{C}([0,\infty))$, with $\omega_n(0) = 0$, and a constant $L_n > 0$ such that for all $x, y \in U_n$, $t, s \in \mathbf{R}$, and $a \in A$,

$$\begin{cases} |f(x,t,a) - f(y,s,a)| \le \omega_n(|x - y| + |t - s|), \\ |c(x,t,a) - c(y,s,a)| \le \omega_n(|x - y| + |t - s|), \\ |h(x) - h(y)| \le \omega_n(|x - y|), \\ |g(x,t,a) - g(y,s,a)| \le L_n(|x - y| + |t - s|), \end{cases}$$

In many applications the functions f, g, and c are defined only on $\overline{\Omega} \times [S,T] \times A$ for some open subset Ω of $\mathcal{H}$ and some constants $S < T$. Our results in this section can be as well applied in this situation by extending f, g, and c to functions on $\mathcal{H}_1 \times A$.

The optimal control problem which we are here concerned with is described as follows.

Here in this section $\mathcal{A}$ denotes the set of all measurable functions $\alpha : \mathbf{R} \to A$. For $(x,t) \in \mathcal{H}_1$ and $\alpha \in \mathcal{A}$ we consider the initial value problem

$$\dot{X}(s) = g(X(s), \alpha(s)) \quad (\, s > t\,), \qquad X(t) = x,$$

the solution of which will be denoted by $X(s; x, t, \alpha)$. It is obvious that the values $\alpha(s)$, with $s \leq t$, is irrelevant to the definition of $X(s; x, t, \alpha)$.

Let Ω and T be a given open subset of $\mathcal{H}$ and a given positive number, respectively. Let Q_T denote the set $\Omega \times (-\infty, T)$.

For $(x,t) \in \overline{Q}_T$ we set

$$\mathcal{A}(x,t) = \{\alpha \in \mathcal{A} \mid X(s; x,t,\alpha) \in \overline{\Omega} \text{ for all } s \in [t,T]\},$$

and define

$$V(x,t) = \inf_{\alpha \in \mathcal{A}(x,t)} \left(\int_t^T e(s; x,t,\alpha) f(X(s), s, \alpha(s)) ds + e(T; x,t,\alpha) h(X(T)) \right),$$

where $X(s) := X(s; x,t,\alpha)$ and

$$e(s; x,t,\alpha) = \exp\Big\{ - \int_t^s c(X(\tau), \tau, \alpha(\tau)) d\tau \Big\}.$$

Since the values $\alpha(s)$, with $s \leq t$ or $s \geq T$, are irrelevant to the definition of $V(x,t)$ and so we may regard the map $\alpha \in \mathcal{A}(x,t)$ as a map defined on $[t,T]$. We call the function V the value function of our control problem.

As before we make an assumption regarding the directions of the vector fields $\{g(x,t,a) \mid a \in A\}$.

(4.3) For each $n \in \mathbf{N}$ there are a map $\xi_n : U_n \times \mathbf{R} \to \mathcal{H}$ and a constant $\delta_n > 0$ having the properties:

a) ξ_n is bounded and Lipschitz continuous on $U_n \times \mathbf{R}$;

b) $\xi_n(x,t) \in \operatorname{co} \{g(x,t,a) \mid a \in A\}$ for $(x,t) \in (\partial\Omega \cap U_n) \times \mathbf{R}$;

c) $B(y + s\xi_n(x,t), \delta_n s) \subset \overline{\Omega}$ for $(x,t) \in (\partial\Omega \cap U_n) \times \mathbf{R}$, $y \in \overline{\Omega} \cap U_n$, and $0 \leq s \leq \delta_n$.

Proposition 4.1. *Assume that* (4.1)-(4.3) *hold. Then the set* $\mathcal{A}(x,t)$ *is non-empty for all* $(x,t) \in \overline{\Omega} \times (-\infty, T)$. *Moreover, the value function* V *is bounded on* $\overline{\Omega} \times [S,T]$ *for all* $S \in (-\infty, T)$.

As before the proof of this proposition is based on the following lemma.

Lemma 4.2. *Assume that (4.1)–(4.3) hold. Let $n \in \mathbf{N}$ and ξ_n be the map from (4.3). Then for each $\varepsilon > 0$ there is a constant $\sigma > 0$ such that for each $(z, \tau) \in (\partial\Omega \cap U_n) \times \mathbf{R}$ there is an $\alpha \in \mathcal{A}$ for which*

$$X(s; x, t, \alpha) \in B(x + (s-t)\xi_n(z, \tau), \varepsilon(s-t)) \quad (\,(x,t) \in B(z, \tau; \sigma),\ s \in (t, t+\sigma]\,).$$

Proof. This is a consequence of Lemma 3.3. In order to see this, let us introduce the notation:

$$V_n = U_n \times \mathbf{R}, \quad Y(s; x, t, \alpha) = (X(s; x, t, \alpha), s),$$

$$g_1(x, t, a) = (g(x, t, a), 1), \quad \text{and} \quad \eta_n(x, t) = (\xi_n(x, t), 1),$$

for $(x, t) \in \mathcal{H}_1$, $\alpha \in \mathcal{A}$, $a \in A$, $s \geq t$, and $n \in \mathbf{N}$, where ξ_n is the map from (4.3). Note that if we set $Y(s) := Y(s; x, t, \alpha)$, then

$$\dot{Y}(s) = g_1(Y(s), \alpha(s)) \quad (s > t), \qquad Y(t) = (x, t).$$

With $\mathcal{H}_1$, V_n, g_1, η_n, and $\Omega \times \mathbf{R}$ replacing $\mathcal{H}$, U_n, g, ξ_n, and Ω, respectively, conditions (3.1), (3.2), and (3.4) are satisfied. Therefore, Lemma 3.3 yields that for each $\varepsilon > 0$ there is a $\sigma > 0$ such that for every $(z, \tau) \in (\overline{\Omega} \cap U_n) \times \mathbf{R}$ there is an $\alpha \in \mathcal{A}$ for which

$$Y(t+s; x, t, \alpha) \in B((x, t) + s\eta_n(z, \tau), \varepsilon s) \quad (\,(x,t) \in B(z, \tau; \sigma),\ s \in (0, \sigma]\,),$$

which implies that

$$X(s; x, t, \alpha) \in B(x + (s-t)\xi_n(z, \tau), \varepsilon(s-t)) \quad (\,(x,t) \in B(z, \tau; \sigma),\ s \in (t, t+\sigma]\,).$$

Thus, the proof is complete. □

With regard to the proof of Proposition 4.1, it can be proved that $\mathcal{A}(x, t) \neq \emptyset$ for all $(x, t) \in \overline{\Omega} \times (-\infty, T)$ by repeating the arguments in the proof of Proposition 3.1 with Lemma 4.2 in place of Lemma 3.3, the proof of which we do not present here. The rest of the proof can be done by observing that for $(x, t) \in \overline{Q}_T$ and $\alpha \in \mathcal{A}(x, t)$,

$$\left| \int_t^T e(s) f(X(s), s, \alpha(s)) ds + e(T) h(X(T)) \right| \leq e^{(T-t)\|c\|_\infty} \big((T-t)\|f\|_\infty + \|h\|_\infty \big),$$

where $e(s) := e(s; x, t, \alpha)$ and $X(s) := X(s; x, t, \alpha)$.

In what follows we write

$$H(x,t,r,p)=\sup_{a\in A}\{c(x,t,a)r-\langle g(x,t,a),p\rangle-f(x,t,a)\}\quad ((x,t,r,p)\in\overline{\Omega}\times\mathbf{R}\times\mathbf{R}\times\mathcal{H}).$$

Theorem 4.3. *Assume that* (4.1)–(4.3) *hold. Then the function* $u(x,t):=V(x,t)$ *is a viscosity solution of*

$$\begin{cases}-u_t(x,t)+H(x,t,u(x,t),Du(x,t))\geq 0 & ((x,t)\in\overline{\Omega}\times(-\infty,T)),\\ -u_t(x,t)+H(x,t,u(x,t),Du(x,t))\leq 0 & ((x,t)\in Q_T).\end{cases}\tag{4.4}$$

Furthermore fix $n\in\mathbf{N}$ *and* $S\in\mathbf{R}$, *with* $S<T$, *let* ξ_n *be the map from* (4.3), *and set* $K:=\|c\|_\infty\sup_{\overline{\Omega}\times(S,T)}|V|+\|f\|_\infty$. *Then* u *satisfies*

$$-u_t(x,t)-\langle\xi_n(x,t),Du(x,t)\rangle\leq K\quad((x,t)\in(\partial\Omega\cap U_n)\times(S,T))$$

in the viscosity sense.

Again, the arguments in the proof of Theorem 3.2 can be applied to proving this theorem with minor modifications, and thus we leave it to the reader to give the details of the proof.

Theorem 4.4. *Assume that* (4.1)–(4.3) *hold. For each* $n\in\mathbf{N}$ *and* $S\in\mathbf{R}$, *with* $S<T$, *the function* $u(x,t):=V(x,t)$ *is uniformly continuous on* $(\overline{\Omega}\cap U_n)\times[S,T]$.

Proof. We will use Theorem 2.1 to prove this theorem.

To this end, we first observe that for all $(x,t)\in\overline{Q}_T$,

$$|u(x,t)-h(x)|\leq\sup_{\alpha\in\mathcal{A}(x,t)}\left\{\int_t^T e(s)|f(X(s),s,\alpha(s))|ds+|e(T)h(X(T))-h(x)|\right\},$$

where $X(s):=X(s;x,t,\alpha)$ and $e(s):=e(s;x,t,\alpha)$. Since

$$X(s)\in B(x,\|g\|_\infty(T-t))\quad(t\leq s\leq T),$$

we deduce that for all $n\in\mathbf{N}$,

$$\lim_{r\searrow 0}\sup\{|u(x,t)-h(x)|\mid x\in\overline{\Omega}\cap U_n,\ T-r<t<T\}=0.\tag{4.5}$$

We fix $n\in\mathbf{N}$ and will show that

$$\lim_{r\searrow 0}\sup\{u(x,t)-u(y,s)\mid(x,t),(y,s)\in(\overline{\Omega}\cap U_n)\times[T-n,T],\ |x-y|+|t-s|<r\}\leq 0.\tag{4.6}$$

Set $S = T - n - 1$, $\lambda = 1 + \|c\|_\infty$, and $v(x,t) = u(x,t)e^{\lambda t}$, and observe that v is a viscosity solution of

$$\begin{cases} \lambda v(x,t) - v_t(x,t) + e^{\lambda t} H(x,t,e^{-\lambda t}v(x,t), e^{-\lambda t}Dv(x,t)) \le 0 & ((x,t) \in \Omega \times (S,T)), \\ \lambda v(x,t) - v_t(x,t) + e^{\lambda t} H(x,t,e^{-\lambda t}v(x,t), e^{-\lambda t}Dv(x,t)) \ge 0 & ((x,t) \in \overline{\Omega} \times (S,T)), \end{cases}$$

and of

$$\lambda v(x,t) - \langle \eta_n(x,t), (v_t(x,t), Dv(x,t)) \rangle \le e^{-\lambda t} K \quad ((x,t) \in \partial\Omega \times (S,T)),$$

where $K := \|c\|_\infty \sup_{\overline{\Omega}\times[S,T]} |V| + \|f\|_\infty$.

As in the proof of Theorem 3.5, we set

$$\zeta(x) = \operatorname{dist}(x, U_n) \quad (x \in \mathcal{H})$$

and

$$w(x,t) = v(x,t) + \zeta(x) + \|g\|_\infty + 1 \quad ((x,t) \in \overline{Q}_T),$$

and then observe that w is a viscosity solution of

$$\lambda w(x,t) - w_t(x,t) + e^{-\lambda t} H(x,t,e^{\lambda t}w(x,t), e^{\lambda t}Dw(x,t)) \ge 1 \quad ((x,t) \in \overline{\Omega} \times (S,T)).$$

Fix $k \in \mathbf{N}$ so large that

$$\zeta(x) > 2 \sup_{\overline{\Omega}\times[S,T]} |v| \quad (x \in \partial U_k).$$

Set $\Omega_1 = (\Omega \cap U_k) \times (S,T)$ and $I = \overline{\Omega \cap U_k} \times \{S,T\}$. Note that the function

$$H_1(x,t,r,p,q) := \lambda r - q + e^{-\lambda t} H(x,t,e^{\lambda t}r, e^{\lambda t}p)$$

on $\overline{\Omega}_1 \times \mathbf{R} \times \mathcal{H}_1$ satisfies (2.3)–(2.5).

Finally we choose a function $j \in \mathrm{C}^1(\mathbf{R})$ so that $j'(t) \le 0$ for all $t \in \mathbf{R}$, $j(t) = 0$ for all $t \ge S+1$, and $j(S) > 2\sup_{\overline{\Omega}_1} |v|$, set

$$z(x,t) = w(x,t) + j(t) \quad ((x,t) \in \overline{\Omega}_1),$$

and see that z is a viscosity solution of

$$H_1(x,t,z(x,t), Dz(x,t), z_t(x,t) \ge 1 \quad ((x,t) \in (\overline{\Omega} \cap U_k) \times (S,T)).$$

Because of (4.5) and our choice of k and j, we find that

$$\lim_{r \searrow 0} \sup\{v(x,t) - z(y,s) \mid (x,t),(y,s) \in \overline{\Omega}_1,\ \operatorname{dist}((x,t),I) < r,\ |x-y| + |t-s| < r\} \le 0.$$

Now, applying Theorem 2.1, we obtain

$$\lim_{r \searrow 0} \sup\{v(x,t) - z(x,y) \mid (x,t),(y,s) \in \overline{\Omega}_1,\ |x-y| + |t-s| < r\} \le 0,$$

which ensures that (4.6) holds. □

Theorem 4.5. *Assume that* (4.1)–(4.3) *hold. Let* $S < T$ *be a constant. Let* $u : \overline{\Omega} \times (S,T] \to \mathbf{R}$ *and* $v : \overline{\Omega} \times (S,T] \to \mathbf{R}$ *be, respectively, a bounded viscosity solution of*

$$-u_t(x,t) + H(x,t,u(x,t),Du(x,t)) \le 0 \quad (\,(x,t) \in \Omega \times (S,T)\,),$$

and a bounded viscosity solution of

$$-u_t(x,t) + H(x,t,u(x,t),Du(x,t)) \le 0 \quad (\,(x,t) \in \overline{\Omega} \times (S,T)\,).$$

Suppose that u *is uniformly continuous on* $(\overline{\Omega} \cap U_n) \times (S,T]$ *and*

$$\lim_{r \searrow 0} \sup\{u(x,T) - v(y,s) \mid x,y \in \overline{\Omega} \cap U_n,\ T - r < s \le T,\ |x-y| < r\} \le 0$$

for all $n \in \mathbf{N}$. *Then*

$$\lim_{r \searrow 0} \sup\{u(x,t) - v(y,s) \mid (x,t),(y,s) \in (\overline{\Omega} \cap U_n) \times (S,T],\ |x-y| + |t-s| < r\} \le 0.$$

This theorem can be proved along the line of the proof of Theorem 3.6 with additional technicalities, most of which can be found in the proof of Theorem 4.4. We thus omit giving the details of the proof here.

References

1. M. Arisawa and P.-L. Lions, Continuity of admissible trajectories for state constraints control problems. Discrete Contin. Dynam. Systems **2** (1996), no. 3, 297–305.
2. M. Arisawa, H. Ishii, and P.-L. Lions, Un contre-exemple pour la théorie du contrôle ergodique en dimension infinie. C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 1, 37–41.
3. M. Arisawa, Some ergodic problems for Hamilton-Jacobi equations in Hilbert space. Differential Integral Equations **9** (1996), no. 1, 59–70.
4. M. Bardi and I. Capuzzo-Dolcetta, Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.
5. I. Capuzzo-Dolcetta and P.-L. Lions, Hamilton-Jacobi equations with state constraints. Trans. Amer. Math. Soc. **318** (1990), no. 2, 643–683.
6. M. G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67.
7. M. G. Crandall and P.-L. Lions, Hamilton-Jacobi equations in infinite dimensions. I. Uniqueness of viscosity solutions. J. Funct. Anal. **62** (1985), no. 3, 379–396.
8. M. G. Crandall and P.-L. Lions, Hamilton-Jacobi equations in infinite dimensions. II. Existence of viscosity solutions. J. Funct. Anal. **65** (1986), no. 3, 368–405.
9. M. G. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations in infinite dimensions. IV. Hamiltonians with unbounded linear terms. J. Funct. Anal. **90** (1990), no. 2, 237–283.
10. M. G. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations in infinite dimensions. V. Unbounded linear terms and B-continuous solutions. J. Funct. Anal. **97** (1991), no. 2, 417–465.
11. M. G. Crandall and P.-L. Lions, Hamilton-Jacobi equations in infinite dimensions. VI. Nonlinear A and Tataru's method refined. Evolution equations, control theory, and biomathematics (Han sur Lesse, 1991), 51–89, Lecture Notes in Pure and Appl. Math., 155, Dekker, New York, 1994.
12. I. Ekeland and G. Lebourg, Generic Fréchet-differentiability and perturbed optimization problems in Banach spaces. Trans. Amer. Math. Soc. **224** (1976), no. 2, 193–216.
13. H. Ishii, A generalization of a theorem of Barron and Jensen, preprint.
14. H. Ishii and S. Koike, A new formulation of state constraint problems for first-order PDEs. SIAM J. Control Optim. **34** (1996), no. 2, 554–571.

15. H. Ishii, P. Loreti, and M. E. Tessitore, An invariance property for stochastic differential equations and its characterizations via Hamilton-Jacobi-Bellman equations, preprint.
16. M. Kocan and P. Soravia, A viscosity approach to infinite-dimensional Hamilton-Jacobi equations arising in optimal control with state constraints. SIAM J. Control Optim. **36** (1998), no. 4, 1348–1375
17. J.-M. Lasry and P.-L. Lions, A remark on regularization in Hilbert spaces. Israel J. Math. **55** (1986), no. 3, 257–266.
18. J. T. Schwartz, Nonlinear functional analysis. Notes by H. Fattorini, R. Nirenberg and H. Porta, with an additional chapter by Hermann Karcher. Notes on Mathematics and its Applications. Gordon and Breach Science Publishers, New York-London-Paris, 1969.
19. H. M. Soner, Optimal control with state-space constraint. I. SIAM J. Control Optim. **24** (1986), no. 3, 552–561.
20. Ch. Stegall, Optimization of functions on certain subsets of Banach spaces. Math. Ann. **236** (1978), no. 2, 171–176.
21. D. Tataru, Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms. J. Math. Anal. Appl. **163** (1992), no. 2, 345–392.