Convexified Gauss curvature flow of bounded open sets in an anisotropic external field: a stochastic approximation and PDE

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Abstract

We construct a random crystalline (or polyhedral) approximation of a convexified Gauss curvature flow of bounded open sets in an anisotropic external field. We also show that a weak solution to the PDE which describes the motion of a bounded open set is unique and is a viscosity solution of it.

1 Introduction

Gauss curvature flow is known as a mathematical model of the wearing process of a convex stone rolling on a beach and has been studied by many authors (see e.g. [2, 3, 6, 7, 11, 14, 24]).

In the last few years we have been generalizing the theory of Gauss curvature flow to a class of nonconvex sets.

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In [17] we studied the existence and the uniqueness of a viscosity solution to the PDE that describes the time evolution of a nonconvex graph by a convexified Gauss curvature (see (1.10) for PDE).

In [19] we proposed and studied the discrete stochastic approximations of evolving functions which are generalizations of those considered in [17], and proved the existence and the uniqueness of a weak solution to the PDE which appears in the continuum limit of discrete stochastic processes, and discussed under what conditions a weak solution to the PDE is a viscosity solution of it.

In [20] we studied the existence and the uniqueness of the motion (or time evolution) of a nonconvex compact set which evolves by a convexified Gauss curvature in $\mathbf{R}^N$ ($N \geq 2$), by the level set approach in the theory of viscosity solutions (see e.g. [5, 10, 23] for the level set approach).

We introduce the notion of the motion of a smooth oriented closed hypersurface by a convexified Gauss curvature.

Let M be a smooth oriented closed hypersurface in $\mathbf{R}^N$ and e be a smooth vector field over M of unit normal vectors. For $x \in M$, let T_xM denote the tangent space of M at x, and let $A_x : T_xM \mapsto T_xM$ denote the Weingarten map at x defined by the following:

$$A_x(v) = -D_v e \quad \text{for } v \in T_xM, \tag{1.1}$$

where $D_v e$ denotes the derivative of e with respect to v. Recall that the principal curvatures $\kappa_1, \cdots, \kappa_n$ ($n := N-1$) of M at x are the eigenvalues of the symmetric map A_x and the Gauss curvature $K(x)$ of M at x is given by $\det A_x$.

Let C be the convex hull $\text{co}\, M$ of M. We define $\sigma : M \mapsto \{0, 1\}$ by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in M \cap \partial C, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

and call $\sigma(x)K(x)$ the *convexified* Gauss curvature of M at x.

The motion of a smooth oriented closed hypersurface by a convexified Gauss curvature is the curvature flow:

$$v = -\sigma K \nu, \tag{1.3}$$

where ν denotes the unit outward normal vector on the surface and v denotes the velocity of the surface.

Let $(A_x)_+$ denote the positive part of the symmetric map A_x. $K_+(x) := \det\{(A_x)_+\}$ is called the *positive part* of the Gauss curvature of M at x, and the following holds:

$$\sigma(x)K(x) = \sigma(x)K_+(x). \tag{1.4}$$

Remark 1 For $x \in M$,

$$\det\{(A_x)_+\} = \begin{cases} \det A_x & \text{if } A_x \text{ is nonnegative definite,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

The crystalline (or polyhedral) approximation of a smooth simple closed convex curve which evolves as the curvature flow was considered by Girão and is useful in the numerical analysis (see [13] and the references therein). We refer to [12] and the references therein for the recent development of this topics.

When $N = 2$, the discrete stochastic approximation of the curvature flow of smooth simple closed convex curves was given in [18] where the model and the approach are completely different from those in this paper.

In this paper we propose and study the discrete stochastic approximation of a convexified Gauss curvature flow of bounded open sets in an anisotropic external field. Our result in this paper is the first one in case $N \geq 3$, among random and nonrandom results, which gives a crystalline approximation of the motion of a bounded open set in $\mathbf{R}^N$ by Gauss curvature.

We briefly describe what we proved in [19], and then discuss the results in this paper more precisely to compare a convexified Gauss curvature flow of graphs with that of closed hypersurfaces.

For $x \in \mathbf{R}^n$ and $u : \mathbf{R}^n \mapsto \mathbf{R}$, the following set is called the subdifferential of u at x:

$$\partial u(x) := \{p \in \mathbf{R}^n : u(y) - u(x) \geq p \cdot (y - x) \text{ for all } y \in \mathbf{R}^n\}, \tag{1.6}$$

where $\cdot$ denotes the inner product in $\mathbf{R}^n$.

Alexandrov-Bakelman's generalized curvature introduced in the following played a crucial role in [19].

Definition 1 (see e.g. [4, section 9.6]). Let $R \in L^1(\mathbf{R}^n : [0,\infty), dx)$ and $u \in C(\mathbf{R}^n)$. For $A \in B(\mathbf{R}^n)$(:=Borel σ-field of $\mathbf{R}^n$), put

$$w(R,u,A) := \int_{\cup_{x\in A}\partial u(x)} R(y)dy \quad (A \in B(\mathbf{R}^n)). \tag{1.7}$$

Let $T \in [0,\infty]$ and $R \in L^1(\mathbf{R}^n : [0,\infty), dx)$. We showed the existence and the uniqueness of a solution $u \in C([0,T)\times\mathbf{R}^n)$ to the following equation (see [19, Theorem 1]): for any $\varphi \in C_o(\mathbf{R}^n)$ and any $t \in [0,T)$,

$$\int_{\mathbf{R}^n} \varphi(x)(u(t,x)-u(0,x))dx = \int_0^t ds \int_{\mathbf{R}^n} \varphi(x)w(R,u(s,\cdot),dx). \tag{1.8}$$

The existence of a continuous solution to (1.8) was given by the continuum limit of the infinite particle systems $\{(Z_m(t,z))_{z\in\mathbf{Z}^n/m}\}_{t\geq 0}$ that satisfies the following: for any $t \geq 0$ and any $z \in \mathbf{Z}^n/m$,

$$P(Z_m(t+\Delta t,z)-Z_m(t,z)>0) = m^n E[w(R,\hat{Z}_m(t,\cdot),\{z\})]\Delta t + o(\Delta t) \tag{1.9}$$

as $\Delta t \to 0$ $(m \geq 1)$, where $\hat{Z}_m(t,\cdot)$ denotes a convex envelope of the function $z \mapsto Z_m(t,z)$, i.e., the graph of the boundary of the convex hull, in $\mathbf{R}^N$, of the set $\{(z,y)|z \in \mathbf{Z}^n/m, y \geq Z_m(t,z)\}$.

In [19, Theorem 2], we proved that a continuous solution u to (1.8) sweeps in time $t>0$ a region with volume given by $t\cdot w(R,u(0,\cdot),\mathbf{R}^n)$, and that, for continuous solutions u and v to (1.8) with $v(0,\cdot)=\hat{u}(0,\cdot)$, $\hat{u}(t,\cdot)$ is different from $v(t,\cdot)$ at time $t>0$ in general if $u(0,\cdot)\neq\hat{u}(0,\cdot)$.

We also showed that a continuous solution to (1.8) is a viscosity solution of the following PDE (see [19, Theorem 3]):

$$\partial_t u(t,x) = \chi(u,Du(t,x),t,x)R(Du(t,x))\mathrm{Det}_+(D^2u(t,x)) \tag{1.10}$$

$((t,x) \in (0,\infty)\times\mathbf{R}^n)$, where $Du(t,x) := (\partial u(t,x)/\partial x_i)_{i=1}^n$, $D^2u(t,x) := (\partial^2 u(t,x)/\partial x_i\partial x_j)_{i,j=1}^n$,

$$\chi(u,p,t,x) := \begin{cases} 1 & \text{if } p \in \partial u(t,x), \\ 0 & \text{otherwise} \end{cases}$$

($\partial u(t,x)$ denotes the subdifferential of the function $x \mapsto u(t,x)$). Conversely, we discussed under what conditions a viscosity solution to (1.10) is a solution to (1.8).

Remark 2 When $R(p) = (1+|p|^2)^{-(n+1)/2}$,

$$(1+|Du(t,x)|^2)^{-1/2}\chi(u, Du(t,x), t, x)R(Du(t,x))\mathrm{Det}_+(D^2u(t,x))$$

can be considered as the convexified Gauss curvature of $\{(y, u(t,y))|y \in \mathbf{R}^N\}$ at x if we consider $\{(y,z)|y \in \mathbf{R}^N, z \geq u(t,y)\}$ as the inside of the hypersurface $\{(y, u(t,y))|y \in \mathbf{R}^N\}$.

Next we briefly discuss what we study in this paper.

Let F be a closed convex set in $\mathbf{R}^N$. For $x \in \partial F$, put

$$N_F(x) := \{p \in \mathbf{S}^{N-1}|F \subset \{y| < y-x, p > \leq 0\}\},$$

where $< \cdot, \cdot >$ denotes the inner product in $\mathbf{R}^N$.

To consider a convexified Gauss curvature flow of bounded open sets by the level set approach, we introduce new types of measures.

Definition 2 Let u be a bounded function from a subset of $\mathbf{R}^N$ to $\mathbf{R}$, and $R \in L^1(\mathbf{S}^{N-1} : [0,\infty), d\mathcal{H}^{N-1})$, where $d\mathcal{H}^{N-1}$ denotes a $(N-1)$-dimensional Hausdorff outer measure.
(i). Let $r \in \mathbf{R}$. For $B \in B(\mathbf{R}^N)$, put

$$\omega_r(R, u, B) := \int_{N_{co\ u^{-1}([r,\infty))^-}(\partial(co\ u^{-1}([r,\infty)))\cap B)} R(p)d\mathcal{H}^{N-1}(p), \tag{1.11}$$

where A^- denotes the closure of the set A.
(ii). For $B \in B(\mathbf{R}^N)$, put

$$\mathbf{w}(R, u, B) := \int_{\mathbf{R}} dr\omega_r(R, u, B), \tag{1.12}$$

provided the right hand side is well defined.

When it is not confusing, we write $\omega_r(R, u, dx) = \omega_r(u, dx)$ and $\mathbf{w}(R, u, dx) = \mathbf{w}(u, dx)$ for the sake of simplicity.

The existence and the uniqueness of a solution to the following equation is given in section 2.

Definition 3 Let $T \in [0,\infty]$ and $R \in L^1(\mathbf{S}^{N-1} : [0,\infty), d\mathcal{H}^{N-1})$. A family of bounded open sets $\{D(t)\}_{t\in[0,T)}$ in $\mathbf{R}^N$ is called a convexified Gauss curvature flow in an $(R$-$)$anisotropic external field on $[0,T)$ if

$$D(t) = (co\ D(t)) \cap D(0) \quad \text{for } t \in [0, T), \tag{1.13}$$

and if the following holds: for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \in [0, T)$,

$$\int_{\mathbf{R}^N} \varphi(x)(I_{D(0)}(x) - I_{D(t)}(x))dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x)\omega_1(R, I_{D(s)}(\cdot), dx). \tag{1.14}$$

We also show the existence and the uniqueness of a solution $u \in C_b([0, T) \times \mathbf{R}^N)$ to the following: for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \in [0, T)$,

$$\int_{\mathbf{R}^N} \varphi(x)(u(0, x) - u(t, x))dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x)\mathbf{w}(R, u(s, \cdot), dx). \tag{1.15}$$

The existence of $\{I_{D(t)}\}_{t\geq 0}$ in Definition 3 is given by the continuum limit of a class of particle systems $\{(Y_m(t, z))_{z\in\mathbf{Z}^N/m}\}_{t\geq 0}$ that satisfies the follows: for any $t \geq 0$ and any $z \in \mathbf{Z}^N/m$,

$$P(Y_m(t + \Delta t, z) - Y_m(t, z) < 0) = m^N E[\omega_1(Y_m(t, \cdot), \{z\})]\Delta t + o(\Delta t) \tag{1.16}$$

as $\Delta t \to 0$ $(m \geq 1)$ (see Theorem 1 in section 2).

The existence and the uniqueness of a solution to (1.15) will be given by the continuum limit of the linear combinations of solutions to (1.14) with $D(0) = u(0, \cdot)^{-1}((r, \infty))$ for $r \in \mathbf{R}$ (see Corollary 2 in section 2).

We also discuss the properties of $\{D(t)\}_{t\geq 0}$ in Definition 3 (see Theorem 2 in section 2).

For $p \in \mathbf{R}^N$ and a $N \times N$-symmetric real matrix X, put

$$G(p, X) := \begin{cases} |p| \det{}_+ \left(-(I - \bar{p} \otimes \bar{p})\frac{X}{|p|}(I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p}\right) & \text{if } p \neq o, \\ 0 & \text{if } p = o \end{cases} \tag{1.17}$$

(see (1.4) for notation), where $\bar{p} := p/|p|$.

Suppose that a smooth oriented hypersurface M in $\mathbf{R}^N$ is given by $M = \{y \in \mathbf{R}^N \mid \varphi(y) = a,\ D\varphi(y) \neq o\}$ for some $\varphi \in C^2(\mathbf{R}^N)$ and $a \in \mathbf{R}$, and that the vector field e is given by $e_x = D\varphi(x)/|D\varphi(x)|$. Regard the tangent space, T_xM, as the orthogonal complement of e_x, and let $E_x :=$ span e_x and id_{E_x} denote the identity map on E_x. Then the map

$$A_x \oplus \mathrm{id}_{E_x} : \mathbf{R}^N \equiv T_xM \oplus E_x \to T_xM \oplus E_x$$

has a matrix representation

$$-(I - \bar{p} \otimes \bar{p}) \frac{X}{|p|} (I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p},$$

with $p = D\varphi(x)$ and $X = D^2\varphi(x)$. Therefore,

$$K(x) = \det \left(-(I - \bar{p} \otimes \bar{p}) \frac{X}{|p|} (I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p} \right), \tag{1.18}$$

$$K_+(x) = \frac{G(p, X)}{|p|}. \tag{1.19}$$

For $\{D(t)\}_{t \geq 0}$ in Definition 3, we show that $I_{D(t)}(x)$ and $I_{D(t)^-}(x)$ are respectively a viscosity supersolution and a viscosity subsolution of the following PDE (see Theorem 3 in section 2):

$$\partial_t u(t,x) + R\Big(\frac{Du(t,x)}{|Du(t,x)|}\Big)\sigma^-(u, Du(t,x), t, x) G(Du(t,x), D^2u(t,x)) = 0 \tag{1.20}$$

($(t,x) \in (0,\infty) \times \mathbf{R}^N$). Here

$$\sigma^-(u,p,t,x) := \begin{cases} 1 & \text{if } u(t,\cdot) < u(t,x) \text{ on } H(p,x) \text{ and } p \in \mathbf{R}^N \setminus \{o\}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.21}$$

where

$$H(p,x) := \{y \in \mathbf{R}^N \backslash \{x\} | < y - x, p > \leq 0\}. \tag{1.22}$$

Moreover, we show that a continuous solution to (1.15) is a viscosity solution of (1.20) (see Corollary 3 in section 2).

In [21], we will study the uniqueness of a viscosity solution to (1.20), from which we conclude that a viscosity solution to (1.20) with a bounded continuous initial data is a unique solution to (1.15).

Since $G(p, X)$ is singular at $p = o$, the standard definition of a viscosity solution (see [8]) is not appropriate for (1.20). We take the definition of a viscosity solution to (1.20) from [22].

We first introduce the set of admissible test functions. We denote by $\mathcal{F}$ the set of all functions $f \in C^2([0,\infty))$ for which $f'' > 0$ on $(0,\infty)$ and

$$\lim_{r \downarrow 0} \frac{f(r)}{r^N} = 0. \tag{1.23}$$

Let Ω be an open subset of $(0,\infty) \times \mathbf{R}^N$. A function $\varphi \in C^2(\Omega)$ is called admissible in Ω if for any $(\hat{t},\hat{x}) \in \Omega$ for which $D\varphi$ vanishes, there exists $f \in \mathcal{F}$ such that as $(t,x) \to (\hat{t},\hat{x})$,

$$|\varphi(t,x) - \varphi(\hat{t},\hat{x}) - \partial_t\varphi(\hat{t},\hat{x})(t-\hat{t})| \leq f(|x-\hat{x}|) + o(|t-\hat{t}|). \tag{1.24}$$

We denote by $\mathcal{A}(\Omega)$ the set of all admissible functions in Ω.

Remark 3 $f(r) = r^{N+1} \in \mathcal{F}$ and $\varphi(t,x) = f(|x-\hat{x}|) \in \mathcal{A}((0,\infty) \times \mathbf{R}^N)$ for any $\hat{x} \in \mathbf{R}^N$.

Definition 4 (Viscosity solution) Let $0 < T \leq \infty$ and set $\Omega := (0,T) \times \mathbf{R}^N$, and put $R(o/|o|) := 0$.
(i). A function $u \in \mathrm{LSC}(\Omega)$ is called a viscosity supersolution of (1.20) in Ω if whenever $\varphi \in \mathcal{A}(\Omega)$, $(s,y) \in \Omega$, and $u - \varphi$ attains a local minimum at (s,y), then

$$\partial_t\varphi(s,y) + R\Big(\frac{D\varphi(s,y)}{|D\varphi(s,y)|}\Big)\sigma^+(u, D\varphi(s,y), s, y)G(D\varphi(s,y), D^2\varphi(s,y)) \geq 0, \tag{1.25}$$

where

$$\sigma^+(u,p,s,y) := \begin{cases} 1 & \text{if } u(s,\cdot) \leq u(s,y) \text{ on } H(p,y) \text{ and } p \in \mathbf{R}^N \setminus \{o\}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.26}$$

(ii). A function $u \in \mathrm{USC}(\Omega)$ is called a viscosity subsolution of (1.20) in Ω if whenever $\varphi \in \mathcal{A}(\Omega)$, $(s,y) \in \Omega$, and $u - \varphi$ attains a local maximum at (s,y), then

$$\partial_t\varphi(s,y) + \sigma^-(u, D\varphi(s,y), s, y)R\Big(\frac{D\varphi(s,y)}{|D\varphi(s,y)|}\Big)G(D\varphi(s,y), D^2\varphi(s,y)) \leq 0. \tag{1.27}$$

(iii). A function $u \in C(\Omega)$ is called a viscosity solution of (1.20) in Ω if it is a viscosity supersolution and a viscosity subsolution of (1.20) in Ω.

Remark 4 $\sigma^+(u,p,s,y) \geq \sigma^-(u,p,s,y)$ for all $u : \Omega \mapsto \mathbf{R}$ and all $(p,s,y) \in \mathbf{R}^N \times \Omega$.

Let $\mathcal{A}_0(\Omega)$ denote the set of all $\phi_1(t) + \phi_2(x) \in \mathcal{A}(\Omega)$ such that $x \mapsto G(D\phi_2(x), D^2\phi_2(x))$ is continuous in Ω. Then one can repalce, in Definition 5, $\mathcal{A}(\Omega)$ by $\mathcal{A}_0(\Omega)$ (see [20]).

Remark 5 For any $f \in \mathcal{F}$ and $\hat{x} \in \mathbf{R}^N$, $\varphi(t,x) = f(|x - \hat{x}|) \in \mathcal{A}_0((0,\infty) \times \mathbf{R}^N)$.

In section 2 we state our main results which will be proved in section 4. In section 3 we give technical lemmas.

2 Main Result

In this section we give our main result.

We give two assumptions to state the stochastic process which approximates the solution to (1.13)-(1.14).

(A.0). D is a bounded open set in $\mathbf{R}^N$.
(A.1). $R \in L^1(\mathbf{S}^{N-1} : [0,\infty), d\mathcal{H}^{N-1})$ and $||R||_{L^1(\mathbf{S}^{N-1})} = 1$.

Take $K > 0$ so that $co\ D \subset [-K+1, K-1]^N$. For $m \geq 1$, put

$$\mathcal{S}_m := \{I_A : [-K,K]^N \cap (\mathbf{Z}^N/m) \mapsto \{0,1\} | A \subset [-K,K]^N \cap \mathbf{Z}^N/m\}, \tag{2.1}$$

$$D_m := D \cap (\mathbf{Z}^N/m). \tag{2.2}$$

For x, $z \in \mathbf{Z}^N/m$ and $v \in \mathcal{S}_m$, put

$$v_{m,z}(x) := \begin{cases} v(x) & \text{if } x \neq z, \\ 0 & \text{if } x = z \end{cases}$$

; and for $f : \mathcal{S}_m \mapsto \mathbf{R}$, put

$$A_m f(v) := m^N \sum_{z \in [-K,K]^N \cap (\mathbf{Z}^N/m)} \omega_1(R, v, \{z\})\{f(v_{m,z}) - f(v)\}. \tag{2.3}$$

Let $\{Y_m(t,\cdot)\}_{t\geq 0}$ be a Markov process on $\mathcal{S}_m$ $(m \geq 1)$, with the generator A_m, such that $Y_m(0,z) = I_{D_m}(z)$ $(z \in [-K,K]^N \cap (\mathbf{Z}^N/m))$. For $(t,x) \in [0,\infty) \times [-K,K]^N$, put also

$$X_m(t,x) := I_{(co\ Y_m(t,\cdot)^{-1}(1))^o \cap D}(x), \tag{2.4}$$

where A^o denotes the interior of the set $A \subset \mathbf{R}^N$.

Then $\{X_m(t,\cdot)\}_{t\geq 0}$ is a stochastic process on

$$\mathcal{S} := \{f \in L^2([-K,K]^N) : ||f||_{L^2([-K,K]^N)} \leq (2K)^N\} \tag{2.5}$$

which is a complete separable metric space by the metric

$$d_{\mathcal{S}}(f,g) := \sum_{k=1}^{\infty} \frac{\max(| < f-g, e_k >_{L^2([-K,K]^N)} |, 1)}{2^k}. \tag{2.6}$$

Here $\{e_k\}_{k\geq 1}$ denotes a complete orthonomal basis of $L^2([-K,K]^N)$.

The following is our main result.

Theorem 1 Suppose that (A.0)-(A.1) hold. Then there exists a unique solution $\{D(t)\}_{t\geq 0}$ to (1.13)-(1.14) with $D(0) = D$ on $[0,\infty)$ such that $I_{D(\cdot)}(\cdot) \in C([0,\infty) : L^2([-K,K]^N))$ and that the following holds: for any $\gamma > 0$,

$$\lim_{m\to\infty} P(\sup_{t\geq 0} ||X_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \geq \gamma) = 0. \tag{2.7}$$

We recall Hausdorff metric of compact sets A and $B \subset \mathbf{R}^N$:

$$d_H(A,B) := \max(\max_{p\in A} dist(p,B), \max_{q\in B} dist(q,A)). \tag{2.8}$$

As a corollary, we obtain

Corollary 1 Suppose that (A.0)-(A.1) hold and that D is convex. Then for a unique solution $\{D(t)\}_{t\geq 0}$ to (1.13)-(1.14) with $D(0) = D$ on $[0,\infty)$, the following holds: for any $T \in [0, \mathrm{Vol}(D))$ and any $\gamma > 0$,

$$\lim_{m\to\infty} P(\sup_{0\leq t\leq T} d_H(co\ Y_m(t,\cdot)^{-1}(1), D(t)) \geq \gamma) = 0. \tag{2.9}$$

We introduce the assumption on the initial function in the equation (1.15).

(A.2). $h \in C_b(\mathbf{R}^N)$. For any $r \in \mathbf{R}^N$, the set $h^{-1}((r,\infty))$ is bounded or $\mathbf{R}^N$.

Then one can easily obtain the following from Theorem 1.

Corollary 2 Suppose that (A.1)-(A.2) hold. Then there exists a unique continuous solution $\{u(t,\cdot)\}_{t\geq 0}$ to (1.15) with $u(0,\cdot) = h(\cdot)$ on $[0,\infty)$. In addition, for any $r \in \mathbf{R}$, $\{u(t,\cdot)^{-1}((r,\infty))\}_{t\geq 0}$ is a unique solution to (1.13)-(1.14) with $D(0) = h^{-1}((r,\infty))$ on $[0,\infty)$.

The following theorem collects some of elementary properties of solutions to (1.13)-(1.14).

Theorem 2 Suppose that (A.0)-(A.1) hold. Let $\{D(t)\}_{t\geq 0}$ be a unique solution to (1.13)-(1.14) with $D(0) = D$ on $[0,\infty)$. Then the following holds.
(a) $t \mapsto D(t)$ is nonincreasing on $[0,\infty)$.
(b) For any $t \leq T^* := \mathrm{Vol}(D(0))$,

$$\mathrm{Vol}(D(0)\backslash D(t)) = t. \tag{2.10}$$

(c) $D(t) = \emptyset$ for $t \geq T^*$.
(d) Let $\{D_1(t)\}_{t\geq 0}$ be a solution to (1.13)-(1.14) on $[0,\infty)$ such that $D_1(0)$ is a bounded, convex, open set which contains D. Then

$$D(t) \subset D_1(t) \quad \text{for all } t \geq 0, \tag{2.11}$$

where the equality holds if and only if $D(0) = D_1(0)$.

Under

(A.3). $R \in C(S^{N-1} : [0,\infty))$,

we give the relation between the solution to (1.13)-(1.14) and the viscosity solution of (1.20).

Theorem 3 Suppose that (A.0)-(A.1) and (A.3) hold. Then for a unique solution $\{D(t)\}_{t\geq 0}$ to (1.13)-(1.14) with $D(0) = D$ on $[0,\infty)$, $I_{D(t)}(x)$ and $I_{D(t)^-}(x)$ is a viscosity supersolution and a viscosity subsolution to (1.20) in $(0,\infty) \times \mathbf{R}^N$, respectively.

As a corollary, we obtain

Corollary 3 Suppose that (A.1)-(A.3) hold. Then a solution $\{u(t,\cdot)\}_{t\geq 0}$ to (1.15) with $u(0,\cdot) = h(\cdot)$ on $[0,\infty)$ is a viscosity solution to (1.20) in $(0,\infty) \times \mathbf{R}^N$.

3 Lemma

In this section we give lemmas which will be used in the next section.

We extend $Y_m(t,\cdot)$ as a function on $\mathbf{R}^N$ so that

$$\overline{Y}_m(t,x) = \begin{cases} 0 & (x \in D^c \cap (\mathbf{Z}^N/m)), \\ Y_m(t,[mx]/m) & (x = (x_i)_{i=1}^N \in \mathbf{R}^N), \end{cases} \tag{3.1}$$

where $[mx] := ([mx_i])_{i=1}^N$ and $[mx_i]$ denotes an integer part of mx_i.

Remark 6 For $z \in \mathbf{Z}^N/m$,

$$Y_m(t,z) = \frac{1}{m^N} \int_{\{x \in \mathbf{R}^N | [mx] = mz\}} \overline{Y}_m(t,x)dx.$$

Lemma 1 Suppose that (A.0)-(A.1) hold. Then $\{\overline{Y}_m(\cdot,\cdot)\}_{m \geq 1}$ is tight in $D([0,\infty) : \mathcal{S})$, and any weak limit point of $\{\overline{Y}_m(\cdot,\cdot)\}_{m \geq 1}$ belongs to the set $C([0,\infty) : \mathcal{S})$.

(Proof). Since $\mathcal{S}$ is compact and since $t \mapsto \overline{Y}_m(t,x)$ is nonincreasing for any $x \in \mathbf{R}^N$, we only have to show the following (see [9, p. 129, Corollary 7.4 and p. 148, Theorem 10.2]): for any $\eta > 0$ and $T > 0$, there exists $\delta > 0$ such that for any i for which $1 \leq i \leq [T/\delta] + 1$,

$$\lim_{m \to \infty} P(||\overline{Y}_m(i\delta,\cdot) - \overline{Y}_m((i-1)\delta,\cdot)||_{L^1([-K,K]^N)} \geq \eta) = 0. \tag{3.2}$$

Indeed, for any s and t for which $(i-1)\delta \leq s \leq t \leq i\delta$,

$$\overline{Y}_m(s,x) - \overline{Y}_m(t,x) = 0 \text{ or } 1,$$

and

$$\begin{aligned} d_{\mathcal{S}}(\overline{Y}_m(t,\cdot), \overline{Y}_m(s,\cdot))^2 &\leq ||\overline{Y}_m(t,\cdot) - \overline{Y}_m(s,\cdot)||^2_{L^2([-K,K]^N)} \\ &= ||\overline{Y}_m(i\delta,\cdot) - \overline{Y}_m((i-1)\delta,\cdot)||_{L^1([-K,K]^N)}. \end{aligned}$$

For $\delta < \eta/2$ and $m \geq 1$, by Chebychev's inequality and Itô's formula (see [15]),

$$
\begin{aligned}
& P(||\overline{Y}_m(i\delta,\cdot) - \overline{Y}_m((i-1)\delta,\cdot)||_{L^1([-K,K]^N)} \geq \eta) && (3.3) \\
\leq \; & \left(\frac{2}{\eta}\right)^2 E[| \sum_{z \in D_m} (Y_m(i\delta, z) - Y_m((i-1)\delta, z)) \frac{1}{m^N} \\
& \quad + \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s,\cdot), D_m) ds|^2] \\
= \; & \left(\frac{2}{\eta}\right)^2 m^{-N} E[\int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s,\cdot), D_m) ds] \\
\leq \; & \left(\frac{2}{\eta}\right)^2 m^{-N} \delta \to 0 \quad \text{ as } m \to \infty
\end{aligned}
$$

(see (2.2) for notation). Indeed,

$$
\begin{aligned}
& ||\overline{Y}_m(i\delta,\cdot) - \overline{Y}_m((i-1)\delta,\cdot)||_{L^1([-K,K]^N)} \\
= \; & - \sum_{z \in D_m} (Y_m(i\delta, z) - Y_m((i-1)\delta, z)) \frac{1}{m^N} - \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s,\cdot), D_m) ds \\
& + \int_{(i-1)\delta}^{i\delta} \omega_1(Y_m(s,\cdot), D_m) ds.
\end{aligned}
$$

Q. E. D.

Remark 7 In (3.3), if $Y_m(s,\cdot) \equiv 0$, then $\omega_1(Y_m(s,\cdot), D_m) = 0$.

Lemma 2 Suppose that (A.0)-(A.1) hold. Then there exist a subsequence $\{m_k\}_{k\geq 1} \subset \mathbf{N}$ and stochastic processes $\{\overline{Y}_{1,m_k}(\cdot,\cdot)\}_{k\geq 1}$ on a probability space $(\Omega_1, \mathbf{B}_1, P_1)$ such that the probability law of $\{\overline{Y}_{1,m_k}(\cdot,\cdot)\}_{k\geq 1}$ is the same as that of $\{\overline{Y}_{m_k}(\cdot,\cdot)\}_{k\geq 1}$, and such that $\{\overline{Y}_{1,m_k}(\cdot,\cdot)\}_{k\geq 1}$ is convergent in $D([0,\infty) : \mathcal{S})$, P_1-almost surely, and such that the following holds P_1-almost surely: for any $T > 0$ and $\varphi \in C([-K,K]^N)$

$$
\begin{aligned}
& \sup_{0 \leq t \leq T} | \sum_{z \in D_{m_k}} \varphi(z)(Y_{1,m_k}(t,z) - Y_{1,m_k}(0,z)) \frac{1}{m_k^N} && (3.4) \\
& \quad + \int_0^t \sum_{z \in D_{m_k}} \varphi(z) \omega_1(Y_{1,m_k}(s,\cdot), \{z\}) ds| \to 0 \quad \text{ as } k \to \infty.
\end{aligned}
$$

Here Y_{1,m_k} is defined by $\overline{Y}_{1,m_k}$ in the same way as in Remark 6.

(Proof). By Lemma 1 and Skorohod's Theorem (see [9, p. 102, Theorem 1.8]), there exist a subsequence $\{m_{0,k}\}_{k\ge1}\subset \mathbf{N}$ and stochastic processes $\{\overline{Y}_{1,m_{0,k}}(\cdot,\cdot)\}_{k\ge1}$ on a probability space $(\Omega_1,\mathbf{B}_1,P_1)$ such that the probability law of $\{\overline{Y}_{1,m_{0,k}}(\cdot,\cdot)\}_{k\ge1}$ is the same as that of $\{\overline{Y}_{m_{0,k}}(\cdot,\cdot)\}_{k\ge1}$, and such that $\{\overline{Y}_{1,m_{0,k}}(\cdot,\cdot)\}_{k\ge1}$ is convergent in $D([0,\infty):\mathcal{S})$, P_1-almost surely.

As in (3.3), by Doob-Kolmogorov's inequality (see [15]), for any $T>0$ and $\varphi\in C([-K,K]^N)$

$$E_1[\sup_{0\le t\le T}|\sum_{z\in D_{m_{0,k}}}\varphi(z)(Y_{1,m_{0,k}}(t,z)-Y_{1,m_{0,k}}(0,z))\frac{1}{(m_{0,k})^N} + \int_0^t\sum_{z\in D_{m_{0,k}}}\varphi(z)\omega_1(Y_{1,m_{0,k}}(s,\cdot),\{z\})ds|^2]\to 0 \quad \text{as } k\to\infty. \tag{3.5}$$

Since a L^2-convergent sequence of random variables has an almost surely-convergent subsequence, and since $C([-K,K]^N)$ is separable, one can complete the proof by the diagonal method.

Q. E. D.

When it is not confusing, we write $\overline{Y}_{1,m_k}=\overline{Y}_{m_k}$ and $Y_{1,m_k}=Y_{m_k}$ on $(\Omega_1,\mathbf{B}_1,P_1)$ for the sake of simplicity.

Take $x_0\in D$ and $r_0>0$ so that $U_{4r_0}(x_0):=\{y\in\mathbf{R}^N:|x_0-y|<4r_0\}\subset D$, and put $U_0:=U_{2r_0}(x_0)$. Then

$$V_0:=\inf_{x\in\partial U_0}\mathrm{Vol}(U_{3r_0}(x_0)\cap H(x_0-x,x))>0. \tag{3.6}$$

Put, on $(\Omega_1,\mathbf{B}_1,P_1)$,

$$\tau_m:=\inf\{t>0|Y_{1,m}(t,z)=0 \text{ for some } z\in(\mathbf{Z}^N/m)\cap U_0\}. \tag{3.7}$$

Then the following holds.

Lemma 3 Suppose that (A.0)-(A.1) hold. Then

$$P_1(V_0\le\liminf_{k\to\infty}\tau_{m_k}\le\limsup_{k\to\infty}\tau_{m_k}\le\mathrm{Vol}(D))=1. \tag{3.8}$$

(Proof). By (3.4), for any $t >$ Vol(D),

$$\begin{aligned}
&\limsup_{k\to\infty}\{\min(\tau_{m_k},t)\} &(3.9)\\
=\ &\limsup_{k\to\infty}\int_0^{\min(\tau_{m_k},t)}\omega_1(Y_{m_k}(s,\cdot),D_{m_k})ds\\
\le\ &\limsup_{k\to\infty}\sum_{z\in D_{m_k}}(Y_{m_k}(0,z)-Y_{m_k}(\min(\tau_{m_k},t),z))\frac{1}{m_k^N}\le \mathrm{Vol}(D)
\end{aligned}$$

P_1- almost surely. We also have

$$\begin{aligned}
V_0 &\le \liminf_{k\to\infty}\sum_{z\in D_{m_k}}(Y_{m_k}(0,z)-Y_{m_k}(\tau_{m_k},z))\frac{1}{m_k^N} &(3.10)\\
&\le \liminf_{k\to\infty}\int_0^{\tau_{m_k}}\omega_1(Y_{m_k}(s,\cdot),D_{m_k})ds=\liminf_{k\to\infty}\tau_{m_k}
\end{aligned}$$

P_1- almost surely.

Q. E. D.

The following lemma can be proved in the same way as in [4, section 5.2] and the proof is omitted.

Lemma 4 Suppose that (A.1) holds. Let F and $F_m(m\ge 1)$ be closed convex sets in $\mathbf{R}^N$ such that ∂F and $\partial F_m(m\ge 1)$ are closed hypersurfaces and such that $d_H(F_m,F)\to 0$ as $m\to\infty$. Then $\omega_1(I_{F_m}(\cdot),dx)$ weakly converges to $\omega_1(F,dx)$ as $m\to\infty$, that is, for any $\varphi\in C_o(\mathbf{R}^N)$,

$$\lim_{m\to\infty}\int_{\mathbf{R}^N}\varphi(x)\omega_1(I_{F_m}(\cdot),dx)=\int_{\mathbf{R}^N}\varphi(x)\omega_1(I_F(\cdot),dx). \tag{3.11}$$

We denote by $X(\cdot,\cdot)\in C([0,\infty):\mathcal{S})$ the P_1-a.s. limit of $\overline{Y}_{1,m_k}(\cdot,\cdot)$ as $k\to\infty$. Then we have

Lemma 5 Suppose that (A.0)-(A.1) hold. Then there exists a solution $\{D(t)\}_{t\in[0,V_0)}$ to (1.13)-(1.14) on $[0,V_0)$ such that the following holds P_1-almost surely:

$$X(t,x)=I_{D(t)}(x),\quad dx-a.e.\quad \text{for all } t\in[0,V_0). \tag{3.12}$$

(Proof). For $p \in \mathbf{S}^{N-1}$, let $C(x_0, r_0; p)$ denote a semi-infinite cylinder

$$\{x_0 + rp + x : r \geq 0, |x| \leq r_0, < x, p >= 0, x \in \mathbf{R}^N\}$$

which can be obtained by moving a $(N-1)$-dimensional ball

$$\{x_0 + x : |x| \leq r_0, < x, p >= 0, x \in \mathbf{R}^N\}$$

in the positive direction of p.

Take $p_1, \cdots, p_{k_0} \in \mathbf{S}^{N-1}$ for some $k_0 \in \mathbf{N}$ so that

$$co\ D \subset \cup_{i=1}^{k_0} C(x_0, r_0; p_i).$$

For $i = 1, \cdots, k_0$, take $\{q_{i1}, \cdots, q_{i(N-1)}\}$ so that $\{q_{i1}, \cdots, q_{i(N-1)}, p_i\}$ is an orthonormal basis in $\mathbf{R}^N$, and put

$$C_{m_k}(t) := co\ Y_{m_k}(t, \cdot)^{-1}(1). \tag{3.13}$$

For $x = (x_k)_{k=1}^{N-1} \in \mathbf{R}^{N-1}$ for which $|x| \leq 2r_0$, put also

$$\tilde{X}_{m_k,i}(t, x) := -\sup\{r > 0 | x_0 + rp_i + \sum_{j=1}^{N-1} q_{ij} x_j \in C_{m_k}(t)\}. \tag{3.14}$$

Then $\tilde{X}_{m_k,i}(t, \cdot)$ is a bounded convex function on $\{x \in \mathbf{R}^{N-1} : |x| \leq 7r_0/4\}$ for $t \in [0, \tau_{m_k})$ if $m_k \geq 8N^{1/2}/r_0$.

It is known that the set of bounded convex functions with the same domain is compact as the set of continuous functions defined on K for every compact subset K of the interior of their domain (see [4, section 3.3]).

Therefore, by Lemma 3 and the diagonal method, there exists a subsequence $\{\tilde{X}_{\tilde{m}_k,i}(t, \cdot)\}_{k\geq 1}$ of $\{\tilde{X}_{m_k,i}(t, \cdot)\}_{k\geq 1}$ and a convex function $\tilde{X}_i(t, \cdot)$ such that for any $t \in \mathbf{Q} \cap [0, V_0)$ and $i = 1, \cdots, k_0$,

$$\lim_{k\to\infty} \sup_{x \in \mathbf{R}^{N-1}, |x| \leq 3r_0/2} |\tilde{X}_{\tilde{m}_k,i}(t, x) - \tilde{X}_i(t, x)| = 0 \tag{3.15}$$

(Notice that $\{\tilde{m}_k\}_{k\geq 1}$ can be random.).

It is clear that there exists a nonincreasing family of compact convex sets $\{\tilde{C}(t)\}_{t \in \mathbf{Q} \cap [0, V_0)}$ such that for any $t \in \mathbf{Q} \cap [0, V_0)$,

$$\lim_{k\to\infty} d_H(C_{\tilde{m}_k}(t), \tilde{C}(t)) = 0, \tag{3.16}$$

$$\tilde{X}_i(t,x) = -\sup\{r>0 | x_0 + rp_i + \sum_{j=1}^{N-1} q_{ij}x_j \in \tilde{C}(t)\}$$

for all $i = 1, \cdots, k_0$, and $x = (x_k)_{k=1}^{N-1} \in \mathbf{R}^{N-1}$ for which $|x| \le 3r_0/2$.

In particular,

$$D \subset \tilde{C}(0) \quad \text{(by (2.2))}, \tag{3.17}$$
$$\lim_{k\to\infty} ||X_{1,\tilde{m}_k}(t,\cdot) - I_{\tilde{C}(t)^o\cap D}(\cdot)||_{L^2([-K,K]^N)} = 0$$

for all $t \in \mathbf{Q} \cap [0, V_0)$, where $X_{1,\tilde{m}_k}$ is defined by $Y_{1,\tilde{m}_k}$ in the same way as in (2.4). When it is not confusing, we write $X_{1,\tilde{m}_k} = X_{\tilde{m}_k}$ on $(\Omega_1, \mathbf{B}_1, P_1)$ for the sake of simplicity.

The following also holds: for all $t \in [0, V_0) \cap \mathbf{Q}$,

$$\lim_{k\to\infty} ||\overline{Y}_{\tilde{m}_k}(t,\cdot) - X_{\tilde{m}_k}(t,\cdot)||_{L^2([-K,K]^N)} = 0. \tag{3.18}$$

Indeed, if $X_m(t,x) \ne \overline{Y}_m(t,x)$, then

$$dist(x, \partial(C_m(t)^o \cap D)) \le \frac{N^{1/2}}{m}$$

; and by (3.16), the volume of the $N^{1/2}/\tilde{m}_k$-neighborhood of the set $\partial D \cup \partial C_{\tilde{m}_k}(t)$ converges to zero as $k \to \infty$ for $t \in [0, V_0) \cap \mathbf{Q}$.

For $t \in [0, V_0) \backslash \mathbf{Q}$, put

$$\tilde{C}(t) := \cap_{s\in\mathbf{Q}\cap[0,t)} \tilde{C}(s). \tag{3.19}$$

Then, by (3.17)-(3.19), the following holds P_1-a.s.:

$$X(t,x) = I_{\tilde{C}(t)^o\cap D}(x), \quad dx - a.e, \quad \text{for all } t \in [0, V_0), \tag{3.20}$$

since $\{\overline{Y}_{\tilde{m}_k}\}_{k\ge1}$ is a subsequence of a convergent sequence $\{\overline{Y}_{m_k}\}_{k\ge1}$ and since $X \in C([0,\infty) : \mathcal{S})$ is the P_1-a.s. limit, in $D([0,\infty) : \mathcal{S})$, of $\overline{Y}_{m_k}$ as $k \to \infty$, and since $\{\tilde{C}(t)\}_{t\in[0,V_0)\cap\mathbf{Q}}$ is nonincreasing in t.

Put

$$D(t) := \tilde{C}(t)^o \cap D. \tag{3.21}$$

Then (1.13) holds for all $t \in [0, V_0)$, since $D = D(0)$ by (3.17) and since

$$D(t) \supset \{\text{co } (\tilde{C}(t)^o \cap D)\} \cap D = (\text{co } D(t)) \cap D \supset D(t) \cap D = D(t).$$

On $[0, V_0)$,

$$\omega_1(I_{\tilde{C}(t)}(\cdot), dx) = \omega_1(I_{D(t)}(\cdot), dx) \quad dt - a.e., \tag{3.22}$$

since

$$\tilde{C}(t) \backslash (\text{co } D(t))^- \subset \tilde{C}(t) \backslash D(t)^- \subset D^c$$

by (3.21), where D^c denotes a complement of D, and since

$$\int_0^{V_0} ds \omega_1(I_{\tilde{C}(s)}(\cdot), D^c) = \int_{D^c} (I_{D(0)}(x) - I_{D(V_0)}(x)) dx = 0$$

by (3.4), (3.20) and Lemma 4. Here we used the fact that (3.16) holds except for at most countably many $t \in [0, V_0)$.

Indeed, $t \mapsto C_{\tilde{m}_k}(t)$ is nonincreasing and (3.16) holds for all $t \in \mathbf{Q} \cap [0, V_0)$. Therefore, if $C_{\tilde{m}_k}(t)$ does not converge to $\tilde{C}(t)$ as $k \to \infty$, then $(\tilde{C}(t) \backslash \tilde{C}(t+))^o$ is not empty and has a positive Lebesgue measure by (3.19), where $\tilde{C}(t+) := \cup_{s>t} \tilde{C}(s)$. Besides, $(\tilde{C}(t) \backslash \tilde{C}(t+))^o$ are disjoint for different t.

By (3.4), Lemma 4, (3.20)-(3.22), (1.14) holds for all $t \in [0, V_0)$ since (3.16) holds except for at most countably many $t \in [0, V_0)$ as we mentioned above.

Q. E. D.

The following lemma implies the uniqueness of the solution to (1.13)-(1.14).

Lemma 6 Suppose that (A.1) hold. For $T > 0$, if $\{D_i(t)\}_{0 \le t < T}$ $(i = 1,\ 2)$ are solutions to (1.13)-(1.14) on $[0, T)$ for which $D_1(0) \subset D_2(0)$, then $D_1(t) \subset D_2(t)$ for all $t \in [0, T)$. In particular, for all $t \in [0, \min(\text{Vol}(D_1(0)), T))$,

$$d_H(D_1(t), D_2(t)^c) \ge d_H(D_1(0), D_2(0)^c). \tag{3.23}$$

(Proof). For each $t \geq 0$, put

$$\tilde{D}(t) := D_1(t)^- \cap D_2(t)^c, u_i(t,\cdot) := I_{D_i(t)}(\cdot) \quad u_i^-(t,\cdot) := I_{D_i(t)^-}(\cdot),$$

$$N_i(t) := \cup_{x\in\partial\tilde{D}(t)\cap\partial D_i(t)}\{p \in S^{N-1} | \sigma^+(u_i, -p, t, x) = 1\}$$

($i = 1, 2$). Then $N_2(t) \subset N_1(t)$.

Take a nondecreasing sequence $\{\eta_n\}_{n\geq 1}$ of nondecreasing C^1-functions such that

$$\eta_n(r) = 0 \quad \text{for all } r \leq 0, \quad \eta_n(r) = 1 \quad \text{for all } r \geq \frac{1}{n}, \tag{3.24}$$

and for $r \in \mathbf{R}$, put

$$\zeta_n(r) = \int_0^r \eta_n(s)ds. \tag{3.25}$$

Then since $t \mapsto u_i(t,x)$ and $t \mapsto u_i^-(t,x)$ are respectively right and left continuous for any $x \in \mathbf{R}^N$, for $t < \min(\mathrm{Vol}(D_1(0)), T)$ and $x \in \mathbf{R}^N$,

$$\begin{aligned} & \zeta_n(u_1^-(t,x) - u_2(t,x) - 1) - \zeta_n(u_1^-(0,x) - u_2(0,x)) \\ = & \int_0^t \zeta_n(u_1^-(s,x) - u_2(s,x) - s/t)(u_1^-(ds,x) - u_2(ds,x)) \\ & -\frac{1}{t}\int_0^t \eta_n(u_1^-(s,x) - u_2(s,x) - s/t)ds. \end{aligned} \tag{3.26}$$

Since $\zeta_n \geq 0$ and $D_1(0) \subset D_2(0)$, we have

$$\begin{aligned} 0 \leq & \int_0^t ds \int_{\mathbf{R}^N} \zeta_n(u_1^-(s,x) - u_2(s,x) - s/t) \\ & \times(\omega_1(u_2(s,\cdot), dx) - \omega_1(u_1(s,\cdot), dx)) \\ & -\frac{1}{t}\int_0^t ds \int_{\mathbf{R}^N} \eta_n(u_1^-(s,x) - u_2(s,x) - s/t)dx \\ \to & \int_0^t (1 - s/t)(\omega_1(u_2(s,\cdot), \tilde{D}(s)) - \omega_1(u_1(s,\cdot), \tilde{D}(s)))ds \\ & -\frac{1}{t}\int_0^t ds \int_{\tilde{D}(s)} dx \quad (\text{as } n \to \infty) \\ \leq & -\frac{1}{t}\int_0^t ds \int_{\tilde{D}(s)} dx, \end{aligned} \tag{3.27}$$

which implies the first assertion of this lemma.

Suppose that (3.23) dose not hold. Then there exists $a \in (0, d_H(D_1(0), D_2(0)^c))$ such that

$$\inf\{d_H(D_1(t), D_2(t)^c) | t \in [0, \min(\mathrm{Vol}(D_1(0)), T))\} < a.$$

Take $p_a \in \mathbf{S}^{N-1}$ and $t_a \in [0, \min(\mathrm{Vol}(D_1(0)), T))$ so that

$$ap_a + D_1(t_a) \not\subset D_2(t_a).$$

Since $ap_a + D_1(0) \subset D_2(0)$ and $\{ap_a + D_1(t)\}_{0 \le t < T}$ is a solution to (1.13)-(1.14) on $[0, T)$, this contradicts the first assertion of this lemma.

Q. E. D.

Take $\varphi \in C^2(\mathbf{R}^N)$ for which $D\varphi(x_o) \neq 0$ for some $x_o \in \mathbf{R}^N$.

Let I_N denote a $N \times N$-identity matrix and put

$$f_N := \frac{D\varphi(x_o)}{|D\varphi(x_o)|}, \quad (g_1 \cdots g_N) := I_N - f_N \otimes f_N.$$

Take $\{f_1, \cdots, f_{N-1}\}$ so that $\{f_1, \cdots, f_N\}$ is an orthonormal basis of $\mathbf{R}^N$. Then the following holds.

Lemma 7 (i) $< g_i, f_N >= 0$ $(1 \le i \le N)$.
(ii) For i for which $\partial_i \varphi(x_o) := \partial \varphi(x_o)/\partial x_i \neq 0$,

$$g_i = -\sum_{k \neq i} \frac{\partial_k \varphi(x_o)}{\partial_i \varphi(x_o)} g_k.$$

(iii) $span(g_1, \cdots, g_N) = span(f_1, \cdots, f_{N-1})$.
(iv) $D(D\varphi(x_o)/|D\varphi(x_o)|)(\mathbf{R}^N) \subset span(g_1, \cdots, g_N)$. As a mapping on $span(g_1, \cdots, g_N)$, eigenvalues and eigenvectors of $D(D\varphi(x_o)/|D\varphi(x_o)|)$ are the same as those of $(g_1 \cdots g_N)(D^2\varphi(x_o)/|D\varphi(x_o)|)(g_1 \cdots g_N)$. In particular, all eigenvalues of $D(D\varphi(x_o)/|D\varphi(x_o)|)$ are real.
(v) If eigenvalues $\lambda_1 \le \cdots \le \lambda_{N-1}$ of $-D(D\varphi(x_o)/|D\varphi(x_o)|)$ as a mapping on $span(g_1, \cdots, g_N)$ are nonnegative, then

$$\Pi_{i=1}^{N-1} \lambda_i = \frac{G(D\varphi(x_o), D^2\varphi(x_o))}{|D\varphi(x_o)|}. \tag{3.28}$$

(Proof). It is easy to see that (i) and (ii) hold. Take i for which $\partial_i\varphi(x_o) \neq 0$. Then, by (i) and (ii), we only have to show, to prove (iii), that $\{g_j\}_{j\neq i}$ is independent. Suppose that for $j = 1, \cdots, N$,

$$\sum_{k\neq i} \lambda_k \Big(\delta_{kj} - \frac{\partial_k\varphi(x_o)\partial_j\varphi(x_o)}{|D\varphi(x_o)|^2}\Big) = 0. \tag{3.29}$$

Putting $j = i$ in (3.29), we obtain

$$\sum_{k\neq i} \lambda_k \frac{\partial_k\varphi(x_o)\partial_i\varphi(x_o)}{|D\varphi(x_o)|^2} = 0,$$

from which

$$\sum_{k\neq i} \lambda_k \partial_k\varphi(x_o) = 0. \tag{3.30}$$

Putting $j \neq i$ in (3.29), we obtain

$$\lambda_j - \partial_j\varphi(x_o) \sum_{k\neq i} \lambda_k \frac{\partial_k\varphi(x_o)}{|D\varphi(x_o)|^2} = 0,$$

from which $\lambda_j = 0$ for $j \neq i$, by (3.30).

We prove (iv). It is easy to see that

$$D\Big(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\Big) = (g_1 \cdots g_N)\frac{D^2\varphi(x_o)}{|D\varphi(x_o)|}. \tag{3.31}$$

Hence

$$D\Big(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\Big)(\sum_{i=1}^N x_i g_i) = \lambda \sum_{i=1}^N x_i g_i$$

if and only if

$$(g_1 \cdots g_N)\frac{D^2\varphi(x_o)}{|D\varphi(x_o)|}(g_1 \cdots g_N)(\sum_{i=1}^N x_i g_i) = \lambda \sum_{i=1}^N x_i g_i,$$

since

$$(g_1 \cdots g_N)^2 = (g_1 \cdots g_N). \tag{3.32}$$

Put $P := (f_1 \cdots f_N)$ and $Q := (f_1 \cdots f_{N-1})$. The proof of (v) is devided into the following.

(STEP I) The eigenvalues of

$$-(I_N - f_N \otimes f_N)\frac{D^2\varphi(x_o)}{|D\varphi(x_o)|}(I_N - f_N \otimes f_N) + f_N \otimes f_N$$

are those of

$$\begin{pmatrix} -Q^*D(\frac{D\varphi(x_o)}{|D\varphi(x_o)|})Q & o \\ o^* & 1 \end{pmatrix}.$$

(STEP II) The eigenvalues of $Q^*D(D\varphi(x_o)/|D\varphi(x_o)|)Q$ are those of $D(D\varphi(x_o)/|D\varphi(x_o)|)$ on $span(g_1, \cdots, g_N)$.
(Proof of Step I). For $\lambda \in \mathbf{R}$, denoting by P^* the transposed matrix of P,

$$\begin{aligned} & \det\Big(-(I_N - f_N \otimes f_N)\frac{D^2\varphi(x_o)}{|D\varphi(x_o)|}(I_N - f_N \otimes f_N) + f_N \otimes f_N - \lambda I_N\Big) \\ = \ & \det\Big(-\begin{pmatrix} I_{N-1} & o \\ o^* & o \end{pmatrix} P^*\frac{D^2\varphi(x_o)}{|D\varphi(x_o)|}P \begin{pmatrix} I_{N-1} & o \\ o^* & o \end{pmatrix} + \begin{pmatrix} O & o \\ o^* & 1 \end{pmatrix} - \lambda I_N\Big) \\ = \ & \det\Big(\begin{pmatrix} -Q^*\frac{D^2\varphi(x_o)}{|D\varphi(x_o)|}Q & o \\ o^* & 1 \end{pmatrix} - \lambda I_N\Big) \end{aligned}$$

since

$$P^*P = I_N, \quad P\begin{pmatrix} O & o \\ o^* & 1 \end{pmatrix}P^* = f_N \otimes f_N.$$

(3.31) completes the proof since $< f_i, f_N >= 0$ if $i \neq N$.
(Proof of Step II). Let $x = (x_i)_{i=1}^{N-1} \in \mathbf{R}^{N-1}$ and $\lambda \in \mathbf{R}$. Suppose that

$$Q^*D\Big(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\Big)Qx = \lambda x. \tag{3.33}$$

Then

$$QQ^*D\Big(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\Big)(\sum_{1\le i\le N-1} x_i f_i) = \lambda \sum_{1\le i\le N-1} x_i f_i$$

and henceforth by (3.31),

$$D\Big(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\Big)\sum_{1\le i\le N-1} x_i f_i = \lambda \sum_{1\le i\le N-1} x_i f_i \tag{3.34}$$

since, by (iii),

$$QQ^*(I_N - f_N \otimes f_N) = I_N - f_N \otimes f_N.$$

It is easy to see that (3.34) implies (3.33).

Q. E. D.

For $i = 1, \cdots, N$, put

$$y_i(x) := \left((\delta_{ij} - 1)\frac{\partial_j \varphi(x)}{|D\varphi(x)|} + \delta_{ij}\varphi(x)\right)_{j=1}^{N}.$$

Then

Lemma 8 Suppose that all eigenvalues of $D(D\varphi(x_o)/|D\varphi(x_o)|)$ are nonpositive. Then, for $i = 1, \cdots, N$,

$$\frac{\partial_i \varphi(x_o)}{|D\varphi(x_o)|} G(D\varphi(x_o), D^2\varphi(x_o)) = \det(Dy_i(x_o)). \tag{3.35}$$

(Proof). For the sake of simplicity, we assume that $i = N$.

We first consider the case when $\partial_N \varphi(x_o) \neq 0$. By (ii) in Lemma 7, it is easy to see that the following holds:

$$\begin{pmatrix} I_{N-1} & o \\ -\frac{D\varphi(x_o)^*}{\partial_N \varphi(x_o)} & \end{pmatrix} Dy_N(x_o) = D\Big(-\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\Big) + \begin{pmatrix} O \\ -D\varphi(x_o)^* \end{pmatrix}. \tag{3.36}$$

By (i) and (iv) in Lemma 7, the eigenvalues and eigenvectors of $D(-D\varphi(x_o)/|D\varphi(x_o)|)$ on $span(g_1, \cdots, g_N)$ are real and are also those of the left hand side (l.h.s. for short) of (3.36).

We show that all eigenvalues of the l.h.s. of (3.36) are those of $D(-D\varphi(x_o)/|D\varphi(x_o)|)$ on $span(g_1, \cdots, g_N)$ and $-\partial_N \varphi(x_o)$.

By (i) and (iv) in Lemma 7, there exists an invariant subspace, which contains f_N, with an eigenvalue λ of the l.h.s. of (3.36).

Take $\ell \geq 1$ such that

$$\left(\begin{pmatrix} I_{N-1} & o \\ -\frac{D\varphi(x_o)^*}{\partial_N \varphi(x_o)} & \end{pmatrix} Dy_N(x_o) - \lambda\right)^{\ell} f_N = o. \tag{3.37}$$

Then $(-\partial_N \varphi(x_o) - \lambda)^{\ell} f_N \in span(g_1, \cdots, g_N)$ since

$$f_N = \frac{|D\varphi(x_o)|}{\partial_N \varphi(x_o)}((\delta_{jN})_{j=1}^N - g_N).$$

Hence $\lambda = -\partial_N \varphi(x_o)$ by (i) in Lemma 7.

Suppose that $\partial_N \varphi(x_o) = 0$. Then, by (3.31) and (i) in Lemma 7, for $x \in \mathbf{R}^N$,

$$< f_N, Dy_N(x_o)x >= \left\langle f_N, D\Big(\frac{D\varphi(x_o)}{|D\varphi(x_o)|}\Big)x \right\rangle = 0, \tag{3.38}$$

which implies that $Dy_N(x_o)(\mathbf{R}^N)$ is at most (N-1)-dimensional and henceforth (3.35) holds.

Q. E. D.

4 Proof

In this section we prove the results in section 2.
(Proof of Theorem 1). By Lemmas 1-6, there exists a unique (nonrandom) solution $\{D(t)\}_{0\le t<V_0}$ (see (3.6) for notation) of (1.13)-(1.14) on $[0,V_0)$ such that $I_{D(\cdot)} \in C([0,V_0) : \mathcal{S})$ and that the following holds: for any $T \in [0,V_0)$ and $\gamma > 0$,

$$\lim_{m\to\infty} P(\sup_{0\le t\le T} d_{\mathcal{S}}(\overline{Y}_m(t,\cdot), I_{D(t)}(\cdot)) \ge \gamma) = 0. \tag{4.1}$$

Therefore

$$\lim_{m\to\infty} P(\sup_{0\le t\le T} ||\overline{Y}_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \ge \gamma) = 0, \tag{4.2}$$

since, for $m \ge 1$ and $t \in [0,T]$,

$$\begin{aligned} & ||\overline{Y}_m(t,\cdot) - I_{D(t)}(\cdot)||^2_{L^2([-K,K]^N)} \\ = & \int_{[-K,K]^N} (\overline{Y}_m(t,x) - 2\overline{Y}_m(t,x) I_{D(t)}(x) + I_{D(t)}(x)) dx. \end{aligned}$$

We prove that the following holds:

$$\lim_{m\to\infty} P(\sup_{0\le t\le T} ||X_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \ge \gamma) = 0. \tag{4.3}$$

For any s and t for which $0 \le s < t \le T$,

$$\begin{aligned} & ||X_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \qquad (4.4) \\ \le & ||X_m(t,\cdot) - X_m(s,\cdot)||_{L^2([-K,K]^N)} + ||X_m(s,\cdot) - \overline{Y}_m(s,\cdot)||_{L^2([-K,K]^N)} \\ & + ||\overline{Y}_m(s,\cdot) - I_{D(s)}(\cdot)||_{L^2([-K,K]^N)} + ||I_{D(s)}(\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)}. \end{aligned}$$

Let $U_{-N^{1/2}/m}(D) := \{x \in D | dist(x, D^c) > N^{1/2}/m\}$. Then

$$\begin{aligned} & ||X_m(t,\cdot) - X_m(s,\cdot)||^2_{L^2([-K,K]^N)} = ||X_m(t,\cdot) - X_m(s,\cdot)||_{L^1([-K,K]^N)} \qquad (4.5) \\ \le & 2^N \sum_{z\in D_m} (Y_m(s,z) - Y_m(t,z)) \frac{1}{m^N} + \mathrm{Vol}(D \backslash U_{-N^{1/2}/m}(D)) \end{aligned}$$

(see (2.2) for notation). Indeed, if $x = (x_i)_{i=1}^N \in U_{-N^{1/2}/m}(D) \backslash (co\ Y_m(t,\cdot)^{-1}(1))$, then $Y_m(t,z) = 0$ for some $z = (z_i)_{i=1}^N \in \mathbf{Z}^N/m$ for which $|x_i - z_i| \le 1/m$ for all $i = 1, \cdots, N$.

In the same way as in (3.5), by (4.5), for any $\gamma > 0$, there exists $\delta > 0$ such that the following holds: for any $s \in [0, T-\delta]$,

$$\lim_{m\to\infty} P(\sup_{s \le s_1 \le s+\delta} ||X_m(s_1,\cdot) - X_m(s,\cdot)||_{L^2([-K,K]^N)} \ge \gamma) = 0. \tag{4.6}$$

Since, for any $t \in [0, V_0)$, any subsequence of $\{C_m(t)\}_{m\ge 1}$ has a convergent subsequence (see the the proof of Lemma 5),

$$\lim_{m\to\infty} ||\overline{Y}_m(t,\cdot) - X_m(t,\cdot)||_{L^2([-K,K]^N)} = 0 \tag{4.7}$$

for all $t \in [0, V_0)$, P_1-almost surely (see the discussion after (3.18)). Hence, for any $\gamma > 0$,

$$\lim_{m\to\infty} P(||\overline{Y}_m(s,\cdot) - X_m(s,\cdot)||_{L^2([-K,K]^N)} \ge \gamma) = 0. \tag{4.8}$$

$I_{D(\cdot)} \in C([0, V_0) : L^2([-K,K]^N))$, since

$$||I_{D(s)}(\cdot) - I_{D(t)}(\cdot)||^2_{L^2([-K,K]^N)} = \int_{[-K,K]^N} I_{D(s)}(x)dx - \int_{[-K,K]^N} I_{D(t)}(x)dx,$$

and since $t \mapsto \int_{[-K,K]^N} I_{D(t)}(x)dx$ is continuous on $[0, V_0)$.

(4.2) and the discussion after (4.3) show that (4.3) is true.

Recall Lemmas 2-3 and the notations therein. For $T < V_0$, take $x_0 \in D(T)$ and r_0 so that $U_{4r_0}(x_0) \subset D(T)$. For sufficiently large $k \ge 1$,

$$U_{3r_0}(x_0) \subset (co\ Y_{m_k}(T,\cdot)^{-1}(1))^o \cap D, \quad P_1 - a.s.,$$

since

$$\lim_{k\to\infty} ||X_{m_k}(T,\cdot) - I_{D(T)}(\cdot)||_{L^2([-K,K]^N)} = 0, \quad P_1 - a.s.$$

by Lemma 2 and (4.7) (see the discussion below (4.2)). Hence in the same way as in Lemma 3,

$$\begin{aligned} V_0 &\le \liminf_{k\to\infty} \sum_{z\in D_{m_k}} (Y_{m_k}(T,z) - Y_{m_k}(\tau_{m_k}, z))\frac{1}{m_k^N} \\ &\le \liminf_{k\to\infty} (\tau_{m_k} - T) \quad P_1 - a.s., \end{aligned} \tag{4.9}$$

which implies that (4.3) holds for $T < 2V_0$. Repeating the same procedure as above and then letting $r_0 \downarrow 0$, (4.3) holds for all $T < T^* := \mathrm{Vol}(D)$.

Put

$$D(t) = \emptyset \quad \text{for } t \ge T^*. \tag{4.10}$$

Then $I_{D(\cdot)} \in C([0,\infty) : L^2([-K,K]^N))$ and $\{D(t)\}_{t\ge 0}$ is a unique solution to (1.13)-(1.14) on $[0,\infty)$ by Lemma 6, since $t \mapsto I_{D(t)}$ is nonincreasing and since

$$\mathrm{Vol}(D(t)) = \mathrm{Vol}(D(0)) - t \downarrow 0, \quad \text{as } t \uparrow T^*, \tag{4.11}$$

by (1.14).

We prove (2.7). Take a sufficiently small positive ε so that

$$\mathrm{Vol}(D(t)) \le \left(\frac{\gamma}{4}\right)^2 \quad \text{for } t \ge t_\varepsilon := T^* - \varepsilon. \tag{4.12}$$

Then

$$\begin{aligned} & P(\sup_{t\ge 0} ||X_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \ge \gamma) \\ \le\ & P(\sup_{0\le t\le t_\varepsilon} ||X_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \ge \gamma) \\ & \quad + P(\sup_{t\ge t_\varepsilon} ||X_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \ge \gamma) \\ \le\ & 2P(\sup_{0\le t\le t_\varepsilon} ||X_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \ge \frac{\gamma}{2}) \to 0 \quad (\text{as } m \to \infty) \end{aligned} \tag{4.13}$$

since for $t \ge t_\varepsilon$,

$$\begin{aligned} & ||X_m(t,\cdot) - I_{D(t)}(\cdot)||_{L^2([-K,K]^N)} \\ \le\ & ||X_m(t_\varepsilon,\cdot)||_{L^2([-K,K]^N)} + ||I_{D(t_\varepsilon)}(\cdot)||_{L^2([-K,K]^N)} \\ \le\ & ||X_m(t_\varepsilon,\cdot) - I_{D(t_\varepsilon)}(\cdot)||_{L^2([-K,K]^N)} + 2||I_{D(t_\varepsilon)}(\cdot)||_{L^2([-K,K]^N)}. \end{aligned}$$

Q. E. D.

(Proof of Corollary 1). Since D is convex,

$$(co\ Y_m(t,\cdot)^{-1}(1))^o \cap D = (co\ Y_m(t,\cdot)^{-1}(1))^o =: D_m(t).$$

For $T < T^*$, take $x_0 \in D(T)$ and r_0 so that $U_{4r_0}(x_0) \subset D(T)$ (see (3.6) for notation). Then, for sufficiently large m, $U_{3r_0}(x_0) \subset D_m(0)$.

Consider cones

$$\text{cone}(x) := \text{co}\ (\{x\} \cup U_0^-) \quad (x \in D^-),$$

and for $r > 0$, put

$$\begin{aligned} V(r) &:= \inf_{x \in \partial D} \text{Vol}(\text{cone}(x) \cap H(x_0 - x, x + r(x_0 - x))), & (4.14)\\ V_m(r) &:= \inf_{x \in \partial D_m(0)} \text{Vol}(\text{cone}(x) \cap H(x_0 - x, x + r(x_0 - x))). & (4.15) \end{aligned}$$

Then for $\gamma > 0$ and sufficiently large $m \geq 1$, by Theorem 1,

$$\begin{aligned} & P(\sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma) & (4.16)\\ \leq\ & P(||I_{D_m(T)}(\cdot) - I_{D(T)}(\cdot)||^2_{L^2([-K,K])} \geq V_0)\\ & \quad + P(U_0 \subset D_m(T), \sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma)\\ \to\ & 0 \quad (\text{as } m \to \infty). \end{aligned}$$

Indeed,

$$\begin{aligned} & P(U_0 \subset D_m(T), \sup_{0 \leq t \leq T} d_H(D_m(t), D(t)) \geq \gamma)\\ \leq\ & P(\sup_{0 \leq t \leq T} ||I_{D_m(t)}(\cdot) - I_{D(t)}(\cdot)||^2_{L^2([-K,K])} \geq \min(V(\gamma), V_m(\gamma))), \end{aligned}$$

and $V_m(\gamma) \geq V(\gamma)$ for all $m \geq 1$.

Q. E. D.

(Proof of Corollary 2). For $r \in \mathbf{R}$, let $\{D_r(t)\}_{t \geq 0}$ denote a unique solution of (1.13)-(1.14) with $D_r(0) = h^{-1}((r, \infty))$ on $[0, \infty)$. Notice that

$$D_r(\cdot) := \begin{cases} \mathbf{R}^N & \text{if } r < \inf\{h(x)|x \in \mathbf{R}^N\}, \\ \emptyset & \text{if } r \geq \sup\{h(x)|x \in \mathbf{R}^N\}. \end{cases} \tag{4.17}$$

Put

$$u(t,x) := \sup\{r \in \mathbf{R}|x \in D_r(t)\}. \tag{4.18}$$

Then, for all $t \geq 0$ and $r \in \mathbf{R}$ for which $D_r(t) \neq \emptyset$, $\mathbf{R}^N$,

$$u(t,\cdot)^{-1}((r,\infty)) = D_r(t), \tag{4.19}$$

since $D_r(t) = D_{r+}(t) := \cup_{\tilde{r}>r} D_{\tilde{r}}(t)$ by (1.13).

Indeed, $D_r(0) = D_{r+}(0)$; and if $\tilde{r} - r$ is positive and is sufficiently small, then $D_{\tilde{r}}(t) \neq \emptyset$ by (b) in Theorem 2, and

$$\int_{\mathbf{R}^N} (I_{D_{\tilde{r}}(t)}(x) - I_{D_r(t)}(x))dx = \int_{\mathbf{R}^N} (I_{D_{\tilde{r}}(0)}(x) - I_{D_r(0)}(x))dx.$$

By Lemma 6 and (4.19), u is continuous.

For $m \geq 1$, put

$$\begin{aligned} k_{m,1} &:= [m \sup\{h(y)|y \in \mathbf{R}^N\}], \\ k_{m,0} &:= [m \inf\{h(y)|y \in \mathbf{R}^N\}] - 1. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k_{m,0} \leq k \leq k_{m,1}} \frac{k}{m}(I_{D_{\frac{k}{m}}(t)}(x) - I_{D_{\frac{k+1}{m}}(t)}(x)) \qquad (4.20) \\ = &\sum_{k_{m,0} < k \leq k_{m,1}} \frac{1}{m} I_{D_{\frac{k}{m}}(t)}(x) - \frac{k_{m,1}+1}{m} I_{D_{\frac{k_{m,1}+1}{m}}(t)}(x) + \frac{k_{m,0}}{m} I_{D_{\frac{k_{m,0}}{m}}(t)}(x). \end{aligned}$$

Since $I_{D_{\frac{k_{m,1}+1}{m}}(t)}(x) \equiv 0$ and since $I_{D_{\frac{k_{m,0}}{m}}(t)}(x) \equiv 1$, the following holds: for any $\varphi \in C_o(\mathbf{R}^N)$ and any $t \geq 0$,

$$\int_{\mathbf{R}^N} \varphi(x)[\sum_{k_{m,0} \leq k \leq k_{m,1}} \frac{k}{m}(I_{D_{\frac{k}{m}}(0)}(x) - I_{D_{\frac{k+1}{m}}(0)}(x)) \tag{4.21}$$

$$
\begin{aligned}
& - \sum_{k_{m,0}\le k\le k_{m,1}} \frac{k}{m}(I_{D_{\frac{k}{m}}(t)}(x) - I_{D_{\frac{k+1}{m}}(t)}(x))]dx \\
= & \int_0^t ds[\sum_{k_{m,0}<k\le k_{m,1}} \frac{1}{m}\int_{\mathbf{R}^N} \varphi(x)\omega_1(I_{D_{\frac{k}{m}}(s)}(\cdot), dx)].
\end{aligned}
$$

Letting $m \to \infty$ in (4.21), one can show that u is a solution to (1.15) by Lemma 4, since co $D_{\frac{[mr]+1}{m}}(s) \to$ co $D_r(s)$ as $m \to \infty$ for $r \in [\inf\{u(s,y)|y \in \mathbf{R}^N\}, \sup\{u(s,y)|y \in \mathbf{R}^N\})$, provided $D_r(s) \neq \emptyset$, $\mathbf{R}^N$.

Let $v \in C([0,\infty) \times \mathbf{R}^N)$ be a solution to (1.15) with $v(0,\cdot) = h(\cdot)$. Then for $n \geq 1$, $r \in [\inf\{h(y)|y \in \mathbf{R}^N\}, \sup\{h(y)|y \in \mathbf{R}^N\})$, and $\varphi \in C_o(\mathbf{R}^N)$ and $t \geq 0$,

$$
\begin{aligned}
& \int_{\mathbf{R}^N} \varphi(x)\{\eta_n(v(0,x)-r) - \eta_n(v(t,x)-r)\}dx \\
= & \int_0^t ds \int_{\mathbf{R}} \frac{d\eta_n(\tilde{r}-r)}{d\tilde{r}} d\tilde{r} \int_{\mathbf{R}^N} \varphi(x)\omega_{\tilde{r}}(v(s,\cdot), dx)
\end{aligned}
\tag{4.22}
$$

(see (3.24) for notation). Let $n \to \infty$ in (4.22). Then we see that $\tilde{D}_r(t) := v(t,\cdot)^{-1}((r,\infty))$ is a solution to (1.14) on $[0,\infty)$ by Lemma 4 and the continuity of v.

We prove that $v(t,\cdot)^{-1}((r,\infty))$ satisfies (1.13). For $x \in (\text{co } \tilde{D}_r(t)) \cap \tilde{D}_r(0)$, take $\delta > 0$ so that $U_\delta(x) \subset (\text{co } \tilde{D}_r(t)) \cap \tilde{D}_r(0)$. Then $U_\delta(x) \subset \text{co } \tilde{D}_r(s)$ for all $s \leq t$. Hence, by (1.14), for any $\varphi \in C_o(\mathbf{R}^N)$ such that $\varphi \equiv 0$ in $U_\delta(x)^c$.

$$
\int_{\mathbf{R}^N} \varphi(x)\{I_{\tilde{D}_r(0)} - I_{\tilde{D}_r(t)}\}dx = \int_0^t ds \int_{\mathbf{R}^N} \varphi(x)\omega_1(I_{\tilde{D}_r(s)}(\cdot), dx) = 0, \tag{4.23}
$$

which implies that $x \in (U_\delta(x) \subset)\tilde{D}_r(t)$. Hence (1.13) holds.

The uniqueness of u follows from that of $D_r(\cdot)$ for all r.

Q. E. D.

Theorem 2 is an easy consequence of Theorem 1 and Lemma 6 and we omit the proof.

(Proof of Theorem 3).
(Step I). We first show that $u(t,x) := I_{D(t)}(x)$ is a viscosity supersolution of (1.20) in $(0,\infty) \times \mathbf{R}^N$.

Let $\psi \in \mathcal{A}((0,\infty)\times \mathbf{R}^N)$ and assume that $u-\psi$ attains a local minimum at $(t_0,x_0)\in (0,\infty)\times \mathbf{R}^N$. Without loss of generality, we may assume that $u(t_0,x_0)=\psi(t_0,x_0)$ and that $u(t,x)>\psi(t,x)$ for all $(t,x)\in (0,\infty)\times \mathbf{R}^N\setminus \{(t_0,x_0)\}$ (see [8]).

If $x_0\notin \partial(co\ D(t_0))\cap \partial D(t_0)$, then $\partial_t\psi(t_0,x_0)\geq 0$.

Indeed, $t\mapsto u(t,x_0)$ is constant if t_0-t is a sufficiently small positive number, from which $\psi(t_0,x_0)>\psi(t,x_0)$ for such t.

Suppose that $x_0\in \partial(co\ D(t_0))\cap\partial D(t_0)$. Then $u(t_0,x_0)=0$, and $D\psi(t_0,x_0)=o$ or $\sigma^+(u,D\psi(t_0,x_0),t_0,x_0)=1$.

Indeed, if $D\psi(t_0,x_0)\neq o$, then for y for which $y+x_0\notin H(D\psi(t_0,x_0),x_0)$ and for $r>0$, by the mean value theorem, there exists $\theta\in(0,1)$ such that

$$u(t_0,x_0+ry)>\psi(t_0,x_0+ry)=\psi(t_0,x_0)+r<D\psi(t_0,x_0+\theta ry),y>>0,$$

provided r is sufficiently small, by the continuity of $D\psi$.

(Case 1). We first consider the case when $D\psi(t_0,x_0)=o$. We may assume that there exist $f\in\mathcal{F}$ and $\varphi_1\in C^2((0,\infty))$ such that

$$\psi(t,x)=-f(|x-x_0|)-\varphi_1(t) \tag{4.24}$$

(see [21]).

For $A>0$ and $m\geq 2$, put

$$\psi_{m,A}(t,x)=\psi(t,x)-A\{|t-t_0|^2+|x-x_0|^m\}. \tag{4.25}$$

Then

$$\partial_t\psi_{m,A}(t_0,x_0)=\partial_t\psi(t_0,x_0),\quad D\psi_{m,A}(t_0,x_0)=D\psi(t_0,x_0), \tag{4.26}$$

and

$$\begin{aligned} U^+_{m,A,\varepsilon} &:= \{(t,x)\in(0,\infty)\times\mathbf{R}^N|\psi_{m,A}(t,x)+\varepsilon>u(t,x)\} \\ &\subset U_{(2\varepsilon/A)^{1/m}}((t_0,x_0)) \end{aligned} \tag{4.27}$$

$(\varepsilon\in(0,A))$, and the following holds: for $t\geq 0$,

$$\lim_{x\to x_0} G(D\psi_{N,A}(t,x),D^2\psi_{N,A}(t,x))=NA. \tag{4.28}$$

We argue by contradiction. We consider $\psi_{N,A}$ instead of ψ. When it is not confusing, we omit ${}_{N,A}$ for the sake of simplicity.

Assume that the following holds:

$$\partial_t \psi(t_0, x_0) < 0. \tag{4.29}$$

By reselecting $A > 0$ sufficiently small and $\varepsilon > 0$ sufficiently small compared to A if necessary, we may assume that

$$\partial_t \psi(t,x) + R\Big(\frac{D\psi(t,x)}{|D\psi(t,x)|}\Big) G(D\psi(t,x), D^2\psi(t,x)) + \varepsilon < 0 \quad \text{on } U_\varepsilon^+, \tag{4.30}$$

and that

$$U_\varepsilon^+ = \cup_{t>0}\{t\} \times (\psi(t,\cdot)^{-1}((-\varepsilon,\infty)) \cap D(t)^c). \tag{4.31}$$

We may also assume that $x \mapsto \psi(t,x)$ is strictly concave on U_ε^+ and henceforth $x \mapsto (\psi(s,x), D\psi(s,x)/|D\psi(s,x)|)$ is one-to-one on some neighborhood of $\partial\psi(s,\cdot)^{-1}((-\varepsilon,\infty)) \cap D(s)^c$, provided $\psi(s,\cdot)^{-1}((-\varepsilon,\infty)) \cap D(s)^c \neq \emptyset$.

Indeed, if $\psi(s,\cdot)^{-1}((-\varepsilon,\infty)) \cap D(s)^c \neq \emptyset$, then $-\varepsilon$ is not the maximum of $\psi(s,\cdot)$ on $\psi(s,\cdot)^{-1}((-\varepsilon,\infty)) \cap D(s)^c$ and hence $D\psi(s,\cdot) \neq o$ on some neighborhood of $\partial\psi(s,\cdot)^{-1}((-\varepsilon,\infty)) \cap D(s)^c$.

For $t \geq 0$,

$$\begin{aligned} &\int_{\mathbf{R}^N} (\zeta_k(\eta_m(\psi(t,x)+\varepsilon) - u(t,x)) \\ &\qquad -\zeta_k(\eta_m(\psi(0,x)+\varepsilon) - u(0,x)))dx \\ = \ &\int_{\mathbf{R}^N} dx \int_0^t (-\zeta_k(\eta_m(\psi(s,x)+\varepsilon) - u(s,x)) u(ds,x) \\ &\qquad +\eta_k(\eta_m(\psi(s,x)+\varepsilon) - u(s,x)) \frac{d\eta_m(\psi(s,x)+\varepsilon)}{dr} \partial_s \psi(s,x) ds) \end{aligned} \tag{4.32}$$

(see (3.24)-(3.25) for notation).

Letting $k \to \infty$ in (4.32), by (4.31),

$$\begin{aligned} 0 \ \leq \ &\int_0^t ds \{ \int_{\psi(s,\cdot)^{-1}((-\varepsilon,\infty)) \cap D(s)^c} \eta_m(\psi(s,x)+\varepsilon) \omega_1(u(s,\cdot), dx) \\ &\qquad + \int_{\psi(s,\cdot)^{-1}((-\varepsilon,-\varepsilon+1/m)) \cap D(s)^c} \frac{d\eta_m(\psi(s,x)+\varepsilon)}{dr} \partial_s \psi(s,x) dx \} \end{aligned} \tag{4.33}$$

For s for which $\psi(s,\cdot)^{-1}((-\varepsilon,-\varepsilon+1/m))\cap D(s)^c\neq\emptyset$ and sufficiently large $m\geq 1$, by Lemma 8,

$$\begin{aligned}
&\int_{\psi(s,\cdot)^{-1}((-\varepsilon,-\varepsilon+1/m))\cap D(s)^c}\frac{d\eta_m(\psi(s,x)+\varepsilon)}{dr}\partial_s\psi(s,x)dx \qquad (4.34)\\
<\ & -\int_{-\varepsilon}^{-\varepsilon+1/m}\frac{d\eta_m(r+\varepsilon)}{dr}dr\int_{\{\frac{-D\psi(s,x)}{|D\psi(s,x)|}:x\in\partial\psi(s,\cdot)^{-1}((r,\infty))\cap D(s)^c\}}(R(p)\\
&\quad +\varepsilon\sup\{G(D\psi(s,x),D^2\psi(s,x)):(s,x)\in U_\varepsilon^+\}^{-1})d\mathcal{H}^{N-1}(p)\\
\to\ & -\int_{\cup_{r>-\varepsilon}\{\frac{-D\psi(s,x)}{|D\psi(s,x)|}:x\in\partial\psi(s,\cdot)^{-1}((r,\infty))\cap D(s)^c\}}(R(p)+\varepsilon\sup\{G(D\psi(s,x)\\
&\quad ,D^2\psi(s,x)):(s,x)\in U_\varepsilon^+\}^{-1})d\mathcal{H}^{N-1}(p) \quad (\text{as } m\to\infty).
\end{aligned}$$

(4.33)-(4.34) contradicts to

$$\begin{aligned}
&\{p\in\mathbf{S}^{N-1}:\sigma^+(u,-p,s,x)=1 \text{ for some } x\in\psi(s,\cdot)^{-1}((-\varepsilon,\infty))\cap D(s)^c\}\\
\subset\ & \cup_{r>-\varepsilon}\Big\{-\frac{D\psi(s,x)}{|D\psi(s,x)|}:x\in\partial\psi(s,\cdot)^{-1}((r,\infty))\cap D(s)^c\Big\}
\end{aligned}$$

since

$$\eta_m(\psi(s,x)+\varepsilon)\to 1 \quad \text{if } x\in\psi(s,\cdot)^{-1}((-\varepsilon,\infty)), \text{ as } m\to\infty.$$

(Case 2). Next we consider the case when $\sigma^+(u,D\psi(t_0,x_0),t_0,x_0)=1$. By (ii)-(iv) in Lemma 7, all eigenvalues of $-D(D\psi(t_0,x_0)/|D\psi(t_0,x_0)|)$ are non-negative since the function $x\mapsto\psi(t_0,x)$ takes a maximum $\psi(t_0,x_0)$ on the set $\{x_0+y\in\mathbf{R}^N|<y,D\psi(t_0,x_0)>=0\}$.

For $A>0$, all eigenvalues of $-D(D\psi_{2,A}(t_0,x_0)/|D\psi_{2,A}(t_0,x_0)|)$ as a mapping on the set $\{y\in\mathbf{R}^N|<y,D\psi_{2,A}(t_0,x_0)>=0\}$ are greater than or equal to $2A/|D\psi(t_0,x_0)|$ (see (3.31)-(3.32)) since, in Lemma 7, 1 and $f_1,\cdots,f_{N-1}$ are a eigenvalue and eigenvectors of $(g_1\cdots g_N)$, respectively.

We argue by contradiction. Assume that the following holds:

$$\partial_t\psi(t_0,x_0)+R\Big(\frac{D\psi(t_0,x_0)}{|D\psi(t_0,x_0)|}\Big)G(D\psi(t_0,x_0),D^2\psi(t_0,x_0))<0. \qquad (4.35)$$

We consider $\psi_{2,A}$ instead of ψ. When it is not confusing, we omit ${}_{2,A}$ for the sake of simplicity. By reselecting A, $\varepsilon>0$ if necessary, we may assume that (4.30)-(4.31) hold.

One can also assume, in $U_{(2\varepsilon/A)^{1/2}}((t_0,x_0))$, that $\partial_i\psi(s,x) \neq 0$ and all eigenvalues of $-D(D\psi(s,x)/|D\psi(s,x)|)$ as a mapping on the set $\{y \in \mathbf{R}^N| < y, D\psi(s,x) >= 0\}$ are greater than or equal to $A/|D\psi(t_0,x_0)|$, and $x \mapsto y_i(s,x)$ is one-to-one for some $i \in \{1,\cdots,N\}$ by the inverse function theorem and (v) in Lemma 7, and Lemma 8.

In the same way as in (4.32)-(4.34), we obtain a contradiction.

(Step II). We show that $u^-(t,x) = I_{D(t)^-}(x)$ is a viscosity subsolution of (1.20).

Let $\psi \in \mathcal{A}((0,\infty)\times\mathbf{R}^d : \mathbf{R}^d)$ and assume that $u^- - \psi$ attains a maximum at $(t_0,x_0) \in (0,\infty)\times\mathbf{R}^d$. We may assume as well that $u^-(t_0,x_0) = \psi(t_0,x_0)$, so that $u^-(t,x) < \psi(t,x)$ for all $(t,x) \in (0,\infty)\times\mathbf{R}^d \setminus \{(t_0,x_0)\}$ (see [8]).

Since $t \mapsto u^-(t,x)$ is nonincreasing, $\partial_t\psi(t_0,x_0) \leq 0$.

Hence we only have to consider the case when the following holds: $D\psi(t_0,x_0) \neq o$, and

$$\sigma^-(u^-, D\psi(t_0,x_0), t_0, x_0) = 1, \quad R\Big(\frac{D\psi(t_0,x_0)}{|D\psi(t_0,x_0)|}\Big)G(D\psi(t_0,x_0), D^2\psi(t_0,x_0)) > 0.$$

In particular, $u^-(t_0,x_0) = 1$. By adding to ψ the function $(t,x) \mapsto A\{|t-s|^2 + |x-y|^2\}$, with a sufficiently small $A > 0$, if necessary, we may assume that

$$U_\varepsilon^- := \{(t,x) \in (0,\infty)\times\mathbf{R}^d | \psi(t,x) - \varepsilon < u^-(t,x)\} \quad (\varepsilon > 0) \tag{4.36}$$

is contained in the set $U_{(\varepsilon/A)^{1/2}}((t_0,x_0))$.

We argue by contradiction. Assume that the following holds:

$$\partial_t\psi(t_0,x_0) + R\Big(\frac{D\psi(t_0,x_0)}{|D\psi(t_0,x_0)|}\Big)G(D\psi(t_0,x_0), D^2\psi(t_0,x_0)) > 0. \tag{4.37}$$

By reselecting $\varepsilon > 0$ if necessary, we may assume that

$$\partial_t\psi(t,x) + R\Big(\frac{D\psi(t,x)}{|D\psi(t,x)|}\Big)G(D\psi(t,x), D^2\psi(t,x)) - \varepsilon > 0, \tag{4.38}$$

and $u^-(t,x) = 1$ on U_ε^- by the continuity of ψ.

Put $\tilde{\eta}_m(r) = \eta_m(r + 1/m)$ for $r \in \mathbf{R}$ and $m \geq 1$. In the same way as in (Step I), considering $u^-(t,x) - \tilde{\eta}_m(\psi(t,x) - 1 - \varepsilon)$ instead of $\eta_m(\psi(t,x) + \varepsilon) - u(t,x)$, we obtain a contradiction.

Q. E. D.

(Proof of Corollary 3).
We first show that u is a viscosity supersolution of (1.20) in $(0,\infty) \times \mathbf{R}^N$. Let $\varphi \in \mathcal{A}((0,\infty) \times \mathbf{R}^N)$ and assume that $u - \varphi$ attains a minimum at $(t_0, x_0) \in (0,\infty) \times \mathbf{R}^N$. We may assume that $u(t_0, x_0) = \varphi(t_0, x_0)$, so that $u(t,x) > \varphi(t,x)$ for all $(t,x) \in (0,\infty) \times \mathbf{R}^N \setminus \{(t_0, x_0)\}$ (see [8]). By subtracting a constant, we may assume that $\varphi \leq u < 0$.

Put $r_0 := \varphi(t_0, x_0)$ and

$$u_r(t,x) := I_{u^{-1}(t,\cdot)((r,0))}(x) \quad (r < 0). \tag{4.39}$$

Then

$$u_{r_0}(t,x) \geq \frac{\varphi(t,x)}{|r_0|} + 1 \qquad \text{for all } (t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{4.40}$$

where the equality holds if and only if $(t,x) = (t_0, x_0)$.

Since u_r is a viscosity supersolution of (1.20) in $(0,\infty) \times \mathbf{R}^N$ by Corollary 2 and Theorem 3, and since

$$\sigma^+(u_{r_0}, D(\varphi(t_0, x_0)/|r_0| + 1), t_0, x_0) = \sigma^+(u, D\varphi(t_0, x_0), t_0, x_0),$$

(1.25) holds.

Next we show that u is a viscosity subsolution of (1.20) in $(0,\infty) \times \mathbf{R}^N$. Let $\varphi \in \mathcal{A}((0,\infty) \times \mathbf{R}^d)$ and assume that $u - \varphi$ attains a maximum at $(t_1, x_1) \in (0,\infty) \times \mathbf{R}^d$. We may assume as well that $u(t_1, x_1) = \varphi(t_1, x_1)$, so that $u(t,x) < \varphi(t,x)$ for all $(t,x) \in (0,\infty) \times \mathbf{R}^d \setminus \{(t_1, x_1)\}$ (see [8]).

By adding a constant, we may assume that $\varphi \geq u > 0$.

Put $r_1 := \varphi(t_1, x_1)$ and

$$u_r^-(t,x) := I_{u^{-1}(t,\cdot)([r,\infty))}(x).$$

Then

$$u_{r_1}^-(t,x) \leq \frac{\varphi(t,x)}{r_1} \qquad \text{for all } (t,x) \in (0,\infty) \times \mathbf{R}^N,$$

where the equality holds if and only if $(t, x) = (t_1, x_1)$.

Since u_r^- is a viscosity subsolution of (1.20) in $(0, \infty) \times \mathbf{R}^N$ by Corollary 2 and Theorem 3, and since

$$\sigma^-(u_{r_1}^-, D(\varphi(t_1, x_1)/r_1), t_1, x_1) = \sigma^-(u, D\varphi(t_1, x_1), t_1, x_1),$$

(1.27) holds.

Q. E. D.

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